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Bound on the counting function for the eigenvalues of an infinite multistratified acoustic strip

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Summary

Let $\mathcal{N}(\mu)$ be the counting function of the eigenvalues associated with the self-adjoint operator $-\nabla(\rho(x, z)\nabla\cdot)$ in the domain $\Omega = \mathbb{R}\times]0, h[$, $h > 0$, with Neuman or Dirichlet conditions at $z = 0$, $z = h$. If $\rho = 1$ in the exterior of a bounded rectangular region \mathcal{O} , that is, for $|x|$ large, then $\mathcal{N}(\mu)$ is known to be sublinear: the proof consists in the spectral analysis of a quadratic form obtained from a Green formula for $-\nabla(\rho(x, z)\nabla\cdot)$ on \mathcal{O} . In our case, the medium is multistratified: the function $\rho(x, z)$ satisfies $\rho(x, z) = \rho(z)$ for $|x|$ large. Since the direct use of the previous proof fails, we modify the quadratic form and obtain the estimate $\mathcal{N}(\mu) \leq C\mu^{3/2}$.

1 Introduction and main results

Consider the propagation of acoustic waves in a perturbed stratified medium described by the wave equation

$$\frac{\partial^2}{\partial t^2}u(x, z, t) - \nabla(\rho(x, z)\nabla u(x, z, t)) = 0$$

where $t \in \mathbb{R}$, $(x, z) \in \Omega = \mathbb{R} \times]0, h[$, $h > 0$. The function $\rho(x, z)$ is the square of the celerity of acoustic waves in the strip Ω . The asymptotic properties of $u(\cdot, x, z)$ for large t can be derived from the spectral analysis of the self-adjoint operator A defined by $Au := -\nabla \cdot \rho(x, z)\nabla u$ with domain $D(A) := \{u \in H_\Omega \mid Au \in L^2(\Omega) \text{ and } (\rho \partial_z u)_{z=0} = 0\}$ where $H_\Omega := \{v \in H^1(\Omega) \mid v(x, h) = 0\}$ and $H^1(\Omega)$ denotes the usual Sobolev space. In this way the waves satisfy a Neumann condition at $z = 0$ and a Dirichlet condition at $z = h$.

The function $\rho(x, z)$ is real-valued, measurable and satisfies the following conditions: $\rho, \rho^{-1} \in L^\infty(\Omega)$ with $0 < \rho_{min} \leq \rho \leq \rho_{max} < \infty$, and $\rho(x, z) = \rho_{\pm\infty}(z)$ for $\pm x > M$ where $M \geq 0$. If $\rho(x, z) = \rho_{\pm\infty}(z)$ almost everywhere in Ω , the medium is said to be "unperturbed", the operator A "free", and we then put $A_\pm = A$. The spectrum $\sigma(A_\pm)$ of A_\pm is well-known [2, 3, 9]. It is reduced to the essential spectrum $\sigma_{ess}(A_\pm) = [S_1^\pm(A_\pm), +\infty[$, where the number $S_1^\pm(A_\pm)$ is the lower bound of $\mathcal{S}(A_\pm)$, the discrete set of thresholds (see also [2]).

In the general case A is considered as a perturbation of the free operators A_+ and A_- coupled to each other. Thus the spectrum of A consists of two parts. The first is the absolutely continuous spectrum $\sigma_{ac}(A)$ which coincides with the essential spectrum: $\sigma_{ess}(A) = \sigma_{ess}(A_+) \cup \sigma_{ess}(A_-) = [S_1(A), +\infty[$, where $S_1(A) := \min(S_1^+(A_+), S_1^-(A_-))$. The second, possibly void, is the point spectrum $\sigma_p(A) \subset [\rho_{min}, +\infty[$. We prove that $\sigma_p(A)$ is a discrete set, and hence improve [9] where it is shown that the eigenvalues of A , counted with their multiplicity, cannot have a finite accumulation point, except maybe to the left at points of $\mathcal{S}(A_-) \cup \mathcal{S}(A_+)$. This last set is conveniently denoted by $\mathcal{S}(A)$ and called "the set of thresholds" for the operator A . In fact one needs to know the behaviour of the resolvent near the real axis and near thresholds. This question is partially solved by the limiting absorption principle developed in [2, 9], where the following Hilbert spaces equipped with obvious norms are introduced:

$$\begin{aligned} L^{2,s}(\Omega) &= \{u \in L^2_{loc}(\Omega) \mid (1+x^2)^{\frac{s}{2}}u(x, z) \in L^2(\Omega)\} \\ H^{1,s}(\Omega) &= \{u \in L^{2,s}(\Omega) \mid \nabla u \in (L^{2,s}(\Omega))^2\}, \end{aligned}$$

for any real s . As proved in [9], the operator $(A - \zeta)^{-1}$ defined for $\zeta \in \mathbb{C}_+ := \{\zeta \in \mathbb{C} \mid \Im m \zeta > 0\}$ extends continuously to $\zeta = \mu \in Z^C := \mathbb{R} \setminus (\mathcal{S}(A) \cup$

$\sigma_p(A)$) as an operator $R_A^+(\mu) \in B(L^{2,s}(\Omega), L^{2,-s}(\Omega))$ which is equipped with the uniform topology of norms, for any $s > \frac{1}{2}$. The investigation of the analytic properties of $R_A^+(\cdot)$ is the next step to confirm that the point spectrum of A is discrete. Consider now the counting function $\mathcal{N}_A(\mu) := \#\sigma_p(A) \cap [0, \mu]$. Estimates of $\mathcal{N}_A(\mu)$ are well-known when $\rho_{+\infty}(z)$ and $\rho_{-\infty}(z)$ are constant:

$$\mathcal{N}_A(\mu) \leq C\mu + o(\mu^{\frac{1}{2}}) \text{ as } \mu \rightarrow +\infty \quad (1)$$

and the optimal value for C is known (see [6, 8, 1]). To do this, note that if μ is an eigenvalue of A with the eigenmode ϕ , then μ is an eigenvalue of some operator $G(\mu)$, with the eigenmode $u = \phi|_{\mathcal{O}}$, restriction of ϕ to the domain $\mathcal{O} :=]-M, M[\times]0, h[$ ⁽¹⁾. The expression of $G(\mu)$ is obtained from a Green formula on \mathcal{O} for A , using a Dirichlet–Neumann operator $T(\mu)$ (cf. section 2).

However, when the medium is really stratified, such a method fails. That is why we modify the operator $G(\mu)$ in section 3. The problem is then more complicated, but we obtain the following estimate:

Theorem 1.1

$$\mathcal{N}_A(\mu) \leq C^* \mu^{3/2} + o(\mu) \text{ as } \mu \rightarrow +\infty \quad (2)$$

where

$$\left\{ \begin{array}{l} C^* := C_+^* + C_-^* \\ C_{\pm}^* := \frac{1}{8} M \rho_{M,\min}^{-1} R_{\rho}^{\pm} \rho_{\pm\infty,m}^{-\frac{1}{2}} \\ \rho_{M,\min} := \inf \text{ess } \rho(x, z) \text{ in } \mathcal{O} \\ \rho_{\pm\infty,M} := \sup \text{ess } \rho_{\pm\infty}(z) \\ R_{\rho}^{\pm} : \text{ the lowest integer greater than or equal to } \sqrt{2} \left(\frac{\rho_{\pm\infty,M}}{\rho_{\pm\infty,m}} \right)^2 \\ \rho_{\pm\infty,m} := \inf \text{ess } \rho_{\pm\infty}(z) . \end{array} \right. \quad (3)$$

In addition, the remainder $o(\mu)$ in (2) is bounded by $C \cdot \max(M, 1)(\mu + 1)$ where C does not depend on M .

This result still holds for any Dirichlet or Neumann boundary conditions at $z = 0$, $z = h$.

The paper is composed of two parts. Section 2 is about the Dirichlet–Neumann operators $T(\zeta)$. In [9] it is proven that the mapping $T(\cdot)$ defined on $\overline{\mathbb{C}}_+$ is continuous. We show here the analyticity of $T(\cdot)$ and give an explicit formula for $T'(\mu)$.

In the second part, section 3, we prove that the point spectrum of A is discrete. In fact by another method we are near to recovering some results of [4] about

¹the case $M = 0$ is trivial since $\sigma_p(A)$ happens to be void

the meromorphic continuation of the resolvent of A through the real axis, and we complete the proof in [9]. Then we prove theorem 1.1.

We shall use the following notations: D_t denotes $\frac{\partial \cdot}{\partial t}$ for the variable t , and $\|\cdot\|_{r,X}$ the usual norm of the Sobolev space $H^r(X)$.

2 The Dirichlet–Neumann operator

2.1 Definition of the thresholds

There are two ways to "reduce" the operator A_{\pm} and thus two ways to introduce the set of thresholds.

First let us consider the operator $A_{\pm,z} := -\frac{d}{dz}(\rho_{\pm\infty}\frac{d}{dz}\cdot)$ self-adjoint on $L^2(]0, h[)$, with domain $D^{\pm} := \{v \in H^1(]0, h[) \mid A_{\pm,z}v \in L^2(]0, h[), \rho_{\pm\infty}v'(0) = v(h) = 0\}$. It has compact resolvent and discrete spectrum which consists of positive eigenvalues: the thresholds $S_1^{\pm} < S_2^{\pm} \dots < S_n^{\pm} \dots$.

It is more convenient to consider for real μ the self-adjoint operator $A_{\pm,red} := -(\rho_{\pm\infty})^{-1}(\frac{d}{dz}\rho_{\pm\infty}\frac{d}{dz} + \mu)$ on $L^2(]0, h[, \rho_{\pm\infty}(z)dz)$, with domain $D_{\pm,red} := \{v \in H^1(]0, h[) \mid A_{\pm,red}v \in L^2(]0, h[), \rho_{\pm\infty}v'(0) = v(h) = 0\}$. Its spectrum is discrete and consists of an increasing sequence $\{K_n^{\pm}(\mu)\}_{n \geq 1}$ of eigenvalues, associated with an orthonormal basis $U_n^{\pm}(\mu; \cdot)$. In fact $K_n^{\pm}(\mu)$ vanishes if and only if $\mu = S_n^{\pm}$ (cf. figure 1).

We set $\mathcal{S}(A) := \mathcal{S}(A_-) \cup \mathcal{S}(A_+)$ where $\mathcal{S}(A_{\pm})$ denotes the set of thresholds of A_{\pm} . By setting $\lambda^{\frac{1}{2}}$ the square root of $\lambda \in \mathbb{C}$ such that $\arg(\lambda^{\frac{1}{2}}) \in]-\pi/2, \pi/2]$, and by using the spectral representation of the operator $A_{\pm,red}$, we can define the square root $A_{\pm,red}^{\frac{1}{2}}$ of $A_{\pm,red}$. The eigenvalues of $A_{\pm,red}^{\frac{1}{2}}$ are $\sqrt{K_n^{\pm}} \equiv ik_n^{\pm}(\mu) \in i\mathbb{R}_+$ for $1 \leq n \leq N^{\pm}(\mu)$ and $\sqrt{K_n^{\pm}} \equiv \theta_n^{\pm}(\mu) \in \mathbb{R}_+$ for $n > N^{\pm}(\mu)$. We also put $k_n^{\pm} := i\theta_n^{\pm}$ for $n > N^{\pm}$.

Let us define the bounded Dirichlet–Neumann operator $T^{\pm}(\mu)$ from $\tilde{H}^{\pm} := D(A_{\pm,red}^{1/4})$ ⁽²⁾ into its antidual space $\tilde{H}^{\pm'}$ by:

$$\langle T^{\pm}(\mu)\varphi, \varphi \rangle := -(\varphi | A_{\pm,red}^{\frac{1}{2}} \varphi)_{\pm} = \sum_{n \geq 1} ik_n^{\pm} |\varphi_n^{\pm}|^2, \quad \forall \varphi \in D(A_{\pm,red}^{\frac{1}{2}})$$

where $(\cdot | \cdot)_{\pm}$ is the scalar product in $L^2(]0, h[, \rho_{\pm\infty}(z)dz)$ and $\varphi_n^{\pm} := (\varphi | U_n^{\pm})_{\pm}$. One sets $T(\mu) := T^-(\mu) \oplus T^+(\mu)$ as a bounded operator from the Hilbert sum $\tilde{H} := \tilde{H}^- \oplus \tilde{H}^+$ into its antidual space \tilde{H}' . By setting $\|\varphi\|_{\frac{1}{2}} := \langle T(0)\varphi, \varphi \rangle^{\frac{1}{2}}$, (resp. $\|\varphi\|_{-\frac{1}{2}} := \langle \varphi, T(0)^{-1}\varphi \rangle^{\frac{1}{2}}$), one defines a norm on \tilde{H} (resp. on \tilde{H}' , the antidual space of \tilde{H}) which does not depend on μ . Note that the trace

²note that \tilde{H}^{\pm} does not depend on μ

operator γ is continuous and onto from $H_o := \{v \in H^1(\mathcal{O}) \mid v|_{z=h} = 0\}$ into \tilde{H} (cf. [9]).

Remark 2.1 *Similar definitions hold for $A_{\pm, red}$ with boundary conditions of Dirichlet or Neumann type at $z = 0, z = h$.*

2.2 Characterization of $R_A^+(\mu)$ and $\ker(A - \mu)$

It is usual to study R_A^+ via the operator $T(\mu)$. In fact one has

Proposition 2.1 *Let $\mu \in Z^C$, let $f \in L^2(\Omega)$ with support in $\bar{\mathcal{O}}$. Then the function $\phi := R_A^+(\mu)f \in D(A)_{loc} \cap L^{2,-s}(\Omega)$ is determined by:*

$$\phi(x, z) = \begin{cases} W^\pm(\mu)\gamma^\pm u(|x| - M, z) & \text{for } \pm x > M \\ u(x, z) & \text{for } |x| < M. \end{cases} \quad (4)$$

where

- γ^\pm is the trace operator from H_o into \tilde{H}^\pm
- the operator $W^\pm(\mu)$ is defined (for any real μ) on \tilde{H}^\pm by:

$$W^\pm(\mu)\varphi(x, z) := \sum_{n \geq 1} \varphi_n e^{ik_n^\pm x} U_n^\pm(z)$$

- $u := \phi|_{\mathcal{O}}$ is the unique ⁽³⁾ solution in H_o of the following variational problem:

$$\forall v \in H_o, b(\mu; u, v) = \int_{\mathcal{O}} f \bar{v} \, dx \, dz \quad (5)$$

where for any $\mu \in \mathbb{R}$, $b(\mu; \cdot, \cdot)$ is the continuous sesquilinear form on $H_o \times H_o$:

$$b(\mu; u, v) := \int_{\mathcal{O}} \{ \rho \nabla u \nabla \bar{v} - \mu u \bar{v} \} \, dx \, dz - \langle T(\mu)\gamma u, \gamma v \rangle .$$

The main results of [9] on the point spectrum of A are resumed by

Proposition 2.2 *Let $\mu \in \sigma_p(A) \cap [S_{N^\pm}^\pm, S_{N^\pm+1}^\pm[$ and $\phi \in D(A)$. Then the two following statements are equivalent:*

- 1) ϕ does not vanish and $A\phi = \mu\phi$.
- 2) With the notations of proposition 2.1, ϕ is determined by the relations (4), where u is a non-trivial solution of the homogenous problem (5) (i.e $f = 0$), and $\varphi := \gamma u$ satisfies $\varphi_n^\pm = 0$ for $1 \leq n \leq N^\pm(\mu)$.

³because $\mu \notin Z$

If the above conditions 1) and 2) hold, then μ and u are associated eigenlements of the unbounded self-adjoint operator $G(\mu)$ on $L^2(\mathcal{O})$, characterized by the following quadratic form $Q(\mu)$ on H_0 :

$$Q(\mu)(u) := \int_{\mathcal{O}} \rho |\nabla u|^2 dx dz - \langle T_R(\mu) \gamma u, \gamma u \rangle$$

where $T_R(\mu)$ denotes the real part of $T(\mu)$: $T_R(\mu) := \frac{1}{2}(T(\mu) + T(\mu)^*)$, and $T(\mu)^*$ the adjoint of $T(\mu)$.

This method is successful for a homogenous medium (i.e $\rho_{\pm\infty}$ independent of z), in order to compute eigenvalues or to estimate $\mathcal{N}_A(\mu)$. Each eigenvalue $\lambda_n(\mu)$ of $G(\mu)$ is a function of μ whose regularity comes from that of $T(\cdot)$. Particularly with regard to analytic regularity. The same concerning the regularity of $R_A^+(\cdot)$. This is the interest of the following section.

2.3 Analyticity of the family $\{T(\mu)\}_\mu$

The main results of this part are theorems 2.2 and 2.3 which render precise the analytic continuation of $T(\cdot)$. Finally an explicit representation of the derivative $T'(\mu)$ is given.

For the sake of simplicity, we assume that $A = A_+ = A_-$ and we suppress the indices $+$ and $-$. In particular, we write \tilde{H} and ρ_∞ instead of \tilde{H}^\pm and $\rho_{\pm\infty}$. Setting $\Omega^+ :=]0, +\infty[\times]0, h[$ and defining γu as the trace of u on $\Sigma := \{0\} \times]0, h[$, note that for any $u \in H^1(\Omega^+)$ satisfying $\gamma u = 0$, the function u can be uniquely extended to the Hilbert space H_i^1 defined by

$$\begin{aligned} H_i^1 &:= L_i^2 \cap H^1(\Omega) , \text{ with} \\ L_i^2 &:= \{u \in L^2(\Omega) \mid u(x, z) = -u(-x, z) \text{ almost everywhere in } \Omega\} . \end{aligned}$$

This continuation will be still denoted by u .

For any fixed $\omega \in \mathbb{C}^*$ with $0 < \arg(\omega) < \pi/2$, one defines the following operator $B(\omega)$ with domain $D(\omega)$:

$$\begin{aligned} B(\omega) &:= \mathcal{B}(\omega) \equiv D_z \rho_\infty(z) D_z + \omega^{-2} \rho_\infty(z) D_x^2 \text{ and} \\ D(\omega) &:= \{u \in H_i^1 \mid \mathcal{B}(\omega) u \in L_i^2, (\rho_\infty D_z u)|_{z=0} = u|_{z=h} = 0\}. \end{aligned}$$

Note that if $u \in H_i^1$ then $\gamma u = 0$. If in addition $u \in D(\omega)$, then $B(\omega)u \in L_i^2$. The operator $B(\omega)$ is closed and unbounded on L_i^2 , but not symmetric. Its domain is dense in L_i^2 , and we shall see in the proof of theorem 2.1) that it does not depend on ω : $D(\omega) = L_i^2 \cap D(A)$.

Theorem 2.1 *The resolvent set of $B(\omega)$ contains the domains \mathbb{C}_+ and $\mathbb{R} \setminus \mathcal{S}(A)$.*

Proof

□

Remark 2.2 *The above proof shows that $\xi^2 \hat{u} \in L^2(\Omega)$. In particular $D_x^2 u$ and $D_z \rho_\infty D_z u$ belong to $L^2(\Omega)$. Thus $D(\omega) = L_i^2 \cap D(A)$ is independent of ω .*

Let $\varphi \in \tilde{H}$, and $\mu \in \mathbb{R} \setminus \mathcal{S}(A)$. Let us consider the following vector u_μ in $H^1(\Omega^+)$

$$u_\mu(x, z) := \sum_{n \geq 1} \varphi_n e^{ik_n(\mu)\omega x} U_n(\mu; z) \text{ with } \varphi := \sum_{n \geq 1} \varphi_n(\mu) U_n(\mu; z).$$

One easily checks that $\mathcal{B}(\omega)u_\mu = \mu u_\mu$ in $L^2(\Omega^+)$, and

$$\left(\rho_\infty \frac{\partial u_\mu}{\partial x}\right)|_\Sigma = \omega T(\mu)\varphi \quad (6)$$

Here T means T^+ or T^- . Setting $v_\mu := u_\mu - u_0$, one has $\gamma v_\mu = 0$ and $(\mathcal{B}(\omega) - \mu)v_\mu = \mu u_0$. The functions u_0 and v_μ being uniquely extended to L_i^2 , one has:

$$v_\mu \in D(\omega) \text{ and } v_\mu = \mu (B(\omega) - \mu)^{-1} u_0$$

which is analytic on \mathbb{C}_+ according to μ . Since $u_\mu = u_0 + v_\mu$ and

$$T(\mu)\varphi = T(0)\varphi + \omega^{-1} \left(\rho_\infty \frac{\partial v_\mu}{\partial x}\right)|_\Sigma,$$

one has

Theorem 2.2 *The family $\{T(\mu)\}_\mu$ defined for $\mu \in \mathbb{R}$ admits an analytic continuation in $\overline{\mathbb{C}_+} \setminus \mathcal{S}(A)$. In addition, $T(\mu) - T(0)$ is a relatively compact perturbation of $T(0)$, since this operates from \tilde{H} into itself (see remark 2.2).*

Let us now define for $N \geq 1$ the operator $T^N(\mu) \in B(\tilde{H}, \tilde{H}')$ by:

$$T^N(\mu)\varphi := T(\mu)\varphi + \begin{cases} -ik_N(\mu)\varphi_N(\mu)\rho_\infty U_N(\mu; \cdot) & \text{if } \mu \geq S_N \\ \theta_N(\mu)\varphi_N(\mu)\rho_\infty U_N(\mu; \cdot) & \text{if } \mu \leq S_N \end{cases} \quad (7)$$

By applying the theory of Kato on analytic perturbations (cf. [7]) to the family of operators A_{red} , one proves that U_N and K_N are analytic in $\mu \in \mathbb{R}$. Thus T^N is analytic on $\mathbb{R} \setminus \mathcal{S}(A)$, since K_N never vanishes on this set. In addition there exists a complex domain V containing S_N such that $(T^N)|_{S_{N-1}, S_N[}$ can be analytically extended onto $V \cap \overline{\mathbb{C}_+} \setminus [S_N, +\infty[$ as an operator \tilde{T}^N , similarly to the function $(K_N)^{\frac{1}{2}}$. Because $\lim \theta_N(\zeta) = -ik_N(\mu)$, one can check that $\lim \tilde{T}^N(\zeta) = T^N(\mu)$, as $\zeta \rightarrow \mu \in]S_N, S_{N+1}[$, with $\zeta \in V \cap \mathbb{C}_+$. For $\mu \in \mathbb{R}$ consider the following characterization of the adjoint operator $T(\mu)^*$ of $T(\mu)$:

$$T(\mu)^*\varphi = - \sum_{1 \leq n \leq N} ik_n(\mu)\varphi_n \rho_\infty U_n(\mu; \cdot) - \sum_{n > N} \theta_n(\mu)\varphi_n \rho_\infty U_n(\mu; \cdot).$$

It admits an analytic continuation to $\mathbb{C}_- := \{\zeta \in \mathbb{C} \mid \bar{\zeta} \in \mathbb{C}_+\}$, and $T(\zeta)^* = \overline{T(\bar{\zeta})}$. It is then not hard to prove that $(T^N)_{\parallel S_{N-1}, S_N[}$ can be uniquely continued into $V' \cap \overline{\mathbb{C}_-}$, where V' is some open complex domain containing S_N .

Because $\lim_{\zeta \rightarrow \mu} \theta_N(\zeta) = ik_N(\mu)$ as $\zeta \rightarrow \mu \in]S_N, S_{N+1}[$ with $\zeta \in V' \cap \mathbb{C}_-$, one obtains under these constraints: $\lim_{\zeta \rightarrow \mu} \tilde{T}^N(\zeta) = T^N(\mu)$.

Choosing V bounded, the operator T^N is then analytic in $V \cap V' \setminus \{S_N\}$ and bounded in V . Thus it is analytic in the neighbourhood of S_N . Let $\theta \in \mathbb{R}$, set $\mathcal{D}(\theta) := \{\zeta \in \mathbb{C} \mid \arg(\zeta - S_N) = \theta\}$, choose $a > 0$ small enough to have $B_{a,N} \subset V \cap V'$, and set $\mathcal{C}(\theta) := \{K_N(\zeta) \mid \zeta \in \mathcal{D}(\theta) \cap B_{a,N}\}$. As the following estimate holds uniformly in $B_{a,N}$:

$$K_N(\zeta) = (S_N - \zeta)|K'_N(S_N)| + 0(|\zeta - S_N|^2)$$

where $K'_N(S_N) = -\|U_N(S_N; \cdot)\|_{0,]0,h[}^2 < 0$, $\mathcal{C}(\theta)$ is then a cut in the set $K_N(B_{a,N})$ and there exists an analytic determination of $K_N(\zeta)^{\frac{1}{2}}$ on $K_N(B_{a,N}) \setminus \mathcal{C}(\theta)$. This result completes theorem 2.2. Moreover one has

Theorem 2.3 *The mapping $\zeta \rightarrow T(\zeta)$ defined for $\zeta \in \mathbb{C}_+$ can be analytically continued into a neighbourhood of the real axis with branching points S_N , $N \geq 1$. This analytic continuation has the following form:*

$$T(\zeta) = T_N(\zeta) + \sum_{n=1}^N \sqrt{\zeta - S_n} T_{1,n}(\zeta)$$

where $\sqrt{\zeta - S_n}$ is defined by the condition $\sqrt{\zeta - S_n} > 0$ for $\zeta > S_n$; the operators $T_N(\zeta)$ and $T_{1,n}(\zeta)$ ($n \leq N$) belong to $B(\tilde{H}, \tilde{H}')$, and the range of $T_{1,n}(\zeta)$ is one. For any integer n , the function $\zeta \rightarrow T_{1,n}$ is holomorphic in a neighbourhood V_n of \mathbb{R} and the function $\zeta \rightarrow T_n(\zeta)$ is holomorphic in $V_n \setminus [S_{n+1}, +\infty[$.

Proof

Let us set $T_N(\mu)\varphi := \sum_{n>N} ik_n \varphi_n \rho_\infty U_n(\cdot)$ for $\mu \in \mathbb{R}$, $N \in \mathbb{N}$. The required property for T_N comes from the properties of T^n (defined by (7)) for $1 \leq n \leq N$. The conclusion is straightforward. \square

Remark 2.3 *If $\mu \in [S_N, S_{N+1}[$, then $T_N(\mu)$ coincides with the real part $T_R(\mu)$.*

2.4 Calculation of $T'(\mu)$:

For $\mu, \lambda \in \mathbb{R}$, one has $(B(\omega) - \mu)(v_\mu - v_\lambda) = (\mu - \lambda)u_\lambda$. The derivative of v_λ at $\lambda = \mu \in \mathbb{R} \setminus \mathcal{S}(A)$ is then:

$$q_\mu := \frac{dv_\mu}{d\mu} = (B(\omega) - \mu)^{-1} u_\mu \quad (\text{where } u_\mu \in L_i^2). \quad (8)$$

This implies

$$T'(\mu)\varphi = \omega^{-1}(\rho_\infty \frac{\partial q_\mu}{\partial x})|_\Sigma . \quad (9)$$

Let us fix μ in $]S_N, S_{N+1}[$, and suppress the corresponding indices to simplify: $u_\mu := u$, $q_\mu := q$ etc. Setting

$$\tilde{u}(x, z) := \sum_{n \geq 1} \bar{\varphi}_n e^{ik_n \omega x} U_n(z) \quad (10)$$

we get $\gamma \tilde{u} = \bar{\varphi}$, $(\mathcal{B}(\omega) - \mu)\tilde{u} = 0$. The Green formula

$$\int_{\Omega^+} u \tilde{u} \, dx \, dz = \int_{\Omega^+} (B(\omega) - \mu) q \tilde{u} \, dx \, dz = \omega^{-2} \int_\Sigma (\rho_\infty \frac{\partial q}{\partial x}) \gamma \tilde{u} \, dz$$

gives

$$\langle T'(\mu)\varphi, \varphi \rangle = \omega \int_{\Omega^+} u \tilde{u} \, dx \, dz.$$

The last value, denoted by $J(\varphi)$, is independent of ω . A short calculation gives

$$J(\varphi) = - \sum_{n, m \geq 1} \frac{\varphi_n \bar{\varphi}_m}{ik_n + ik_m} a_{n, m} , \text{ with } a_{n, m} := \int_0^h U_n(z) U_m(z) \, dz \quad (11)$$

In particular one has $\langle T'_N(\mu)\varphi, \varphi \rangle = \Re e(J(\varphi))$ which is non-negative for $\varphi_n(\mu) = 0$, $1 \leq n \leq N$. In fact in this case $\Re e(J(\varphi))$ is the square norm in $L^2(\Omega^+, dx \, dz)$ of the vector $\sum_{n > N} \varphi_n e^{-\theta_n x} U_n(z)$.

3 Counting of the point spectrum of A

3.1 Absence of accumulation point of eigenvalues

The following theorem proved by another method in [?] completes the result in [9]:

Theorem 3.1 *The point spectrum of A is discrete.*

The proof uses the non-negativity of $T(\mu)$ and $\Re e(J(\varphi))$ (see section 2).

Proof

□

3.2 Counting the eigenvalues of A

This part is devoted to the proof of theorem 1.1, which is also valid for Dirichlet or Neumann boundary conditions. One denotes by $\mathcal{N}_A(\mu)$ the finite number of eigenvalues of A (counted with their order of multiplicity) less or equal to μ .

Proof

One proceeds in three steps. It is assumed until the second step that $\rho_{+\infty} = \rho_{-\infty} =: \rho_\infty$. The indices $+$ and $-$ are thus suppressed until we deal with the general case in the third step.

1. Setting $S_0 := 0$, recall that (cf. section 2) if for some $N \geq 0$, $\mu \in [S_N, S_{N+1}[$ is an eigenvalue of A associated with the eigenmode ϕ , then $u = \phi|_{\mathcal{O}}$ is a non-trivial solution in H_o of the following equations:

$$\forall v \in H_o, b(\mu; u, v) = 0$$

and $(\gamma u)_n(\mu)$ is null for $1 \leq n \leq N$. Thus (μ, u) is a pair of eigenvalue and eigenmode for the unbounded self-adjoint operator $G(\mu)$ on $L^2(\mathcal{O})$, which is associated with the following quadratic form $Q(\mu)$ defined on H_o :

$$Q(\mu)(u) := \int_{\mathcal{O}} \rho |\nabla u|^2 dx dz - \langle T_R(\mu) \gamma u, \gamma u \rangle + t(\mu) (V(\mu) \gamma u | \gamma u)$$

where $t(\mu)$ is an arbitrary real function, $V(\mu)$ is the finite range operator defined by $V(\mu)\varphi := \sum_{n=1}^N \varphi_n(\mu) U_n(\mu; \cdot)$, and $(\cdot | \cdot)$ denotes the scalar product in $L^2(]0, h[, \rho_\infty(z) dz)$.

Let us consider a subdivision $0 = \mu_0 < \mu_1 < \dots < \mu_k \dots$ of \mathbb{R}_+ which contains the thresholds. The number of intervals $[\mu_k, \mu_{k+1}]$ contained in $[S_n, S_{n+1}]$ is R_n . On the interval $J_k :=]\mu_k, \mu_{k+1}] \subset [S_N, S_{N+1}]$, we choose a non-negative, differentiable, non-increasing function $t(\mu)$ satisfying

$$(i) \quad Q'(\mu) \leq 0 .$$

Lemma 3.1 *Denoting by $\mathcal{N}_A(J)$ the number of eigenvalues of A in the set $J \subset \mathbb{R}_+$, one has under condition (i):*

$$\mathcal{N}_A(J_k) \leq \rho_{M, \min}^{-1} C_M \mu_{k+1} + \max(M, 1) 0(\mu_{k+1}^{\frac{1}{2}})$$

where C_M depends only on M and the remainder $0(\mu_{k+1}^{\frac{1}{2}})$ is independent of M .

Proof

□

Lemma 3.2 *If the condition (i) is satisfied and if the sequence $\{R_n\}_n$ is bounded, then the required estimate holds:*

$$\mathcal{N}_A(\mu) \leq \frac{1}{2}R \rho_{M,\min}^{-1} \rho_{\infty,m}^{-\frac{1}{2}} C_M \mu^{3/2} + \max(M, 1)0(\mu) \quad \text{as } \mu \rightarrow +\infty.$$

where R is a bound for $\{R_n\}_n$ and the remainder $0(\mu)$ is independent of M .

Proof

□

2. On condition that we find the adequate subdivision $\{\mu_k\}$ and function $t(\mu)$, theorem 1.1 is proved.

Lemma 3.3 *Let $\mu \in J_k \subset [S_N, S_{N+1}]$, $\mu \notin \mathcal{S}(A)$. For any $u \in H^1(\mathcal{O})$ one has*

$$-Q'(\mu)[u] \geq \rho_{\infty,M}^{-1} b^2 - (C_{2,N} + C_{1,N}t(\mu))ab - t'(\mu)a^2$$

with the notations:

$$\left\{ \begin{array}{l} a := \left(\sum_{n=1}^N |\varphi_n|^2 \right)^{\frac{1}{2}} \\ b := \left(\sum_{m>N} \frac{|\varphi_m|^2}{\theta_m} \right)^{\frac{1}{2}} \\ \varphi := \gamma u \\ C_{1,N} := 2 \rho_{\infty,m}^{-1} \rho_{\infty,M}^{3/4} (S_{N+1} - S_N)^{-3/4} \\ C_{2,N} := 2\sqrt{2} \rho_{\infty,m}^{-1} \rho_{\infty,M}^{1/4} (S_{N+1} - S_N)^{-1/4} \end{array} \right. \quad (12)$$

Proof

□

□

Remark 3.1 *Some additional calculations show that the numerical constant $1/8$ in (3) can be improved. However the estimates (2) on $C_{1,N}$ and $C_{2,N}$ are optimal. Putting $\varphi_n := 0$ if and only if $n \notin \{N, N+1\}$, this is easily checked.*

Remark 3.2 *The use of C_M (instead of its present value $M/4$) generalizes the results to a non-rectangular domain \mathcal{O} . In fact, the case of a non-rectilinear strip Ω can be treated too (cf. [5] for example).*

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