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## Wigner–Souriau translations and Lorentz symmetry of chiral fermions

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## ABSTRACT

Chiral fermions can be embedded into Souriau's massless spinning particle model by “enslaving” the spin, viewed as a gauge constraint. The latter is not invariant under Lorentz boosts; spin enslavement can be restored, however, by a Wigner–Souriau (WS) translation, analogous to a compensating gauge transformation. The combined transformation is precisely the recently uncovered twisted boost, which we now extend to finite transformations. WS-translations are identified with the stability group of a motion acting on the right on the Poincaré group, whereas the natural Poincaré action corresponds to action on the left.

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## 1. Introduction

Semiclassical chiral fermions of spin  $\frac{1}{2}$  can be described by the phase-space action

$$S = \int \left( (\mathbf{p} + e\mathbf{A}) \cdot \frac{d\mathbf{x}}{dt} - h - \mathbf{a} \cdot \frac{d\mathbf{p}}{dt} \right) dt, \quad h = |\mathbf{p}| + e\phi(\mathbf{x}), \quad (1.1)$$

where  $\mathbf{a}(\mathbf{p})$  is a vector potential for the “Berry monopole” in  $\mathbf{p}$ -space,  $\nabla_{\mathbf{p}} \times \mathbf{a} = \frac{\hat{\mathbf{p}}}{2|\mathbf{p}|^2}$ ,  $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$ ,  $\mathbf{p} \neq 0$ . Here  $\mathbf{A}(\mathbf{x})$  and  $\phi(\mathbf{x})$  are [static] vector and scalar potentials and  $e$  is the electric charge [1–6]. A distinctive feature is that spin is “enslaved” to the momentum, i.e.,

$$\mathbf{s} = \frac{1}{2} \hat{\mathbf{p}}. \quad (1.2)$$

An intriguing aspect of the model is its *lack of manifest Lorentz symmetry*. Recently [5], it was shown, though, that modifying the dispersion relation in (1.1) as  $h = |\mathbf{p}| + e\phi(\mathbf{x}) + \frac{\hat{\mathbf{p}} \cdot \mathbf{B}}{2|\mathbf{p}|}$  yields a theory which is covariant w.r.t. Lorentz transformations. Turning off the external field, their expression # (6) reduces to

$$\delta \mathbf{x} = \boldsymbol{\beta} t + \boldsymbol{\beta} \times \frac{\hat{\mathbf{p}}}{2|\mathbf{p}|}, \quad \delta \mathbf{p} = |\mathbf{p}| \boldsymbol{\beta}, \quad \delta t = \boldsymbol{\beta} \cdot \mathbf{x}, \quad (1.3)$$

where  $\boldsymbol{\beta}$  is an infinitesimal Lorentz boost. This formula has also been found, independently [6], by relating the chiral fermion to Souriau's model of a relativistic massless spinning particle [6,7].

The chiral and the Souriau systems [7] have seemingly different degrees of freedom: the first one has “enslaved” spin, while the latter has “unchained” spin represented by a vector  $\mathbf{s}$ , which may not satisfy (1.2). However, the freedom specific to the massless case of applying what we call a *Wigner–Souriau (WS) translation*, Eq. (2.7) below,<sup>1</sup> allows one to eliminate the additional degrees of freedom of the Souriau framework, so that the two systems have identical spaces of motions [6,7]. This enabled us to “export” the natural Lorentz symmetry to the chiral system, yielding (1.3).

In this Note we derive (1.3) in another, rather more intuitive way, namely by *embedding* the chiral fermion model directly into the Souriau model, namely by viewing spin enslavement, (1.2), as a *gauge condition* with WS translations and viewed as gauge transformations. Then our clue is that a natural Lorentz boost does not leave the constraint (1.2) invariant; the “gauge” condition (1.2) can, however, be restored by a WS translation, yielding, once again, (1.3). This is reminiscent of what happens in *gauge theories*, where a space–time transformation can be a symmetry when the variation of the vector potential can be compensated by a suitable gauge transformation [12].

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<sup>1</sup> Wigner–Souriau translations (called “Z-shifts” in [6]) were known to Wigner [8] and to Penrose [9]. Their use in the semiclassical framework was advocated by Souriau [7]. They were overlooked by most present authors with the notable exception of Stone et al. [4], who also suggested viewing them as gauge transformations.

Further insight is gained in Section 5 which clarifies the geometry hidden behind: while the natural Poincaré action corresponds to the left-action of the Poincaré group on itself, WS translations correspond to the right-action of the stability group of a motion.

We conclude our Letter by some remarks about non-commutativity.

## 2. Symplectic description of the chiral and the massless spinning models

Both the chiral and the Souriau models can conveniently be described within Souriau's framework [6,7], where the classical motions are identified with curves or surfaces in some evolution space, and are tangent to the kernel of a closed two-form; see [6] for details. We limit our considerations to the free case; coupling to external fields has been discussed in the literature, see [6,13].

We first consider the [free] chiral model (1.1). It has been shown [6] that the associated variational problem admits an alternative geometric formulation. To that end, we introduce the seven-dimensional evolution space  $V^7 = T(\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}$  described by triples  $(\mathbf{x}, \mathbf{p}, t)$ , and endow it with the two-form  $\sigma$  defined by

$$\sigma = \omega - dh \wedge dt \quad \text{where } \omega = dp_i \wedge dx^i - \frac{1}{4|\mathbf{p}|^3} \epsilon^{ijk} p_i dp_j \wedge dp_k, \\ h = |\mathbf{p}|. \quad (2.1)$$

The two-forms  $\omega$  and  $\sigma$  are closed, since  $\nabla_{\mathbf{p}} \cdot \mathbf{b} = 0$ . The kernel of  $\sigma$  defines an integrable distribution, whose leaves [integral manifolds] can be viewed as generalized solutions of the variational problem. Here, the kernel is one-dimensional and a curve  $(\mathbf{x}(t), \mathbf{p}(t), t)$  is tangent to it iff the equations of motion,

$$\frac{d\mathbf{x}}{dt} = \widehat{\mathbf{p}}, \quad \frac{d\mathbf{p}}{dt} = 0, \quad (2.2)$$

are satisfied [6]; the solution is plainly  $\mathbf{p} = \mathbf{p}_0 = \text{const}$ ,  $\mathbf{x}(t) = \mathbf{x}_0 + \widehat{\mathbf{p}}t$ ,  $\mathbf{x}_0 = \text{const}$ , i.e., the motion is in the  $\widehat{\mathbf{p}}$  direction with the velocity of light.

The Souriau model admits a similar description [6]. Restricting ourselves again to the free case, the evolution space here is nine-dimensional and is described by

$$V^9 = \left\{ R, P \in \mathbb{R}^{3,1}, S \in \mathfrak{o}(3,1) \mid P_\mu P^\mu = 0, P^4 > 0, S_{\mu\nu} P^\nu = 0, \right. \\ \left. S_{\mu\nu} S^{\mu\nu} = \frac{1}{2} \right\}. \quad (2.3)$$

The evolution space is endowed with the closed two-form

$$\sigma = -dP_\mu \wedge dR^\mu - 2dS^\mu_\lambda \wedge S^\lambda_\rho dS^\rho_\mu. \quad (2.4)$$

A (3+1)-decomposition can be introduced by writing, in a Lorentz frame,  $R = (\mathbf{r}, t)$ , and  $P = (\mathbf{p}, |\mathbf{p}|)$  where  $\mathbf{p} \neq 0$ . The components  $S_{\mu\nu}$  of the spin tensor can in turn be split into space and space-time components,

$$S_{ij} = \epsilon_{ijk} s^k, \quad S_{j4} = \Sigma_j \quad \text{where } \Sigma = \widehat{\mathbf{p}} \times \mathbf{s}, \quad (2.5)$$

the 3-vector  $\mathbf{s}$  being interpreted as the spin in the chosen Lorentz frame. Note that  $\Sigma$  is not an independent variable, so that a point of  $V^9$  can be labeled by  $(\mathbf{r}, t, \mathbf{p}, \mathbf{s})$ . Then the [free] equations of motion associated with the kernel of  $\sigma$  in (2.4) are expressed as,<sup>2</sup>

$$-\mathbf{p} \cdot \dot{\mathbf{r}} + |\mathbf{p}| \dot{t} = 0, \quad \dot{\mathbf{p}} = 0, \quad \dot{\mathbf{s}} = \mathbf{p} \times \dot{\mathbf{r}}, \quad (2.6)$$

and can be deduced from Eq. (3.6) in [6]. A particular solution of (2.6) is obtained by *embedding the chiral solution above into the spin-extended evolution space*  $V^9$  by identifying  $\mathbf{r}$  with  $\mathbf{x}$  in (2.2) and completing [trivially] with a constant spin vector,  $\mathbf{p} = \mathbf{p}_0 = \text{const}$ ,  $\mathbf{r}(t) = \mathbf{r}_0 + \widehat{\mathbf{p}}t$ ,  $\mathbf{r}_0 = \text{const}$ ,  $\mathbf{s}(t) = \mathbf{s}_0 = \text{const}$ . A remarkable feature of Eqs. (2.6) is that, for an arbitrary 3-vector  $\mathbf{W}$ , the transformation we refer to as a *Wigner–Souriau (WS) translation* [4,6–9],

$$\mathbf{r} \rightarrow \mathbf{r} + \mathbf{W}, \quad t \rightarrow t + \widehat{\mathbf{p}} \cdot \mathbf{W}, \quad \mathbf{p} \rightarrow \mathbf{p}, \quad \mathbf{s} \rightarrow \mathbf{s} + \mathbf{p} \times \mathbf{W} \quad (2.7)$$

takes a solution of (2.6) into another, equivalent one. The kernel of (2.4) is invariant under WS-translations and is in fact three-dimensional, swept by the images of the embedded solutions.

The spin vector here,  $\mathbf{s}$ , is not necessarily “enslaved”, i.e., may not be parallel to the momentum,  $\mathbf{p}$ . The spin constraint in (2.3) implies nevertheless that the projection of the spin onto the momentum and the perpendicular component  $\Sigma$  satisfy,

$$\mathbf{s} \cdot \widehat{\mathbf{p}} = \frac{1}{2}, \quad \Sigma = \widehat{\mathbf{p}} \times \mathbf{s}, \quad (2.8)$$

respectively, see (2.5). From the WS action above we infer that  $\Sigma \rightarrow \Sigma + \widehat{\mathbf{p}} \times (\mathbf{p} \times \mathbf{W})$ . It follows that  $\Sigma$  can be *eliminated*: choosing  $\mathbf{W} = \Sigma/|\mathbf{p}|$  carries  $\Sigma$  to zero.

The Poincaré group acts naturally on  $V^9$ , namely according to [6,7]

$$\begin{cases} \delta \mathbf{r} = \boldsymbol{\omega} \times \mathbf{r} + \boldsymbol{\beta} t + \boldsymbol{\gamma}, \\ \delta t = \boldsymbol{\beta} \cdot \mathbf{r} + \varepsilon, \\ \delta \mathbf{p} = \boldsymbol{\omega} \times \mathbf{p} + \boldsymbol{\beta} |\mathbf{p}|, \\ \delta \mathbf{s} = \boldsymbol{\omega} \times \mathbf{s} - \boldsymbol{\beta} \times \Sigma, \end{cases} \quad (2.9)$$

where  $\boldsymbol{\omega}$ ,  $\boldsymbol{\beta}$ ,  $\boldsymbol{\gamma}$ ,  $\varepsilon$  are identified with infinitesimal rotations, boosts, translations and time-translations; their action duly projects to Minkowski space-time as the natural one. In what follows, we focus our attention at boosts; WS-translations will be studied further in Section 5.

## 3. Embedding the chiral system into the massless spinning model

Now we embed the evolution space of the chiral model,  $V^7$ , into that of the massless spinning particle,  $V^9$ . We note first that, by (2.8), spin enslavement, (1.2), is equivalent to

$$\Sigma = \widehat{\mathbf{p}} \times \mathbf{s} = 0, \quad (3.1)$$

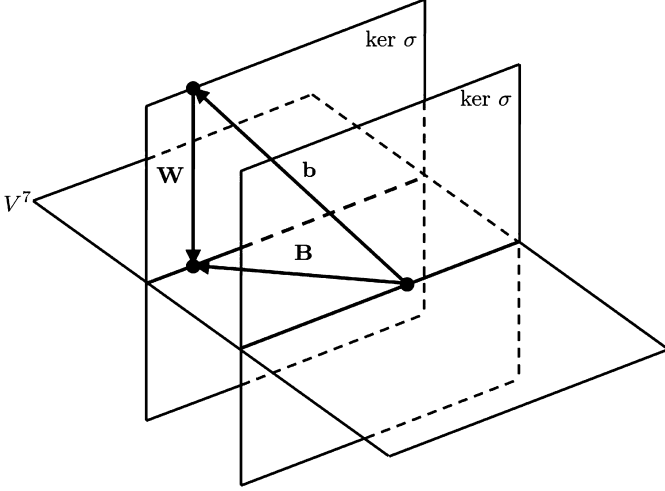
which, viewed as a constraint, defines a seven-dimensional submanifold of  $V^9$  that we parametrize with  $\mathbf{r}, \mathbf{p}, t$  and denote (with a suggestive abuse of notation) still  $V^7$ . Eqs. (2.6) and (2.8) imply that  $\Sigma = |\mathbf{p}|(\widehat{\mathbf{p}}\dot{t} - \dot{\mathbf{r}})$ . Requiring  $\Sigma = 0$  is therefore consistent with the dynamics: the motions of the chiral system lie in the intersection of the three-dimensional characteristic leaves of  $V^9$  with the surface  $V^7$  defined by spin enslaving; they remain therefore motions also for the extended dynamics.<sup>3</sup> A chiral motion is embedded into  $V^9$  by respecting the gauge condition (1.2), namely as  $\gamma(t) = (\mathbf{r}(t) = \mathbf{x}(t), \mathbf{p}(t), \mathbf{s}(t) = \frac{1}{2}\widehat{\mathbf{p}}(t))$ .

## 4. Lorentz boost actions

We consider now an infinitesimal Lorentz boost,  $\boldsymbol{\beta}$ . For  $\mathbf{s} = \frac{1}{2}\widehat{\mathbf{p}}$  we have  $\delta_{\boldsymbol{\beta}}(\mathbf{p} \times \mathbf{s}) = \frac{1}{2}\boldsymbol{\beta} \times \mathbf{p}$ , which does *not* vanish in general: *spin*

<sup>2</sup> These equations can also be obtained in a Wess–Zumino framework [10], namely as the variational equations of the Lagrangian # (3.1) of [10]; the latter is in fact the one-form (B.1) in [6], whose exterior derivative is the two-form  $d(\mu_0 \cdot g^{-1}dg)$ .

<sup>3</sup> Alternatively, the restriction of the free two-form (2.4) of  $V^9$  to  $V^7$  is (2.1).



**Fig. 1.** Enslaving the spin amounts to embedding the chiral evolution space  $V^7$  into that,  $V^9$ , of the massless spinning particle. A chiral motion thus becomes the intersection of  $V^7$  with the characteristic leaf of  $\ker \sigma$  (depicted as a vertical plane). Enslavement is not invariant under a (natural) boost  $\mathbf{b}$ , but a suitable compensating Wigner–Souriau translation  $\mathbf{W}$  allows us to re-enslave the spin. The combination of the two transformations yields the twisted boost  $\mathbf{B}$  in (1.3).

and momentum do not remain parallel even if they were so initially: embedding the chiral system into the Souriau model through spin enslavement is not boost-invariant: natural boost symmetry is broken by spin enslavement. However, as dictated by the analogy with gauge symmetries [12], let us apply an infinitesimal WS-translation

$$\delta_{\mathbf{W}} \mathbf{r} = \mathbf{W}, \quad \delta_{\mathbf{W}} t = \hat{\mathbf{p}} \cdot \mathbf{W}, \quad \delta_{\mathbf{W}} \mathbf{p} = 0, \quad \delta_{\mathbf{W}} \mathbf{s} = \mathbf{p} \times \mathbf{W} \quad \text{with } \mathbf{W} = \frac{\boldsymbol{\beta} \times \hat{\mathbf{p}}}{2|\mathbf{p}|}. \quad (4.1)$$

Then  $\delta_{\mathbf{W}} \mathbf{s} = \frac{1}{2}(\boldsymbol{\beta} - \hat{\mathbf{p}}(\hat{\mathbf{p}} \cdot \boldsymbol{\beta}))$ , implying that the combined transformation  $\delta = \delta_{\boldsymbol{\beta}} + \delta_{\mathbf{W}}$  does preserve spin enslavement,  $\delta(\mathbf{p} \times \mathbf{s}) = 0$ . The transformation of  $V^7$  generated by  $\delta$  is precisely the twisted boost (1.3).<sup>4</sup>

So far we studied infinitesimal actions only. But our strategy is valid also for finite transformations, as we show it now. Firstly, from Appendix C of [6] we infer the action of a finite boost  $\mathbf{b}$  on the evolution space  $V^9$ , namely

$$\begin{cases} \mathbf{r}' = \mathbf{r} + (\gamma - 1)(\hat{\mathbf{b}} \cdot \mathbf{r})\hat{\mathbf{b}} + \gamma t\mathbf{b}, \\ t' = \gamma(\mathbf{b} \cdot \mathbf{r} + t), \\ \mathbf{p}' = \mathbf{p} + \gamma|\mathbf{p}|\mathbf{b} + (\gamma - 1)(\hat{\mathbf{b}} \cdot \mathbf{p})\hat{\mathbf{b}}, \\ \mathbf{s}' = \mathbf{s} + (\gamma - 1)\hat{\mathbf{b}} \times (\mathbf{s} \times \hat{\mathbf{b}}) - \gamma \mathbf{b} \times \boldsymbol{\Sigma}, \end{cases} \quad (4.2)$$

which is consistent with the infinitesimal action (2.9). Then, starting with enslaved spin,  $\mathbf{s} = \frac{1}{2}\hat{\mathbf{p}}$ , we find,

$$\boldsymbol{\Sigma}' = \frac{\mathbf{p}' \times \mathbf{s}'}{|\mathbf{p}'|} = \frac{1}{2}(\gamma^2|\mathbf{b}| + (\gamma^2 - 1)\hat{\mathbf{b}} \cdot \hat{\mathbf{p}}) \frac{\hat{\mathbf{b}} \times \mathbf{p}}{|\mathbf{p}'|} \quad (4.3)$$

which does not vanish in general: spin is unchained. At last, the finite WS-translation (2.7) with  $\mathbf{W} = \boldsymbol{\Sigma}'/|\mathbf{p}'|$  restores the validity of (1.2): spin is re-enslaved,  $\mathbf{s}'' = \frac{1}{2}\hat{\mathbf{p}}'$ . The combined transformation for finite boosts,

$$\mathbf{r}'' = \mathbf{r} + \gamma t\mathbf{b} + (\gamma - 1)(\hat{\mathbf{b}} \cdot \mathbf{r})\hat{\mathbf{b}} + \frac{\boldsymbol{\Sigma}'}{|\mathbf{p}'|}, \quad (4.4)$$

completed with  $t'' = t'$  and  $\mathbf{p}'' = \mathbf{p}'$  where  $t'$ ,  $\mathbf{p}'$  and  $\boldsymbol{\Sigma}'$  are given in (4.2) and (4.3), respectively. The infinitesimal action is (1.3) as it should be. In conclusion, Fig. 1 is valid also for finite boosts.

We now turn to the space of motions [6,7], defined as the quotient of the evolution space by the characteristic foliation of  $\sigma$ ; we denote it by  $M$ . The equations of motion of the spin-extended system, (2.6), imply that

$$\tilde{\mathbf{x}}(t) = \mathbf{r}(t) - \hat{\mathbf{p}}t + \frac{\boldsymbol{\Sigma}}{|\mathbf{p}|} \quad (4.5)$$

is in fact a constant of the motion,  $d\tilde{\mathbf{x}}/dt = 0$ . It can be used therefore to label the motion, i.e., a characteristic leaf in  $V^9$ . The conserved momentum,  $\mathbf{p}$ , is another good coordinate set on  $M$ , which is six-dimensional, and whose points can therefore be labeled by  $\tilde{\mathbf{x}}$  and  $\mathbf{p} \neq 0$ .

In [6], we derived the twisted boost (1.3) from the Poincaré action on the space of motions. Conversely, the latter can be obtained from our construction here. Boosting in  $V^9$  according to (2.9) we find that the preceding terms combine to yield

$$\delta_{\boldsymbol{\beta}} \tilde{\mathbf{x}} = \frac{1}{2}\boldsymbol{\beta} \times \frac{\hat{\mathbf{p}}}{|\mathbf{p}|} - \hat{\mathbf{p}}(\boldsymbol{\beta} \cdot \tilde{\mathbf{x}}). \quad (4.6)$$

Now WS-translations act in turn trivially on the space of motions,  $\delta_{\mathbf{W}} \tilde{\mathbf{x}} = \delta_{\mathbf{W}} \mathbf{p} = 0$ , since they move each characteristic leaf within itself. Completing (4.6) with  $\delta_{\boldsymbol{\beta}} \mathbf{p} = |\mathbf{p}|\boldsymbol{\beta}$  cf. (2.9), we end up with the (boost-)action on the space of motions, – Eq. (3.19) in [6]. At last, applying (4.2) to each term in (4.5) allows us to confirm the finite action on the space of motions, Eq. # (3.24) in [6].

## 5. The geometry of Wigner–Souriau transformations

Do WS-translations belong to the Poincaré group? To answer this question, let us first briefly summarize Souriau’s construction for “elementary systems” [meaning that the group acts symplectically and transitively] [7].

The connected component of the Poincaré group,  $G$ , can be identified with the group of matrices  $g = \begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix}$  where  $L$  is a Lorentz transformation,  $L \in H$  (the connected Lorentz group), and  $C$  a Minkowski-space vector. The Poincaré Lie algebra,  $\mathfrak{g}$ , is hence spanned by the matrices  $X = \begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix}$  where the infinitesimal Lorentz transformations and translations,  $\Lambda \in \mathfrak{so}(3, 1)$ , and  $\Gamma \in \mathbb{R}^{3,1}$ , respectively, are such that, in a given Lorentz frame,  $\Lambda = \begin{pmatrix} j(\boldsymbol{\omega}) & \boldsymbol{\beta} \\ \boldsymbol{\beta}^T & 0 \end{pmatrix}$  and  $\Gamma = \begin{pmatrix} \boldsymbol{\gamma} \\ \varepsilon \end{pmatrix}$  with  $\boldsymbol{\omega}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^3$  interpreted as an infinitesimal rotation, boost, and space-translation, while  $\varepsilon \in \mathbb{R}$  stands for an infinitesimal time-translation. Here  $j(\boldsymbol{\omega})$  is the matrix of cross product in 3-space,  $j(\boldsymbol{\omega})\mathbf{x} = \boldsymbol{\omega} \times \mathbf{x}$ .

The Poincaré group acts on itself on the left,  $g \rightarrow hg$ , for  $h \in G$ , and the quotient of  $G$  by the subgroup of Lorentz transformations,  $G/H$ , can be identified with Minkowski space-time. This left-action of  $G$  on itself projects to the natural action, infinitesimally the two upper lines in (2.9), of the Poincaré group on space-time.

The Poincaré group acts on the dual Lie algebra by the coadjoint representation, defined by  $\text{Coad}_g \mu \cdot X = \mu \cdot g^{-1}Xg$ , for all  $X \in \mathfrak{g}$ . We denote here by  $\mu = (M, P) \in \mathfrak{g}^*$  a “moment” of the Poincaré group, and by  $\mu \cdot X = \frac{1}{2}M_{\mu\nu}\Lambda^{\mu\nu} - P_{\mu}\Gamma^{\mu}$  its contraction with  $X \in \mathfrak{g}$ , see [7].

Contracting the Maurer–Cartan one-form,  $g^{-1}dg$ , with an arbitrary fixed element  $\mu_0$  of the dual of Lie algebra, yields a real one-form on the group  $G$ ; we denote by  $\sigma = d(\mu_0 \cdot g^{-1}dg)$  its exterior derivative. As a general fact, the characteristic leaves of the two-form  $\sigma$ , defined by its kernel, are identified with the classical motions [11] associated with  $\mu_0$ . Indeed, the space of all motions,

<sup>4</sup> The order is irrelevant, since WS-shifts and boosts commute.

$M$ , of an elementary system for the group  $G$  is interpreted by Souriau [7] as the orbit of some basepoint  $\mu_0$  under the coadjoint action,  $M = \text{Coad}_G \mu_0 \approx G/G_0$ , where  $G_0$  is the stability subgroup of  $\mu_0$ . The coadjoint orbit,  $M$ , is hence canonically endowed with the symplectic structure defined by the Kostant–Kirillov–Souriau two-form  $\omega$ , the image of  $\sigma$  under the projection  $G \rightarrow M$ .

In our case and following [7], the basepoint can readily be chosen as  $\mu_0 = (M_0, P_0)$  with  $M_0 = s \begin{pmatrix} j(\hat{\mathbf{p}}_0) & 0 \\ 0 & 0 \end{pmatrix}$  and  $P_0 = |\mathbf{p}_0| \begin{pmatrix} \hat{\mathbf{p}}_0 \\ 1 \end{pmatrix}$  with  $\hat{\mathbf{p}}_0 = \mathbf{e}_3$ ; here  $s = \frac{1}{2}$  is the scalar spin and the positive constant  $|\mathbf{p}_0|$  is the energy. Then a straightforward calculation shows that the Poincaré Lie algebra element  $(\Lambda, \Gamma)$  with parameters  $\boldsymbol{\omega}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^3, \varepsilon \in \mathbb{R}$  leaves the chosen basepoint  $\mu_0$  invariant whenever

$$\boldsymbol{\omega} = \boldsymbol{\beta} \times \hat{\mathbf{p}}_0 + \lambda \hat{\mathbf{p}}_0, \quad \boldsymbol{\gamma} = -\frac{1}{2} \frac{\boldsymbol{\beta} \times \hat{\mathbf{p}}_0}{|\mathbf{p}_0|} + \varepsilon \hat{\mathbf{p}}_0. \quad (5.1)$$

The stability Lie algebra,  $\mathfrak{g}_0$ , is therefore four-dimensional, parametrized by  $(\boldsymbol{\beta}, \varepsilon, \lambda)$ , where  $\boldsymbol{\beta} \perp \hat{\mathbf{p}}_0$ ,  $\varepsilon \in \mathbb{R}$ , and  $\lambda \in \mathbb{R}$  represents an infinitesimal rotation around  $\hat{\mathbf{p}}_0$ . We note that the evolution space  $V^9$  is in fact the quotient of the Poincaré group by rotations around  $\hat{\mathbf{p}}_0$ , and (2.9) above is in fact the projection of the infinitesimal left-action of the group  $G$  to  $V^9 \approx G/\text{SO}(2)$ , whereas the two-form  $d(\mu_0 \cdot g^{-1}dg)$  projects as (2.4), as anticipated by the notation.

Remember now that the Poincaré group also acts on itself from the right,  $g \rightarrow gh^{-1}$ , for  $h \in G$ ; its infinitesimal right-action is therefore given by matrix multiplication,  $\delta_X g = -gX$ , where  $X = \delta h$  at  $h = 1$ . Choosing in particular  $X \in \mathfrak{g}_0$ , the action on the group reads  $\delta_X \begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -L\Lambda & -L\Gamma \\ 0 & 0 \end{pmatrix}$  and  $\Gamma = \begin{pmatrix} \boldsymbol{\gamma} \\ \varepsilon \end{pmatrix}$ , where  $\boldsymbol{\gamma}$  and  $\boldsymbol{\omega}$  are as in (5.1). This 4-parameter vector field belongs, at each point  $g \in G$ , to the kernel of the two-form  $\sigma$ ; in fact, it generates its kernel [7]. Renaming the Poincaré translation  $C$  as  $R = (\mathbf{r}, t)$ , a space-time event, shows that the right-action on  $G$  of a vector  $X$  from the stability algebra  $\mathfrak{g}_0$  yields,

$$\delta_X \mathbf{r} = \frac{1}{2} \frac{\boldsymbol{\beta} \times \hat{\mathbf{p}}}{|\mathbf{p}|} - \varepsilon \hat{\mathbf{p}} \quad \text{and} \quad \delta_X t = -\varepsilon. \quad (5.2)$$

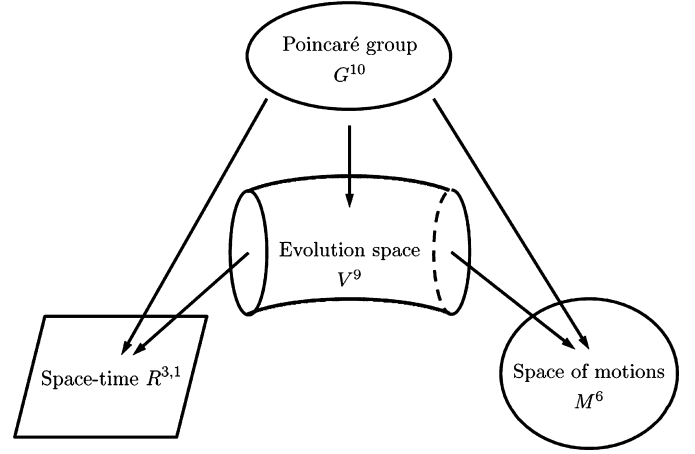
This transformation satisfies the condition  $\hat{\mathbf{p}} \cdot \delta_X \mathbf{r} = \delta_X t$  required for an infinitesimal WS-translation; conversely, any WS-translation  $\mathbf{W}$  is of the form (5.2), with  $\boldsymbol{\beta}$  perpendicular to  $\mathbf{p}$  and  $\lambda$  arbitrary. Therefore (with a slight abuse) we will refer to  $\mathfrak{g}_0$  acting from the right as a WS-translations.

Comparing now (5.2) with (4.1) allows us to conclude that while a boost  $\boldsymbol{\beta}$  acting on the left unchains the spin, the latter is re-enslaved by a WS-translation with the same boost,  $\boldsymbol{\beta}$ , acting from the right.

It is readily verified that the right-action of an  $X$  from the stability algebra  $\mathfrak{g}_0$  acts as  $\delta_X \mathbf{p} = 0$  and  $\delta_X \mathbf{s} = \frac{1}{2} \hat{\mathbf{p}} \times (\hat{\mathbf{p}} \times \boldsymbol{\beta})$ , consistently with the infinitesimal action. Thus, after rotations around  $\hat{\mathbf{p}}_0$  are factored out, the right-action of the stability subalgebra  $\mathfrak{g}_0$  projects to  $V^9$  as the WS translations. In conclusion, Wigner–Souriau translations are Poincaré transformations, – but which act on the right and not on the left as natural ones do. Projecting further down to the space of motions,  $M$ , they act trivially. The various spaces are shown, symbolically, in Fig. 2.

## 6. Relation to non-commutativity

The spaces of motions of both the chiral and the Souriau systems carry the symplectic structure  $\omega$  in (2.1) [with  $\tilde{\mathbf{x}}$  replacing  $\mathbf{x}$ ] [6]. Translating into Poisson bracket language, we have  $\{\xi^\alpha, \xi^\beta\} = \omega^{\alpha\beta}(\xi)$  where  $(\omega^{\alpha\beta}) = -(\omega_{\alpha\beta})^{-1}$  are the coefficients of the inverse of the symplectic form  $\omega = \frac{1}{2} \omega_{\alpha\beta}(\xi) d\xi^\alpha \wedge d\xi^\beta$ . Therefore the components of the space of motions coordinates  $\tilde{\mathbf{x}}$  do not Poisson-commute,



**Fig. 2.** The evolution space,  $V^9$ , is the quotient of the Poincaré group,  $G^{10}$  by rotations around  $\hat{\mathbf{p}}_0$ . The space of motions is identified with a coadjoint orbit  $M^6 \approx G^{10}/G_0$  of  $G^{10}$ , where  $G_0$ , the stability group of the basepoint  $\mu_0$ . The left-action of  $G^{10}$  on itself projects to the natural Poincaré action on both the space of motions and Minkowski space-time, consistently with projecting first to  $V^9$  and then to  $M^6$  and  $\mathbb{R}^{3,1}$ , respectively.

$$\{\tilde{x}^i, \tilde{x}^j\} = \frac{1}{2} \epsilon^{ijk} \frac{p_k}{|\mathbf{p}|^3}. \quad (6.1)$$

Putting  $t = 0$  and  $\boldsymbol{\Sigma} = 0$  into (4.5) shows that the chiral coordinate  $\mathbf{x}$  satisfies the same commutation relation. Non-commutativity is thus a consequence of spin, cf. [15–18]; chiral fermions provide just another example of non-commutative mechanics.

It is worth stressing that the non-commutativity is *unrelated* to WS translations. Non-commutativity arises in fact due to the twisted symplectic structure of the *space of motions* (modeled by the phase space); but the WS translations act on the evolution space *before projecting, trivially*, to the space of motions.

In the operator context a Bopp-shift transforms the commutation relations to canonical ones [19]. In the (semi)classical context we are interested in it amounts to introducing new, canonical coordinates, eliminating spatial non-commutativity altogether. Now Darboux's theorem [20] guarantees that this is always possible locally. In detail, we redefine the coordinates as

$$\tilde{\mathbf{X}} = \tilde{\mathbf{x}} + \mathbf{a}(\mathbf{p}), \quad \tilde{\mathbf{P}} = \mathbf{p}, \quad (6.2)$$

where  $\mathbf{a}(\mathbf{p})$  is a “Berry” vector potential. Then, in terms of  $\tilde{\mathbf{P}}$  and  $\tilde{\mathbf{X}}$ , the symplectic form  $\omega$  in (2.1) has the canonical expression  $\omega = d\tilde{P}_i \wedge d\tilde{X}^i$ , so that  $\{\tilde{X}^i, \tilde{X}^j\} = \{\tilde{P}_i, \tilde{P}_j\} = 0$  and  $\{\tilde{P}_i, \tilde{X}^j\} = \delta_i^j$ . Spin seems thus eliminated. This is *not* possible globally, though, since any  $\mathbf{a}$  is singular along a Dirac string, as it is well-known.

The variation of the Berry vector potential under a boost  $\boldsymbol{\beta} \in \mathbb{R}^3$  is then found to be  $\delta_{\boldsymbol{\beta}} \mathbf{a} = -\boldsymbol{\beta} \times \frac{\hat{\mathbf{p}}}{2|\mathbf{p}|} - (\mathbf{a} \cdot \boldsymbol{\beta}) \hat{\mathbf{p}} + \nabla_{\mathbf{p}}(|\mathbf{p}| \boldsymbol{\beta} \cdot \mathbf{a})$ . It follows that, in terms of the Darboux coordinates (6.2), the infinitesimal action of boosts is given by

$$\delta_{\boldsymbol{\beta}} \tilde{\mathbf{X}} = -(\boldsymbol{\beta} \cdot \tilde{\mathbf{X}}) \frac{\tilde{\mathbf{P}}}{|\tilde{\mathbf{P}}|} + \nabla_{\tilde{\mathbf{P}}}(|\tilde{\mathbf{P}}| \boldsymbol{\beta} \cdot \mathbf{a}), \quad \delta_{\boldsymbol{\beta}} \tilde{\mathbf{P}} = \boldsymbol{\beta} |\tilde{\mathbf{P}}|. \quad (6.3)$$

Redefining analogously the chiral coordinate as  $\mathbf{X} = \mathbf{x} + \mathbf{a}$  and  $\mathbf{P} = \mathbf{p}$  allows us to infer

$$\begin{aligned} \delta_{\boldsymbol{\beta}} \mathbf{X} &= \boldsymbol{\beta} t - (\boldsymbol{\beta} \cdot \mathbf{a}) \hat{\mathbf{P}} + \nabla_{\mathbf{P}}(|\mathbf{P}| \boldsymbol{\beta} \cdot \mathbf{a}) = \boldsymbol{\beta} t + |\mathbf{P}| \nabla_{\mathbf{P}}(\boldsymbol{\beta} \cdot \mathbf{a}), \\ \delta_{\boldsymbol{\beta}} \mathbf{P} &= \boldsymbol{\beta} |\mathbf{P}|, \end{aligned} \quad (6.4)$$

which is reminiscent of but still different from the natural action (2.9).



## 7. Conclusion

In this Letter, we re-derived the twisted Lorentz symmetry (1.3) of chiral fermions by embedding the theory of Refs. [1–5] into Souriau's massless spinning model [6,7] by spin enslavement, (1.2), viewed as a gauge fixing. The latter is not boost invariant, but enslavement can be restored by a suitable compensating WS-translation, which is analogous to a gauge transformation [12]. Our formula (4.4) extends (1.3) to finite transformations.

The motions of the extended model are not mere curves but 3-planes, swept by WS-translations. The massless relativistic particle is delocalized [4,6–10] and behaves rather as a 3-brane.

Coupling to an external electromagnetic field breaks the WS “gauge” symmetry, so that spin can not be enslaved. The motions then take place along curves: the particle gets localized [6]. New results [21] indicate however that while a WS translation is *unobservable* for an infinite plane wave, it is *observable* for a wave packet. Intuitively, forming and keeping together a wave packet behaves as a sort of interaction.

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