



HAL
open science

A note on “Re-examining the law of iterated expectations for Choquet decision makers”

André Lapied, Pascal Toquebeuf

► **To cite this version:**

André Lapied, Pascal Toquebeuf. A note on “Re-examining the law of iterated expectations for Choquet decision makers”. 2024. hal-01499621

HAL Id: hal-01499621

<https://amu.hal.science/hal-01499621>

Preprint submitted on 11 Jan 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



A note on "Re-examining the law of iterated expectations for Choquet decision makers"

ANDRE LAPIED, PASCAL TOQUEBEUF

www.tepp.eu

A note on "Re-examining the law of iterated expectations for Choquet decision makers"

André Lapied* Pascal Toquebeuf†

November 4, 2011

Abstract

This note completes the main result of [Zimper A., (2010) Re-examining the law of iterated expectations for Choquet decision makers. *Theory and decision*, DOI 10.1007/s11238-010-9221-8], by showing that additional conditions are needed in order the law of iterated expectations to hold true for Choquet decision makers. Due to the comonotonic additivity of Choquet expectations, the equation

$$E[f, \nu(d\omega)] = E[E[f(\omega_{i,j}), \nu(A_{i,j}|A_i)], \nu(A_i)],$$

is valid only when the act f is comonotonic with its dynamic form, that we name "conditional certainty equivalent act".

Keywords: Choquet; Capacities; Updating; Law of iterated expectations

JEL Classification : D81

1 Introduction

The Choquet model enlarges the standard approach thanks to a generalized notion of probability, called Choquet capacity. The extension of Choquet expectations to a dynamic set-up is a subject of matter. Whereas several updating rules have been proposed in the literature, no one allows preferences to exhibit a recursive structure as, for instance the multiple priors model (see Epstein and Schneider 2003). Such a feature would be suitable for applications as well as normative reasons.

Zimper (2010) uses the updating rule for Choquet capacities proposed by Sarin and Wakker (1998). He claims that it allows the law of iterated expectations to hold for Choquet decision makers. In this note, we give counter-examples to his theorem. They show that if the criterion is a Choquet expectation with respect to a non-additive capacity, then the law of iterated expectations does not hold in general. Nevertheless, there are particular cases where Zimper's theorem is valid. We identify them.

*Paul Cézanne University, GREQAM and IDEP.

†University of Maine, GAINS, TEPP-CNRS, pascal.toquebeuf@gmail.com.

2 Notations and definitions

Let Ω be a *state space* endowed with a σ -algebra denoted by \mathcal{F} , so that (Ω, \mathcal{F}) is a measurable space. Elements of \mathcal{F} are called events and for any $A \in \mathcal{F}$, the event $\Omega \setminus A$ is noted A^c . We will consider throughout random variables (or acts) $f : \Omega \rightarrow \mathbb{R}$ s.t. f is a \mathcal{F} -measurable function taking only finite values. The set of such functions is denoted by \mathcal{A} . A Choquet capacity is a set function $\nu : \mathcal{F} \rightarrow [0, 1]$ such that (i) $\nu(\emptyset) = 0$, $\nu(\Omega) = 1$, and (ii) $\forall A, B \in \mathcal{F}, A \subset B \Rightarrow \nu(A) < \nu(B)$.

In order to define the Choquet expectation of an act f w.r.t. ν , denoted by $E[f, \nu]$ we associate to any r.v. $f \in \mathcal{A}$ the coarsest finite partition over Ω , $\mathcal{P} = \{A_1, \dots, A_m\}$, to which f is measurable and ordered. Hence the Choquet expectation of $f \equiv (x_1, A_1; \dots; x_m, A_m)$, w.l.o.g. $x_1 < \dots < x_m$, can be written as

$$E[f, \nu] = x_1 + \sum_{i=2}^m [x_i - x_{i-1}] \cdot \nu(\cup_{j=i}^m A_j) \quad (1)$$

$$= \sum_{i=1}^m x_i \cdot [\nu(A_i \cup \dots \cup A_m) - \nu(A_{i+1} \cup \dots \cup A_m)] \quad (2)$$

where $[\nu(A_i \cup \dots \cup A_m) - \nu(A_{i+1} \cup \dots \cup A_m)]$ is called a *decision weight*. For any $A \in \mathcal{F}$, $D_f[A]$ refers to the dominating event, that is the event that dominates A when the valued act is f , such that

$$D_f[A] = \{\omega \in A^c | \forall \omega' \in A, f(\omega) \geq f(\omega')\} \quad (3)$$

Then the Choquet expectation of f , with

$$\tilde{\nu}_f(A_i) = \nu(A_i \cup D_f[A_i]) - \nu(D_f[A_i]) \quad (4)$$

can be rewritten as

$$E[f, \nu] = \sum_{i=1}^m x_i \cdot \tilde{\nu}_f(A_i) \quad (5)$$

A crucial concept in this model is the comonotonicity.

Definition 1. *Two random variables f and g are comonotonic if and only if $\forall \omega, \omega' \in \Omega$, $[f(\omega) - f(\omega')][g(\omega) - g(\omega')] \geq 0$.*

On the opposite, f and g are *antimonotonic* if \geq is replaced by \leq . An important property of the Choquet expectation is its *comonotonic additivity*, which states that if f is comonotonic with g , then

$$E[f, \nu] + E[g, \nu] = E[f + g, \nu] \quad (6)$$

Indeed, in this case, f and g use the same decision weight.

In the Choquet framework, new information is integrated in the decision process by means of an updating rule that specifies the way of calculate the conditional capacity $\nu(\cdot | \cdot)$. For instance, if the decision maker is informed that the "right" state is in event B , then the conditional Choquet expectation of a r.v. $f \equiv (x_1, A_1; \dots; x_m, A_m)$, w.l.o.g. $x_1 < \dots < x_m$, is given by :

$$E[f, \nu(\cdot | B)] = \sum_{i=1}^m x_i \cdot [\nu(A_i \cup \dots \cup A_m | B) - \nu(A_{i+1} \cup \dots \cup A_m | B)] \quad (7)$$

where $\nu(\cdot|B)$ is a conditional capacity. If we use the updating rule proposed by Sarin and Wakker (1998) for the decision weight $\tilde{\nu}_f(\cdot)$, then

$$\tilde{\nu}_f(A|B) = \frac{\nu((A \cap B) \cup D_f[A \cap B]) - \nu(D_f[A \cap B])}{\nu(B \cup D_f[B]) - \nu(D_f[B])} \quad (8)$$

hence

$$E[f, \nu(\cdot|B)] = \sum_{i=1}^m x_i \cdot \tilde{\nu}_f(A_i|B) \quad (9)$$

Following Zimper, we consider a two-stage filtration $\mathcal{G} = \{\mathcal{G}_t, t = 0, 1, 2\}$, that is an increasing sequence of events, such that

$$\mathcal{G}_0 = \{\Omega, \emptyset\}, \mathcal{G}_1 = \{\Omega, \emptyset, A_1, \dots, A_m\}$$

and

$$\mathcal{G}_2 = \{\Omega, \emptyset, A_1, \dots, A_m, A_{1,1}, \dots, A_{1,m_1}, \dots, A_{m,1}, \dots, A_{m,m_m}\}$$

with $(\cup_{j=1}^{m_i} A_{i,j}) = A_i$.

Finally, recall the definition of the law of iterated expectations for the Choquet case :

Definition 2. *Let $f \in \mathcal{A}$ be a \mathcal{G}_2 -measurable act. The law of iterated expectations holds for f and ν if and only if*

$$E[f, \nu] = E[E[f, \nu(\cdot|A_i)], \nu(A_i)] \quad (10)$$

To understand this law, consider the act g such that $g(\omega_{i,j}) = E[f, \nu(\cdot|A_i)]$ for all $i = 1, \dots, m$ and $j = 1, \dots, m_i$. We call such an act the *conditional certainty equivalent act* of f . The law of iterated expectations means that the DM does not care about the timing of resolution of uncertainty, such that she is indifferent between the act f and its "dynamic" form. This law is trivially verified by the Bayesian model. Nevertheless, Choquet preferences are generally perceived as unable to perform it. The main result of Zimper (2010) states that Choquet preferences may satisfy the law of iterated expectations if conditional capacities are given by formula (8). In the next section, we show that his result needs additional conditions to hold true.

3 Result

We argue that the result of Zimper (2010), stating that the law of iterated expectations is valid for Choquet maximizers when the conditional capacity is derived from formula (8), does not hold in general. It can be seen by means of a simple example.

Let $\Omega = \{\omega_1, \dots, \omega_4\}$, with first stage events $A_1 = \{\omega_1, \omega_3\}$ and $A_2 = \{\omega_2, \omega_4\}$. Let f be a random variable such that $\forall i = 1, \dots, 4, f(\omega_i) = i$ and $\nu(\cdot)$ be a capacity such that :

$$\begin{aligned} \forall i, j, k = 1, \dots, 4, i \neq j \neq k, \nu(\{\omega_i, \omega_j, \omega_k\}) &= \frac{1}{2} \\ \forall i, j = 1, \dots, 4, i \neq j, \nu(\{\omega_i, \omega_j\}) &= \frac{1}{3} \\ \forall i = 1, \dots, 4, \nu(\{\omega_i\}) &= \frac{1}{4} \end{aligned}$$

The Choquet expectation of f is

$$E[f, \nu] = 1 + \nu(\{\omega_2, \omega_3, \omega_4\}) + \nu(\{\omega_3, \omega_4\}) + \nu(\{\omega_4\}) = \frac{25}{12} \quad (11)$$

and its conditional expectations are

$$E[f, \nu(\cdot|A_1)] \quad (12)$$

$$= \frac{\nu(\{\omega_1\} \cup D_f[\{\omega_1\}]) - \nu(D_f[\{\omega_1\}])}{\nu(A_1 \cup D_f[A_1]) - \nu(D_f[A_1])} + 3 \cdot \frac{\nu(\{\omega_3\} \cup D_f[\{\omega_3\}]) - \nu(D_f[\{\omega_3\}])}{\nu(A_1 \cup D_f[A_1]) - \nu(D_f[A_1])} \quad (13)$$

$$= \frac{1 - \nu(\{\omega_3\} \cup A_2)}{\nu(A_1 \cup \{\omega_4\}) - \nu(\{\omega_4\})} + 3 \cdot \frac{\nu(\{\omega_3, \omega_4\}) - \nu(\{\omega_4\})}{\nu(A_1 \cup \{\omega_4\}) - \nu(\{\omega_4\})} = 3 \quad (14)$$

and

$$E[f, \nu(\cdot|A_2)] = 2 \cdot \frac{\nu(\{\omega_3\} \cup A_2) - \nu(\{\omega_3, \omega_4\})}{\nu(A_2)} + 4 \cdot \frac{\nu(\{\omega_4\})}{\nu(A_2)} = 4 \quad (15)$$

Therefore, the unconditional expectation of the conditional certainty equivalent act is

$$E[E[f, \nu(\cdot|A_i)], \nu(A_i)] = 3 + \nu(A_2) = \frac{10}{3} \quad (16)$$

and then, from eq. (11) and (16),

$$E[f, \nu] \neq E[E[f, \nu(\cdot|A_i)], \nu(A_i)] \quad (17)$$

in contradiction with the law of iterated expectations.

Furthermore, it is straightforward that:

Remark 1. *The formula (8) is not an update rule for Choquet capacities.*

Stated otherwise, the conditional decision weight obtained by applying the Sarin and Wakker update to $\tilde{\nu}_f(\cdot)$ may be superior to 1, hence the conditional set function $\nu(\cdot|A_i)$ is not normalized thus it is not a Choquet capacity. To see it, consider eq. (14) in the previous example and observe that

$$\tilde{\nu}_f(\{\omega_1\}|A_1) = \frac{1 - \nu(\{\omega_3\} \cup A_2)}{\nu(A_1 \cup \{\omega_4\}) - \nu(\{\omega_4\})} > 1$$

The previous example shows that the law of iterated expectations does not necessarily holds if the capacity is not additive. Nevertheless, it does for Choquet expectations in some cases. Specifically, it turns out to be when f and its conditional certainty equivalent act are comonotonic. Indeed, in this case, the same decision weight will be used to value these two random variables. Consider again the previous example and let $B_1 = \{\omega_1, \omega_2\}$ and $B_2 = \{\omega_3, \omega_4\}$ be first stage events. Then, conditional expectations of f are

$$E[f, \nu(\cdot|B_1)] = \frac{1 - \nu(\{\omega_2, \omega_3, \omega_4\})}{1 - \nu(B_2)} + 2 \cdot \frac{\nu(\{\omega_2, \omega_3, \omega_4\}) - \nu(B_2)}{1 - \nu(B_2)} = \frac{5}{4} \quad (18)$$

and

$$E[f, \nu(\cdot|B_2)] = 3 \cdot \frac{\nu(B_2) - \nu(\{\omega_4\})}{\nu(B_2)} + 4 \cdot \frac{\nu(\{\omega_4\})}{\nu(B_2)} = \frac{15}{4} \quad (19)$$

hence the conditional certainty equivalent act of f is g such that:

$$\begin{cases} g(\omega) = 5/4 & \text{when } \omega \in B_1; \\ g(\omega) = 15/4 & \text{when } \omega \in B_2. \end{cases}$$

The Choquet expectation of g is $E[g, \nu] = 25/12$ and it is equal to the one of f . Further, the Sarin and Wakker update rule reduces to the Dempster-Shafer (pessimistic) update rule

$$\nu(A|B_1) = \frac{\nu((A \cap B_1) \cup B_2) - \nu(B_2)}{1 - \nu(B_2)} \quad (20)$$

since $D_f[A \cap B_1] = D_f[B_1] = B_2$ when $A = \{\omega_2, \omega_3, \omega_4\}$, or to the Bayes (optimistic) update rule

$$\nu(A|B_2) = \frac{\nu(A \cap B_2)}{\nu(B_2)} \quad (21)$$

since $D_f[A \cap B_2] = D_f[B_2] = \emptyset$ when $A = \{\omega_4\}$. It implies that the conditional set function $\nu(\cdot|A_i)$ is a Choquet capacity. This result is linked to the one of Chateauneuf et al. (2001). They showed that if f and its conditional certainty equivalent act g are comonotonic, then Choquet expectations have a recursive structure if the DM uses the optimistic update rule conditionally to the "good" event and the pessimistic update rule conditionally to the "bad" event. It can be seen as an extension of the f -Bayesian approach of Gilboa and Schmeidler (1993).

More generally, let $f \equiv (x_{1,1}, A_{1,1}; \dots; x_{m,m}, A_{m,m})$ and $x_{1,1} < \dots < x_{m,m}$. In this case, f is comonotonic with its conditional certainty equivalent act g since $E[f, \nu(\cdot|A_1)] < \dots < E[f, \nu(\cdot|A_m)]$ by monotonicity of conditional Choquet expectations. The Choquet expectation of f is

$$E[f, \nu] = \sum_{i=1}^m \sum_{j=1}^{m_i} x_{i,j} \cdot \tilde{\nu}_f(A_{i,j}) \quad (22)$$

and the Choquet expectation of g , that is a constant act on each A_i ($g(\omega_{i,1}) = \dots = g(\omega_{i,m_i})$), is

$$\begin{aligned} E[g, \nu] &= \sum_{i=1}^m g(\omega_i) \cdot \tilde{\nu}_g(A_i) \\ &= \sum_{i=1}^m E[f, \nu(\cdot|A_i)] \cdot \tilde{\nu}_g(A_i) \\ &= \sum_{i=1}^m \left[\sum_{j=1}^{m_i} \frac{\tilde{\nu}_f(A_{i,j})}{\tilde{\nu}_f(A_i)} \cdot x_{i,j} \right] \cdot \tilde{\nu}_g(A_i) \end{aligned}$$

By comonotonic additivity of the Choquet integral, $\tilde{\nu}_f(A_i) = \tilde{\nu}_g(A_i)$ for all $i = 1, \dots, m$, hence $E[f, \nu] = E[g, \nu]$. We state it generally:

Theorem 1. *If any \mathcal{G}_2 -measurable function $f : \Omega \rightarrow \mathbb{R}$ is comonotonic with its conditional certainty equivalent act $g(\cdot)$ such that*

$$g(\omega_{i,j}) = E[f, \nu(\cdot|A_i)]$$

for all $i = 1, \dots, m$ and $j = 1, \dots, m_i$, then, under the Sarin and Wakker update rule, the law of iterated expectations holds.

Proof. It is sufficient to observe that the proof of Zimper (2010) holds true for these cases, as illustrated before. \square

This theorem explains why the example of Zimper (in section 4) works properly. It is implicit in his paper that the comonotonic condition between the valued act and the conditional certainty equivalent act holds. Such a condition is called "nest-monotonicity" by Koida (2010).

4 Conclusion

From an axiomatic point of view, the claims of this note can be explained by the inability of Choquet expectations to simultaneously satisfy *consequentialism* and *dynamic consistency* (see Lapied and Toquebeuf 2010). Both axioms seem to be necessary to the law of iterated expectations to universally hold. But then the result of Yoo (1991) holds, too, and the capacity has to be additive.

acknowledgements

We wish thank J.P. Lefort for give us the opportunity to consider the work of Zimper (2010). We also thank two anonymous referees for their valuable comments.

References

- [1] Chateauneuf A., Kast R. and Lapied A., Conditionning capacities and Choquet integrals : the role of comonotony, *Theory and Decision*, 51, 367-386 (2001).
- [2] Epstein L. and Schneider M., Recursive multiple priors, *Journal of Economic Theory*, 113, 1-31 (2003).
- [3] Koida N., 2010, Nest-monotonic two-stage acts and exponential probability capacities, *Economic Theory*, DOI: 10.1007/s00199-010-0551-0.
- [4] Lapied A. and Toquebeuf P., Atemporal non-expected utility preferences, dynamic consistency and consequentialism, *Economics Bulletin*, 30, 1661-1669 (2010).
- [5] Sarin R. and Wakker P., Revealed likelihood and Knightian uncertainty, *Journal of Risk and Uncertainty*, 16, 223-250 (1998).
- [6] Yoo, K. R., The iterative law of expectations and non-additive probability measure, *Economics Letters*, 37, 145-149 (1991).
- [7] Zimper A., Re-examining the law of iterated expectations for Choquet decision makers, *Theory and Decision*, DOI 10.1007/s11238-010-9221-8 (2010).

12-1. What drives Health Care Expenditure in France since 1950? A time-series study with structural breaks and non-linearity approaches

Thomas Barnay, Olivier Damette

12-2. How to account for changes in the size of Sports Leagues: The Iso Competitive Balance Curves

Jean-Pascal Gayant, Nicolas Le Pape

12-3. Hedonic model of segmentation with horizontal differentiated housing

Masha Maslianskaia-Pautrel

12-4. Stricter employment protection and firms' incentives to train: The case of French older workers

Pierre-Jean Messe, Bénédicte Rouland

12-5. Advantageous Semi-Collusion Revisited: A Note

Kai Zhao

12-6. Entry mode choice and target firm selection: private and collective incentive analysis

Kai Zhao

12-7. Optimal Unemployment Insurance for Older Workers

Jean-Olivier Hairault, François Langot, Sébastien Ménard, Thepthida Sopraseuth

12-8. Job Polarization in Aging Economies

Eva Moreno - Galbis, Thepthida Sopraseuth

11-1. The French "Earned Income Supplement" (RSA) and back-to-work incentives

Denis Anne, Yannick L'Horty

11-2. The effect of place of residence on access to employment: a field experiment on qualified young job applicants in Ile-de-France

Yannick L'Horty, Emmanuel Duguet, Loïc du Parquet, Pascale Petit, Florent Sari

11-3. Why is there a faster return to work near the border?

Jonathan Bougard

11-4. Residential Discrimination and Ethnic Origin: An experimental assessment in the Paris suburbs

Emmanuel Duguet, Yannick L'Horty, Pascale Petit

11-5. The Fateful Triangle : Complementarities between product, process and organisational innovation in the UK and France

Gérard Ballot, Fathi Fakhfakh, Fabrice Galia, and Ammon Salter

11-6. How important is innovation? A Bayesian factor-augmented productivity model on panel data

Georges Bressona, Jean-Michel Etienne, Pierre Mohnen

11-7. Fiscal Shocks in a Two Sector Open Economy

Olivier Cardi, Romain Restout

11-8. Productivity, Capital and Labor in Labor-Managed and Conventional Firms

Fathi Fakhfakh, Virginie Pérotin, Mónica Gago

11-9. What is the Natural Weight of the Current Old ?

Damien Gaumont, Daniel Leonard

11-10. Routinization-Biased Technical Change, Globalization and Labor Market Polarization: Does Theory Fit the Facts?

Jaewon Jung, Jean Mercenier

The TEPP Institute

The CNRS **Institute for Labor Studies and Public Policies** (the TEPP Institute, FR n°3435 CNRS) gathers together research centres specializing in economics and sociology:

- **l'Equipe de Recherche sur les Marchés, l'Emploi et la Simulation** (Research Team on Markets, Employment and Simulation), **ERMES**, University of Paris II Panthéon-Assas
- the **Centre d'Etudes des Politiques Economiques de l'université d'Evry** (Research Centre focused on the analysis of economic policy and its foundations and implications), **EPEE**, University of Evry Val d'Essonne
- the **Centre Pierre Naville** (Research on Work and Urban Policies), **CPN**, University of Evry Val d'Essonne
- **l'Equipe de Recherche sur l'Utilisation des Données Temporelles en Economie** (Research Team on Use of Time Data in Economics), **ERUDITE**, University of Paris-Est Créteil and University of Paris-Est Marne-la-Vallée
- the **Groupe d'Analyse des Itinéraires et des Niveaux Salariaux** (The Group on Analysis of Wage Levels and Trajectories), **GAINS**, University of the Maine

The TEPP Institute brings together 147 researchers and research professors and 100 PhD students who study changes in work and employment in relation to the choices made by firms and analyse public policies using new evaluation methods.