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Abstract

The logic of Conditional Beliefs (*CDL*) has been introduced by Board, Baltag and Smets to reason about knowledge and revisable beliefs in a multi-agent setting. In this paper both the semantics and the proof theory for this logic are studied. First, a natural semantics for *CDL* is defined in terms of neighbourhood models, a multi-agent generalisation of Lewis' spheres models, and it is shown that the axiomatization of *CDL* is sound and complete with respect to this semantics. Second, it is shown that the neighbourhood semantics is equivalent to the original one defined in terms of plausibility models, by means of a direct correspondence between the two types of models. On the basis of neighbourhood semantics, a labelled sequent calculus for *CDL* is obtained. The calculus has strong proof-theoretic properties, in particular admissibility of contraction and cut, and it provides a decision procedure for the logic. Furthermore, its semantic completeness is used to obtain a constructive proof of the finite model property of the logic. Finally, it is shown that other doxastic operators can be easily captured within neighbourhood semantics. This fact provides further evidence of the naturalness of the neighbourhood semantics for the analysis of epistemic/doxastic notions.[1]

1 Introduction

Modal epistemic logic has been studied for a long time in formal epistemology, computer science, and notably in artificial intelligence. In this logic, to each agent i is associated a knowledge modality K_i, so that the formula K_iA expresses that "agent i knows A." Through agent-indexed modal operators, epistemic logic can be used to reason about the mutual knowledge of a set of agents. The logic has been further extended by other modalities to encode various types of combined knowledge of agents (e.g. common knowledge). However, knowledge is not the only propositional attitude, and belief is equally significant to reason about epistemic interaction among agents. Board (2004), and then

[1] This work is a substantial extension of the article by the same authors *The logic of conditional belief: neighbourhood semantics and sequent calculus* in L. Beklemishev, S. Demri and A. Máté Editors, AIML 2016, vol. 11, pp 322-341.

Baltag and Smets (2006; 2008; 2008), have proposed a logic called *CDL* (Conditional Doxastic Logic) for modelling both belief and knowledge in a multi-agent setting. The essential feature of beliefs is that they are *revisable* whenever the agent learns new information. To capture the revisable nature of beliefs, *CDL* contains the conditional belief operator $Bel_i(C|B)$, the meaning of which is that agent i would believe C in case she learnt B. The conditional belief has an hypothetical meaning: if agent i learnt B, she would believe that C held in the state of the world *before* the act of learning B, i.e. the connective is employed to represent how an agent would react in response to an hypothetical situation. For this reason Baltag and Smets qualify this logic as "static" in contrast to "dynamic" epistemic logic, where the very act of learning (by some form of announcement) may change the agent's beliefs. The logic *CDL* in itself is used as the basic formalism to study further dynamic extensions of epistemic logics, determined by several kinds of epistemic/doxastic actions. Notice that both unconditional beliefs and knowledge can be defined in *CDL*: Bel_iB (agent i believes B) as $Bel_i(B|\top)$, K_iB (agent i knows B) as $Bel_i(\bot|\neg B)$, the latter meaning that i considers impossible (inconsistent) to learn $\neg B$.

The logic of conditional belief has been significantly employed in game theory (Stalnaker, 1998). This logic is suitable to describe game models, i.e. idealized *static* models which represent games. In this setting, the operators of simple belief and knowledge account for a player's doxastic and epistemic attitudes, whereas the conditional belief operator is employed to represent the choices a player maintains as possible at a certain stage, i.e. the strategies a player would apply in response to other player's choices.

More generally, the conditional belief operator is suited to represent the states of belief an agent would form in response to an hypothetical situation; thus, *CDL* is able to give a complete representation of an agent's epistemic and doxastic attitudes at a given moment of time. In order to illustrate the difference between the conditional belief operator $Bel_i(B|A)$ and the simple belief operator $Bel_i(A \supset B)$, consider the following (modified) example from Stalnaker (1998). Let agent i have the belief that Jones is a coward, formalized as $Bel_iC(j)$. Now, we want to express the fact that if the agent is to learn that Jones has been sent to battle, he would no longer believe that he is a coward (since only brave men are sent to battle). If we expressed this fact with the simple belief operator we would end up in a contradiction, because from $\neg Bel_i(S(j) \supset C(j))$ we conclude $\neg Bel_iC(j)$. However, if we express it as $\neg Bel_i(C(j)|S(j))$, we do not end up in contradiction, since $\neg Bel_iC(j)$ cannot be derived (this can be verified using either the axiom system in Subsection 2.1 or the sequent calculus in Section 3).

The axiomatization of the operator Bel_i in *CDL* internalises the well-known AGM postulates of belief revision[2]. This is something we include, without modification, in our treatment. However, differently from what has been previously done in the literature, we provide a semantics for *CDL* based on *neighbourhood* models. These models are often used in the interpretation of non-normal modal logics; in the present setting they can be seen as a multi-agent generalization of Lewis' sphere models for counterfactual logics.

[2]We cannot mention here the vast literature on the relation between belief revision, conditional logics, the Ramsey Test, and Gärdenfors Triviality Result.

In these models to each world x and agent i is associated a set $I_i(x)$ of nested sets of worlds; each set $\alpha \in I_i(x)$ represents, so to say, a relevant piece of information that can be used to establish the truth of an epistemic/doxastic statement. We provide a direct completeness proof of the axiomatization for *CDL* with respect to our semantics.

In the literature the semantic interpretation of *CDL* is usually defined in terms of epistemic plausibility models, where to each agent i is associated an equivalence relation $\sim_i$ and a well-founded pre-order $\preceq_i$ on worlds. The former relation models knowledge and is used to interpret epistemic indistinguishability of states, whereas the latter relation models the conditional belief. To this aim, the relation assesses the relative plausibility of worlds according to an agent i; then, it holds that i believes B conditionally on A in a world x if B holds in *the most plausible worlds* accessible from x in which A holds, the "most plausible worlds" for an agent i being the $\preceq_i$-minimal ones. This semantic approach has been dominant in the studies of *CDL*; in addition to Board (2004) and Baltag and Smets (2008) we mention works by Pacuit (2013), van Ditmarsch et al. (2008) and Demey (2011).

We prove that the semantics defined in terms of neighbourhood models is equivalent to the one defined with epistemic plausibility models. This result does not come as totally unexpected: for the mono-agent case, it was suggested by Board (2004), Pacuit (2013), Marti et al. (2013), based on an old result about the correspondence between partial orders and Alexandroff topologies (Alexandroff, 1937). We detail the correspondence for the multi-agent case. We argue that neighbourhood models provide by themselves a terse interpretation of the epistemic and doxastic modalities, abstracting away the relational information specified in plausibility models. Moreover, it is worth noticing that in these models the interpretation of unconditional beliefs and knowledge results in the standard universal/existential neighbourhood modalities.

Up to this moment, the logic *CDL* has been studied only from a semantic point of view, and no proof system or calculus has been given. Our main goal is to provide one. On the basis of neighbourhood semantics we develop a labelled sequent calculus, called **G3CDL**, following the general methodology of Negri (2005) to develop labelled calculi for modal logics. Similarly to Negri and Olivetti (2015), the calculus **G3CDL** makes use of world and neighbourhood labels, thereby importing the semantics, limited to the essential, into the syntax. In **G3CDL**, each connective is handled by symmetric left/right rules, whereas the properties of neighbourhood models are handled by additional rules independent of the language of *CDL*. The resulting calculus is analytic and enjoys strong proof-theoretical properties, the most important being admissibility of cut and contraction, which we prove syntactically. We show that the adoption of a standard strategy for the calculus **G3CDL** provides a decision procedure for *CDL*. We also prove the semantic completeness of the calculus: it is possible to extract from a failed derivation a finite countermodel of the initial formula. This result combined with the soundness of the calculus yields a constructive proof of the finite model property of *CDL*.

The paper is organized as follows: In Section 2 we present the logic *CDL*, an axiomatization, and neighbourhood models for it. In Section 3 we give the rules of sequent calculus **G3CDL**, and in Section 4 we provide proofs of soundness, termination and

completeness. Section 5 contains the proof of equivalence between preferential models and neighbourhood models for *CDL*. Finally, in the last section we take into account other belief operators studied in the literature, such as strong and safe belief (Baltag and Smets, 2008), and we extend both the interpretation in neighbourhood models and the sequent calculus to cover also these cases. The fact that we can easily accommodate these further operators in neighbourhood semantics gives further evidence of the naturalness of this semantics in the analysis of epistemic and doxastic notions.

2 The logic of conditional beliefs

In this section, we first recall an axiomatization for the logic of conditional beliefs and present a new semantics, given in terms of neighbourhood models, for this logic. We then prove soundness and completeness of the axiomatization with respect to this class of models.

In the literature, the semantics of *CDL* is usually defined in terms of epistemic plausibility models. A presentation of these models and a proof of equivalence between plausibility and neighbourhood models will be shown in Section 5.

2.1 Axiomatization

The language of *CDL* is defined from a denumerable set of atoms Atm by means of propositional connectives and the conditional operator Bel_i, where the index i ranges over a set of agents $\mathcal{A}$. In the following, P denotes an atomic formula and i an agent. The formulas of the language are generated according to the following definition:

$$A \ := \ P \mid \bot \mid \neg B \mid B \wedge C \mid B \vee C \mid B \supset C \mid Bel_i(C|B)$$

In the following, let $\wedge$ and $\vee$ bind stronger than $\supset$ and Bel_i. The conditional belief operator $Bel_i(C|B)$ is read as "agent i believes C, given B". As mentioned in the introduction, we may define the modalities of unconditional belief and knowledge in terms of conditional belief as follows:

$Bel_i A =_{def} Bel_i(A|\top)$ (belief)
$K_i A =_{def} Bel_i(\bot|\neg A)$ (knowledge)

An equivalent second-order characterization of knowledge is $K_i A$ iff for all B we have $Bel(A|B)$, meaning that A will persist as a belief no matter what is learnt. Observe that in the definition of the operator of conditional belief the "given B" part is to be interpreted as "in case B is added to the set of belief". In other words, B is to be intended as a new belief, and not as a new knowledge. Interpreting B as knowledge would lead to a circularity in the definition of knowledge in terms of conditional belief; furthermore, this operation is illegitimate, since $Bel_i(K_i B|B)$ is not derivable in the system. An axiomatization of *CDL* has been discussed in Board (2004), Pacuit (2013), Baltag and Smets (2008). We present an alternative axiomatization, $\mathcal{H}_{CDL}$, equivalent to the one in Baltag and Smets (2008).

The double implication $A \supset\subset B$ is defined in the standard way as $(A \supset B) \wedge (B \supset A)$.

(AX.0) Any axiomatization of classical propositional calculus including modus ponens

(AX.1) If $\vdash B$, then $\vdash Bel_i(B|A)$

(AX.2) If $\vdash A \supset\subset B$, then $\vdash Bel_i(C|A) \supset\subset Bel_i(C|B)$ *epistemization rule*

(AX.3) $(Bel_i(B|A) \wedge Bel_i(B \supset C|A)) \supset Bel_i(C|A)$ *distribution axiom*

(AX.4) $Bel_i(A|A)$ *success axiom*

(AX.5) $Bel_i(B|A) \supset (Bel_i(C|A \wedge B) \supset\subset Bel_i(C|A))$ *minimal change I*

(AX.6) $\neg Bel_i(\neg B|A) \supset (Bel_i(C|A \wedge B) \supset\subset Bel_i(B \supset C|A))$ *minimal change II*

(AX.7) $Bel_i(B|A) \supset Bel_i(Bel_i(B|A)|C)$ *positive introspection*

(AX.8) $\neg Bel_i(B|A) \supset Bel_i(\neg Bel_i(B|A)|C)$ *negative introspection*

(AX.9) $A \supset \neg Bel_i(\bot|A)$ *consistency*

Note that Axiom 6 can be equivalently replaced by the following axioms:

$$(AX.6a)\ \neg Bel_i(\neg B|A) \supset (Bel_i(C|A) \supset Bel_i(C|A \wedge B)$$
$$(AX.6b)\ Bel_i(C|A \wedge B) \supset Bel_i(B \supset C|A)$$

In terms of Belief Revision, the above axioms may be understood as a sort of epistemic and internalized version of the AGM postulates. Some remarks are in order (we refer to Board, 2004 for a deeper discussion): Distribution axiom (3) and epistemization rule (2) express deductive closure of beliefs. Success axiom (4) ensures that the learned information is included in the set of beliefs. Axioms (5) and (6) encode the *minimal change principle*, a basic assumption of belief revision (see the correspondence with AGM postulates K*7 and K*8). Axiom (9) ensures that learning a true information cannot lead to inconsistent beliefs (it roughly corresponds to AGM K*5). Observe also that it is possible to derive the standard S5 characterization of knowledge from the above axioms:

$$K_iA \supset A \qquad K_iA \supset K_iK_iA \qquad \neg K_iA \supset K_i\neg K_iA$$

We denote by $\vdash_{\mathcal{H}_{CDL}}$ derivability in $\mathcal{H}_{CDL}$, so $\vdash_{\mathcal{H}_{CDL}} A$ means that A is a theorem in $\mathcal{H}_{CDL}$.

2.2 Neighbourhood semantics

We introduce a semantics for *CDL* based on neighbourhood models, or N-models for short. As explained in the introduction, these are a multi-agent version of the *sphere models* introduced by Lewis (1973) for the logic of counterfactuals.

Definition 2.1. Let $\mathcal{A}$ be a set of agents. A *multi-agent neighbourhood model* has the form $\mathcal{M} = \langle W, \{I\}_{i\in\mathcal{A}}, [\![\ ]\!]\rangle$ where W is a nonempty set;[3] for each $i \in \mathcal{A}$, I_i is a neighbourhood function $I_i : W \to \mathcal{P}(\mathcal{P}(W))$ that assigns a collection of sets of worlds to each world in W; $[\![\ ]\!] : Atm \to \mathcal{P}(W)$ is the propositional evaluation.
For $i \in \mathcal{A}$, $x \in W$, I_i satisfies the following properties:

[3] As in the sphere models semantics, W can be thought as the set of *possible worlds*.

- *Nonemptiness*: $\forall\alpha \in I_i(x)\,.\,\alpha \neq \emptyset$;
- *Nesting*: $\forall\alpha,\beta \in I_i(x)\,.\,\alpha \subseteq \beta$ *or* $\beta \subseteq \alpha$;
- *Total reflexivity*: $\exists\alpha \in I_i(x)\,.\,x \in \alpha$;
- *Local absoluteness*: If $\alpha \in I_i(x)$ and $y \in \alpha$ then $I_i(x) = I_i(y)$;
- *Strong closure under intersection*: If $S \subseteq I_i(x)$ and $S \neq \emptyset$ then $\bigcap S \in S$.

The truth conditions for formulas of the language are given inductively by extending the evaluation function $[\![\ ]\!]$ as follows:

- For the Boolean case we have the standard clauses, $[\![A \wedge B]\!] \equiv [\![A]\!] \cap [\![B]\!]$, $[\![\neg A]\!] \equiv W - [\![A]\!]$, $[\![A \vee B]\!] \equiv [\![A]\!] \cup [\![B]\!]$, $[\![A \supset B]\!] \equiv (W - [\![A]\!]) \cup [\![B]\!]$;
- $x \in [\![Bel_i(B|A)\ ]\!]$ iff $(\forall\alpha \in I_i(x)\,.\,\alpha \cap [\![A]\!] = \emptyset)$ *or* $(\exists\beta \in I_i(x)\,.\,\beta \cap [\![A]\!] \neq \emptyset$ *and* $\beta \subseteq [\![A \supset B]\!])$

A formula A is *valid* in $\mathcal{M}$ if $[\![A]\!] = W$. We say that A *is valid in the class of neighbourhood models* if A is valid in every neighbourhood model $\mathcal{M}$.

Observe that total reflexivity entails that every $I_i(x)$ is non-empty, whereas strong closure under intersection always holds in finite models, because of non-emptiness and nesting.
Notational convention: We often write $\mathcal{M}, x \Vdash A$, meaning $x \in [\![A]\!]$. This is further shortened to $x \Vdash A$ whenever $\mathcal{M}$ is unambiguous. Then, we use the local forcing relations introduced in Negri (2017b):

$$\alpha \Vdash^{\forall} A \text{ iff } \forall y \in \alpha\,.\,y \Vdash A \text{ and } \alpha \Vdash^{\exists} A \text{ iff } \exists y \in \alpha\,.\,y \Vdash A$$

With this notation, the truth condition of conditional belief Bel_i becomes:

$$x \Vdash Bel_i(B|A) \text{ iff } (\forall\alpha \in I_i(x)\,.\,\alpha \Vdash^{\forall} \neg A) \textit{ or } (\exists\beta \in I_i(x)\,.\,\beta \Vdash^{\exists} A \textit{ and } \beta \Vdash^{\forall} A \supset B)$$

It is worth noticing that with the notation just introduced, the semantic definition of the unconditional belief and knowledge operators can be stated as follows:

$$x \Vdash Bel_i B \text{ iff } \exists\beta \in I_i(x)\,.\,\beta \Vdash^{\forall} B \quad \text{and} \quad x \Vdash K_i B \text{ iff } \forall\beta \in I_i(x)\,.\,\beta \Vdash^{\forall} B$$

It can be easily shown that the axiomatization is *sound* with respect to neighbourhood semantics:

Theorem 2.1. For any formula A, if $\vdash_{\mathcal{H}_{CDL}} A$, then A is valid in the class of neighbourhood models.

Proof. By induction on the length of the derivation of A defined in the standard way. As an example, we show validity of Axiom 6a, Axiom 7 and Axiom 9.
$(AX.6a)$ $\neg Bel_i(\neg B|A) \supset (Bel_i(C|A) \supset Bel_i(C|A \wedge B)$. Let us assume that there is a model $\mathcal{M}$ which satisfies the antecedent but does not satisfy the consequent of the axiom at world x. Thus, assume $\mathcal{M}, x \Vdash \neg Bel_i(\neg B|A)$, $\mathcal{M}, x \Vdash Bel_i(C|A)$ and $\mathcal{M}, x \nVdash Bel_i(C|A \wedge B)$. We now have the following:

1. $\exists \alpha \in I_i(x) \,.\, \alpha \Vdash^{\exists} A$
2. $\forall \delta \in I_i(x) \,.\, \delta \Vdash^{\exists} A \;\rightarrow\; \delta \Vdash^{\exists} A \wedge B$
3. $(\forall \alpha \in I_i(x) \alpha \Vdash^{\forall} \neg A)$ *or* $(\exists \beta \in I_i(x) \,.\, \beta \Vdash^{\exists} A$ *and* $\beta \Vdash^{\forall} A \supset C)$
4. $\exists \alpha \in I_i(x) \,.\, \alpha \Vdash^{\exists} A \wedge B$
5. $\forall \delta \in I_i(x) \,.\, \delta \Vdash^{\exists} A \wedge B \;\rightarrow\; \delta \Vdash^{\exists} A \wedge B \wedge \neg C$

The first disjunct of 3. does not hold, since it contradicts 1. From the second disjunct of 3, we have that there exists a β_0 such that $\beta_0 \Vdash^{\exists} A$. From 2, we have that $\beta_0 \Vdash^{\exists} A \wedge B$. Then, from 5. we have that $\beta_0 \Vdash^{\exists} A \wedge B \wedge \neg C$. Thus, there exists $y \in \beta_0$ such that $y \Vdash A \wedge B \wedge \neg C$. From 3. we have that all the worlds w in β_0 it holds that $w \Vdash A \supset C$, thus we have a contradiction.
$(AX.7)$ $Bel_i(B|A) \supset Bel_i(Bel_i(B|A)|C)$. Again, suppose $\mathcal{M}, x \Vdash Bel_i(B|A)$ and $\mathcal{M}, x \nVdash Bel_i(Bel_i(B|A)|C)$. Thus,

1. $(\forall \alpha \in I_i(x) \,.\, \alpha \Vdash \neg A)$ *or* $(\exists \beta \in I_i(x) \,.\, \beta \Vdash^{\exists} A$ *and* $\beta \Vdash^{\forall} A \supset B)$
2. $\exists \alpha \in I_i(x) \,.\, \alpha \Vdash^{\exists} C$
3. $\forall \beta \in I_i(x) \,.\, \beta \Vdash^{\exists} C \;\rightarrow\; (\beta \Vdash^{\exists} C$ *and* $\neg Bel_i(B|A)$

From 3. we have 4. $\exists y \in \beta.\, y \Vdash C$ and $y \Vdash \neg Bel_i(B|A)$, i.e. 5. $\exists \gamma \in I_i(y) \,.\, \gamma \Vdash^{\exists} A$ and 6. $\forall \delta \in I_i(y).\, \delta \Vdash^{\exists} A \;\rightarrow\; \delta \Vdash^{\exists} A \wedge \neg B$. By the absoluteness condition applied to 4., since $\beta \in I_i(x)$ and $y \in \beta$, we have $I_i(x) = I_i(y)$. Observe that the first disjunct of 1. does not hold, since it contradicts 5. Thus, the second disjunct of 1. holds, and we have that 7. $\forall \beta \in I_i(x) = I_i(y) \,.\, \beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \supset B$. By 6, $\exists y \in \beta \,.\, y \Vdash A$ and $y \Vdash A \wedge \neg B$, in contradiction with 7. $(AX.9)$ $A \supset \neg Bel_i(\bot|A)$. Suppose $\mathcal{M}, x \Vdash A$ and $\mathcal{M}, x \Vdash Bel_i(\bot|A)$. Thus, $\forall \alpha \in I_i(x) \,.\, \alpha \Vdash^{\forall} \neg A$ or $\exists \beta \in I_i(x) \,.\, \beta \Vdash^{\exists} A$ *and* $\beta \Vdash^{\forall} A \supset \bot$. By total reflexivity the first disjunct does not hold, since $\mathcal{M}, x \Vdash A$ and $\exists \alpha \in I_i(x) \,.\, x \in \alpha$. The second disjunct is contradictory: we have that $\exists y \in \beta \,.\, y \Vdash A$, and that $y \Vdash A \supset \bot$; thus, $y \Vdash \bot$.

□

2.3 Direct completeness proof

The purpose of this section is to show the following:

Theorem 2.2. For any formula A, if A is valid in the class of neighbourhood models, then $\vdash_{\mathcal{H}_{CDL}} A$.

We shall prove the contrapositive: If $\nvdash_{\mathcal{H}_{CDL}} A$, then A is not valid in the class of neighbourhood models. We introduce standard notions and lemmas.

Definition 2.2. Given $S \subseteq \mathcal{L}_{CDL}$, we say that S is *inconsistent* if it has a finite subset $\{B_1, \ldots B_n\} \subseteq S$ such that $\vdash_{\mathcal{H}_{CDL}} B_1 \wedge \ldots \wedge B_n \supset \bot$. We say that S is *consistent* if it is not inconsistent. We say that $S \subseteq \mathcal{L}_{CDL}$ is *maximal consistent* if it is consistent and for any formula $A \notin S$, $S \cup \{A\}$ is inconsistent. Let $MAXCONS(\mathcal{L}_{CDL})$ denote the set of maximal consistent sets of $\mathcal{L}_{CDL}$.

Lemma 2.3. Let $S \subseteq \mathcal{L}_{CDL}$ be consistent, then there exists $X \in MAXCONS(\mathcal{L}_{CDL})$ such that $S \subseteq X$.

Proof. Standard: Let $A_0, A_1, \ldots, A_n \ldots$ be an enumeration of all formulas of $\mathcal{L}_{CDL}$. Define a sequence of sets $X_0 = S$, $S_{i+1} = S_i \cup \{A_i\}$ if A_i is consistent with S_i, and $S_{i+1} = S_i$ if not. Then define $X = \bigcup_i X_i$ and we prove that X is consistent and maximal. □

Lemma 2.4. Let X be in $MAXCONS(\mathcal{L}_{CDL})$. Then the following properties hold:

(i) For any formula A, either $A \in X$ or $\neg A \in X$
(ii) $A \wedge B \in X$ iff $A \in X$ and $B \in X$
(iii) $A \vee B \in X$ iff $A \in X$ or $B \in X$
(iv) $A \in X$ and $A \supset B \in X$ implies $B \in X$
(v) If $\vdash_{\mathcal{H}_{CDL}} A$ then $A \in X$

The following lemma contains a list of theorems of *CDL* tacitly used in subsequent proofs.

Lemma 2.5. The following are derivable in *CDL*:

(1) $Bel_i(B|A) \wedge Bel_i(C|A) \supset Bel_i(B \wedge C|A)$
(2) $Bel_i(\bot|A \vee B) \supset (Bel_i(\bot|A) \wedge Bel_i(\bot|B))$
(3) $Bel_i(\bot|A) \supset Bel_i(\neg A|A \vee B)$
(4) If $\vdash_{\mathcal{H}_{CDL}} A \supset B$ then $\vdash_{\mathcal{H}_{CDL}} Bel_i(B|A)$
(5) $Bel_i(\neg D|C \vee D) \supset Bel_i(\neg D|C)$
(6) $Bel_i(D|C) \supset Bel_i(\bot|\neg Bel_i(D|C))$
(7) $\neg Bel_i(D|C) \supset Bel_i(\bot|Bel_i(D|C))$
(8) $(\neg Bel_i(\neg A|A \vee B) \wedge Bel_i(\neg A|A \vee C)) \supset Bel_i(\neg B|B \vee C)$

Proof. (1). We have $\vdash B \supset (C \supset B \wedge C)$, so by Axiom 2, $Bel_i(B \supset (C \supset B \wedge C)|A)$. By Axiom 3 (twice) and the assumptions we obtain $Bel(B \wedge C|A)$.
(2). It suffices to show that $Bel_i(\bot|A \vee B) \supset Bel_i(\bot|A)$. Since $A \supset (A \vee B)$, by (3) we have (a) $Bel_i(A \vee B|A)$. Then, from the antecedent $Bel_i(\bot|A \vee B)$ we have $(b) Bel_i(\bot \supset A|A \vee B)$, by propositional reasoning. Apply Axiom 2 to (a) and (b) to obtain $Bel_i(\bot|A \vee B) \supset Bel_i(\bot|A)$.
(3). From the antecedent by Axiom 3 we have $(a) Bel_i(\neg A|A)$. Then, again by Axiom 3 we have (b) $Bel_i(A \vee B|A)$. Applying Axiom 5 to (a) and (b) we have (c) $Bel_i(\neg A|A \wedge (A \vee B))$. From Axiom 6.b and propositional reasoning we have (d) $Bel_i(A \vee B|A)$. Apply Axiom 5 to (a) and (d) to obtain (e) $Bel_i(\neg A|\neg A \wedge (A \vee B))$. By Axiom 6.b applied to (c) and (e) yields $Bel_i(\neg A|(A \vee \neg A) \wedge (A \vee B))$, thus $Bel_i(\neg A|A \vee B)$.
(4). By Axiom 1, $\vdash A \supset B$ gives $\vdash Bel_i(A \supset B|A)$. By Axiom 4 we also have $\vdash Bel_i(A|A)$, and by Axiom 3 we conclude that $Bel_i(B|A)$.
(5). By Axiom 4 we have $Bel_i(C \vee D|C \vee D)$. By propositional reasoning, we also have (a) $Bel_i(\neg D \supset C|C \vee D)$. Apply Axiom 3 to (a) and to the antecedent $Bel_i(\neg D|C \vee D)$ to get (b) $Bel_i(C|C \vee D)$. Then, apply Axiom 5 to the antecedent and (b), and obtain (c) $Bel_i(\neg D|C \wedge (C \vee D))$. Formula $Bel_i(C \vee D|C)$ is derivable. Apply Axiom 5 again to (c) and (4) and obtain the consequent $Bel_i(\neg D|C)$.
(6). From Axiom 7 we obtain

(a) $Bel_i(D|C) \supset Bel_i(Bel_i(D|C)|\neg Bel_i(D|C))$

By Axiom 4 we have (b) $Bel_i(\neg Bel_i(D|C)|\neg Bel_i(D|C))$. Then, apply Axiom 5 to (a) and (b) and obtain (c) $Bel_i(D|C) \supset Bel_i(\neg Bel_i(D|C)|Bel_i(D|C) \wedge \neg Bel_i(D|C))$. Finally, from (c) we have by Axiom 1 that $Bel_i(D|C) \supset Bel_i(\bot|Bel_i(D|C))$.
(7). From Axiom 8 we obtain

(a) $\neg Bel_i(D|C) \supset Bel_i(\neg Bel_i(D|C)|Bel_i(D|C))$

Then, Axiom 4 gives (b) $Bel_i(Bel_i(D|C)|Bel_i(D|C))$. Apply (1) to (a) and (b) and obtain (c) $\neg Bel_i(D|C) \supset Bel_i(Bel_i(D|C) \wedge \neg Bel_i(D|C)|Bel_i(D|C)$. Therefore, we have $\neg Bel_i(D|C) \supset Bel_i(\bot|Bel_i(D|C))$.
(8). We prove the following equivalent formulation: $(Bel(\neg A|A \vee C) \wedge \neg Bel(\neg B|B \vee C)) \supset Bel(\neg A|A \vee B)$. First, let us prove the following: *i*) $Bel(\neg A|A \vee C) \supset Bel(\neg A|A \vee B \vee C)$. It holds that (a) $Bel(A \vee B \vee C|A \vee B)$. Apply Axiom 5 to (a) and the antecedent of *i*) and obtain (b) $Bel(\neg A|(A \vee C) \wedge (A \vee B \vee C)$. The following holds for F arbitrary formula: by Axiom 4, $Bel(\neg A \wedge F|\neg A \wedge F)$, and $Bel(\neg A|\neg A \wedge F)$. Let $F = \neg B \wedge (A \vee B \vee C)$. Thus we have $Bel(\neg A|\neg A \wedge \neg B \wedge (A \vee B \vee C))$, from which by propositional reasoning we have c) $Bel(\neg A|\neg(A \vee B) \wedge (A \vee B \vee C))$. From (b), (c) and Axiom 5 we have:

(d) $Bel(\neg A|(\neg(A \vee B) \wedge (A \vee B \vee C)) \vee ((A \vee B) \wedge (A \vee B \vee C))$

By propositional reasoning, this is equivalent to $Bel(\neg A|\neg(A \vee B) \vee (A \vee B) \wedge (A \vee B \vee C)$, which is equivalent to $Bel(\neg A|A \vee B \vee C)$.
Then, we prove *ii*) $\neg Bel(\neg B|B \vee C) \supset \neg Bel(\neg(A \vee B)|A \vee B \vee C)$. We prove the contrapositive: $Bel(\neg(A \vee B)|A \vee B \vee C) \supset Bel(\neg B|B \vee C)$. From the antecedent, derive (e) $Bel(\neg A|A \vee B \vee C)$, and (f) $Bel(\neg B|A \vee B \vee C)$. Apply Axiom 5 to the first premiss and (e), and obtain (g) $Bel(\neg(A \vee B)|B \vee C)$. Apply again Axiom 5, to (f) and (g), to get (h) $Bel(\neg B|(B \vee C) \wedge (A \vee B \vee C))$. Application of the same axiom to (g) and (h) yields (l) $Bel(A \vee B \vee C|B \vee C)$. A final application of Axiom 5 to (h) and (l) yields the desired conclusion $Bel(\neg B|B \vee C)$.
We can now proceed with the proof. Apply *i*) to the first conjunct of the antecedent $Bel(\neg A|A \vee C)$ to obtain (a') $Bel(\neg A|A \vee B \vee C)$. Apply *ii*) to the second conjunct of the antecedent $\neg Bel(\neg B|B \vee C)$ and obtain (b') $\neg Bel(\neg(A \vee B)|A \vee B \vee C)$. Applying Axiom 6 to ($a'$) and ($b'$) yields $Bel(\neg A|(a \vee B \vee C) \wedge (A \vee B))$. Application of the same axiom to this formula and to the derivable formula $Bel(A \vee B \vee C|A \vee B)$ yields the desired conclusion $Bel(\neg A|A \vee B)$.

□

Our goal is to build a canonical neighbourhood model $\mathcal{M}$ such that for any set of formulas S, if S is consistent then it is satisfiable in $\mathcal{M}$. To this regard:

- The worlds W are the maximal consistent sets: $W = MAXCONS(\mathcal{L}_{CDL})$;
- The propositional evaluation is defined in the obvious way; for an atom P:

$$[\![P]\!] = \{X \in MAXCONS(\mathcal{L}_{CDL}) \mid P \in X\}.$$

We have to define the neighbourhoods $I_i(X)$ for an element $X \in W$ (and this is the hard part). We proceed similarly to Lewis (1973), defining the notion of an 'implausible' set of formulas with respect to X. Then, each implausible set S with respect to X will provide a neighbourhood of X, namely the set of elements of $MAXCONS(\mathcal{L}_{CDL})$ which do not contain any formula in S.

Definition 2.3. Let $S \subseteq \mathcal{L}_{CDL}$ and $X \in MAXCONS(\mathcal{L}_{CDL})$. Define S to be an *implausible set* with respect to an agent i and a maximal consistent set X whenever the following conditions hold:

(i) For any formula A, if $Bel_i(\bot|A) \in X$ then $A \in S$;
(ii) If $A \in S$ and $B \notin S$ then $Bel_i(\neg A|A \vee B) \in X$.

We denote by $IMPLA_i(X)$ the set of all implausible sets S with respect to X and i.

Intuitively, condition (i) means that S contain all formulas that lead agent i to believe an absurdity, whereas condition (ii) means that for each $A \in S$ and $B \notin S$, agent i considers B strictly more plausible than A, that is, if i learns $A \vee B$ then she would believe $\neg A$, (whence she would believe B, since from $Bel_i(\neg A|A \vee B)$ follows $Bel_i(B|A \vee B)$).

Lemma 2.6. The following hold:

(i) If $S_1, S_2 \in IMPLA_i(X)$ then $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$;
(ii) $\mathcal{L}_{CDL} \in IMPLA_i(X)$;
(iii) Let $S \in IMPLA_i(X)$ with $S \neq \mathcal{L}_{CDL}$; for any A, if $\vdash_{\mathcal{H}_{CDL}} A$ then $A \notin S$;
(iv) $IMPLA_i(X)$ has a smallest element:

$$S_X^{min} = \{A \in \mathcal{L}_{CDL} | Bel_i(\bot|A) \in X\}.$$

Proof.

(i) Suppose the contrary and let $A \in S_1 \setminus S_2$ and $B \in S_2 \setminus S_1$; by condition (ii) in Definition 2.3 we get $Bel_i(\neg A|A \vee B) \in X$ and $Bel_i(\neg B|A \vee B) \in X$. Using axioms of *CDL*, we have $Bel_i(\neg(A \vee B)|A \vee B) \in X$, and since $Bel_i(A \vee B|A \vee B) \in X$, we get $Bel_i(\bot|A \vee B) \in X$. This implies both $Bel_i(\bot|A) \in X$ and $Bel_i(\bot|B) \in X$, violating condition (i) of definition of implausible set for both S_1 and S_2.
(ii) Obvious, since the antecedent of condition (ii) in Definition 2.3 is always false.
(iii) Suppose the contrary: let $\vdash_{\mathcal{H}_{CDL}} A$ and $A \in S$. Since $S \neq \mathcal{L}_{CDL}$, let $B \notin S$. Then by (ii) we have (1) $Bel_i(\neg A|A \vee B) \in X$. Since $\vdash_{\mathcal{H}_{CDL}} A$ we also have $A \in X$ and $\vdash Bel_i(\bot\ |\neg A)$ and therefore $Bel_i(\bot\ |\neg A) \in X$ so (2) $Bel_i(A|A \vee B) \in X$ by (3) of Lemma 2.5. By (1) and (2) $Bel_i(\bot|A \vee B) \in X$, which implies $Bel_i(\bot|A) \in X$, whence we obtain $A \notin X$, thus a contradiction.
(iv) It suffices to show that S_X^{min} satisfies condition (ii) of the definition of implausible set. Let $A \in S_X^{min}$ then $Bel_i(\bot|A) \in X$, whence for any B, by (3) of Lemma 2.5, $Bel_i(\neg A|A \vee B) \in X$.

□

For any set $S \subseteq \mathcal{L}_{CDL}$ we define:

$$CO(S) = \{Y \in MAXCONS(\mathcal{L}_{CDL}) \,|\, Y \cap S = \emptyset\}$$
$$I_i(X) = \{CO(S) |\ S \in IMPLA_i(X) \text{ and } S \neq \mathcal{L}_{CDL}\}$$

It trivially holds that $CO(\mathcal{L}_{CDL}) = \emptyset$; furthermore, it can be proved that if $S \in IMPLA_i(X)$ and $S \neq \mathcal{L}_{CDL}$ then $CO(S) \neq \emptyset$ (the proof is similar to the one of the following Lemma 2.7). Observe that the largest neighbourhood is $CO(S_X^{min})$ which contains all Y that do not contain any formula considered "impossible" for X.

The following lemma is similar to Lewis' *Cosphere Lemma* (Lewis, 1973), and will be widely used in the sequel.

Lemma 2.7. Let $\alpha \in I_i(X)$ with $\alpha = CO(S)$ for some $S \in IMPLA_i(X)$. Then for any formula A it holds that $A \in S$ if and only if for all $Y \in \alpha$ it holds $A \notin Y$ (thus $\neg A \in Y$).

Proof. To prove direction ($\Rightarrow$), suppose $A \in S$ then by definition of $\alpha = CO(S)$, for all $Y \in \alpha$ it holds $A \notin Y$.

To prove direction ($\Leftarrow$), suppose that for all $Y \in \alpha = CO(S)$ it holds that $A \notin Y$, and by reductio ad absurdum that $A \notin S$. Let us consider the set $\{\neg B \,|\, B \in S\}$. Suppose first that $\{\neg B \,|\, B \in S\} \cup \{A\}$ is consistent. Then for some $Z \in MAXCONS(\mathcal{L}_{CDL})$, we have $\{\neg B \,|\, B \in S\} \cup \{A\} \subseteq Z$ (Lemma 2.3). We get that $Z \cap S = \emptyset$, so that $Z \in \alpha = CO(S)$. But since $A \in Z$, we have a contradiction with the hypothesis. Thus $\{\neg B \,|\, B \in S\} \cup \{A\}$ is inconsistent; this means that there is a finite set $\{\neg B_1, \ldots, \neg B_n\}$ such that:

$$\vdash_{\mathcal{H}_{CDL}} (\neg B_1 \wedge \ldots \wedge \neg B_n) \supset \neg A$$

which is the same as

$$\vdash_{\mathcal{H}_{CDL}} A \supset (B_1 \vee \ldots \vee B_n).$$

It follows that

(1) $Bel_i(B_1 \vee \ldots \vee B_n | A) \in X$

Then, since each $B_k \in S$ and $A \notin S$, by condition (ii) of Definition 2.3 we have $Bel_i(\neg B_k | A \vee B_k) \in X$. This implies that $Bel_i(\neg B_k | A) \in X$ for each (i), whence

(2) $Bel_i(\neg(B_1 \vee \ldots \vee \neg B_n) | A) \in X$

But (1) and (2) imply $Bel_i(\bot | A) \in X$. Thus by condition (i) $A \in S$ against the assumption $A \notin S$. □

We are finally ready for the main result. Let us define the canonical model $\mathcal{M} = \langle W, I_i, [\![\ ]\!] \rangle$, where $W = MAXCONS(\mathcal{L}_{CDL})$, and $[\![\ ]\!]$, I_i are defined as before. We prove that $\mathcal{M}$ is indeed a multi-agent neighbourhood model and that it gives correctly the truth condition for formulas.

The only property we do not show is *strong closure under intersection*, because we do not (yet) know whether this property holds in the canonical model. However, this

property is irrelevant for completeness, since (1) the axioms of *CDL* are valid in models which do not necessarily satisfy this property, as is shown in the proof of Theorem 2.1, and (2) by the finite model property, shown through the sequent calculus, it follows that if a formula A is satisfiable in a neighbourhood model then A is satisfiable in a finite model, that in itself satisfies the strong intersection property. Thus the class of formulas which are valid in models that satisfy the strong intersection property is the same as the class of formulas that are valid in models that do not necessarily satisfy this property. No formula can distinguish between models that satisfy and those that do not satisfy the strong intersection property. The situation could be different if we considered *strong* completeness, where we are concerned about derivability of logical consequences of an infinite theory, not just of valid formulas.

Proposition 2.8. The model $\mathcal{M} = \langle W, I_i, [\![\ ]\!] \rangle$ defined above is a neighbourhood model.

Proof. We show that the properties of nonemptiness, nesting, total reflexivity, and local absoluteness hold in the model.

Nonemptiness: If $\alpha \in I_i(X)$ we want to show that $\alpha \neq \emptyset$. Let $\alpha = CO(S)$ for some $S \in IMPLA_i(X)$ ($S \neq \mathcal{L}_{CDL}$). We proceed similarly to the ($\Leftarrow$) direction of Lemma 2.7: we consider the set $\{\neg B | B \in S\}$, and prove that it is consistent (by contradiction); thus, there is a $Y \in MAXCONS(\mathcal{L}_{CDL})$ such that $\{\neg B | B \in S\} \subseteq Y$, from which $Y \in \alpha$.

Nesting: Let $\alpha, \beta \in I_i(X)$. Then for some $S_1, S_2 \in IMPLA_i(X)$, $\alpha = CO(S_1)$ and $\beta = CO(S_2)$. By Lemma 2.6, either $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$. In the former case $\beta \subseteq \alpha$, in the latter $\alpha \subseteq \beta$.

Total reflexivity: Given $X \in W$, let us consider the set $S_X^{min} = \{A \in \mathcal{L}_{CDL} | Bel_i(\bot|A) \in X\} \in IMPLA_i(X)$. If $A \in S_X^{min}$ then $Bel_i(\bot|A) \in X$; thus by Axiom (9) $\neg A \in X$, whence $A \notin X$. We have shown that $S_X^{min} \cap X = \emptyset$, thus $X \in CO(S_X^{min})$.

Local absoluteness: Let $\alpha \in I_i(X)$ and $Y \in \alpha$; we have to show that $I_i(X) = I_i(Y)$. To this purpose it is enough to show that $IMPLA_i(X) = IMPLA_i(Y)$. To prove this it suffices to show that for any formulas C, D we have $Bel_i(D|C) \in X$ if and only if $Bel_i(D|C) \in Y$, since the conditions (i) and (ii) in the definition of implausible set only involve formulas of this form (including the particular case of $D = \bot$). We know that $\alpha \subseteq CO(S_X^{min})$. From $Bel_i(D|C) \in X$, and from (7), (8) of Lemma 2.5 it follows that $Bel_i(\bot|\neg Bel_i(D|C)) \in X$ Thus $\neg Bel_i(D|C)) \in S_X^{min}$, and since $Y \in CO(S_X^{min})$, we have that $\neg Bel_i(D|C) \notin Y$, so that $Bel_i(D|C) \in Y$. Conversely, suppose that $Bel_i(D|C) \notin X$, then $\neg Bel_i(D|C) \in X$, thus also $Bel_i(\bot|Bel_i(D|C)) \in X$. We have that $Bel_i(D|C) \in CO(S_X^{min})$, and since $Y \in CO(S_X^{min})$ we finally obtain $Bel_i(D|C) \notin Y$. □

Here is the main proposition.

Proposition 2.9. Given the canonical model $\mathcal{M} = \langle W, I_i, [\![\ ]\!] \rangle$ defined above, for any formula A and any $X \in W$, we have $X \Vdash A$ if and only if $A \in X$.

Proof. By induction on the complexity of A, defined as follows (see also Definition 3.1): $\mathtt{w}(P) = \mathtt{w}(\bot) = 1$; $\mathtt{w}(\neg A) = \mathtt{w}(A) + 2$; $\mathtt{w}(A \circ B) = \mathtt{w}(A) + \mathtt{w}(B) + 1$ for $\circ$ conjunction, disjunction, or implication; $\mathtt{w}(Bel_i(B|A)) = \mathtt{w}(A) + \mathtt{w}(B) + 2$.

The base case (A is atomic) holds by definition. The inductive cases of Boolean combinations easily follow by the properties of maximal consistent sets. The only interesting case is the one of $A = Bel_i(D|C)$.

Suppose that $X \Vdash Bel_i(D|C)$. Thus either (1) for each $\alpha \in I_a(X)$, $\alpha \Vdash^{\forall} \neg C$ or (2) there is there is $\alpha \in I_a(X)$ such that $\alpha \Vdash^{\exists} C$ and $\alpha \Vdash^{\forall} C \supset D$. In case (1), let us consider $\alpha = CO(S_X^{min})$. We have that for all $Y \in \alpha$, $Y \nVdash C$, thus by inductive hypothesis, $C \notin Y$. By Lemma 2.7, we get $C \in S_X^{min}$, thus $Bel_i(\bot|C) \in X$, whence also $Bel_i(D|C) \in X$. In case (2), let $\alpha = CO(S)$ for some $S \in IMPLA_i(X)$. Then, since $\alpha \Vdash^{\exists} C$ for some $Y \in \alpha$, we have $Y \Vdash C$; thus by inductive hypothesis, $C \in Y$. By Lemma 2.7, $C \notin S$. On the other hand $\alpha \Vdash^{\forall} C \supset D$, that is $\alpha \Vdash^{\forall} \neg(C \wedge \neg D)$, similarly to case (1). Employing Lemma 2.7 and the inductive hypothesis, we get that $(C \wedge \neg D) \in S$. Since $C \notin S$, we have that $Bel_i(\neg(C \wedge \neg D)|C \vee (C \wedge \neg D)) \in X$. But this implies that $Bel_i(\neg(C \wedge \neg D)|C) \in X$, that is $Bel_i(C \supset D)|C) \in X$, and finally $Bel_i(D|C) \in X$.

Conversely, suppose that $Bel_i(D|C) \in X$. We distinguish different cases.

Case (1). Suppose that $Bel_i(\bot|C) \in X$. Consider the largest neighbourhood $\alpha = CO(S_X^{min})$, we have that $C \in S_X^{min}$ then for all $Y \in \alpha$ we have $C \notin Y$, so that by inductive hypothesis, $Y \Vdash \neg C$, thus $\alpha \Vdash^{\forall} \neg C$, but this also holds for any other $\beta \in I_i(X)$, since $\beta \subseteq \alpha$. We can conclude that $X \Vdash Bel_i(D|C)$.

Case (2). Suppose that $Bel_i(\bot|C) \notin X$. Subcase (2.1). Suppose that $Bel_i(\bot|C \wedge \neg D) \in X$. Then again consider $\alpha = CO(S_X^{min})$, we have that $C \notin S_X^{min}$, thus by Lemma 2.7 for some $Y \in \alpha$, $C \in Y$. Thus by inductive hypothesis $Y \Vdash C$, so that $\alpha \Vdash^{\exists} C$. On the other hand $C \wedge \neg D \in S_X^{min}$ and reasoning as in (case 1), we finally get $\alpha \Vdash^{\forall} C \supset D$. We have shown that $X \Vdash Bel_i(D|C)$.

Subcase (2.2). Suppose that $Bel_i(\bot|C \wedge \neg D) \notin X$. This is the most difficult case. Let us consider the following set:

$$S = \{E \in \mathcal{L}_{CDL} | \neg Bel_i(\neg(C \wedge \neg D)|(C \wedge \neg D) \vee E) \in X \text{ or } Bel_i(\bot|E) \in X\}$$

We first show that *a*) $C \wedge \neg D \in S$: to see this suppose on the contrary that it does not, then $Bel_i(\neg(C \wedge \neg D)|(C \wedge \neg D)) \in X$. We obtain that $Bel_i(\bot|(C \wedge \neg D)) \in X$, against the hypothesis of subcase (2.2).

We also show that *b*) $C \notin S$. Suppose on the contrary that $C \in S$; since $Bel_i(\bot|C) \notin X$, it must be

$$\neg Bel_i(\neg(C \wedge \neg D)|(C \wedge \neg D) \vee C) \in X$$

But $C \equiv (C \wedge \neg D) \vee C$, thus we have $\neg Bel_i(\neg(C \wedge \neg D)|C) \in X$, that is $\neg Bel_i(C \supset D|C) \in X$, so that finally $\neg Bel_i(D|C) \in X$ against the hypothesis $Bel_i(D|C) \in X$.

We now show that $S \in IMPLA_i(X)$. Clearly S satisfies condition (i). We want to show that S satisfies also condition (ii). To this purpose let $G \in S$ and $H \notin S$. Since $G \in S$, we have: $Bel_i(\bot|G) \in X$ or $\neg Bel_i(\neg(C \wedge \neg D)|(C \wedge \neg D) \vee G) \in X$.
In the former case we get $Bel_i(\neg G|H \vee G) \in X$ by (4) of Lemma 2.5 fulfilling condition (ii). Otherwise we have (1) $\neg Bel_i(\neg(C \wedge \neg D)|(C \wedge \neg D) \vee G) \in X$. We have that $H \notin S$, which means that: $Bel_i(\bot|H) \notin X$ and (3) $Bel_i(\neg(C \wedge \neg D)|(C \wedge \neg D) \vee H) \in X$.

From (1) and (2) we obtain (by (8) of Lemma 2.5) again $Bel_i(\neg G|H \vee G) \in X$. Thus $S \in IMPLA_i(X)$.

Let us consider $\beta = CO(S)$. We have that $C \notin S$ and $C \wedge \neg D \in S$, as shown above in a) and b). By Lemma 2.7 we have for some $Y \in \beta$, $C \in Y$, whence by inductive hypothesis $Y \Vdash C$ and $\beta \Vdash^{\exists} C$. Similarly by Lemma 2.7 for all $Y \in \beta$, $C \wedge \neg D \notin Y$, whence by inductive hypothesis for all $Y \in \beta$ $Y \Vdash \neg(C \wedge \neg D)$, that is $Y \Vdash C \supset D$, that is $\beta \Vdash^{\forall} C \supset D$. We have shown that $X \Vdash Bel_i(D|C)$.

□

We conclude the proof of the completeness theorem in the standard way. Suppose that $\not\vdash_{\mathcal{H}_{CDL}} A$; then there is $X \in MAXCONS(\mathcal{L}_{CDL})$ such that $\neg A \in X$ and $A \notin X$. We consider the canonical model $\mathcal{M} = \langle W, I_i, [\![\ ]\!] \rangle$, we have that $X \in W$ and by the above proposition $X \not\Vdash A$. Thus A is not valid in $\mathcal{M}$.

3 Sequent calculus

The neighbourhood semantics is used to generate a *labelled* sequent calculus, **G3CDL**, for *CDL*; this generation follows the methodology established in Negri (2005) of internalizing possible worlds semantics into the syntax of a contraction-free sequent system. A neighbourhood semantics is, however, more general than a standard (relational) possible worlds semantics, and there are non-trivial issues to be faced when the internalization method is applied. A methodological discussion on the stages needed to find the rules of a well-behaved sequent calculus starting from the meaning explanation of the logical constants in terms of a neighbourhood semantics is given in Negri (2017a). The steps needed to establish the structural properties of sequent calculi based on neighbourhood semantics are carried over for some basic non-normal modal systems in Negri (2017b).

The calculus **G3CDL** displays two kinds of labels: labels for worlds $x, y \ldots$ and labels for neighbourhoods $a, b \ldots$, as in the ground calculus for neighbourhood semantics introduced in Negri (2017b).

The meaning of the expressions employed in the calculus is defined as follows, where on the right side we assume that x ranges over possible worlds and a, b over neighbourhoods of possible worlds:

$$
\begin{array}{lcl}
a \Vdash^{\exists} A & \equiv & \exists x \,.\, x \in a \ \textit{and}\ x \Vdash A \\
a \Vdash^{\forall} A & \equiv & \forall x \,.\, x \in a \rightarrow x \Vdash A \\
x \Vdash_i B|A & \equiv & \exists c \,.\, c \in I_i(x) \ \textit{and}\ c \Vdash^{\exists} A \ \textit{and}\ c \Vdash^{\forall} A \supset B \\
x : Bel_i(B|A) & \equiv & \forall a \in I_i(x) \,.\, a \Vdash^{\exists} A \rightarrow (\exists b \in I_i(x) \,.\, b \Vdash^{\exists} A \ \textit{and}\ b \Vdash^{\forall} A \supset B)
\end{array}
$$

Here $\Vdash$ denotes the forcing condition of neighbourhood semantics; to distinguish the semantic notion and its syntactic counterpart and for the sake of a more compact notation, we use a colon in the labelled calculus.

The rules of **G3CDL** can be found in Figure 3. The propositional rules are those of a classical propositional system, decorated with labels. The rules for local forcing are defined as in Negri and Olivetti (2015).

Initial sequents

$x : P, \Gamma \Rightarrow \Delta, x : P$

Rules for local forcing

$$\frac{x \in a, \Gamma \Rightarrow \Delta, x : A}{\Gamma \Rightarrow \Delta, a \Vdash^{\forall} A} \; R\Vdash^{\forall} \textit{(x fresh)} \qquad \frac{x : A, x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta} \; L\Vdash^{\forall}$$

$$\frac{x \in a, \Gamma \Rightarrow \Delta, x : A, a \Vdash^{\exists} A}{x \in a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A} \; R\Vdash^{\exists} \qquad \frac{x \in a, x : A, \Gamma \Rightarrow \Delta}{a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta} \; L\Vdash^{\exists} \textit{(x fresh)}$$

Propositional rules

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A \wedge B, \Gamma \Rightarrow \Delta} \; L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, x : A \quad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \wedge B} \; R\wedge$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \quad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta} \; L\vee \qquad \frac{\Gamma \Rightarrow \Delta, x : A \vee B}{\Gamma \Rightarrow \Delta, x : A, x : B} \; R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad x : B, \Gamma \Rightarrow \Delta}{x : A \supset B, \Gamma \Rightarrow \Delta} \; L\supset \qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \supset B} \; R\supset$$

$$\frac{}{\bot, \Gamma \Rightarrow \Delta} \; L\bot$$

Rules for conditional belief

$$\frac{a \in I_i(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_i B|A}{\Gamma \Rightarrow \Delta, x : \mathit{Bel}_i(B|A)} \; RB\textit{(a fresh)}$$

$$\frac{a \in I_i(x), x : \mathit{Bel}_i(B|A), \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A \quad x \Vdash_i B|A, a \in I_i(x), x : \mathit{Bel}_i(B|A), \Gamma \Rightarrow \Delta}{a \in I_i(x), x : \mathit{Bel}_i(B|A), \Gamma \Rightarrow \Delta} \; LB$$

$$\frac{a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\exists} A \quad a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\forall} A \supset B}{a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A} \; RC$$

$$\frac{a \in I_i(x), a \Vdash^{\exists} A, a \Vdash^{\forall} A \supset B, \Gamma \Rightarrow \Delta}{x \Vdash_i B|A, \Gamma \Rightarrow \Delta} \; LC\textit{(a fresh)}$$

Rules for inclusion

$$\frac{a \subseteq a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \; \mathit{Ref} \qquad \frac{c \subseteq a, c \subseteq b, b \subseteq a, \Gamma \Rightarrow \Delta}{c \subseteq b, b \subseteq a, \Gamma \Rightarrow \Delta} \; \mathit{Tr} \qquad \frac{x \in a, a \subseteq b, x \in b, \Gamma \Rightarrow \Delta}{x \in a, a \subseteq b, \Gamma \Rightarrow \Delta} \; L\subseteq$$

Rules for semantic conditions

$$\frac{a \subseteq b, a \in I_i(x), b \in I_i(x), \Gamma \Rightarrow \Delta \quad b \subseteq a, a \in I_i(x), b \in I_i(x), \Gamma \Rightarrow \Delta}{a \in I_i(x), b \in I_i(x), \Gamma \Rightarrow \Delta} \; S$$

$$\frac{x \in a, a \in I_i(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \; T \textit{ (a fresh)}$$

$$\frac{a \in I_i(x), y \in a, b \in I_i(x), b \in I_i(y), \Gamma \Rightarrow \Delta}{a \in I_i(x), y \in a, b \in I_i(x), \Gamma \Rightarrow \Delta} \; A_1 \qquad \frac{a \in I_i(x), y \in a, a \in I_i(y), \Gamma \Rightarrow \Delta}{a \in I_i(x), y \in a, \Gamma \Rightarrow \Delta} \; A_2$$

Figure 1: Rules of **G3CDL**

Each semantic condition on neighbourhood models (Definition 2.1) is in correspondence with a rule in the calculus. Rule (S) corresponds to the property of nesting in Definition 2.1; (T) corresponds to total reflexivity, and (A) to local absoluteness. As for nonemptiness, the property is expressed by the rules for local forcing. The property of strong closure under intersection need not be expressed, because the property holds in finite models and we shall prove that the logic has the finite model property. Moreover, we wish to obtain a calculus in which the contraction rule is height-preseving admissible (Negri, 2005). To this purpose, a few rules keep their principal formula in their premisses: $(L \Vdash^{\forall})$, $(R \Vdash^{\exists})$, (LB) and (RC). Some extra care is needed for rules that may have instances with a duplication of atomic formulas in their conclusion. For such instances of rules, a rule featuring duplicated formulas contracted into one single formula both in the premiss(es) and in the conclusion has to be added to the calculus. This requirement is called the *closure condition* and usually results in some rules to be added to the calculus (possibly none). In our calculus, the rules which are potentially subject to the closure condition are (S) (sphere nesting) and (A_1) (absoluteness). For (S), the instance with the duplication is

$$\frac{a \subseteq a, a \in I_i(x), a \in I_i(x), \Gamma \Rightarrow \Delta \quad a \subseteq a, a \in I_i(x), a \in I_i(x), \Gamma \Rightarrow \Delta}{a \in I_i(x), a \in I_i(x), \Gamma \Rightarrow \Delta} S$$

and the contracted instance is

$$\frac{a \subseteq a, a \in I_i(x), \Gamma \Rightarrow \Delta \quad a \subseteq a, a \in I_i(x), \Gamma \Rightarrow \Delta}{a \in I_i(x), \Gamma \Rightarrow \Delta} S^*$$

This rule does not need to be added to the calculus because it reduces (with two identical premisses one of which is superfluous) to an instance of *Ref.*
As for (A_1), the instance with a duplication has the following form

$$\frac{a \in I_i(x), y \in a, a \in I_i(x), a \in I_i(y), \Gamma \Rightarrow \Delta}{a \in I_i(x), y \in a, a \in I_i(x), \Gamma \Rightarrow \Delta} A_1$$

whereas the contracted instance is rule (A_2); observe that (A_2) is not an instance of any of the pre-existing rules of the calculus so it has to be explicitly added in order to satisfy the closure condition and thus ensure admissibility of contraction.

Example 3.1. We show a derivation of the left-to-right direction of axiom (6). We omit in the final derivation the derivable left premisses of rule (RC) in $\mathcal{D}$ and of rule (LB), as well as the derivable right premiss of $L\supset$.

$$\mathcal{D}: \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y : A \cdots \Rightarrow \ldots y : A \qquad y : B \cdots \Rightarrow \ldots y : B}{y : A, y : B, y \in b, c \in I_i(x), c \Vdash^{\exists} A, b \in I_i(x) \cdots \Rightarrow \ldots y : A \wedge B} R\wedge}{y : A, y : B, y \in b, c \in I_i(x), c \Vdash^{\exists} A, b \in I_i(x) \cdots \Rightarrow \ldots b \Vdash^{\exists} A \wedge B} R\Vdash^{\exists}}{y \in b, c \in I_i(x), c \Vdash^{\exists} A, b \in I_i(x) \cdots \Rightarrow \ldots b \Vdash^{\exists} A \wedge B, y : A \supset \neg B} R\supset, R\supset}{c \in I_i(x), c \Vdash^{\exists} A, b \in I_i(x) \cdots \Rightarrow \ldots b \Vdash^{\exists} A \wedge B, b \Vdash^{\forall} A \supset \neg B} R\Vdash^{\forall}}{c \in I_i(x), c \Vdash^{\exists} A, b \in I_i(x) \cdots \Rightarrow \ldots b \Vdash^{\exists} A \wedge B, x \Vdash_i \neg B | A} RC}{b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset C, a \Vdash^{\exists} A \wedge B \cdots \Rightarrow \ldots x : Bel_i(\neg B | A), b \Vdash^{\exists} A \wedge B} RB$$

$\mathcal{E}$:

$$
\cfrac{\cfrac{\cfrac{\cfrac{z : A \cdots \Rightarrow \ldots z : A \qquad z : c \cdots \Rightarrow \ldots z : C}{z : A \supset C, z : A, z : B, z \in b, b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset C, a \Vdash^{\exists} A \wedge B, \cdots \Rightarrow \ldots z : C}\ L\supset}{z : A, z : B, z \in b, b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset C, a \Vdash^{\exists} A \wedge B \cdots \Rightarrow \ldots z : C}\ L\Vdash^{\forall}}{z \in b, b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset C, a \Vdash^{\exists} A \wedge B \cdots \Rightarrow \ldots z : (A \wedge B) \supset C}\ R\supset, L\wedge}{b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset C, a \Vdash^{\exists} A \wedge B \cdots \Rightarrow \ldots b \Vdash^{\forall} (A \wedge B) \supset C}\ R\Vdash^{\forall}
$$

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\vdots\ \mathcal{D} \qquad \vdots\ \mathcal{E}}{b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset C, a \in I_i(x), a \Vdash^{\exists} A \wedge B, x : Bel_i(C|A) \Rightarrow x : Bel_i(\neg B|A), x \Vdash_i C|A \wedge B}\ RC}{x \Vdash_i C|A, a \in I_i(x), a \Vdash^{\exists} A \wedge B, x : Bel_i(C|A) \Rightarrow x : Bel_i(\neg B|A), x \Vdash_i C|A \wedge B}\ LC}{a \in I_i(x), a \Vdash^{\exists} A \wedge B, x : Bel_i(C|A) \Rightarrow x : Bel_i(\neg B|A), x \Vdash_i C|A \wedge B}\ LB}{x : Bel_i(C|A) \Rightarrow x : Bel_i(\neg B|A), x : Bel_i(C|A \wedge B)}\ RB}{x : \neg(Bel_i(\neg B|A)), x : Bel_i(C|A) \Rightarrow x : Bel_i(C|A \wedge B)}\ L\supset
$$

3.1 Rules for knowledge and simple belief

As recalled in Section 2.1, the modal operators of knowledge and simple belief can be defined in terms of the conditional belief operator: $K_i A = Bel_i(\bot|\neg A)$ and $Bel_i A = Bel_i(A|\top)$. By adopting these definitions, we can extend **G3CDL** by the rules displayed below which correspond to the interpretation of these two modalities in neighbourhood semantics.

Rules for knowledge and simple belief

$$
\frac{a \in I_i(x), \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}{\Gamma \Rightarrow \Delta, x : K_i A}\ LK\,(a\ fresh) \qquad \frac{a \in I_i(x), x : K_i A, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{a \in I_i(x), x : K_i A, \Gamma \Rightarrow \Delta}\ RK
$$

$$
\frac{a \in I_i(x), \Gamma \Rightarrow \Delta, x : Bel_i A, a \Vdash^{\forall} A}{a \in I_i(x), \Gamma \Rightarrow \Delta, x : Bel_i A}\ LSB \qquad \frac{a \in I_i(x), a \Vdash^{\forall} A \Rightarrow \Delta}{x : Bel_i A, \Gamma \Rightarrow \Delta}\ RSB\,(a\ fresh)
$$

These rules are *admissible* in **G3CDL**, i.e. whenever the premiss is derivable, also the conclusion is. This is shown through a derivation that uses rules of **G3CDL** and other rules (such as weakening and cut) which will be later shown to be admissible. By way of example, we show the case of rule RK.

$$
\cfrac{\cfrac{a \in I_i(x), x : Bel_i(\bot|\neg A), a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{a \in I_i(x), x : Bel_i(\bot|\neg A), \Gamma \Rightarrow \Delta, a \Vdash^{\exists} \neg A}\ * \qquad \cfrac{\cfrac{\cfrac{\cfrac{\ldots y : \neg A \Rightarrow y : \neg A \ldots \qquad y : \bot \Rightarrow \ldots}{y : \neg A \supset \bot, y : \neg A, y : \neg A \ldots \Gamma \Rightarrow \Delta}\ L\supset}{y \in b, y : \neg A, b \Vdash^{\forall} \neg A \supset \bot \ldots \Gamma \Rightarrow \Delta}\ L\Vdash^{\forall}}{b \in I_i(x), b \Vdash^{\exists} \neg A, b \Vdash^{\forall} \neg A \supset \bot \ldots \Gamma \Rightarrow \Delta}\ L\Vdash^{\exists}}{x \Vdash_i \bot|\neg A \ldots \Gamma \Rightarrow \Delta}\ LC}{a \in I_i(x), x : Bel_i(\bot|\neg A), \Gamma \Rightarrow \Delta}\ LB
$$

In the above derivation, the left premiss of LB is derivable from the premiss of RK; the right premiss of LB is derivable from initial sequents. The step denoted by $(*)$ is justified by the rules for negation.

3.2 Structural properties

In this section we prove the principal structural properties of the calculus, among which the admissibility of cut. Admissibility of cut is a fundamental property, as it ensures that the calculus is consistent (whence the logic); moreover, it ensures the subformula property, meaning that no new formulas are introduced in backwards proof search of a given sequent.

The syntactic proof of cut admissibility requires to establish several properties, which are also important from a computational viewpoint. Basically, we have to show that the structural rules are height-preserving admissible, that label substitution is admissible and that logical rules are invertible.

We start by defining a notion of weight of labelled formulas:[4]

Definition 3.1. The *label* of formulas of the form $x : A$ is x. The *label* of formulas of the form $a \Vdash^{\forall} A$ and $a \Vdash^{\exists} A$ is a. The *label* of a formula $\mathcal{F}$ will be denoted by $l(\mathcal{F})$. The *pure part* of a labelled formula $\mathcal{F}$ is the part without the label and without the forcing relation, either local ($\Vdash^{\exists}$, $\Vdash^{\forall}$) or worldwise (:) and will be denoted by $p(\mathcal{F})$.

The *weight* of a labelled formula $\mathcal{F}$ is the pair $(\mathtt{w}(p(\mathcal{F})), \mathtt{w}(l(\mathcal{F})))$, where:

(i) For all world labels x and all neighbourhood labels a, $\mathtt{w}(x) = 0$, $\mathtt{w}(a) = 1$;
(ii) $\mathtt{w}(P) = \mathtt{w}(\bot) = 1$; $\mathtt{w}(\neg A) = \mathtt{w}(A) + 2$; $\mathtt{w}(A \circ B) = \mathtt{w}(A) + \mathtt{w}(B) + 1$ for $\circ$ conjunction, disjunction, or implication; $\mathtt{w}(B|A) = \mathtt{w}(A) + \mathtt{w}(B) + 2$; $\mathtt{w}(Bel_i(B|A)) = \mathtt{w}(B|A) + 1$.

Weights of labelled formulas are ordered lexicographically.

It is clear from the definition of weight that the weight gets decreased if we move from a formula labelled by a neighbourhood label to the same formula labelled by a world label, or if we move (regardless of the label) to a formula with a pure part of strictly smaller weight.

Lemma 3.1. Sequents of the following form are derivable in **G3CDL** for arbitrary neighbourhoods labels a, b and formulas A and B:

(i) $a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A$
(ii) $a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A$
(iii) $x \Vdash_i B|A, \Gamma \Rightarrow \Delta, x \Vdash_i B|A$
(iv) $x : A, \Gamma \Rightarrow \Delta, x : A$

Proof. All cases are proved by simultaneous induction on formula weight.
(i) We have the following inference

$$\dfrac{\dfrac{x : A, x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, x : A}{x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, x : A}\, L\Vdash^{\forall}}{a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}\, R\Vdash^{\forall}$$

The topsequent is derivable by inductive hypothesis because $\mathtt{w}(x : A) < \mathtt{w}(a \Vdash^{\forall} A)$.

[4] A different notion of weight, which does not take labels into account, will be introduced and used in 5.2 to show that the "new" and the "old" semantics have the same class of valid formulas.

(ii) Similar, with $L \Vdash^{\exists}$ and $R \Vdash^{\exists}$ in place of $R \Vdash^{\forall}$ and $L \Vdash^{\forall}$, respectively, using $\mathtt{w}(x : A) < \mathtt{w}(a \Vdash^{\exists} A)$.

(iii) By the derivation

$$\dfrac{\begin{array}{c} a \in I_i(x), a \Vdash^{\exists} A, a \Vdash^{\forall} A \supset B, \Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\exists} A \\ \vdots \qquad\qquad a \in I_i(x), a \Vdash^{\exists} A, a \Vdash^{\forall} A \supset B, \Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\forall} A \supset B \end{array}}{\dfrac{a \in I_i(x), a \Vdash^{\exists} A, a \Vdash^{\forall} A \supset B, \Gamma \Rightarrow \Delta, x \Vdash_i B|A}{x \Vdash_i B|A, \Gamma \Rightarrow \Delta, x \Vdash_i B|A}\, LC}\, RC$$

Both topsequents are derivable by inductive hypothesis, since $\mathtt{w}(a \Vdash^{\exists} A) < \mathtt{w}(x \Vdash_i B|A)$ and $\mathtt{w}(a \Vdash^{\forall} A \supset B) < \mathtt{w}(x \Vdash_i B|A)$.

(iv) By induction on the structure of A. If it is atomic or $\bot$, the sequent is initial or conclusion of $L\bot$. If the outermost connective of A is a conjunction or a disjunction, or an implication, the sequent is derivable by application of the respective rules and the inductive hypothesis. If A is a formula of conditional belief we have

$$\dfrac{\begin{array}{c} a \in I_i(x), a \Vdash^{\exists} A, x : Bel_i(B|A)\Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\exists} A \\ \vdots \qquad\qquad x \Vdash_i B|A, a \in I_i(x), a \Vdash^{\exists} A, x : Bel_i(B|A), \Gamma \Rightarrow \Delta, x \Vdash_i B|A \end{array}}{\dfrac{a \in I_i(x), a \Vdash^{\exists} A, x : Bel_i(B|A)\Gamma \Rightarrow \Delta, x \Vdash_i B|A}{x : Bel_i(B|A), \Gamma \Rightarrow \Delta, x : Bel_i(B|A)}\, RB}\, LB$$

The topsequents are derivable by inductive hypothesis, since $\mathtt{w}(a \Vdash^{\exists} A) < \mathtt{w}(x : Bel_i(B|A))$ and $\mathtt{w}(x \Vdash_i B|A) < \mathtt{w}(x : Bel_i(B|A))$. □

The definition of substitution of labels given in Negri (2005) can be extended in an obvious way—that need not be detailed here—to all the formulas of our language and to neighbourhood labels. With this definition we have, for example, $(a \Vdash^{\exists} A)(b/a) \equiv b \Vdash^{\exists} A$, and $(x \Vdash_i B|A)(y/x) \equiv y \Vdash_i B|A$. The calculus is routinely shown to enjoy the property of height preserving (hp for short) substitution both of world and neighbourhood labels:

Proposition 3.2.

(i) If $\vdash_n \Gamma \Rightarrow \Delta$, then $\vdash_n \Gamma(y/x) \Rightarrow \Delta(y/x)$;
(ii) If $\vdash_n \Gamma \Rightarrow \Delta$, then $\vdash_n \Gamma(b/a) \Rightarrow \Delta(b/a)$.

Proof. By induction on the height of the derivation. If it is 0, then $\Gamma \Rightarrow \Delta$ is an initial sequent or a conclusion of $L\bot$. The same then holds for $\Gamma(y/x) \Rightarrow \Delta(y/x)$ and for $\Gamma(b/a) \Rightarrow \Delta(b/a)$. If the derivation has height $n > 0$, we consider the last rule applied. If $\Gamma \Rightarrow \Delta$ has been derived by a rule without variable conditions, we apply the inductive hypothesis and then the rule. Rules with variable conditions require some care in case the substituted variable coincides with the fresh variable in the premiss. This is the case for the rules $R \Vdash^{\forall}$, $L \Vdash^{\exists}$, RB, LC, T. So, if $\Gamma \Rightarrow \Delta$ has been derived by any of these rules, we apply the inductive hypothesis twice to the premiss, first to replace the fresh variable with another fresh variable different, if necessary, from the one we want to substitute, then to make the substitution, and finally we apply the rule. □

Proposition 3.3. The rules of left and right weakening are hp-admissible in **G3CDL**.

Proof. Straightforward induction, with a similar proviso as in the above proof for rules with variable conditions. □

Next, we prove *hp-invertibility* of the rules of **G3CDL**, i.e. for every rule of the form $\frac{\Gamma'\Rightarrow\Delta'}{\Gamma\Rightarrow\Delta}$, if $\vdash_n \Gamma \Rightarrow \Delta$ then $\vdash_n \Gamma' \Rightarrow \Delta'$, and for every rule of the form $\frac{\Gamma'\Rightarrow\Delta' \quad \Gamma''\Rightarrow\Delta''}{\Gamma\Rightarrow\Delta}$ if $\vdash_n \Gamma \Rightarrow \Delta$ then $\vdash_n \Gamma' \Rightarrow \Delta'$ and $\vdash_n \Gamma'' \Rightarrow \Delta''$.

Lemma 3.4. The following hold in **G3CDL**:

1. If $\vdash_n \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A$ then $\vdash_n x \in a, \Gamma \Rightarrow \Delta, x : A$ for any x.
2. If $\vdash_n x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta$ then $\vdash_n x \in a, x : A, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta$.
3. If $\vdash_n x \in a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A$ then $\vdash_n x \in a, \Gamma \Rightarrow \Delta, x : A, a \Vdash^{\exists} A$.
4. If $\vdash_n a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta$ then $\vdash_n x \in a, x : A, \Gamma \Rightarrow \Delta$ for any x.
5. If $\vdash_n \Gamma \Rightarrow \Delta, x : Bel_i(B|A)$ then $\vdash_n a \in I_i(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_i B|A$ for any a.
6. If $\vdash_n a \in I_i(x), x : Bel_i(B|A), \Gamma \Rightarrow \Delta$ then $\vdash_n a \in I_i(x), x : Bel_i(B|A), \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A$ and $x \Vdash_i B|A, a \in I_i(x), x : Bel_i(B|A), \Gamma \Rightarrow \Delta$.
7. If $\vdash_n a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A$ then $\vdash_n a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\exists} A$ and $\vdash_n a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\forall} A \supset B$.
8. If $\vdash_n x \Vdash_i B|A, \Gamma \Rightarrow \Delta$ then $\vdash_n a \in I_i(x), a \Vdash^{\exists} A, a \Vdash^{\forall} A \supset B, \Gamma \Rightarrow \Delta$ for any a.
9. If $\vdash_n a \in I_i(x), b \in I_i(x), \Gamma \Rightarrow \Delta$ then $\vdash_n a \subseteq b, a \in I_i(x), b \in I_i(x), \Gamma \Rightarrow \Delta$ and $\vdash_n b \subseteq a, a \in I_i(x), b \in I_i(x), \Gamma \Rightarrow \Delta$.
10. If $\vdash_n \Gamma \Rightarrow \Delta$ then $\vdash_n x \in a, a \in I_i(x), \Gamma \Rightarrow \Delta$ for any x and a.
11. If $\vdash_n a \in I_i(x), y \in a, b \in I_i(x), \Gamma \Rightarrow \Delta$ then $\vdash_n a \in I_i(x), y \in a, b \in I_i(x), b \in I_i(y), \Gamma \Rightarrow \Delta$.
12. If $\vdash_n x \in a, a \subseteq b, \Gamma \Rightarrow \Delta$ then $\vdash_n x \in a, a \subseteq b, x \in b, \Gamma \Rightarrow \Delta$.
13. If $\vdash_n \Gamma \Rightarrow \Delta$ then $\vdash_n a \subseteq a, \Gamma \Rightarrow \Delta$.
14. If $\vdash_n c \subseteq b, b \subseteq a, \Gamma \Rightarrow \Delta$ then $\vdash_n c \subseteq a, c \subseteq b, b \subseteq a, \Gamma \Rightarrow \Delta$.

Proof. We show by means of example the proof of (5), by induction on n. *Base case*: Suppose that $\Gamma \Rightarrow \Delta, x : Bel_i(B|A)$ is an initial sequent or conclusion of $L\bot$. In the former case, since $x : Bel_i(B|A)$ is neither of the form $x : P$ nor of the form $x \in a$, we have that $a \in I_i(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_i B|A$ is an initial sequent as well; in the latter case, it is a conclusion of $L\bot$. *Inductive step*: Assume hp-invertibility up to n, and assume $\vdash_{n+1} \Gamma \Rightarrow \Delta, x : Bel_i(B|A)$. If $x : Bel_i(B|A)$ is principal in the last rule of the derivation, then the premiss is of the form $a \in I_i(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_i B|A$, with a fresh, and has a derivation of height n. We obtain the claim for any a by hp-substitution. Otherwise, $x : Bel_i(B|A)$ is not principal, the last rule has one or two premisses of the form $\Gamma' \Rightarrow \Delta', x : Bel_i(B|A)$ of derivation height $\leq n$. If the rule has a variable condition, before proceeding we have to apply an hp-substitution to avoid that the eigenvariable coincides with a. By inductive hypothesis we have $a \in I_i(x), a \Vdash^{\exists} A, \Gamma' \Rightarrow \Delta', x \Vdash_i B|A$ for each premiss, with derivation height at most n. Thus, by application of the same rule, we have $\vdash_{n+1} a \in I_i(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_i B|A$. □

Lemma 3.5. All the propositional rules of **G3CDL** are hp-invertible.

Proof. Similar to the proof for **G3c** (Theorem 3.1.1 in Negri and von Plato, 2001). □

Therefore, as a general result, we have:

Corollary 3.6. All the rules of **G3CDL** are hp-invertible.

Proof. By Lemmas 3.4, 3.5, and 3.3, the latter because of the general form of the neighbourhood rules. □

The rules of contraction of **G3CDL** have the following form, where $\mathcal{F}$ is either a "relational" atom of the form $a \in I(x)$ or of the form $x \in a$, or a labelled formula of one of the forms $x : A$, $a \Vdash^{\forall} A$, $a \Vdash^{\exists} A$, or a formula of the form $x \Vdash_i B|A$ or $x : Bel_i(B|A)$:

$$\frac{\mathcal{F}, \mathcal{F}, \Gamma \Rightarrow \Delta}{\mathcal{F}, \Gamma \Rightarrow \Delta} LCtr \qquad \frac{\Gamma \Rightarrow \Delta, \mathcal{F}, \mathcal{F}}{\Gamma \Rightarrow \Delta, \mathcal{F}} RCtr$$

Theorem 3.7. The rules of left and right contraction are hp-admissible in **G3CDL**.

Proof. By simultaneous induction on the height of derivation for left and right contraction. If $n = 0$ the premiss is either an initial sequent or a conclusion of a zero-premiss rule. In each case, the contracted sequent is also an initial sequent or a conclusion of the same zero-premiss rule. If $n > 0$, consider the last rule used to derive the premiss of contraction. There are two cases, depending on whether the contraction formula is principal or a side formula in the rule.

1. If the contraction formula is not principal in it, both occurrences are found in the premisses of the rule and they have a smaller derivation height. By inductive hypothesis, they can be contracted and the conclusion is obtained by applying the rule to the contracted premisses.

2. If the contraction formula is principal in it, we distinguish two sub-cases:

2.1. The last rule is one in which the principal formulas appear also in the premiss (such as $L\Vdash^{\forall}$, $R\Vdash^{\exists}$, LB, RC, S, A, Tr, $L\subseteq$). In all these cases we apply the inductive hypothesis to the premiss(es) and then the rule. For example, if the last rule use to derive the premiss of contraction is (RC) we have:

$$\frac{a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, x \Vdash_i B|A, a \Vdash^{\exists} A \quad a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, x \Vdash_i B|A, a \Vdash^{\forall} A \supset B}{a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, x \Vdash_i B|A} RC$$

By inductive hypothesis applied to the premiss, of shorter height, we get $a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\exists} A$ and $a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\forall} A \supset B$ and thus by a step of RC we obtain $a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A$, with the same derivation height of the given premiss of contraction.

2.2. The last rule is one in which the active formulas are proper subformulas of the principal formula and possibly relational atoms (such as the rules for &, $\vee$, $\supset$, $R\Vdash^{\forall}$, $L\Vdash^{\exists}$, RB, LC). In all such cases, we apply hp-invertibility to the premiss(es) of the rule so that we have a duplication of formulas at a smaller derivation height, then apply the

inductive hypothesis (as many times as needed) then the rule in question. For example, if the last rule is RB, we have:

$$\frac{a \in I_i(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_i B|A, x : Bel_i(B|A)}{\Gamma \Rightarrow \Delta, x : Bel_i(B|A), x : Bel_i(B|A)}\ RB\ (a\,fresh)$$

Using hp-invertibility of RB we obtain from the premiss a derivation of height $n-1$ of

$$a \in I_i(x), a \in I_i(x), a \Vdash^{\exists} A, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_i B|A, x \Vdash_i B|A$$

By the inductive hypothesis we get a derivation of the same height of $a \in I_i(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_i B|A$ and application of RB gives a derivation of height n of $\Gamma \Rightarrow \Delta, x : Bel_i(B|A)$. □

Theorem 3.8. Cut is admissible in **G3CDL**.

Proof. By double induction, with primary induction on the weight of the cut formula and subinduction on the sum of the heights of derivations of the premisses of cut. The cases in which the premisses of cut are either initial sequents or obtained through the rules for &, ∨, or ⊃ follow the treatment of Theorem 3.2.3 of Negri and von Plato, 2001. For the cases in which the cut formula is a side formula in at least one rule used to derive the premisses of cut, the cut reduction is dealt with in the usual way by permutation of cut, with possibly an application of hp-substitution to avoid a clash with the fresh variable in rules with variable condition. In all such cases the cut height is reduced.

The only cases we shall treat in detail are those with cut formula principal in both premisses of cut and of the form $a \Vdash^{\forall} A$, $a \Vdash^{\exists} A$, $x \Vdash_i B|A$, $x : Bel_i(B|A)$. We thus have the following cases:

1. The cut formula is $a \Vdash^{\forall} A$, principal in both premisses of cut. We have a derivation of the form

$$\dfrac{\dfrac{x \in a, \Gamma \overset{\mathcal{D}}{\Rightarrow} \Delta, x : A}{\Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}\ R\Vdash^{\forall} \qquad \dfrac{y : A, y \in a, a \Vdash^{\forall} A, \Gamma' \Rightarrow \Delta'}{y \in a, a \Vdash^{\forall} A, \Gamma' \Rightarrow \Delta'}\ L\Vdash^{\forall}}{y \in a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ Cut$$

This is converted into the following derivation:

$$\dfrac{\dfrac{\mathcal{D}(y/x)}{y \in a, \Gamma \Rightarrow \Delta, y : A} \qquad \dfrac{\Gamma \Rightarrow \Delta, a \Vdash^{\forall} A \quad y : A, y \in a, a \Vdash^{\forall} A, \Gamma' \Rightarrow \Delta'}{y \in a, y : A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ Cut_2}{y : A, x \in a, \Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta'}\ Cut_1$$
$$\dfrac{y : A, x \in a, \Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta'}{y \in a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ Ctr^*$$

Here $\mathcal{D}(y/x)$ denotes the result of application of hp-substitution to $\mathcal{D}$, using the fact that x is a fresh variable; compared to the original cut, Cut_1 is a cut of reduced height, Cut_2 is one of reduced weight of cut formula, because $\mathtt{w}(y : A) < \mathtt{w}(a \Vdash^{\forall} A)$, and Ctr^* denote repreated applications of hp-admissible contraction steps.

2. The cut formula is $a \Vdash^{\exists} A$, principal in both premisses of cut. The cut is reduced in a way similar to the one in the case above and the inequality to be used on formula weight is $\mathtt{w}(y : A) < \mathtt{w}(a \Vdash^{\exists} A)$.

3. The cut formula is $x \Vdash_i B|A$, principal in both premisses of cut. The premisses of cut are the following:

$$\frac{a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\exists} A \quad a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\forall} A \supset B}{a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A}\ RC$$

$$\frac{\begin{array}{c}\mathcal{D}\\ b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset B, \Gamma' \Rightarrow \Delta'\end{array}}{x \Vdash_i B|A, \Gamma' \Rightarrow \Delta'}\ LC$$

The conclusion of the cut is the sequent $a \in I_i(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. The derivation is converted into the following:

$$\dfrac{\dfrac{\dfrac{a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\exists} A \quad x \Vdash_i B|A, \Gamma' \Rightarrow \Delta'}{a \in I_i(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta', a \Vdash^{\exists} A}\ Cut_1 \qquad (1)}{a \in I_i(x)^3, \Gamma^2, \Gamma'^3 \Rightarrow \Delta^2, \Delta'^3}\ Cut_4}{a \in I_i(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ Ctr^*$$

Here (1) is the derivation:

$$\dfrac{\dfrac{a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\forall} A \supset B \quad x \Vdash_i B|A, \Gamma' \Rightarrow \Delta'}{a \in I_i(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta', a \Vdash^{\forall} A \supset B}\ Cut_2 \qquad \begin{array}{c}\mathcal{D}(a/b)\\ a \in I_i(x), a \Vdash^{\exists} A, a \Vdash^{\forall} A \supset B, \Gamma' \Rightarrow \Delta'\end{array}}{a \in I_i(x)^2, a \Vdash^{\exists} A, \Gamma, \Gamma'^2 \Rightarrow \Delta, \Delta'^2}\ Cut_3$$

Observe that all the four cuts are of reduced height (Cut_1 and Cut_2) or reduced weight (Cut_3 and Cut_4) because $\mathtt{w}(a \Vdash^{\exists} A) < \mathtt{w}(a \Vdash^{\forall} A \supset B) < \mathtt{w}(x \Vdash_i B|A)$.

4. The cut formula is $x : Bel_i(B|A)$, principal in both premisses of cut. The premisses of cut are the following:

$$\frac{\begin{array}{c}\mathcal{D}\\ b \in I_i(x), b \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_i B|A\end{array}}{\Gamma \Rightarrow \Delta, x : Bel_i(B|A)}\ RB$$

$$\frac{a \in I_i(x), x : Bel_i(B|A), \Gamma' \Rightarrow \Delta', a \Vdash^{\exists} A \quad a \in I_i(x), x \Vdash_i B|A, x : Bel_i(B|A), \Gamma' \Rightarrow \Delta'}{a \in I_i(x), x : Bel_i(B|A), \Gamma' \Rightarrow \Delta'}\ LB$$

The conclusion of cut is the sequent $a \in I_i(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. The cut is converted to four smaller cuts as follows:

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x : Bel_i(B|A) \quad a \in I_i(x), x : Bel_i(B|A), \Gamma' \Rightarrow \Delta', a \Vdash^{\exists} A}{a \in I_i(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta', a \Vdash^{\exists} A}\ Cut_2 \qquad (2)}{a \in I_i(x)^3, \Gamma^3, \Gamma'^2 \Rightarrow \Delta^3, \Delta'^2}\ Cut_4}{a \in I_i(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ Ctr^*$$

Here (2) is the derivation:

$$\dfrac{\dfrac{\mathcal{D}(a/b)}{a \in I_i(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_i B|A} \quad \dfrac{\Gamma \Rightarrow \Delta, x : Bel_i(B|A) \quad a \in I_i(x), x \Vdash_i B|A, x : Bel_i(B|A), \Gamma' \Rightarrow \Delta'}{a \in I_i(x), x \Vdash_i B|A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ Cut_3}{a \in I_i(x)^2, a \Vdash^{\exists} A, \Gamma^2, \Gamma' \Rightarrow \Delta^2, \Delta'}\ Cut_1$$

Observe that the two uppermost cuts (Cut_1 and Cut_2) have reduced cut height and the others are cuts on formulas of reduced weight because $\mathtt{w}(a \Vdash^{\exists} A) < \mathtt{w}(x \Vdash_i B|A) < \mathtt{w}(x : Bel_i(B|A))$.

□

With standard Gentzen calculi, admissibility of cut immediately ensures the subformula property and its immediate consequences such as consistency (by underivability of the empty sequent). With the calculus that we have introduced we have similar properties, but we have to be more specific with the precise formulation of the subformula property. This property, taken literally, would impose that any sequent occurring in a derivation of a given sequent $\Gamma \Rightarrow \Delta$ contain only formulas which are subformulas of the formulas in $\Gamma \Rightarrow \Delta$. However, the decomposition of a formula such as $Bel_i(A|B)$ may introduce formula $A|B$, and this latter might introduce $A \supset B$. Neither $A|B$ nor $A \supset B$ are, strictly speaking, subformulas of $Bel_i A|B$ and $A|B$ respectively. Even it these are not strictly subformulas, they are less complex formulas built from subformulas of formulas in the conclusion. However, we cannot yet conclude that the calculus is analytic because there are also the labels to be considered. There are rules, such as *Ref*, that when read bottom-up, may introduce arbitrary labels. It is easy to prove that *Ref* can be restricted to a rule that operates on labels already in the conclusion (by basically the same argument given in Section 8 of **intlog**), thereby justifying the fact that in a derivation in *CDL* all labels are either eigenvariables in rules with freshness condition, or labels already in the conclusion.

If we extend the notion of subformula to cover these cases, we can conclude that the calculus is analytic and has the subformula property.

4 Soundness, termination, and completeness

In this section we shall prove soundness of the calculus with respect to the neighbourhood semantics that we have introduced. For this purpose, we need to interpret labelled sequents in neighbourhood models, which requires a notion of realization that connects the syntactic labels with the semantic elements (possible worlds, neighbourhoods).

Definition 4.1. Let $\mathcal{M} = \langle W, \{I\}_{i \in \mathcal{A}}, [\![\ ]\!] \rangle$ be a neighbourhood model, S a set of world labels, and N a set of neighbourhood labels. An SN-realization over $\mathcal{M}$ consists of a pair of functions (ρ, σ) such that

- $\rho : S \to W$ is a function that assigns to each $x \in S$ an element $\rho(x)$ of W;
- $\sigma : N \to \mathcal{P}(W)$ is a function that assigns to each $a \in N$ an element $\sigma(a)$ of $I(w)$, for some $w \in W$.

Given a sequent $\Gamma \Rightarrow \Delta$, with S, N as above, and (ρ, σ) an SN-realization, we say that $\Gamma \Rightarrow \Delta$ is satisfiable in $\mathcal{M}$ under the SN-realization (ρ, σ) if the following conditions hold:

- $\mathcal{M} \vDash_{\rho,\sigma} a \in I_i(x)$ if $\sigma(a) \in I_i(\rho(x))$ and $\mathcal{M} \vDash_{\rho,\sigma} a \subseteq b$ if $\sigma(a) \subseteq \sigma(b)$;
- $\mathcal{M} \vDash_{\rho,\sigma} x : A$ if $\rho(x) \Vdash A$;
- $\mathcal{M} \vDash_{\rho,\sigma} a \Vdash^{\exists} A$ if $\sigma(a) \Vdash^{\exists} A$ and $\mathcal{M} \vDash_{\rho,\sigma} a \Vdash^{\forall} A$ if $\sigma(a) \Vdash^{\forall} A$;
- $\mathcal{M} \vDash_{\rho,\sigma} x \Vdash_i B|A$ if for some c $\in I_i(\rho(x))$, $c \Vdash^{\exists} A$ and $c \Vdash^{\forall} A \supset B$;
- $\mathcal{M} \vDash_{\rho,\sigma} x \Vdash_i Bel_i(B|A)$ if for all $a \in I_i(\rho(x))$, $a \Vdash^{\forall} A$ or $\mathcal{M} \vDash_{\rho,\sigma} x \Vdash_i B|A$;
- $\mathcal{M} \vDash_{\rho,\sigma} \Gamma \Rightarrow \Delta$ if either $\mathcal{M} \nvDash_{\rho,\sigma} F$ for some formula $F \in \Gamma$ or $\mathcal{M} \vDash_{\rho,\sigma} G$ for some formula $G \in \Delta$.

Then, define $\mathcal{M} \vDash \Gamma \Rightarrow \Delta$ iff $\mathcal{M} \vDash_{\rho,\sigma} \Gamma \Rightarrow \Delta$ for every SN- realization (ρ, σ). A sequent $\Gamma \Rightarrow \Delta$ is said to be *valid* if $\mathcal{M} \vDash \Gamma \Rightarrow \Delta$ holds for every neighbourhood model $\mathcal{M}$, i.e. if $\Gamma \Rightarrow \Delta$ is satisfied for every model $\mathcal{M}$ and for every SN-realization (ρ, σ).

Theorem 4.1 (Soundness)**.** If a sequent $\Gamma \Rightarrow \Delta$ is derivable in the calculus, then it is valid in the class of multi-agent neighbourhood models.

Proof. By induction on the height of the derivation of a sequent $\Gamma \Rightarrow \Delta$. If the height of the derivation is 0, the sequent is initial or conclusion of $L\bot$, and by definition it is valid in the class of multi-agent neighbourhood models. If the height of the derivation is > 0, the sequent $\Gamma \Rightarrow \Delta$ has been derived by one of the rules of the calculus **G3CDL**. We prove that all rules preserve validity from the premisses to the conclusion. We consider in detail the cases in which the last rule applied is one of the rules for conditional belief.

[RC] Suppose the premisses of the rule are valid, whereas the conclusion is not. Thus, there is a model $\mathcal{M}$ and a realization (ρ, σ) that falsify the conclusion, i.e. $\mathcal{M} \vDash_{\rho,\sigma} a \in I_i(x)$, $\mathcal{M} \vDash_{\rho,\sigma} F$ for all $F \in \Gamma$, $\mathcal{M} \nvDash_{\rho,\sigma} G$ for all $G \in \Delta$ and $\mathcal{M} \nvDash_{\rho,\sigma} x \Vdash_i B|A$, i.e. 1) for all $c \in I_i(\rho(x))$ it holds that $c \Vdash^{\exists} \neg A$ or $c \Vdash^{\forall} \neg(A \supset B)$. Since by hypothesis both premisses are valid, it holds that 2) $\mathcal{M} \vDash_{\rho,\sigma} a \Vdash^{\exists} A$ and 3) $\mathcal{M} \vDash_{\rho,\sigma} a \Vdash^{\forall} A \supset B$. However, 2) and 3) cannot simultaneously hold: if 2) holds, then the first term of the disjunction in 1) is not satisfied, and the second term must hold, i.e. for all $c \in I_i(\rho(x))$, $c \Vdash^{\exists} \neg t(A \supset B)$, and this contradicts with 3). A similar contradiction is reached if 3) holds; thus, one of the premisses is not valid.

[LC] Suppose the premiss is valid and the conclusion is not, i.e. $\mathcal{M} \vDash_{\rho,\sigma} x \Vdash_i B|A$, $\mathcal{M} \vDash_{\rho,\sigma} F$ for all $F \in \Gamma$ and $\mathcal{M} \nvDash_{\rho,\sigma} G$ for all $G \in \Delta$, i.e. there exists a $c \in I_i(\rho(x))$ such that $c \Vdash^{\exists} A$ and $c \Vdash^{\forall} A \supset B$. Now define a new interpretation (ρ', σ') such that

$\rho'(x) = \rho(x)$
$\sigma'(b) = c$
$\sigma'(t) = \sigma(t)$, for $t \neq a$

Since the premiss is valid, it is valid under all interpretations; thus, it holds that $\mathcal{M} \vDash_{\rho',\sigma'} c \in I_x$, and that $\mathcal{M} \vDash_{\rho',\sigma'} F$ for all $F \in \Gamma$ and $\mathcal{M} \nvDash_{\rho',\sigma'} G$ for all $G \in \Delta$. It must hold that either $\mathcal{M} \nvDash_{\rho',\sigma'} c \Vdash^{\exists} A$ or $\mathcal{M} \nvDash_{\rho',\sigma'} c \Vdash^{\forall} A \supset B$, which contradicts with $c \Vdash^{\exists} A$ and $c \Vdash^{\forall} A \supset B$.

[RB] Suppose the premiss of RB is valid, whereas the conclusion is not. Then there is a model $\mathcal{M}$ and a realization (ρ, σ) which falsify the conclusion, i.e. $\mathcal{M} \vDash_{\rho,\sigma} F$ for

all formulas $F \in \Gamma$, $\mathcal{M} \nvDash_{\rho,\sigma} G$ for all formulas $G \in \Delta$ and $\mathcal{M} \nvDash_{\rho,\sigma} x : Bel_i(B|A)$. This means $\rho(x) \nVdash Bel_i(B|A)$, i.e. there exists a $b \in I_i(\rho(x))$ such that $b \Vdash^{\exists} A$ and for all $c \in I_i(\rho(x))$, $c \Vdash^{\forall} \neg A$ or $c \Vdash^{\exists} \neg(A \supset B)$. Now consider the premiss of the rule, and define a new realization (ρ', σ') defined as follows:

$\rho'(x) = \rho(x)$
$\sigma'(a) = b$
$\sigma'(t) = \sigma(t)$, for $t \neq a$

The realization (ρ', σ') differs from (ρ, σ) only for the interpretation of the neighbourhood label a, which is the new neighbourhood introduced in the premiss. Consider the model $\mathcal{M}$ defined above, and the new realization (ρ', σ'). It holds that $\mathcal{M} \vDash_{\rho',\sigma'} a \in I_i(x)$, $\mathcal{M} \vDash_{\rho',\sigma'} a \Vdash^{\exists} A$, and $\mathcal{M} \vDash_{\rho',\sigma'} F$ for all formulas $F \in \Gamma$, $\mathcal{M} \nvDash_{\rho',\sigma'} G$ for all formulas $G \in \Delta$. Since the premiss of the rule is valid (hypothesis), it holds that $\mathcal{M} \vDash_{\rho',\sigma'} x \Vdash_i B|A$, which means that for some $b \in I_i(\rho(x))$ it holds that $b \Vdash^{\exists} A$ and $b \Vdash^{\exists} A \supset B$. However, this is a contradiction with what stated above, i.e. that for all $c \in I_i(\rho(x))$, $c \Vdash^{\forall} \neg A$ or $c \Vdash^{\exists} \neg(A \supset B)$.

[LB] Suppose the premisses of the rule are valid, whereas the conclusion is not. Then, there is be a model $\mathcal{M}$ and a realization (ρ, σ) which falsify the conclusion, i.e. $\mathcal{M} \vDash_{\rho,\sigma} a \in I_i(x)$, $\mathcal{M} \vDash_{\rho,\sigma} x : Bel_i(B|A)$, $\mathcal{M} \vDash_{\rho,\sigma} F$ for all $F \in \Gamma$ and $\mathcal{M} \nvDash_{\rho,\sigma} G$ for all $G \in \Delta$. This means that for some $\sigma(a) \in I_i(\rho(x))$, $\rho(x) \vDash x : Bel_i(B|A)$, i.e. 1) for all $b \in I_i(\rho(x))$ either $b \Vdash^{\forall} \neg A$ or there exists $c \in I_i(x)$ such that $c \Vdash^{\exists} A$ and $c \Vdash^{\forall} A \supset B$. Then, since both premisses of the rule are valid (hypothesis) it holds that 2) $\mathcal{M} \vDash_{\rho,\sigma} a \Vdash^{\exists} A$ and 3) $\mathcal{M} \nvDash_{\rho,\sigma} x \Vdash_i B|A$, i.e. 4) for all $c \in I_i(\rho(x))$, $c \Vdash^{\forall} \neg A$ or $c \Vdash^{\exists} \neg(A \supset B)$. Now, 2) and 3) cannot be simultaneously satisfied. Suppose 2) holds; then the first term of the disjunction of 1) is not satisfied, and the second term must hold, i.e. there exists $c \in I_i(x)$ such that $c \Vdash^{\exists} A$ and $c \Vdash^{\forall} A \supset B$. But these conditions are in contradiction with 4). A similar reasoning applies if 2) holds. Thus, one of the premisses is not valid, against the hypothesis.

□

4.1 Termination

We now show that, by adopting a suitable proof search strategy, the calculus yields a decision procedure for *CDL*. We also prove the completeness of the calculus under the same strategy. The adoption of the strategy is not strictly necessary for completeness, but it ensures that we can extract a finite countermodel from an open or failed derivation branch. Although the termination proof has some similarity with the one in Negri and Olivetti (2015), for **G3CDL** it is more difficult because of specific semantic rules, in particular local absoluteness.

As often happens with labelled calculi, the calculus **G3CDL** in itself is non-terminating in the sense that a root-first (i.e. upwards) construction of a derivation may generate infinite branches. Here below is an example (in which we omit writing the derivable left premisses of LB):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{c \in I_i(x), c \Vdash^{\exists} A, c \Vdash^{\forall} A \supset B ... x : Bel_i(B|A) \Rightarrow x \Vdash_i C|A}}{x \Vdash_i B|A, b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset B, a \in I_i(x), a \Vdash^{\exists} A, x : Bel_i(B|A) \Rightarrow x \Vdash_i C|A}\,LC}{b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset B, a \in I_i(x), a \Vdash^{\exists} A, x : Bel_i(B|A) \Rightarrow x \Vdash_i C|A}\,LB}{x \Vdash_i B|A, a \in I_i(x), a \Vdash^{\exists} A, x : Bel_i(B|A) \Rightarrow x \Vdash_i C|A}\,LC}{a \in I_i(x), a \Vdash^{\exists} A, x : Bel_i(B|A) \Rightarrow x \Vdash_i C|A}\,LB}{x : Bel_i(B|A) \Rightarrow x : Bel_i(C|A)}\,RB$$

The loop is generated by the application of rules LB and LC. Our aim is to specify a strategy that ensures termination by preventing any kind of loop. The main point is to avoid redundant (backwards) applications of rules. To specify this notion we need to define saturation conditions associated to each rule.

Definition 4.2. Given a derivation branch $\mathcal{B}$ of the form $\Gamma_0 \Rightarrow \Delta_0, ..., \Gamma_k \Rightarrow \Delta_k, \Gamma_{k+1} \Rightarrow \Delta_{k+1}, ...$ where $\Gamma_0 \Rightarrow \Delta_0$ is the sequent $\Rightarrow x_0 : A$, let $\downarrow \Gamma_k / \downarrow \Delta_k$ denote the union of the antecedents/succedents that occur in the branch from the root $\Gamma_0 \Rightarrow \Delta_0$ up to $\Gamma_k \Rightarrow \Delta_k$.

We say that a sequent $\Gamma \Rightarrow \Delta$ *satisfies the saturation condition for a rule* R if, whenever $\Gamma \Rightarrow \Delta$ contains the principal formulas in the conclusion of R, then it also contains the formulas introduced by *one* of the premisses of R. The saturation conditions for the rules of **G3CDL** are detailed below (the saturation conditions associated to propositional rules are standard and can be found in Negri and Olivetti, 2015).

($L\wedge$) If $x : A \wedge B \in \Gamma$, then $x : A \in \downarrow \Gamma$ and $x : B \in \downarrow \Gamma$;
(Rf) If a is in Γ, Δ then $a \subseteq a$ is in Γ;
(Tr) If $a \subseteq b$ and $b \subseteq c$ are in Γ, then $a \subseteq c$ is in Γ;
($L\subseteq$) If $x \in a$ and $a \subseteq b$ are in Γ, then $x \in b$ is in Γ;
($R \Vdash^{\forall}$) If $a \Vdash^{\forall} A$ is in $\downarrow \Delta$, then for some x there is $x \in a$ in Γ and $x : A$ in $\downarrow \Delta$;
($L \Vdash^{\forall}$) If $x \in a$ and $a \Vdash^{\forall} A$ are in Γ, then $x : A$ is in $\downarrow \Gamma$;
($R \Vdash^{\exists}$) If $x \in a$ is in Γ and $a \Vdash^{\exists} A$ is in Δ, then $x : A$ is in $\downarrow \Delta$;
($L \Vdash^{\exists}$) If $a \Vdash^{\exists} A$ is in $\downarrow \Gamma$, then for some x there is $x \in a$ in Γ and $x : A$ is in $\downarrow \Gamma$;
(RB) If $x : Bel_i(B|A)$ is in $\downarrow \Delta$, then for some a, $a \in I_i(x)$ is in Γ, $a \Vdash^{\exists} A$ is in $\downarrow \Gamma$ and $x \Vdash_i B|A$ is in $\downarrow \Delta$;
(LB) If $a \in I_i(x)$ and $x : Bel_i(B|A)$ are in Γ, then either $a \Vdash^{\exists} A$ is in $\downarrow \Delta$ or $x \Vdash_i B|A$ is in $\downarrow \Gamma$;
(RC) If $a \in I_i(x)$ is in Γ and $x \Vdash_i B|A$ is in Δ, then either $a \Vdash^{\exists} A$ or $a \Vdash^{\forall} A \supset B$ are in $\downarrow \Delta$;
(LC) If $x \Vdash_i B|A$ is in $\downarrow \Gamma$, then for some a, $a \in I_i(x)$ is in Γ, $a \Vdash^{\exists} A$ and $a \Vdash^{\forall} A \supset B$ are in $\downarrow \Gamma$;
(S) If $a \in I_i(x)$ and $b \in I_i(x)$ are in Γ, then $a \subseteq b$ or $b \subseteq a$ are in Γ;
(T) For all x occurring in $\downarrow \Gamma \cup \downarrow \Delta$, for all $i \in \mathcal{A}$ there is an a such that $a \in I_i(x)$ and $x \in a$ are in Γ;

(A) If $a \in I_i(x)$ and $y \in a$ are in Γ, then if $b \in I_i(x)$ is in Γ also $b \in I_i(y)$ is in Γ; if $b \in I_i(y)$ is in Γ also $b \in I_i(x)$ is in Γ.

Furthermore, $\Gamma \Rightarrow \Delta$ is *saturated* if

(Init) There is no $x : P$ in $\Gamma \cap \Delta$;
($L\bot$) there is no $x : \bot$ in Γ;
$\Gamma \Rightarrow \Delta$ satisfies *all* saturation conditions.

To analyse the interdependences between labels in a sequent we introduce the following:

Definition 4.3. Given a branch $\mathcal{B}$ as in Definition 4.2, let a be neighbourhood label and x, y be world labels all occurring in $\downarrow \Gamma_k$, we define:

- $k(x) = min\{t \mid x \text{ occurs in } \Gamma_t\}$; we similarly define $k(a)$.
- $x \rightarrow_g a$ (read "x generates a") if for some $t \leq k$ and $i \in \mathcal{A}$, $k(a) = t$ and $a \in I_i(x)$ occurs in Γ_t.
- $a \rightarrow_g x$ (read "a generates x") if for some $t \leq k$ and $i \in \mathcal{A}$, $k(x) = t$ and $x \in a$ occurs in Γ_t.
- $x \overset{w}{\rightarrow} y$ (read "x generates y") if for some a it holds that $x \rightarrow_g a$ and $a \rightarrow_g y$.

Lemma 4.2. Given a branch $\mathcal{B}$ as in Definition 4.2, the following hold:

(a) The relation $\overset{w}{\rightarrow}$ is acyclic and forms a *tree* with root x_0;
(b) All world labels occurring in $\mathcal{B}$ are nodes of the tree, that is letting $\overset{w}{\rightarrow}^*$ be the *transitive closure* of $\overset{w}{\rightarrow}$, if u occurs in $\downarrow \Gamma_k$, then $x_0 \overset{w}{\rightarrow}^* u$.

Proof. (a) immediately follows from the definition of relation $\rightarrow_g$ and from the sequent calculus rules. As for (b), it is easily proved by induction on $k(u) \leqslant k$. If $k(u) = 0$, then $u = x_0$ and (b) trivially holds. If $k(u) = t > 0$, u does not occur in Γ_{t-1} and u occurs in Γ_t. This means that there exists a v and there exists a b such that $b \in I_i(v)$ occurs in Γ_{t-1}, and that $u \in b$ occurs in Γ_t; thus, $k(v) < k(u)$. By inductive hypothesis, $x_0 \overset{w}{\rightarrow}^* v$; since $v \overset{w}{\rightarrow} u$, also $x_0 \overset{w}{\rightarrow}^* u$ holds. □

We can now define the proof-search strategy. A rule R is said to be *applicable* to a world label x if R is applicable to a labelled formula with label x occurring in the conclusion of a rule. In case of rules A_1 and A_2 of local absoluteness, we say the rules are applied to x (rather than to y).

Definition 4.4. When constructing root-first a derivation tree for a sequent $\Rightarrow x_0 : A$, apply the following strategy:

(i) No rule can be applied to an initial sequent;
(ii) If $k(x) < k(y)$ all rules applicable to x are applied before any rule applicable to y.
(iii) Rule T is applied as the first one to each world label x.
(iv) Rules which do not introduce a new label (static rules) are applied *before* the rules which do introduce new labels (dynamic rules), with the exception of T, as in the previous item;

(v) For each x, y and a, static rules A_1 and A_2 are applied before any other static rule;
(vi) A rule R cannot be applied to a sequent $\Gamma_i \Rightarrow \Delta_i$ if $\downarrow \Gamma_i$ and / or $\downarrow \Delta_i$ satisfy the saturation condition associated to R.

It follows from the strategy that if $x \overset{w}{\to} y$, every rule applicable to x is applied before every rule applicable to y. In the example given before, the loop would have been stopped at the second application root-first of LB, because the application of LB would violate condition (vi): the branch already satisfies the saturation condition for LB, because $x \Vdash_i B|A$ is already in $\downarrow \Gamma$.
As usual, the size of a formula A, denoted by $|A|$, is the number of symbols that occur in A. The size of a sequent $\Gamma \Rightarrow \Delta$ is the sum of all the sizes of the formulas that occur in it.

Lemma 4.3. Given a branch $\mathcal{B}$ as in Definition 4.2 and a world label x, we define $N(x) = \{a \mid x \to_g a\}$ as the set of neighbourhood labels generated by x, and $W(x) = \{y \mid x \overset{w}{\to} y\}$ as the set of world labels generated by x. The size of $N(x)$ and $W(x)$ is finite, more precisely: $|N(x)| = O(n)$ and $|W(x)| = O(n^2)$.

Proof. We first prove that $|N(x)| = O(n)$. By definition, $a \in N(x)$ iff $x \to_g a$, i.e. if there exists t and there exists $i \in \mathcal{A}$ such that a does not occur in Γ_s for all $s \leqslant t$ and $a \in I_i(x)$ belongs to Γ_t. This means that label a has been introduced either by RB or by LC. Therefore x may create as many neighbourhood labels a as there are formulas $x : Bel_i(B|C)$ occurring in $\downarrow \Gamma_k \cup \downarrow \Delta_k$ (plus one neighbourhood introduced by T, total reflexivity) and the number of these formulas is $O(n)$.

We now prove $|W(x)| = O(n^2)$. By definition $y \in W(x)$ iff $x \overset{w}{\to} y$, i.e. iff for some b it holds that $x \to_g b$ and $b \to_g y$. We have just shown that for each x, the number of neighbourhood labels generated by x is $O(n)$. Let us consider $b \to_g y$. By definition, this means that there exists $t < k$, and there exists an $i \in \mathcal{A}$, such that y does not occur in Γ_s for $s \leqslant t$ and $y \in b$ occurs in Γ_{t+1}. There are several ways in which a formula $y \in b$ can be introduced:

Case 1. The formula $y \in b$ is introduced by a formula $b \Vdash^{\exists} C$ that belongs to $\downarrow \Gamma_k$ by application of rule $L \Vdash^{\exists}$. There are two subcases, according to how formula $b \Vdash^{\exists} C$ has been derived: (a) $b \Vdash^{\exists} C$ has been introduced by RB applied to a formula $x : Bel_i(D|C)$ that belongs to $\downarrow \Delta_k$ and (b) $b \Vdash^{\exists} C$ has been introduced by LC applied to a formula $x \Vdash_i D|C$ that belongs to $\downarrow \Gamma_k$. In turn, this formula has been introduced by LB applied to a formula $x : Bel_i(D|C)$ that belongs to $\downarrow \Gamma_k$. In case (a), we notice again that RB can be applied only *once* to each formula $x : Bel_i(D|C)$ that occurs in the consequent, and it generates exactly *one* new neighbourhood label b and one formula $b \Vdash^{\exists} C$. Similarly in case (b) LC can be applied only *once* to $x \Vdash_i D|C$ and generates *one* new neighbourhood label b and *one* formula $b \Vdash^{\exists} C$. By the saturation condition, each formula $x \Vdash_i D|C$ in turn is introduced by LB applied only *once* to one formula $x : Bel_i(D|C)$ that occurs in $\downarrow \Gamma_k$. Now each rule $L \Vdash^{\exists}$ generates exactly *one* new world label for each $b \Vdash^{\exists} C$ that occurs in $\downarrow \Gamma_k$ and, as we have just shown the number of such formulas is bounded by the number of formulas of type $x : Bel_i(D|C)$ that occur

in $\downarrow \Gamma_k$ which is $O(n)$. Therefore we can conclude that the number of new world labels introduced in this case is $O(n)$.

Case 2. The formula $y \in b$ is introduced by a formula $b \Vdash^{\forall} C$ that belongs to $\downarrow \Delta_k$ by application of rule $R \Vdash^{\forall}$. But a formula $b \Vdash^{\forall} C$ may be introduced only by an application of RC to a formula $u \Vdash_i F|E$, where $C = E \supset F \in \downarrow \Delta_k$. In turn, a formula of type $u \Vdash_i F|E$ may be introduced only by an application of RB. Let us consider the set S_b of formulas C such that $S_b = \{C \mid b \Vdash^{\forall} C \text{ belongs to } \downarrow \Delta_k\}$. It holds that:

$$\begin{aligned} S_b &= \{C \mid b \Vdash^{\forall} C \text{ belongs to } \downarrow \Delta_k\} \\ &= \{E \supset F \mid \exists u \exists i \,.\, u \Vdash_i F|E \text{ belongs to } \downarrow \Delta_k\} \\ &= \{E \supset F \mid \exists u \exists i \,.\, u : Bel_i(F|E) \text{ belongs to } \downarrow \Delta_k\} \end{aligned}$$

The cardinality of S_b is the same as the cardinality of the set $\{E \supset F \mid \exists u \exists i \,.\, u : Bel_i(F|E) \text{ belongs to } \downarrow \Delta_k\}$; thus, for each $b \in W(x)$, $|S_b| = O(n)$. In the present case, each $b \in W(x)$ generates $O(n)$ labels.
Then, since $|N(x)| = O(n)$ we finally get that $|W(x)| = O(n^2)$. □

Proposition 4.4. Any derivation branch $\mathcal{B} = \Gamma_0 \Rightarrow \Delta_0, ..., \Gamma_k \Rightarrow \Delta_k, \Gamma_{k+1} \Rightarrow \Delta_{k+1}, ...$ of a derivation starting from $\Gamma_0 \Rightarrow \Delta_0 \equiv \Rightarrow x_0 : A_0$ and is built in accordance with the strategy is finite.

Proof. Let us consider a branch $\mathcal{B}$, and suppose by contradiction that $\mathcal{B}$ is not finite. Let $\Gamma^* = \bigcup_k \Gamma_k$ and $\Delta^* = \bigcup_k \Delta_k$; then, Γ^* is infinite. All labelled formulas in Γ^* are subformulas of A_0; however, the subformulas of A_0 are finitely many (namely they are $O(n)$, where n is the length of A_0); thus Γ^* must contain infinitely many labels. In the light of Lemma 4.3, in particular Γ^* must contain infinitely many *world* labels, since each world label x generates only $O(n)$ neighbourhood labels. Let us consider now the tree determined by the relation $\overset{w}{\rightarrow}^*$ with root x_0. By Lemma 4.2, each label in any Γ_k occurs in the tree, which therefore is *infinite*. By Lemma 4.3, every label in the tree has $O(n^2)$ successors, thus a finite number. By König's lemma, the tree must contain an *infinite path*: $x_0 \overset{w}{\rightarrow} x_1 \overset{w}{\rightarrow} \ldots \overset{w}{\rightarrow} x_t \overset{w}{\rightarrow} x_{t+1} \ldots$, with all x_t being different. We observe that (a) infinitely many x_t must be generated by dynamic rules applied to subformulas of A_0, but (b) these formulas are finitely many, thus there must be a subformula of A_0 which is used infinitely many times to "generate" world labels (or better to generate a neighbourhood label from which a further world label is generated).

There are two cases: this subformula is of type $Bel_i(D|C)$ and occurs in Δ^* or it is of type $\Vdash_i B|A$ and occurs in Γ^* (in this latter case it is not properly a subformulas of A_0 but it is derived form a subformula of A_0).

In the first case, for some x_t we have that $x_t : Bel_i(D|C)$ occurs in some $\Delta_{s(x_t)}$; furthermore, for some a such that $k(a) = s(x_t) + 1$, we have that $a \in I_i(x_t), a \Vdash^{\exists} C \in \Gamma_{s(x_t)+1}$ and $x_t \Vdash_i D|C \in \Delta_{s(x_t)+1}$. Moreover, we have $a \rightarrow_g x_{t+1}$. However, there must be in the sequence an x_r with $r > t$, such that $x_r : Bel_i(D|C)$ occurs in some $\Delta_{s(x_r)}$ and for a new b, that is with $k(b) = s(x_r) + 1$, we have that $(*)$ $b \in I_i(x_r), b \Vdash^{\exists} C$ belongs to $\Gamma_{s(x_r)+1}$, $x_r \Vdash_i D|C$ occurs in $\Delta_{s(x_r)+1}$ and $b \rightarrow_g x_{t+1}$. By the definition of the strategy, we have that $a \in I_i(x_r)$, thus a itself fulfils the saturation condition for RB applied

to $x_r : Bel_i(D|C)$ belongs to $\Delta_{s(x_r)}$. Thus, step $(*)$ violates the strategy and we get a contradiction.

The second case displays a similar situation: for some t, $x_t \Vdash_i D|C$ occurs in some $\Gamma_{s(x_t)}$ and for a new a, with $k(a) = s(x_t) + 1$, we have that $a \in I_i(x_t), a \Vdash^{\exists} C$ occurs in $\Gamma_{s(x_t)+1}$ and $a \Vdash^{\forall} C \supset D$ occurs in $\Gamma_{s(x_t)+1}$. Moreover, we have that $a \rightarrow_g x_{t+1}$. Similarly there must be an x_r in the sequence with $r > t$, such that $x_r \Vdash_i D|C$ occurs in some $\Gamma_{s(x_r)}$ and for a new b, with $k(b) = s(x_r) + 1$, we have that we have that $(**)$ $b \in I_i(x_r), b \Vdash^{\exists} C$ occurs in $\Gamma_{s(x_r)+1}$ and $b \Vdash^{\forall} C \supset D$ occurs in $\Gamma_{s(x_r)+1}$. By definition of the strategy we have that $a \in I_i(x_r)$, thus a itself fulfils the saturation condition for LC applied to $x_r \Vdash_i D|C$ occurring in $\Gamma_{s(x_r)}$, so that step $(**)$ violates the strategy. In both cases we get a contradiction. □

The previous proof actually shows something stronger than termination of each derivation branch. The proof demonstrates that a formula of type $Bel_i(B|A)$ or $x \Vdash_i B|A$ cannot be used twice to generate two world labels that occur in the same path of the label tree associated to the derivation. Therefore, given an initial formula A_0, the number of formulas of type $Bel_i(B|A)$ or $x \Vdash_i B|A$ that can be generated in the derivation of $\Rightarrow x : A_0$ is bounded by $O(n)$, with n length of A_0. As a consequence, we have the following:

Fact 4.1. For a branch of a derivation as described in Proposition 4.4, the height of the derivation tree associated to the derivation is bounded by $O(n)$, where n is the length of A_0.

Termination of proof search under the strategy is now an obvious consequence:

Theorem 4.5. Proof search built in accordance with the strategy for any sequent of the form $\Rightarrow x_0 : A_0$ always comes to an end after a finite number of steps. More precisely, the maximal size of each sequent is $O(n^{4n+2})$, and the maximal length of a derivation branch is bounded by $O(n^{2n+1} \cdot n^{4n+2}) = O(n^{6n+3})$.
Furthermore, each sequent that occurs as a leaf of the derivation tree is either an initial sequent or a saturated sequent.

Proof. Consider a branch of a derivation tree whose root is the sequent $\Rightarrow x_0 : A_0$, and build the *finite* tree structure with all the labels that occur in the derivation. The root of the tree will be the label x_0, and all the other labels that occur in $\downarrow \Gamma_k$ will occur as nodes in the tree. As above, $n = |A_0|$.

By Proposition 4.1 we have that the height of the label tree associated with the derivation is bounded by $O(n)$.

Then, by Lemma 4.3 we have that the number of world labels and of neighbourhood labels that can be generated from each node is finite, and it is bounded by n^2, i.e. it is $O(n^2)$.

Let us consider a derivation tree with root $\Rightarrow x_0 : A_0$. The number of world labels that occur in each branch $\downarrow \Gamma_k$ is bounded by $O(n^{2n})$. The number of neighbourhood labels occurring in $\downarrow \Gamma_k$ is bounded by the number of world labels multiplied by the

maximal number of labels generated by each world, that is at most n. Thus, the number of neighbourhood labels is bounded by $O(n^{2n} \cdot n) = O(n^{2n+1})$.

The maximal size of each sequent occurring in the derivation is given by the maximal number of labelled formulas multiplied by the maximal number of subformulas of A_0, which is bounded by n: thus, $O(n^{2n+1} \cdot n) = O(n^{2n+2})$. However, this measure is not sufficient, since it takes into account only formulas of the form $x : F$, $a \Vdash^{Q} F$ or $x \Vdash_i F|G$. We have to calculate also the number of formulas of the form $y \in b$ and $b \in I_i(x)$ which could have been introduced in the derivation by $L\subseteq$ or T. The cardinality of the set $\{(y \in b) \mid y, b \text{ occurs in } \downarrow \Gamma_k\}$ is given by $n^{2n+1} \cdot n^{2n+1} = n^{4n+2}$. Thus, the maximal size of the sequents is bounded by $O(n^{4n+2})$.

Finally, the maximal length of each derivation branch is calculated by taking into account the maximal size of the sequents and the maximal number of rules which can be applied to it. We have to distinguish between rules which can be applied more than once (rules $L \Vdash^{\forall}$, $R \Vdash^{\exists}$, RC and LB) and rules which can be applied only once (all the others). The rules which can be applied more than once can be applied as many times as the number of labels occurring in the sequent, i.e. $O(n^{2n+1})$. Thus, the maximal length of a derivation branch is bounded by $O(n^{2n+1} \cdot n^{4n+2}) = O(n^{6n+3})$.

To prove the second part of the theorem, consider a branch $\Gamma_0 \Rightarrow \Delta_0, ..., \Gamma_n \Rightarrow \Delta_n$. We have proved in Proposition 4.4 that every branch of a derivation tree is finite. The leaf of the branch will be the sequent $\Gamma_n \Rightarrow \Delta_n$, and no rule is applicable to it; thus, trivially, the sequent is either an initial sequent or it is saturated.

□

From the proof of Theorem 4.5 we have the following:

Proposition 4.6. The validity of a formula A in *CDL* can be decided in *NEXPTIME*.

We know that multi-agent $S5$ is a fragment of *CDL*. By the result in Halpern and Friedman (1994) we immediately obtain that $PSPACE$ is the lower bound for deciding validity of a *CDL* formula. We conjecture that $PSPACE$ is also the upper bound for the logic; we will consider this problem in further research.

4.2 Completeness

We show that the calculus is complete under the terminating strategy of Definition 4.4.

Theorem 4.7. Let $\Gamma \Rightarrow \Delta$ be the upper sequent of a saturated branch $\mathcal{B}$ in a derivation tree. Then there exists a finite countermodel $\mathcal{M}$ to $\Gamma \Rightarrow \Delta$ that satisfies all formulas in $\downarrow \Gamma$ and falsifies all formulas in $\downarrow \Delta$.

Proof. Let $\Gamma \Rightarrow \Delta$ be the upper sequent of a saturated branch $\mathcal{B}$. By theorem 4.5, $\mathcal{B}$ is finite. We construct a model $\mathcal{M}_{\mathcal{B}}$ and an $SN_{\mathcal{B}}$-realization (ρ, σ), and show that the model satisfies all formulas in $\downarrow \Gamma$ and falsifies all formulas in $\downarrow \Delta$. Let

$$S_{\mathcal{B}} = \{x \,|\, x \ \in (\downarrow \Gamma \cup \downarrow \Delta)\} \text{ and } N_{\mathcal{B}} = \{a \,|\, a \in (\downarrow \Gamma \cup \downarrow \Delta).\}$$

Then, associate to each $a \in N_{\mathcal{B}}$ a neighbourhood α_a, such that $\alpha_a = \{y \in S_{\mathcal{B}} \,|\, y \in a \text{ belongs to } \Gamma\}$, thus $\alpha_a \subseteq S_{\mathcal{B}}$. We define a neighbourhood model $\mathcal{M}_{\mathcal{B}} = \langle W, \{I\}_{i\in\mathcal{A}}, [\![\ ]\!]\rangle$ as

- $W = S_{\mathcal{B}}$, i.e. the set W consists of all the labels occurring in the saturated branch $\mathcal{B}$;
- For each $x \in W$, $I_i(x) = \{\alpha_a \,|\, a \in I_i(x) \text{ belongs to } \downarrow\Gamma\}$;
- For P atomic, $[\![P]\!] = \{x \in W \,|\, x : P \text{ belongs to } \downarrow\Gamma\}$.

We first show that:

($*$) If $a \subseteq b$ belongs to Γ, then $\alpha_a \subseteq \alpha_b$.

To this aim, suppose $y \in \alpha_a$. This means that $y \in a$ belongs to Γ; then, by the saturation condition $L \subseteq$ also $y \in b$ belongs to Γ. By definition of the model we have $y \in \alpha_b$, and thus that $\alpha_a \subseteq \alpha_b$.
We now show that $\mathcal{M}_{\mathcal{B}} = \langle W, \{I\}_{i\in\mathcal{A}}, [\![\ ]\!]\rangle$ satisfies the properties of a multi-agent neighbourhood model, namely non-emptiness (trivial), total reflexivity, nesting and local absoluteness. Strong closure under intersection follows from finiteness, cf. the end of this section.

Total reflexivity: According to the saturation condition T, for every x that occurs in $\downarrow\Gamma \cup \downarrow\Delta$ also $a \in I_i(x)$, $x \in a$ occur in Γ; then, by definition of $\mathcal{M}_{\mathcal{B}}$, $\alpha_a \in I_i(x)$ and $x \in \alpha_a$.

Nesting: Suppose $\alpha_a \in I_i(x)$ and $\alpha_b \in I_i(x)$. We want to to show that $\alpha_a \subseteq \alpha_b$ or $\alpha_b \subseteq \alpha_a$. By definition of the model, from $\alpha_a \in I_i(x)$ and $\alpha_b \in I_i(x)$ it follows that $a \in I_i(x)$ and $b \in I_i(x)$ both belong to Γ. From the saturation condition S, we have that $a \subseteq b$ or $b \subseteq a$ belong to Γ and we conclude by the fact ($*$) above.

Local absoluteness: Suppose $\alpha_a \in I_i(x)$ and $y \in \alpha_a$. We want to show that $I_i(x) = I_i(y)$. Suppose $\alpha_b \in I_i(x)$, then by definition of the model we have that $a \in I_i(x)$, $y \in a$ and $b \in I_i(x)$ all belong to Γ. Then, by the saturation condition A, also $b \in I_i(y)$ belongs to Γ; thus, by definition, $\alpha_b \in I_i(y)$ holds. For the opposite inclusion we apply the same reasoning, exploiting the second condition of the saturation condition A.

Next, define a realization (ρ, σ) such that $\rho(x) = x$ and $\sigma(a) = \alpha_a$. We now prove the following, where $\mathcal{F}$ denotes any formula of the language, i.e. $\mathcal{F}$ is $a \in I_i(x)$, $x \in A$, $a \subseteq b$, $x \Vdash^{\forall} A$, $x \Vdash^{\exists} A$, $x \Vdash_i B|A$, $x : A$, $x : Bel_i(B|A)$:

[Claim 1] If $\mathcal{F}$ is in $\downarrow\Gamma$, then $\mathcal{M}_{\mathcal{B}} \vDash \mathcal{F}$;

[Claim 2] If $\mathcal{F}$ is in $\downarrow\Delta$, then $\mathcal{M}_{\mathcal{B}} \nvDash \mathcal{F}$;

The two claims are proved by cases, by induction on the weight of the formula $\mathcal{F}$.

[a] If A is a formula of the form $a \in I_i(x)$, $x \in a$ or $a \subseteq b$, claim 1. holds by definition of $\mathcal{M}_{\mathcal{B}}$, and claim 2. is empty. For the case of $a \subseteq b$, employ the fact ($*$) above.

[b] If A is a labelled atomic formula $x : P$, the claim holds by definition of the model; by the saturation condition ($Init$) no inconsistencies arise. If $A \equiv \bot$, it is not forced

in any model so Claim 2 holds; instead Claim 1 holds by the saturation clause $L\bot$. If A is a conjunction, disjunction or implication, both claims hold for the corresponding saturation conditions and by inductive hypothesis on formulas on smaller weight.

[c] If $A \equiv a \Vdash^{\exists} A$ is in $\downarrow \Gamma$, then by the saturation clause $L \Vdash^{\exists}$ for some x there are $x \in a$, $x : A$ are in $\downarrow \Gamma$. By definition of the model $\mathcal{M}_{\mathcal{B}}$, for some x, $x \in \alpha_a$. Then, since $w(x : A) < w(a \Vdash^{\exists} A)$, apply the inductive hypothesis and obtain $\mathcal{M}_{\mathcal{B}} \vDash x : A$. Therefore, by definition of satisfiability, $\mathcal{M}_{\mathcal{B}} \vDash \alpha_a \Vdash^{\exists} A$.
If $a \Vdash^{\exists} A$ is in $\downarrow \Delta$, then it is also in Δ. Consider an arbitrary world x in α_a. By definition of $\mathcal{M}_{\mathcal{B}}$ we have that $x \in a$ is in Γ; we apply the saturation condition $R \Vdash^{\forall}$ and obtain that $x : A$ is in $\downarrow \Delta$. By inductive hypothesis we have that $\mathcal{M}_{\mathcal{B}} \nvDash x : A$; thus, since this line of reasoning holds for arbitrary x, we can conclude by definition of satisfiability that $\mathcal{M}_{\mathcal{B}} \nvDash \alpha_a \Vdash^{\exists} A$.
The case in which $A \equiv a \Vdash^{\forall} A$ is similar.

[d] If $x \Vdash_i B|A$ is in $\downarrow \Gamma$, then by the saturation condition LC for some i, a it holds that $a \in I_i(x)$ is in Γ, and $a \Vdash^{\exists} A$, $a \Vdash^{\forall} A \supset B$ are in $\downarrow \Gamma$. By inductive hypothesis, $\mathcal{M}_{\mathcal{B}} \vDash \alpha_a \Vdash^{\exists} A$, and $\mathcal{M}_{\mathcal{B}} \vDash \alpha_a \Vdash^{\forall} A \supset B$. By definition, this yields $\mathcal{M}_{\mathcal{B}} \vDash x \Vdash_i B|A$.
If $x \Vdash_i B|A$ is in $\downarrow \Delta$, consider an arbitrary neighbourhood γ_c in $I_i(x)$. Then by definition of $\mathcal{M}_{\mathcal{B}}$ we have that $c \in I_i(x)$ is in Γ; apply the saturation condition RC and obtain that either $c \Vdash^{\exists} A$ or $c \Vdash^{\forall} A \supset B$ is in $\downarrow \Delta$. By inductive hypothesis, either $\mathcal{M} \nvDash \gamma_c \Vdash^{\exists} A$ or $\mathcal{M}_{\mathcal{B}} \nvDash \gamma_c \Vdash^{\forall} A \supset B$. In both cases, by definition $\mathcal{M}_{\mathcal{B}} \nvDash x \Vdash_i B|A$.

[e] If $x : Bel_i(B|A)$ is in $\downarrow \Gamma$, then it is also in Γ. Consider an arbitrary neighbourhood α_a in $I_i(x)$. By definition of $\mathcal{M}_{\mathcal{B}}$ we have that $a \in I_i(x)$ is in Γ; apply the saturation condition LB and conclude that either $a \Vdash^{\exists} A$ is in $\downarrow \Delta$, or $x \Vdash_i B|A$ is in $\downarrow \Gamma$. By inductive hypothesis, it holds that either $\mathcal{M}_{\mathcal{B}} \nvDash \alpha_a \Vdash^{\exists} A$ or $\mathcal{M}_{\mathcal{B}} \vDash x \Vdash_i B|A$. In both cases, by definition $\mathcal{M}_{\mathcal{B}} \vDash x : Bel_i(B|A)$.
If $x : Bel_i(B|A)$ is in $\downarrow \Delta$, by the saturation condition RB for some i, a it holds that $a \in I_i(x)$ is in Γ, $a \Vdash^{\exists} A$ is in $\downarrow \Gamma$ and $x \Vdash_i B|A$ is in $\downarrow \Delta$. By inductive hypothesis, $\mathcal{M}_{\mathcal{B}} \vDash \alpha_a \Vdash^{\exists} A$ and $\mathcal{M}_{\mathcal{B}} \nvDash x \Vdash_i B|A$, thus, by definition, we have $\mathcal{M}_{\mathcal{B}} \nvDash x : Bel_i(B|A)$.
□

The completeness of the calculus is an obvious consequence:

Theorem 4.8. If A is valid then it is provable in **G3CDL**.

Theorem 4.7 together with the soundness of **G3CDL** provides a constructive proof of *the finite model property* of *CDL*: if A is satisfiable in a model (i.e. $\neg A$ is not valid), then, by the soundness of **G3CDL** $\neg A$ is not provable, thus by Theorem 4.7 we can build a finite countermodel that falsifies $\neg A$, i.e. which satisfies A.

5 Relating the old and the new

In this section we shall recall the earlier semantics for *CDL* known from the literature, the semantics of plausibility models. We shall relate this semantics to the neighbourhood semantics we have formerly introduced and prove that they are equivalent, i.e. that

they validate exactly the same formulas. This result provides an alternative proof of the soundness and completeness of the axiomatization of *CDL* with respect to plausibility models.

5.1 The semantics of plausibility models

Epistemic plausibility models are versatile structures that have been used in a variety of different contexts by logicians, game theorists, and computer scientists, as emphasised in the recent survey article by Pacuit (2013). Epistemic plausibility models, here called P-models for short, also come with different names depending on the context of inquiry: Board (2004), for instance, calls them *Belief Revision Structures*.

Epistemic plausibility models are Kripke structures that display for each agent both an equivalence relation over worlds, defining knowledge (as in standard epistemic models) and a plausibility relation, which is used to define beliefs. The intuition is that an agent's beliefs are the propositions that hold in the worlds (state of affairs, scenarios) that the agent considers the most plausible.

Before defining these models, we recall a few preliminary notions. A *pre-order* $\preceq$ over a set W is a reflexive and transitive relation over W. Given $S \subseteq W$, $\preceq$ is *connected* over S if for all $x, y \in S$ either $x \preceq y$ or $y \preceq x$. An *infinite descending* $\preceq$*-chain* over W is a sequence $\{x_n\}_{n\geq 0}$ of elements of W such that for all n, $x_{n+1} \preceq x_n$ but $x_n \not\preceq x_{n+1}$. We say that $\preceq$ is *well-founded* over W if there are no infinite descending $\preceq$-chains over W. Given $S \subseteq W$, let $Min_{\preceq}(S) \equiv \{u \in S \,|\, \forall z \in S \,.\, z \preceq u \rightarrow u \preceq z\}$. Observe that whenever $\preceq$ is connected over S the definition $Min_{\preceq}(S)$ can be simplified to $Min_{\preceq}(S) = \{u \in S \,|\, \forall z \in S \,.\, u \preceq z\}$. Finally, the well-foundedness property can be equivalently stated as: for each $S \subseteq W$ if $S \neq \emptyset$ then $Min_{\preceq}(S) \neq \emptyset$.

Definition 5.1. Let $\mathcal{A}$ be a set of agents; an *epistemic plausibility model* $\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, [\![\ ]\!]\rangle$ consists of a nonempty set W of elements called "worlds"or "states"; for each $i \in \mathcal{A}$, an equivalence relation $\sim_i$ over W (with $[x]_{\sim_i} \equiv \{w \mid w \sim_i x\}$); for each $i \in \mathcal{A}$, a well-founded pre-order $\preceq_i$ over W; a valuation function $[\![\ ]\!] : Atm \rightarrow \mathcal{P}(W)$. The preorder $\preceq_i$ satisfies the following properties:

- *Plausibility implies possibility*: If $w \preceq_i v$ then $w \sim_i v$.
- *Local connectedness*: If $w \sim_i v$ then $w \preceq_i v$ or $v \preceq_i w$ (in other words, $\preceq_i$ is connected over every equivalence class of $\sim_i$).

The truth conditions for Boolean combinations of formulas are the standard ones; the truth condition for the conditional belief operator is the following:

$$[\![Bel_i(B|A)]\!] \equiv \{x \in W \,|\, Min_{\preceq_i}([x]_{\sim_i} \cap [\![A]\!]) \subseteq [\![B]\!]\}$$

A formula A is *valid* in a model $\mathcal{M}$ if $[\![A]\!] = W$ and that A is *valid in the class of epistemic plausibility models* if A is valid in every epistemic plausibility model.

The following proposition, proved by unfolding the definitions, gives an equivalent formulation of the truth condition of the conditional operator Bel_i given in Definition 5.1. From now on, we shall use this formulation.

Proposition 5.1. Given any P-model $\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, [\![\ ]\!]\rangle$, with $x \in W$ we have that $\mathcal{M}, x \Vdash Bel_i(B|A)$ iff:

$$(\forall y\,.\, y \sim_i x \ \rightarrow\ y \Vdash \neg A) \textit{ or } (\exists y \sim_i x\,.\, y \Vdash A \textit{ and } (\forall z\,.\, z \preceq_i y \ \rightarrow\ z \Vdash A \supset B))$$

Proof. (Only if) Assume $\mathcal{M}, x \Vdash Bel_i(B|A)$, that is, $Min_{\preceq_i}([x]_{\sim_i} \cap [A]) \subseteq [B]$. Now, it is either true or false that for all y, $y \sim_i x$ implies $y \Vdash \neg A$: if it is true, we immediately get the result. Else, for some y, $y \sim_i x$ and $y \Vdash A$. Hence $S_A \equiv Min_{\preceq_i}(\{w \mid w \sim_i x, w \Vdash A\}) \neq \emptyset$ from the well-foundedness of $\preceq_i$. Given any $z \in S_A$, given any world y such that $y \sim_i x$ and $y \Vdash A$, we have $z \preceq_i y$ since $\preceq_i$ is a total preordering. Hence $z \Vdash B$ from our initial assumption, so that $z \Vdash A \supset B$.

(If) Assume that for all y, $y \sim_i x$ implies $y \Vdash \neg A$ or there is $y \sim_i x$ such that $y \Vdash A$ and $\forall z, z \preceq_i y$ implies $z \Vdash A \supset B$. If the first disjunct holds, then S_A is empty, which makes the *(If)*-direction trivially true. If the second disjunct holds, then there is some y with $y \sim_i x$ such that $y \Vdash A$ (i.e. S_A is nonempty) and $\forall z, z \preceq_i y$ implies $z \Vdash A \supset B$. Let $w \in S_A$. We then have $w \preceq_i y$ and therefore $w \Vdash A \supset B$. Since $w \in S_A$, we also have $w \Vdash A$, so that $w \Vdash B$ follows, hence the claim $Min_{\preceq_i}([x]_{\sim_i} \cap [A]) \subseteq [B]$. □

Observation 5.1. Recall the definitions of the operators of unconditional belief and knowledge in terms of the conditional belief operator: $Bel_i A =_{def} Bel_i(A|\top)$ and $K_i A =_{def} Bel_i(\bot|\neg A)$. The truth conditions for these operators in plausibility models are the following:

$$[\![Bel_i A]\!] \equiv \{x \in W \mid Min_{\preceq_i}([x]_{\sim_i}) \subseteq [\![A]\!]\}$$
$$[\![K_i A]\!] \equiv \{x \in W \mid [x]_{\sim_i} \subseteq [\![A]\!])\}$$

By Proposition 5.1 it is possible to reformulate the above conditions as follows:

$$\mathcal{M}, x \Vdash Bel_i A \text{ iff } \exists y \sim_i x\,.\, y \Vdash A \textit{ and } (\forall z\,.\, z \preceq_i y \ \rightarrow\ z \Vdash A)$$
$$\mathcal{M}, x \Vdash K_i(A) \text{ iff } \forall y\,.\, y \sim_i x \ \rightarrow\ y \Vdash A$$

5.2 Equivalence between N- and P-models

We now show the equivalence between neighbourhood models, here called N-models, and epistemic plausibility models (P-models). The proofs make use of the basic correspondence between partial orders and topologies dating back to Alexandroff (1937) and recalled in Marti and Pinosio (2013) and Pacuit (2007). However, the result must be adapted to the present setting of multi-agent epistemic and neighbourhood models. The equivalence (Theorem 5.4) is obtained from Theorems 5.2, 5.3 and 2.1. Before proceeding to these results, proved through a suitable induction on *CDL* formulas, we need to introduce the measure of *weight* for such formulas.

Definition 5.2. The *weight* of a *CDL* formula is defined as follows: $w(P) = w(\bot) = 1$; $w(A \circ B) = w(A) + w(B) + 1$ for $\circ = \{\wedge, \vee, \supset\}$; $w(Bel_i(B|A)) = w(A) + w(B) + 3$ (cf. Definition 3.1).

Theorem 5.2. If a formula A is valid in the class P-models, then it is valid in the class of multi-agent N-models.

Proof. Given a N-model $\mathcal{M}_N$ we build an P-model $\mathcal{M}_P$ and we show that for any formula A, if A is valid in $\mathcal{M}_P$ then A is valid in $\mathcal{M}_N$. The result easily follows from this fact.
Let $\mathcal{M}_N = \langle W, \{I\}_{i\in\mathcal{A}}, [\![\ ]\!]\rangle$ be a multi-agent N-model. We construct a P-model $\mathcal{M}_P = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, [\![\ ]\!]\rangle$, by stipulating:

- $x \sim_i y$ iff $\exists\alpha \in I_i(x)\,.\, y \in \alpha$
- $x \preceq_i y$ iff $\forall\alpha \in I_i(y)\,.\, y \in \alpha \ \rightarrow\ x \in \alpha$.

We first show that $\sim_i$ is an equivalence relation.

- *Reflexivity.* By total reflexivity $\exists\alpha \in I_i(x), x \in \alpha$ holds, thus $x \sim_i x$.
- *Symmetry.* Suppose $x \sim_i y$, this means $\exists\alpha \in I_i(x), y \in \alpha$; by local absoluteness we get $I_i(x) = I_i(y)$. By total reflexivity, $\exists\beta \in I_i(x)\,.\, x \in \beta$, thus also $\beta \in I(y)$, and this shows $y \sim_i x$.
- *Transitivity.* Suppose $x \sim_i y$ and $y \sim_i z$, i.e., $\exists\alpha \in I_i(x)\,.\, y \in \alpha$ and $\exists\beta \in I_i(y)\,.\, z \in \beta$; by local absoluteness of I_i we have $I_i(x) = I_i(y)$; therefore $\exists\beta \in I_i(x)\,.\, z \in \beta$, which means $x \sim_i z$.

Next we prove that $\preceq_i$ such as constructed satisfies reflexivity, transitivity, plausibility implies possibility, local connectedness, and well-foundedness:

- *Reflexivity.* Trivial since $\forall\alpha \in I_i(x)\,.\, x \in \alpha \rightarrow x \in \alpha$.
- *Transitivity.* Suppose $x \preceq_i y$ and $y \preceq_i z$, we have 1) $\forall\alpha \in I_i(y)\,.\, y \in \alpha \ \rightarrow\ x \in \alpha$ and 2) $\forall\beta \in I_i(z)\,.\, z \in \beta \ \rightarrow\ y \in \beta$. Let $z \in \beta$. Then, from 2) we have $y \in \beta$ and from 1) $x \in \beta$ follows, i.e. $\forall\beta \in I_i(z)\,.\, z \in \alpha \ \rightarrow\ x \in \beta$ holds. This means $x \preceq_i z$.
- *Local connectedness*: by contradiction suppose that $x \sim_i y$ holds, but that but neither $x \preceq_i y$ nor $y \preceq_i x$ holds. By definition of $\preceq_i$ we have:

  for some $\beta \in I_i(y)$, $y \in \beta$ and $x \notin \beta$
  for some $\gamma \in I_i(x)$, $x \in \gamma$ and $y \notin \gamma$.

  Since $x \sim_i y$, by reflexivity $\exists\alpha \in I_i(x)\,.\, y \in \alpha$, whence by local absoluteness $I_i(y) = I_i(x)$. Thus both $\beta, \gamma \in I_i(x)$ and by nesting $\beta \subseteq \gamma$ or $\gamma \subseteq \beta$ holds. If the former holds we get $y \in \gamma$, if the latter holds $x \in \beta$, in both cases reaching a contradiction.
- *Plausibility implies possibility.* Suppose $x \preceq_i y$; by definition, it holds that $\forall\alpha \in I_i(y)$ if $y \in \alpha$ then $x \in \alpha$. By total reflexivity, there exists $\beta \in I_i(y)\,.\, y \in \beta$, thus we get $x \in \beta$. Therefore we have $\exists\beta \in I_i(y)\,.\, x \in \beta$, which means $y \sim_i x$, whence $x \sim_i y$ by symmetry.
- *Well-foundedness.* If $\mathcal{M}_N$ is finite there is nothing to prove. Suppose then that $\mathcal{M}_N$ is *infinite*. Suppose by contradiction that there is an infinite descending chain $\{z_k\}_{k\geq 0}$, i.e. such that for all k:

  $z_{k+1} \preceq_i z_k$ and $z_k \not\preceq_i z_{k+1}$

Observe that by definition of $\preceq_i$, plausibility implies possibility shown above, and local absoluteness we obtain that for all $k, h \geq 0$, it holds that $I_i(z_k) = I_i(z_h) = \ldots = I_i(z_0)$. Thus by definition of $\preceq_i$, for all $k \geq 0$ since $z_k \not\preceq_i z_{k+1}$, we get that for all $z_k \in \{z_k\}_{k\geq 0}$ there exists $\beta_{z_{k+1}} \in I_i(z_0)$ such that:

$(*)$ $\quad z_{k+1} \in \beta_{z_{k+1}}$ and $z_k \notin \beta_{z_{k+1}}$.

Consider the set $T = \{\beta_{z_{k+1}} \mid z_k \in \{z_k\}_{k\geq 0}\}$. T is nonempty thus by strong closure under intersection it follows that that $\bigcap T \in T$, and also $\bigcap T \neq \emptyset$. Obviously, we have that

$(**)$ $\quad$ for all $\beta \in T$, $\bigcap T \subseteq \beta$.

Since $\bigcap T \in T$, we have $\bigcap T = \beta_{z_{t+1}}$ for some $z_t \in \{z_k\}_{k\geq 0}$. But by using $(*)$ *twice* (namely for z_{t+1} and for z_{t+2}) we have $z_{t+1} \in \beta_{z_{t+1}}$ and $z_{t+1} \notin \beta_{z_{t+2}}$, thus $\bigcap T = \beta_{z_{t+1}} \not\subseteq \beta_{z_{t+2}}$ against $(**)$.

We now prove that for any $x \in W$ and formula A

$$(a)\ \mathcal{M}_N, x \Vdash A \text{ iff } \mathcal{M}_P, x \Vdash A$$

We proceed by induction on the structure of A. The base case (A atomic) holds by definition as $[\![\ ]\!]$ is the same in the two models. For the propositional cases $A = B \wedge C, B \vee C, B \supset C$, statement (a) easily follows by inductive hypothesis. We only consider the case $A = Bel_i(C|B)$. To simplify the notation we write $u \Vdash_P B$ instead of $\mathcal{M}_P, u \Vdash B$ and $u \Vdash_N B$ instead of $\mathcal{M}_N, u \Vdash B$.
[$\Rightarrow$] Suppose that $x \Vdash_N Bel_i(C|B)$. This means that:

$$(\forall \alpha \in I_i(x)\,.\,\alpha \Vdash^{\forall} \neg B) \text{ or } (\exists \beta \in I_i(x)\,.\,\beta \Vdash^{\exists} \text{ and } \beta \Vdash^{\forall} B \supset C).$$

We consider the two cases separately. Suppose first that $\forall \alpha \in I_i(x)\,.\,\alpha \Vdash^{\forall} \neg B$ holds; we show that $\forall y$, $y \sim_i x$ implies $y \Vdash_P \neg B$ holds as well. Let $y \sim_i x$; then, by definition, $\exists \alpha \in I_i(x)\,.\,y \in \alpha$; since $\alpha \Vdash^{\forall} \neg B$ we get $y \Vdash_N \neg B$, thus by inductive hypothesis $y \Vdash_P \neg B$ holds.

Suppose now that $\exists \beta \in I_i(x)\,.\,\beta \Vdash^{\exists} B$ and $\beta \Vdash^{\forall} B \supset C$ hold. We prove that $\exists w\,.\,w \sim_i x$ and $w \Vdash_P B$, and that $\forall z\,.\,z \preceq_i w \rightarrow z \Vdash_P B \supset C$. The hypothesis gives in particular that $\exists \beta \in I_i(x)$ such that $\beta \Vdash^{\exists} B$ whence $\exists w \in \beta w$ such that $w \Vdash_N B$. Thus, $x \sim_i w$ and by inductive hypothesis also $w \Vdash_P B$. Now let $z \preceq_i w$. By definition this means that $\forall \gamma \in I(w)\,.\,w \in \gamma \rightarrow z \in \gamma$. Therefore, since $w \in \beta$, also $z \in \beta$. But we have $\beta \Vdash^{\forall} B \supset C$, thus we get $z \Vdash_N B \supset C$, whence also $z \Vdash_P B \supset C$ by inductive hypothesis.

[$\Leftarrow$] Suppose that $x \Vdash_P Bel_i(C|B)$ holds. This means that:

$$(\forall y\,.\,y \sim_i x \rightarrow y \Vdash_P \neg B) \text{ or } (\exists w\,.\,w \sim_i x \text{ and } w \Vdash_P B \text{ and } (\forall z\,.\,z \preceq_i w \rightarrow z \Vdash_P B \supset C)).$$

As above, there are two cases to consider. Suppose first that $\forall y\,.\,y \sim_i x \rightarrow y \Vdash_P \neg B$. Let $\alpha \in I(x)$ and $u \in \alpha$. By definition $u \sim_i x$, thus by hypothesis $u \Vdash_P \neg B$ and by

inductive hypothesis $u \Vdash_N \neg B$. This means that $\alpha \Vdash^{\forall} \neg B$ (first case of truth definition of Bel_i in neighbourhood models).

Suppose now that there exists w such that $w \sim_i x$ and $w \Vdash_P B$ and $\forall z \,.\, z \preceq_i w$ implies $z \Vdash_P B \supset C$. From $w \sim_i x$ (hypothesis) it follows by definition that $\exists \alpha \in I(x) \,.\, w \in \alpha$. By local absoluteness, $I(x) = I(w)$. Now consider the set $S = \{\beta \in I(x) \,|\, w \in \beta\}$. It holds that $\alpha \in S$, and that $S \neq \emptyset$. Let $\gamma = \cap S$. By strong closure under intersection, $\gamma \in S \subseteq I_i(x)$, thus $\gamma \in I_i(x)$. But $w \in \gamma$ and since we have $w \Vdash_P B$, we also have $w \Vdash_N B$ by inductive hypothesis. We have obtained that $\gamma \Vdash^{\exists} B$. We still have to prove that $\gamma \Vdash^{\forall} B \supset C$. Let $u \in \gamma$ we want to prove that $u \Vdash_N B \supset C$. We first show that $u \preceq_i w$. To this purpose (by definition of $\preceq_i$), let $\delta \in I(w)$ with $w \in \delta$ we have to show that $u \in \delta$: since $I(x) = I(w)$, also $\delta \in I(x)$, whence, $\delta \in S$, so that $\gamma \subseteq \delta$, and therefore $u \in \delta$. Since $u \preceq_i w$ by the hypothesis we have $u \Vdash_P B \supset C$ and finally by inductive hypothesis $u \Vdash_N B \supset C$.

(End of the proof). Suppose that A is valid in $\mathcal{M}_P$. Thus for all $w \in W$, we have $w \Vdash_P A$, and by (a) we have also $w \Vdash_N A$ for all $w \in W$, which means that A is valid in $\mathcal{M}_N$. So we proved that if A is valid in $\mathcal{M}_P$ then A is also valid in $\mathcal{M}_N$. Finally, given a N-model $\mathcal{M}_N$, we build an P-model $\mathcal{M}_P$ as above. By the proof given above, if A is valid in $\mathcal{M}_P$, A is valid in $\mathcal{M}_N$. This concludes the proof.

□

Theorem 5.3. If a formula A is valid in the class of multi-agent N-models, then it is valid in the class of P-models.

Proof. Given a P-model $\mathcal{M}_P$ we build an N-model $\mathcal{M}_N$ and we show that for any formula A, if A is valid in $\mathcal{M}_N$ then A is valid in $\mathcal{M}_P$. The result easily follows from this fact.
Let $\mathcal{M}_P = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, [\![\ ]\!]\rangle$ be an P-model. We build a N-model $\mathcal{M}_N$ as follows. Let $u \in W$, and define its downward closed set $\downarrow^{\preceq_i} u$ according to $\preceq_i$ as $\downarrow^{\preceq_i} u = \{v \in W \,|\, v \preceq_i u\}$ Now we define the model $\mathcal{M}_N = \langle W, \{I\}_{i\in\mathcal{A}}, [\![\ ]\!]\ \rangle$, where for any $x \in W$

$$I_i(x) = \{\downarrow^{\preceq_i} u \,|\, u \sim_i x\}$$

We first show that $\mathcal{M}_N$ is indeed a N-model.

- *Nonemptiness:* Let $\alpha \in I_i(x)$, then $\alpha = \downarrow^{\preceq_i} u$ for some $u \sim_i x$ and since $u \in \downarrow^{\preceq_i} u$, we have $\alpha \neq \emptyset$.
- *Nesting:* Let $\alpha, \beta \in I_i(x)$. Then, $\alpha = \downarrow^{\preceq_i} u$ for some $u \sim_i x$ and $\beta = \downarrow^{\preceq_i} v$ for some $v \sim_i x$. We can conclude $u \sim_i v$, so that by local connectedness we have $u \preceq_i v$ or $v \preceq_i u$. It is immediate to see that this entails $\downarrow^{\preceq_i} u \subseteq \downarrow^{\preceq_i} v$ or $\downarrow^{\preceq_i} v \subseteq \downarrow^{\preceq_i} u$, from which the result follows.
- *Total reflexivity:* Obvious since $x \in \downarrow^{\preceq_i} x$.
- *Local absoluteness:* We first prove the following fact: if $y \sim_i x$ then $I_i(y) = I_i(x)$. Let $y \sim_i x$ and $\downarrow^{\preceq_i} z \in I_i(y)$, then $z \sim_i y$, so that by transitivity $z \sim_i x$, thus $\downarrow^{\preceq_i} z \in I_i(x)$ and hence $I_i(y) \subseteq I_i(x)$. The opposite inclusion $I_i(x) \subseteq I_i(y)$ is

proven in the same way. Let us now prove local absoluteness: suppose $\alpha \in I_i(x)$ and $y \in \alpha$. This means that $\alpha = {\downarrow}^{\preceq_i} u$ for some $u \sim_i x$; since $y \in {\downarrow}^{\preceq_i} u$, we have $y \preceq_i u$ and by plausibility implies possibility $y \sim_i u$ and therefore also $y \sim_i x$. Then we apply the above fact.

- *Closure under intersection:* In the finite case, this property immediately follows from properties nonemptiness and nesting. Let us consider the case when $\mathcal{M}_P$ is infinite. Let $S \subseteq I_i(x)$, $S \neq \emptyset$, with S countable so that $S = \{\alpha_h \mid h \geqslant 0\}$ where $\alpha_h = {\downarrow}^{\preceq_i} x_h$ for $x_h \sim_i x$. We prove that

  $(*)$ $\exists \alpha_h \in S$ such that $\forall \alpha_k \in S \,.\, \alpha_h \subseteq \alpha_k$

  If $(*)$ holds then $\alpha_h = \bigcap S$ and $\alpha_h \in S$ and the proof is over. Suppose by contradiction that $(*)$ does not hold. This means that: 1) $\forall \alpha_h \in S \; \exists \alpha_k \in S \,.\, \alpha_h \nsubseteq \alpha_k$. Thus, by the property of spheres nesting 2) $\forall \alpha_h \in S \; \exists \alpha_k \in S \,.\, \alpha_k \subset \alpha_h$. From 2), by denumerable dependent choice, we can build an infinite (strictly decreasing) chain of neighbourhoods

  $\alpha_1 \supset \alpha_2 \supset \alpha_3 \supset \ldots$

  For every $n \geq 1$ we have by definition that $\alpha_n = {\downarrow}^{\preceq_i} u_n$. Let $v_n \in \alpha_n - \alpha_{n+1}$, $v_{n+1} \in \alpha_{n+1} - \alpha_{n+2}$, etc. We have $v_{n+1} \preceq_i u_{n+1}$ by construction and it is enough to prove that $u_{n+1} \preceq_i v_n$ to conclude by transitivity that $v_{n+1} \preceq_i v_n$. By construction, we have $v_n \npreceq_i u_{n+1}$ and therefore by local connectedness, $u_{n+1} \preceq_i v_n$. Moreover by $v_n \npreceq_i u_{n+1}$ it also follows that $v_n \npreceq_i v_{n+1}$. We have thus an infinitely descending $\preceq_i$-chain of worlds $\{v_n\}_{n \geq 1}$, against the assumption of well-foundedness of W. We reached a contradiction from the negation of $(*)$; therefore, $(*)$ holds.

We now prove that for any $x \in W$ and formula A

$$(b)\ \mathcal{M}_P, x \Vdash A \text{ iff } \mathcal{M}_N, x \Vdash A$$

We proceed by induction on the structure of A. Again, for the base case, A atomic it holds by definition as $[\![\ ]\!]$ is the same in the two models. For the propositional cases $A = B \wedge C, B \vee C, B \supset C$, statement (b) easily follows by inductive hypothesis. We only consider the case $A = Bel_i(C|B)$. As in previous theorem we use the following abbreviations: $u \Vdash_P B$ instead $\mathcal{M}_P, u \Vdash B$ and $u \Vdash_N B$ instead of $\mathcal{M}_N, u \Vdash B$.
[$\Rightarrow$] Suppose that $x \Vdash_P Bel_i(C|B)$. This means that

$$(\forall y \,.\, x \sim_i y \;\rightarrow\; y \Vdash_P \neg B) \textit{ or } (\exists w \,.\, w \sim_i x \textit{ and } w \Vdash_P B \textit{ and } (\forall z \,.\, z \preceq_i w \;\rightarrow\; z \Vdash B \supset_P C))$$

Suppose first that $\forall y \,.\, y \sim_i x \rightarrow y \Vdash_P \neg B$. Take any $\alpha \in I_i(x)$. By definition, $\alpha = {\downarrow}^{\preceq_i} z$, for some $z \sim_i x$. Let $y \in {\downarrow}^{\preceq_i} z$. Then by definition $y \preceq_i z$ and by plausibility implies possibility, $y \sim_i z$; thus by transitivity $y \sim_i x$. By hypothesis we have $y \Vdash_P \neg B$, whence by inductive hypothesis also $y \Vdash_N \neg B$. We showed $\alpha \Vdash^{\forall} \neg B$ for any $\alpha \in I_i(x)$, thus $x \Vdash_N Bel_i(C|B)$ holds (first case of the truth condition).
Suppose now that there is a $w \sim_i x$ such that $w \Vdash_P B$ and $\forall z \,.\, z \preceq_i w \rightarrow z \Vdash_P B \supset C$. Let us consider $\alpha = {\downarrow}^{\preceq_i} w$. By inductive hypothesis $w \Vdash_N B$ and since $w \in {\downarrow}^{\preceq_i} w$ we

obtain $\alpha \Vdash^{\exists} B$. Now consider any $u \in \alpha = {\downarrow}^{\preceq_i} w$. By definition $u \preceq_i w$. Thus by hypothesis $u \Vdash_P B \supset C$, whence by inductive hypothesis also $u \Vdash_N B \supset_N C$. We showed that $\alpha \Vdash^{\forall} B \supset C$.

[$\Leftarrow$] Suppose that $x \Vdash_N Bel_i(C|B)$, this means that

$$(\forall \alpha \in I_i(x) \,.\, \alpha \Vdash^{\forall} \neg B) \ or \ (\exists \beta \in I_i(x) \,.\, \beta \Vdash^{\exists} B \ E \ \beta \Vdash^{\forall} B \supset C)$$

In first case $\forall \alpha \in I_i(x) \,.\, \alpha \Vdash^{\forall} \neg B$ holds. Let $y \sim_i x$, we want to show that $y \Vdash_P \neg B$. Since $y \sim_i x$, we have ${\downarrow}^{\preceq_i} y \in I_i(x)$. Thus by hypothesis ${\downarrow}^{\preceq_i} y \Vdash^{\forall} \neg B$ and $y \Vdash_N \neg B$, whence by inductive hypothesis also $y \Vdash_P \neg B$.

In the second case, there is $\beta \in I_i(x)$ such that $\beta \Vdash^{\exists} B$ and $\beta \Vdash^{\forall} B \supset C$. We prove that for some $u \sim_i x$ we have $u \Vdash_P B$ and for all $v \preceq_i u$ it holds $v \Vdash_P B \supset C$. By definition $\beta = {\downarrow}^{\preceq_i} z$ for some $z \sim_i x$. Since by hypothesis $\beta \Vdash^{\exists} B$ there exists $u \in \beta$ such that $u \Vdash_N B$, whence also $u \Vdash_P B$ by inductive hypothesis. But by definition of β, we have $u \preceq_i z$ and then $u \sim_i x$. Let now $v \preceq_i u$. Then by transitivity $v \in \beta$, and since $\beta \Vdash^{\forall} B \supset C$ we have $v \Vdash_N B \supset C$, whence also $v \Vdash_P B \supset C$ by inductive hypothesis.

(End of the proof). We proved that if A is valid in $\mathcal{M}_N$ then A is also valid in $\mathcal{M}_P$. Suppose that A is valid in $\mathcal{M}_N$. Thus, for all $w \in W$, we have $w \Vdash_N A$, and by (b) we have also $w \Vdash_P A$ for all $w \in W$, which means that A is valid in $\mathcal{M}_P$. Finally, let A be valid in the class of N-models. Then, A is also valid in the class of P-models. Given a P-model $\mathcal{M}_P$, we build an N-model $\mathcal{M}_N$ as above. By hypothesis A is valid in $\mathcal{M}_N$ and for what we have just shown A is valid in $\mathcal{M}_P$. This concludes the proof. □

Putting the two previous theorems together and making use of Theorem 2.1 we finally obtain the following:

Theorem 5.4. A formula A is a theorem of *CDL* if and only if it is valid in the class of plausibility models.

6 Other epistemic and doxastic modalities

Following Baltag and Smets (2008) and Pacuit (2013), we add to *CDL* the doxastic operators of *safe belief* and *strong belief*. These operators can be defined both in terms of epistemic plausibility models and in terms of neighbourhood models. Starting from the neighbourhood models characterization, we give sequent calculus rules for these operators and extend the sequent calculus **G3CDL** to cover also these modalities. Similarly, we define in both models a modal operator $[>]_i$ that expresses a strict order relation, which in turn allows to define two additional modalities: weakly safe belief and the operator of unary revision.

The safe belief operator actually captures the epistemic attitude that corresponds to "Stalnaker's knowledge": according to Stalnaker, knowledge is a doxastic attitude which remains stable in front of belief revision with any *true* information (Baltag and Smets,

2008). The view that in a strong sense of knowledge, the grounds should be conclusive and thus this notion of knowledge be stable under acquisition of further information, was made explicit already by Hintikka (p. 20–21, Hintikka, 1962) following Malcom (1952):

> If someone says "I know that p" in this strong sense of knowledge, he implicitly denies that any further information would have led him to alter his view. He commits himself to the view that he would still persist in saying that p is true (...) even if he knew more than he now knows.

Following Baltag and Smets (2008) we use the term "knowledge" for the strong notion and call instead the present attitude "safe belief". Thus, the intuitive meaning of the safe belief operator $Bel_i^{\text{Safe}}A$ is that agent i safely believes A if and only if A is true, she believes A, and she continues to believe A whatever *true* information is received.

In terms of epistemic plausibility models, the safe belief operator is defined as follows (Baltag and Smets, 2008; Pacuit, 2013):

$$(Safe_P) \quad \mathcal{M}_P, x \Vdash Bel_i^{\text{Safe}}A \text{ iff } \forall y \,.\, y \preceq_i x \;\rightarrow\; \mathcal{M}_P, y \Vdash A$$

We give the following condition in terms of neighbourhood models:

$$(Safe_N) \quad \mathcal{M}_N, x \Vdash Bel_i^{\text{Safe}}A \text{ iff } \exists \alpha \in I_i(x) \,.\, x \in \alpha \textit{ and } \alpha \Vdash^{\forall} A$$

To prove that the two notions correspond to each other, we have to extend the inductive proofs of Theorems 5.2 and 5.3 on the equivalence between epistemic plausibility models and neighbourhood models. More precisely, we have to add a suitable inductive step which takes into account also the strong belief operator. The key fact is expressed in the next proposition.

Proposition 6.1. The extension of preferential models by the truth condition for the safe belief operator, $Safe_P$, gives the same class of valid formulas as the extension of neighbourhood models with condition $Safe_N$.

Proof. Let $\mathcal{M}_P$ be an epistemic plausibility model. We construct a neighbourhood model as in the proof of Theorem 5.2. We now have to prove that

$$(a+)\ \mathcal{M}_P, x \Vdash Bel_i^{\text{Safe}}A \text{ iff } \mathcal{M}_N, x \Vdash Bel_i^{\text{Safe}}A$$

from the assumption that $[\![A]\!]^{\mathcal{M}_N} = [\![A]\!]^{\mathcal{M}_P}$. In order to prove the left-to-right direction, suppose $\mathcal{M}_P, x \Vdash Bel_i^{\text{Safe}}A$, i.e. $\forall y \,.\, y \preceq_i x \;\rightarrow\; y \Vdash A$. This means that $\forall y \in {\downarrow^{\preceq_i} x} \,.\, y \Vdash A$, i.e. ${\downarrow^{\preceq_i}} x \Vdash^{\forall} A$. By construction we have ${\downarrow^{\preceq_i}} x \in I_i(x)$, and therefore $\exists \alpha \in I_i(x) \,.\, x \in \alpha \textit{ and } \alpha \Vdash^{\forall} A$, i.e. $\mathcal{M}_N, x \Vdash Bel_i^{\text{Safe}}A$. As for the other direction of $(a+)$, suppose that $\mathcal{M}_N, x \Vdash Bel_i^{\text{Safe}}A$. This means $\exists x \in I_i(x) \,.\, x \in \alpha \textit{ and } \alpha \Vdash^{\forall} A$. By construction, $\alpha = {\downarrow^{\preceq_i}} z$ for some z, $z \sim_i x$. Since $x \in \alpha$, then $x \in {\downarrow^{\preceq_i}} z$. This implies that ${\downarrow^{\preceq_i}} x \subseteq {\downarrow^{\preceq_i}} z$, and since ${\downarrow^{\preceq_i}} z \Vdash^{\forall} A$, we have a fortiori ${\downarrow^{\preceq_i}} x \Vdash^{\forall} A$, i.e., $\forall y \,.\, y \preceq_i x \;\rightarrow\; y \Vdash A$.

As for the other direction of the proposition, let $\mathcal{M}_N$ be a neighbourhood model. We construct from it a plausibility model $\mathcal{M}_P$ following the procedure described in the proof of Theorem 5.3. We now have to prove that

(b+) $\mathcal{M}_N, x \Vdash Bel_i^{\text{Safe}} A$ iff $\mathcal{M}_P, x \Vdash Bel_i^{\text{Safe}} A$

assuming as hypothesis that $[\![A]\!]^{\mathcal{M}_N} = [\![A]\!]^{\mathcal{M}_P}$. For one direction, suppose that $\mathcal{M}_P, x \Vdash Bel_i^{\text{Safe}} A$. This means that $\forall y (y \preceq_i x \ \rightarrow \ \mathcal{M}_P, y \Vdash A)$, i.e. from the definition of $\mathcal{M}_P$:

$(hp1)$ $(\forall y \forall \beta \in I(x) \,.\, x \in \beta \ \rightarrow \ y \in \beta) \ \rightarrow \ \mathcal{M}_N, y \Vdash A$

We have to prove that $(*)$ $\exists \alpha \in I_i(x) \,.\, x \in \alpha$ *and* $\alpha \Vdash^{\forall} A$. We proceed by absurdum, assuming as hypothesis the negation of $(*)$:

$(hp2)$ $\forall \alpha \in I_i(x) \,.\, x \in \alpha \ \rightarrow \ \alpha \nVdash^{\forall} A$

Let $\Sigma = \{\alpha \in I_i(x) \,|\, x \in \alpha\}$ (i.e. Σ is the principal filter generated by x in $I_i(x)$). By total reflexivity, we have that $\Sigma \neq \varnothing$. Let $\alpha^* = \cap \Sigma$. By the intersection property we have that $\alpha^* \neq \varnothing$, and by the strong intersection property we have that $\alpha^* \in I_i(x)$ (and that $\alpha^* \in \Sigma$ as well). Thus we have that $x \in \alpha^*$, and it holds that $\forall \beta \in I_i(x) \,.\, \alpha^* \subseteq \beta$. By $(hp2)$ we conclude $\alpha^* \nVdash^{\forall} A$; thus, $\exists y \in \alpha^* \,.\, y \nVdash A$.

We now show that $y \preceq_i x$, in order to apply $(hp1)$. Consider an arbitrary $\beta \in I_i(x)$ and suppose $x \in \beta$. Then $\alpha^* \subseteq \beta$ and, if $y \in \alpha^*$, $y \in \beta$, i.e. it holds that $\forall \beta \in I(x) \,.\, x \in \beta \ \rightarrow \ y \in \beta$. Apply $(hp1)$ to conclude $y \Vdash A$ (for arbitrary y), in contraction with $\exists y \in \alpha^* \,.\, y \nVdash A$.

As for the other direction of $(b+)$, suppose that $\mathcal{M}_N \Vdash Bel_i^{\text{Safe}} A$. Thus we have as hypothesis that $\exists \alpha \in I_i(x) \,.\, x \in \alpha$ *and* $\alpha \Vdash^{\forall} A$. We want to prove that $(\forall \beta \in I(x) \,.\, x \in \beta \ \rightarrow \ y \in \beta) \ \rightarrow \ \mathcal{M}_N, y \Vdash A$. Given an arbitrary y, suppose that $\forall \beta \in I_i(x) \,.\, x \in \beta \ \rightarrow \ y \in \beta$; we have to show that $y \Vdash A$. By hypothesis there is an $\alpha_0 \in I_i(x) \,.\, x \in \alpha_0$ *and* $\alpha_0 \Vdash^{\forall} A$. Thus, since $x \in \alpha_0$, also $y \in \alpha_0$ (by hypothesis) and $y \Vdash A$.

□

The notion of strong belief can be found in Stalnaker et al. (1996), where it is called "robust belief", and was treated later by Battigalli and Siniscalchi (2002), Baltag and Smets (2008) and Pacuit (2013). According to Baltag and Smets[5], the strong belief operator can be defined in terms of knowledge and safe belief:

$$Bel_i^{\text{Strong}} A \ \textit{iff} \ Bel_i A \wedge K_i (A \supset Bel_i^{\text{Safe}} A) \qquad (*)$$

Intuitively, a strong belief formula $Bel_i^{\text{Strong}} A$ is saying that an agent i strongly believes A if she believes A, and if she knows that if A is true, then she safely believes A, i.e., A is stable under belief revision with any true information.

This condition can be expressed in terms of epistemic plausibility models. Recall first the truth condition for the unconditional belief operator in plausibility models in

[5]Pacuit provides a slightly different characterization of the operator, always in terms of epistemic plausibility models: $\mathcal{M}_P, x \Vdash Bel_i^{\text{Strong}} A$ *iff* $(\exists y \,.\, y \sim_i x$ *and* $y \Vdash A)$ *and* $([\![A]\!] \cap [x]_{\sim_i} \preceq_i [\![\neg A]\!] \cap [x]_{\sim_i})$, where for $S, S' \subseteq W$, let $S \preceq_i S'$ iff $\forall x \in S \forall y \in S' \,.\, x \preceq_i y$.

Observation 5.1: $\mathcal{M}, x \Vdash Bel_i A$ *iff* $\exists y \sim_i x \,.\, y \Vdash A$ *and* $(\forall z.z \preceq_i y \ \rightarrow \ z \Vdash A)$ [6]. We have:

$$\begin{array}{ll} \mathcal{M}_P, x \Vdash Bel_i^{\text{Strong}} A \textit{ iff} & (\exists y.y \sim_i x \ \& \ (\forall z \,.\, z \preceq_i y \ \rightarrow \ z \Vdash A)) \\ & \& \ (\forall z \,.\, z \sim_i x \ \& \ z \Vdash A \ \rightarrow \ (\forall y \,.\, y \preceq_i z \ \rightarrow \ y \Vdash A)) \end{array}$$

The condition can be translated in terms of neighbourhood models in an immediate way as follows:

$$\begin{array}{ll} \mathcal{M}_N, x \Vdash Bel_i^{\text{Strong}} A \textit{ iff} & (\exists \alpha \in I_i(x) \,.\, \alpha \Vdash^{\forall} A) \ \& \\ & (\forall \beta \in I_i(x) \forall y \in \beta \,.\, y \Vdash A \ \rightarrow \ (\exists \gamma \in I_i(x).y \in \gamma \ \& \ \gamma \Vdash^{\forall} A)) \end{array}$$

The sequent calculus rules for both safe and strong belief can be derived from the definitions of the operators in terms of neighbourhood models. Note that the rules for strong belief introduce the simple and safe belief operators in the premisses, in accordance with the definition of the operator.

$$\frac{x \in a, a \in I_i(x), a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x : Bel_i^{\text{Safe}} A, \Gamma, \Rightarrow \Delta} \, LSF \ (a \textit{ fresh})$$

$$\frac{a \in I_i(x), x \in a, \Gamma \Rightarrow \Delta, x : Bel_i^{\text{Safe}} A, a \Vdash^{\forall} A}{a \in I_i(x), x \in a, \Gamma \Rightarrow \Delta, x : Bel_i^{\text{Safe}} A} \, RSF$$

$$\frac{a \in I_i(x), a \Vdash^{\forall} A, x : K^T A, \Gamma \Rightarrow \Delta}{x : Bel_i^{\text{Strong}} A, \Gamma \Rightarrow \Delta} \, LSG \ (a \textit{ fresh})$$

$$\frac{a \in I_i(x), \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A \quad a \in I_i(x), \Gamma \Rightarrow \Delta, x : K^T A}{a \in I_i(x), \Gamma \Rightarrow \Delta, x : Bel_i^{\text{Strong}} A} \, RSG$$

$$\frac{a \in I_i(x), y \in a, y : A, \Gamma \Rightarrow \Delta, y : Bel_i^{\text{Safe}} A}{\Gamma \Rightarrow \Delta, x : K^T A} \, RK^T \ (y, a \textit{ fresh})$$

$$\frac{a \in I_i(x), y \in a, x : K^T A, \Gamma \Rightarrow \Delta, y : A \quad a \in I_i(x), y \in a, x : K^T A, y : Bel_i^{\text{Safe}} A, \Gamma \Rightarrow \Delta}{a \in I_i(x), y \in a, x : K^T A, \Gamma \Rightarrow \Delta} \, LK^T$$

Observe that characterisation $(*)$ of strong belief is guaranteed by the rules of the calculus, since it is easy to prove that for arbitrary x the following sequents are derivable

$x : Bel_i^{\text{Strong}} A \Rightarrow x : Bel_i A \wedge K_i(A \supset Bel_i^{\text{Safe}} A)$

$x : Bel_i A \wedge K_i(A \supset Bel_i^{\text{Safe}} A) \Rightarrow x : Bel_i^{\text{Strong}} A$

Baltag and Smets (2008) also consider the epistemic modality that expresses a strict order on plausibility models, i.e. the following operator:

$(>_P) \quad \mathcal{M}_P, x \Vdash [>]_i A$ iff $\forall y \,.\, y <_i x \ \rightarrow \ y \Vdash A$

[6] Since the strong belief operator can be defined in terms of the other epistemic operators, we do not explicitly extend the theorem of equivalence between models.

In terms of neighbourhood models, the definition of the $[>]_i$ operator is the following:

$$(>_N) \quad \mathcal{M}_N, x \Vdash [>]_i A \text{ iff } \forall \alpha \in I_i(x) \,.\, x \notin \alpha \;\rightarrow\; \alpha \Vdash^{\forall} A$$

The $[>]_i$ operator is not particularly meaningful by itself; however, it can be used to define the operator of *weakly safe belief* and the (more interesting) operator of *unary revision*, respectively:

$$Bel_i^{Weak} A := A \wedge [>]_i A \qquad\qquad *_i A := A \wedge [>]_i \neg A$$

Observe that $x \Vdash Bel_i^{Weak} A$ holds only if x is a minimal world with to respect to the strict relation $<_i$, where for *minimal* is meant that all smaller worlds do not satisfy A. We will now prove the equivalence of conditions $(>_P)$ and $(>_N)$, thus proving the equivalence of the two classes of models also with respect to this operator. The proof is an extension of those of Theorems 5.2 and 5.3, as in the strong belief operator case.

Proposition 6.2. The definition of the safe belief operator in preferential models, expressed by condition $>_P$, is equivalent to the the definition of the operator in neighbourhood models, expressed by condition $>_N$.

Proof. Suppose we have a plausibility model $\mathcal{M}_P$. We build a neighbourhood model $\mathcal{M}_N$ as described in the proof of Theorem 5.2. We now have to prove the following proposition, assuming as hypothesis that $[\![A]\!]^{\mathcal{M}_N} = [\![A]\!]^{\mathcal{M}_P}$:

$$(a++)\ \mathcal{M}_N, x \Vdash [>]_i A \textit{ iff } \mathcal{M}_P, x \Vdash [>]_i A$$

In order to prove one direction, take as hypothesis $\forall \alpha \in I_i(x) \,.\, x \notin \alpha \;\rightarrow\; \alpha \Vdash^{\forall} A$. We want to prove that $\forall y \,.\, y <_i x \;\rightarrow\; y \Vdash A$. Suppose $y <_i x$. Then, by construction, $x \notin {\downarrow^{\preceq_i}} y$, for ${\downarrow^{\preceq_i}} y = \{u \in W \mid u \preceq y\}$. We have that ${\downarrow^{\preceq_i}} y = \alpha$, for some α. By hypothesis, $\alpha \Vdash^{\forall} A$. Then, since $y \in \alpha$, we have that $y \Vdash A$.

As for the other direction, we assume as hypothesis that $\forall y \,.\, y <_i x \;\rightarrow\; y \Vdash A$, and we want to prove that $\forall \alpha \in I_i(x) \,.\, x \notin \alpha \;\rightarrow\; \alpha \Vdash^{\forall} A$. Suppose $\alpha \in I_i(x)$ and $x \notin \alpha$. By construction, $\alpha = {\downarrow^{\preceq_i}} y$, for some $y \sim_i x$, and $x \notin {\downarrow^{\preceq_i}} y$. Thus, we have $y < x$. By hypothesis, $y \Vdash A$. Since this holds for all y such that $x \notin {\downarrow^{\preceq_i}} y$, and since $y \in {\downarrow^{\preceq_i}} y$, we have that ${\downarrow^{\preceq_i}} y \Vdash^{\forall} A$.

Suppose we have a neighbourhood model $\mathcal{M}_N$. We built a plausibility model $\mathcal{M}_P$ from it, as described in Theorem 5.3. In order to build the plausibility model, we will use the following additional condition:

$$\begin{array}{ll} y \prec_i x \text{ iff} & (1)\ \forall \alpha \in I_i(x) \,.\, x \in \alpha \;\rightarrow\; y \in \alpha \\ & (2)\ \exists \beta \in I_i(x) = I_i(y) \,.\, y \in \beta \textit{ and } x \notin \beta \end{array}$$

We have to prove the following statement, always under the hypothesis $[\![A]\!]^{\mathcal{M}_N} = [\![A]\!]^{\mathcal{M}_P}$:

$$(b++)\ \mathcal{M}_N, x \Vdash [>]_i A \textit{ iff } \mathcal{M}_P, x \Vdash [>]_i A$$

To prove one direction of $(b++)$, suppose $\forall\alpha \in I_i(x)\,.\,x \notin \alpha \;\rightarrow\; \alpha \Vdash^{\forall} A$. We want to show that $\forall y\,.\,y \prec_i x \;\rightarrow\; y \Vdash A$. Suppose $y \prec_i x$. This means that (1) $\forall\alpha \in I_i(x)\,.\,x \in \alpha \;\rightarrow\; y \in \alpha$ and (2) $\exists\beta \in I_i(x) = I_i(y)\,.\,y \in \beta$ *and* $x \notin \beta$. Note that the condition of the equality of spheres in (2) is justified by the following reasoning: by total reflexivity, $y \in \alpha$, and by absoluteness we have $I_i(x) = I_i(y)$. From (2), we have that there exists a sphere β_0 such that $\beta_0 \in I_i(x)$, $y \in \beta_0$ and $x \notin \beta_0$. By hypothesis, we have that $\beta_0 \Vdash^{\forall} A$. Thus, since $y \in \beta_0$, $y \Vdash A$.

As for the other direction, assume that $\forall y\,.\,y \prec_i x \;\rightarrow\; y \Vdash A$. We want to show that $\forall\alpha \in I_i(x)\,.\,x \notin \alpha \;\rightarrow\; \alpha \Vdash^{\forall} A$. Let $\alpha \in I_i(x)$ such that $x \notin \alpha$, and let $u \in \alpha$; we have to show that $u \Vdash A$. Let $\Sigma = \{\gamma \,|\, u \in \beta$ *and* $x \notin \gamma\}$. Since $\alpha \in \Sigma$, $\Sigma \neq \varnothing$. Let $\delta = \cap\Sigma$; by the strong intersection property, $\delta \in I_i(x)$ and $\delta \neq \varnothing$.

It holds that $u \in \delta$. We want to show that $u \prec_i x$, since by hypothesis this implies $u \Vdash A$. Thus, we have to prove that condition (1) and (2) hold. Condition (2) holds by construction; as for (1), let $\beta \in I_i(x)$ such that $x \in \beta$; we have to prove that also $u \in \beta$. By nesting, it holds that either $\beta \subseteq \delta$ or $\delta \subseteq \beta$. The case $\beta \subseteq \delta$ is not possible, since we have set that $x \in \beta$, but by construction we have that $x \notin \delta$. Thus, it must hold that $\delta \subseteq \beta$; since by construction $u \in \delta$, we have $\in \beta$. Thus, by hypothesis we can conclude $u \Vdash A$, and the proposition in proved.

□

It should be possible to extend the calculus **G3CDL** to cover also operators Bel_i^{Weak} and $*_i$. We leave the definition of appropriate rules to the interested reader.

The following informal observation should be useful to get an idea of the motivation behind the definition in neighbourhood models of the operators we have introduced in this section. Let us consider a world x and the set $I_i(x)$ of neighbourhoods associated to it. We can split $I_i(x)$ into two sets, namely:

$$I_i(x)^+ = \{\alpha \in I_i(x) \,|\, x \in \alpha\} \qquad I_i(x)^- = \{\alpha \in I_i(x) \,|\, x \notin \alpha\}$$

These represent, respectively, the set of neighbourhoods to which x belongs and the set of neighbourhoods to which x does not belong. Now recall the four modalities which can be defined in a standard way in neighbourhood models:[7]

$$\begin{array}{lll} x \Vdash \Box^{\forall} A & \textit{iff} & \forall\alpha \in I_i(x)(\alpha \Vdash^{\forall} A) \\ x \Vdash \Box^{\exists} A & \textit{iff} & \exists\alpha \in I_i(x)(\alpha \Vdash^{\forall} A) \\ x \Vdash \Diamond^{\forall} A & \textit{iff} & \forall\alpha \in I_i(x)(\alpha \Vdash^{\exists} A) \\ x \Vdash \Diamond^{\exists} A & \textit{iff} & \exists\alpha \in I_i(x)(\alpha \Vdash^{\exists} A) \end{array}$$

Note that the simple belief operator $x \Vdash Bel_i A$ iff $\exists\alpha \in I_i(x)\,.\,\alpha \Vdash^{\forall} A$ corresponds to the $\Box^{\exists}$ modality, while the knowledge operator $x \Vdash K_i A$ iff $\forall\alpha \in I_i(x)\,.\,\alpha \Vdash^{\forall} A$ corresponds to the $\Box^{\forall}$ modality.

[7] Such modalities are denoted by $[\,]$, $\langle\,\rangle$, $[\,\rangle$, $\langle\,\rangle$ in Pacuit (2013); their proof theory is studied through labelled sequent calculi based on neinghbourhood semantics in Negri (2017b).

Furthermore, all the operators that we have taken into account in this section can be interpreted as one of the above modalities, defined either on $I_i(x)^+$ or $I_i(x)^-$. More precisely, the safe belief operator $x \Vdash Bel_i^{\text{Safe}} A$ iff $\exists \alpha \in I_i(x)(x \in \alpha \text{ and } \alpha \Vdash^{\forall} A)$ corresponds to the $\Box^{\exists}$ modality defined on $I_i(x)^+$; and the strong belief operator is defined on the same set. The $[>]_i$ operator $x \Vdash [>]_i A$ iff $\forall \alpha \in I_i(x) \,.\, x \notin \alpha \;\rightarrow\; \alpha \Vdash^{\forall} A$ corresponds to the $\Box^{\forall}$ modality defined on $I_i(x)^-$. Also the weakly safe belief operator and the unary revision operator are defined on the same set.

This overview gives an idea of the wide variety of modal operators which is possible to define in neighbourhood models. Following Baltag and Smets, 2008, we have restricted our analysis to the operators that should be interesting from an epistemic viewpoint—in principle, however, there are many others.

7 Conclusions and further research

We have proposed a semantics based on neighbourhood models, a multi-agent version of Lewis' sphere models, for the logic *CDL* of doxastic conditional beliefs. On the basis of this neighbourhood semantics, we have developed a labelled sequent calculus **G3CDL** for the logic, following the methodology of Negri (2005; 2017b) and Negri and Olivetti (2015). The calculus **G3CDL** is analytic and enjoys cut elimination and admissibility of the other structural rules as well as invertibility of all its rules. Moreover, on the basis of this calculus, we obtain a decision procedure for the logic under a natural strategy of proof search. The completeness of the calculus is proved by a finite countermodel construction extracted from a failed or open branch of a derivation. The finite countermodel construction provides in itself a constructive proof of the finite model property of the logic. Finally we have shown how to extend the semantic interpretation and the sequent calculus **G3CDL** to other doxastic operators, namely *safe belief* and *strong belief* (Baltag and Smets, 2008).

There are a number of issues which may be objects of further investigation. First, *CDL* is the "static" logic that underlies dynamic extensions by *doxastic actions* (Baltag and Smets, 2008). It should be worth studying whether and how our calculus can be extended to deal also with the dynamic extensions. From a computational side, to the best of our knowledge the exact complexity of *CDL* is not known. We conjecture its upper bound to be PSPACE; however, further investigations are needed to confirm this result. Moreover, some optimization of the search strategy is possible, in particular to reduce the number of labels generated in a derivation. We plan to deal with all these topics in future research.

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