Position-Frequency Analysis and Phase Space

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CONTINUOUS WAVELETS, POSITION-FREQUENCY ANALYSIS AND PHASE SPACE

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Abstract: Continuous wavelet decompositions are described both from the position-frequency and position-scale points of view. These two approaches are illustrated by specific examples. A geometrical approach to the construction of wavelets associated with group representations is finally briefly described.

I. INTRODUCTION:

During the last decade, a new methodology has grown up simultaneously in many apparently weakly connected areas such as mathematical analysis, signal and image processing, numerical analysis, computer vision, computer musics and quantum mechanics. The common idea is to decompose signals, functions or operators into "elementary contributions" or "grains", and to analyze the signals, functions or operators in terms of the coefficients of the corresponding decomposition.

The two basic examples of such a program are respectively wavelet analysis and Windowed Fourier analysis. Let us focus for simplicity on the problem of analysis of functions. In both cases, the set of coefficients is interpreted as a transform of the analyzed function. The wavelet transform was introduced to perform a *position-scale* analysis of functions, and we describe that particular aspect in section II. On the other hand, the goal of windowed Fourier analysis (or Gabor analysis [Ga]) was to provide a local version of Fourier analysis by the introduction of windows that select a neighborhood of a given point of the analyzed function before Fourier-transform it.

Nevertheless, it turns out that wavelet analysis can also be used as a local version of Fourier analysis, as described in sction III. More precisely, it can be used to give a precise meaning to the notions of local frequencies and amplitudes of a signal. It can then be used in some contexts as a position-frequency method.

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Section IV is devoted to a geometrical approach to this particular point. We describe the geometrical status of some of the previously described position-frequency decompositions. It is shown how generalized wavelets can be obtained from irreducible unitary continuous representations of locally compact groups, under some integrability condition. Namely, the representation must be square-integrable with respect to the whole group or to some quotient space of it. Some examples are provided at the end of section IV.

It must be noticed that there is a peculiarly interesting case, namely the case where to the group representation one starts with is associated a coadjoint orbit in the dual of the group, and where the considered coset space is precisely the coadjoint orbit. In such a case, the corresponding wavelet transform (if it exists) is an isometry between the representation space and the space of square-integrable functions on the coadjoint orbit, otherwise stated the space of square-integrable functions on the associated phase space. This is an additional argument for the "position-frequency" aspect of wavelet decompositions.

II. WAVELETS OF CONSTANT SHAPE:

Wavelets of constant shape were introduced by geophysicists in the signal analysis community at the beginning of the eighties [Go.Mo.Gr], and at a more formal level a little bit later [Gr.Mo]. The basic idea was to be able to perform a local analysis (i.e. essentially make convolutions with "well localized" functions[(1)]) at different scales. It turns out that the analysis scheme developed by A. Grossmann and J. Morlet was quite close to a technique developed by A. Calderón and his collaborators for the study of operators defined by singular integrals.

The basic tool is the so-called Calderón's formula, that can be expressed as follows. Let $\psi(x)$ be a $L^1(\mathbf{R})$-function, such that in addition the following condition holds:

$$0 < c_\psi := \int_0^\infty |\hat{\psi}(\xi)|^2 \frac{d\xi}{\xi} < \infty \qquad (II.1)$$

Such an equation basically means that $\hat{\psi}(\xi)$ vanishes continuously at the origin, then that $\psi(x)$ is or zero integral.

With ψ are associated the following elementary wavelets, dilated and shifted copies of ψ

$$\psi_{(b,a)}(x) := \frac{1}{\sqrt{a}} \psi\left(\frac{x-b}{a}\right) \qquad (II.2)$$

Then (see [Gr.Mo] for more details) any Hardy function [(2)] $f \in H^2(\mathbf{R})$ can be decomposed as follows

$$f(x) = \frac{1}{c_\psi} \int_{\mathbf{R}_+^* \times \mathbf{R}} T_f(b,a) \psi_{(b,a)}(x) \frac{da}{a} \frac{db}{a} \qquad (II.3)$$

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(1) Throughout this paper, the term "well localized" is to be interpreted in a somewhat vague sense; "well localized" means "decaying at infinity better than some fixed criterion, for example exponential decay or algebraic decay with some fixed exponent".

(2) We recall here that $H^2(\mathbf{R}) = \left\{ f \in L^2(\mathbf{R}),\ \hat{f}(\xi) = 0\, \forall \xi \leq 0 \right\}$

where the coefficients $T_f(b,a)$ are given by

$$T_f(b,a) :=< f, \psi_{(b,a)} > \qquad (II.4)$$

and form the wavelet transform (or affine wavelet transform) of f. The equality in $(II.3)$ is to be taken in the strong $L^2(\mathbf{R})$-sense. As a consequence, one gets a "wavelet Plancherel formula":

$$||f||^2 = \frac{1}{c_\psi} \int_{\mathbf{R}_+^* \times \mathbf{R}} |T_f(b,a)|^2 \frac{da}{a} \frac{db}{a} \qquad (II.5)$$

Such a formula allows the interpretation of the squared modulus $|T_f(b,a)|^2$ of the wavelet transform as an energy density in the position-scale half-plane. $(II.5)$ then simply expresses the energy conservation.

Calderón's formula can be also applied to $L^2(\mathbf{R})$-functions, but it is necessary to make an additional symmetry assumption on ψ. It is sufficient for instance to assume that $\hat{\psi}(\xi)$ is an even function of ξ.

The n-dimensional generalization of Calderón's formula is quite simple. The analyzing wavelet is now a radial function $\psi(\underline{x}) \in L^1(\mathbf{R})$, such that

$$0 < c_\psi := \int_0^\infty |\hat{\psi}(a\underline{\xi})|^2 \frac{da}{a} < \infty \qquad (II.6)$$

(since $\psi(\underline{x})$ is radial, c_ψ does not depend on $\underline{\xi}$). To $\psi(\underline{x})$ are associated the wavelets $\psi_{(\underline{b},a)}(\underline{x}) := a^{-n/2}\psi\left(\frac{\underline{x}-\underline{b}}{a}\right)$, and any $f \in L^2(\mathbf{R}^n)$ can be decomposed as

$$f(\underline{x}) = \frac{1}{c_\psi} \int_{\mathbf{R}_+^* \times \mathbf{R}^n} T_f(\underline{b},a)\, \psi_{(\underline{b},a)}(\underline{x})\, \frac{da}{a} \frac{d\underline{b}}{a^n} \qquad (II.7)$$

where the wavelet transform $T_f(\underline{b},a)$ is now a function of $n+1$ variables.

$$T_f(\underline{b},a) :=< f, \psi_{(\underline{b},a)} > \qquad (II.8)$$

With such a formula, we are still in the context of position-scale decompositions.

Calderón's formula was introduced by mathematicians [Cal] in order to get an improvement of Littlewood-Paley theory, especially in the context of the classification of functional spaces. To illustrate such applications, let us briefly describe two examples for which wavelets are particularly useful. The first example deals with the characterization of global regularity of functions. Let $0 < \alpha < 1$, and consider the homogeneous Lipschitz space Λ^α: $f \in \Lambda^\alpha$ if there exists some constant C such that

$$|f(x_0) - f(x_0+h)| \le C|h|^\alpha \ \forall h, x_0 \in \mathbf{R} \qquad (II.9)$$

Denote by $||f||_\alpha$ the infimum of all C such that condition $(II.9)$ holds. $||.||_\alpha$ provides Λ^α with a Banach space structure. Let now $\psi(x)$ be an admissible wavelet, such that in addition

$$\int |x|^\alpha |\psi(x)|\, dx < \infty \qquad (II.10)$$

and define $T_f(b,a)$ by $(II.4)$. Let now

$$|||f|||_\alpha = \sup_{(b,a)\in\mathbf{R}\times\mathbf{R}_+^*} a^{-\alpha}|T_f(b,a)| \qquad (II.11)$$

Then it is not very difficult to show (see e.g. [Fr.Ja.We]) that

$$\Lambda^\alpha \cong \{f : \mathbf{R} \to \mathbf{C}\, s.t.\, |||f|||_\alpha < \infty\} \qquad (II.12)$$

and that $|||.|||_\alpha$ and $||.||_\alpha$ are equivalent norms on Λ^α. In other words, the global regularity of an analyzed function can be directly measured on its wavelet coefficients. It is however important to notice that such a result is not a direct consequence of Calderón's formula as it was presented before, since Lipschitz functions are only defined modulo constants. Nevertheless, the main argument is that since they are of zero integral, wavelets also operate modulo constants.

Wavelets actually reveal to be much more precise for the analysis of the regularity properties of functions, since they also the characterization of local regularity properties. Consider the following simple example. Assume that the function $f(x)$ (assumed to be square-integrable for simplicity) is locally Lipschitz of order $0 < \alpha < 1$ in some neighborhood of some point x_0. More precisely, there exist two constants $C > 0$ and X such that

$$|f(x_0) - f(x_0 + h)| \leq C|h|^\alpha \;\forall h < X \qquad (II.13)$$

Then it is follows from simple estimates that

$$|T_f(b,a)| = O(a^\alpha) + O(|x_0 - b|^\alpha) \qquad (II.14)$$

A very interesting point is that it is possible to get a converse theorem, namely to find sufficient conditions on the behaviour of the wavelet transform to ensure that $f(x)$ is locally Lip-α at $x = x_0$. Such sufficient conditions are actually a little bit stronger than $(II.14)$. More precisely, to show that some function $f(x)$ is locally Lip-α at some point x_0, it is necessary to assume that $T_f(b,a) = O(a^\gamma)$ uniformly in b for some $\gamma > 0$, and that $T_f(b,a) = O(a^\alpha + |x_0 - b|^\alpha / \ln|x_0 - b|$. We refer to [Ho.Tc] for a careful analysis of that point, together with other examples of regularity estimation through wavelet analysis.

A very interesting aspect of regularity analysis with wavelets is that mathematical results can be (almost) directly applied to very concrete problems of say signal processing. Let us quote for example the very interesting work of S. Mallat and his collaborators (see [Ma.Hw] for a review) on wavelet analysis of singularities and its applications to edge detection problems in signal and image analysis. In particular, starting from mathematical considerations, they are able to construct fast algorithms for the characterization of the singularities of a signal, and to exploit them for signal compression.

III. WAVELET ANALYSIS AS A LOCAL FOURIER ANALYSIS:

We have shown in the previous section examples in which the local aspects of wavelet analysis was used. Some authors emphasize this aspect by calling wavelet analysis a *position-scale method*, to be distinguished from *position-frequency methods* such as windowed Fourier transform.

Nevertheless, wavelet analysis has been shown in a signal analysis context to provide a very precise local version of Fourier analysis. Indeed, by Plancherel's formula, one has

$$T_f(b,a) = \frac{1}{\sqrt{a}} \int_{\mathbf{R}} f(x) \psi\left(\frac{x-b}{a}\right)^* \, dx \qquad (III.1)$$

$$= \sqrt{a} \int_{\mathbf{R}} \hat{f}(\xi) e^{-i\xi . b} \hat{\psi}(a\xi)^* \, d\xi \qquad (III.2)$$

Then, assuming that $\psi(x)$ is "well localized" around $x = 0$, and that $\hat{\psi}(\xi)$ is "well localized" around $\xi = \omega_0 \neq 0$ (because of $(II.1)$), $T_f(b,a)$ measures a position-frequency content of $f(x)$ in a neighborhood of the point $(b, \omega_0/a)$ in the position frequency plane.

In the context of signal analysis, the signals to be analyzed are in general real-valued functions, and are then completely determined by their positive frequencies. It is then sufficient to work in the $H^2(\mathbf{R})$-context, and then quite convenient to use progressive (or analytic) wavelets, i.e. wavelets $\psi(x)$ such that $\hat{\psi}(\xi) = 0$ for negative values of ξ. Let $\psi(x)$ be such a wavelet.

Let us consider the following simple examples. A direct computation shows that the wavelet transform of $f(x) = \cos(\omega x)$ is given by $T_f(b,a) = e^{i\omega b}\sqrt{a}\hat{\psi}(a\omega)^*$. The main consequence of the progressivity of the wavelet is the absence of beats. Then one has a perfect localization of the energy in the position-scale half-plane, around the horizontal line $a = \omega_0/\omega$.

Consider now a slightly more complicated signal, of the form

$$f(x) = A(x)\cos(\omega x) \qquad (III.3)$$

where $A(x)$ is some slowly varying (with respect to the oscillations at pulsation ω) amplitude function, assumed to be of class $\mathrm{C}^2(\mathbf{R})$. Then it is possible to proceed almost as before. A direct application of Taylor's formula shows that

$$T_f(b,a) = A(b) e^{i\omega b} \sqrt{a}\, \hat{\psi}(a\omega)^* + r(b,a) \qquad (III.4)$$

where $r(b,a)$ is some remainder, bounded as $|r(b,a)| \leq K.a^{-1}.Sup\{|A'(x)|\}$, for some constant K. To get such an estimate, the wavelet $\psi(x)$ must decay enough at infinity so that $\int |x||\psi(x)|\, dx < \infty$. Moreover, if the scale parameter a is precisely equal to the one giving the frequency ω, i.e. if $a = \omega_0/\omega$, then the approximation given by $A(b)e^{i\omega b}\hat{\psi}(a\omega)$ is even better than before, since the remainder $r(b,a)$ is now bounded as $|r(b,a)| \leq K'.a^{-2}.Sup\{|A''(x)|\}$, for some constant K' (here $\psi(x)$ should be such that $\int |x|^2|\psi(x)|dx < \infty$). Contrary to the first example, this is a case where Fourier analysis would be in great difficulty, because of its non locality. In such a case, Fourier analysis would allow the determination of the pulsation ω, but not at all that of the amplitude $A(x)$ (except in very simple academic cases).

Let us now examine the case where the frequency of the analyzed signal is allowed to change with x. Before entering the subject, let us specify a little bit more the notion of instantaneous frequency. To analyze a signal im terms of local amplitude and frequency, it is convenient (see [Pi.Ma] for instance) to write it in the form

$$f(x) = a(x)\cos(\varphi(x)) \qquad (III.5)$$

and to define an instantaneous frequency $\nu_i(x)$ by

$$\nu_i(x) = \frac{1}{2\pi}\frac{d\varphi(x)}{dx} \qquad (III.6)$$

Unfortunately such a presentation is far from unique. Indeed, one may also write $f(x)$ as $2a(x)\sin(\varphi(x)/2)\cos(\varphi(x)/2)$, and then interpret $2a(x)\sin(\varphi(x)/2)$ as the local amplitude and $\varphi(x)/2$ as the local phase. Such a problem can be solved by the introduction of the so-called analytic signal $Z_f(x)$ of $f(x)$, defined by its Fourier transform

$$\widehat{Z_f}(\xi) = 2H(\xi)\hat{f}(\xi) \qquad (III.7)$$

where $H(\xi)$ is the Heaviside step function. $Z_f(x)$ can clearly be analytically continued to the upper half-plane, and (once chosen a determination of the logarithm) written in a unique way as

$$Z_f(x) = A(x)e^{i\phi(x)} \qquad (III.8)$$

where $A(x)$ is a positive real-valued function, and $\phi(x)$ takes values in say $[0, 2\pi[$. (A, ϕ) is called the canonical pair of $f(x)$, and the expression

$$f(x) = A(x)\cos(\phi(x)) \qquad (III.9)$$

is the canonical presentation of $f(x)$. It is easily checked that the instantaneous frequency (as defined in $(III.6)$ with $\varphi(x) = \phi(x)$) coincides with the usual frequency in the case of plane waves.

The problem is then the use of wavelet analysis for the characterization of the canonical pair $(A(x), \phi(x))$ of a signal $f(x)$. To carry on such an analysis, it is necessary to make the same assumptions as before, namely to assume that $A(x)$ is slowly varying compared with the oscillating term $\cos(\phi(x))$. Then, assuming that $\psi(x)$ is progressive and also satisfies the same conditions, the wavelet coefficient $T_f(b, a)$ takes the form of an oscillatory integral. Using very classical approximation techniques for such integrals, it can be shown [De.E.Gu.KM.Tc.To] that $T_f(b, a)$ is locally maximum around a curve in the (b, a) half-plane, called the ridge of the wavelet transform, of equation

$$a = a_r(b) = \frac{\phi'_\psi(0)}{\phi'(b)} \qquad (III.10)$$

Here $\phi_\psi(x)$ is the canonical phase function of the analyzing wavelet $\psi(x)$ (that equals its own analytic signal by assumption). Moreover, the restriction of the wavelet transform to its ridge (the so-called *skeleton of the transform*) has a very simple expression:

$$T_f(b, a_r(b)) = Corr(b)Z_f(b) + r(b) \qquad (III.11)$$

where $Corr(b)$ is a corrective factor, completely determined by the wavelet $\psi(x)$ and the ridge $a_r(b)$, and $r(b)$ is some remainder, whose size can be estimated by standard techniques. Then the knowledge of the ridge of the wavelet transform is a sufficient information to characterize $A(x)$ and $\nu_i(x)$. It turns out that a fast fixed-point algorithm can be built to numerically extract the ridge from the wavelet transform [De.E.Gu.KM.Tc.To]. This algorithm is based on a careful analysis of the phase of the wavelet transform.

Of course, except in very simple cases, Fourier analysis fails to solve such problems. In the cases where the analyzed function is simply of the form $(III.9)$, wavelet analysis is not necessary, since one can directly compute the analytic signal $Z_f(x)$ and then deduce the canonical pair $(A(x), \phi(x))$. However, let us now assume that the signal to be analyzed is a linear combination of components of the form $(III.9)$. One is clearly interested in the individual canonical pairs of the components, and the global analytic signal can't give such an answer. Now, since wavelet transform is a linear transform, the wavelet transform of the composite signal is the sum of the wavelet transforms of the components, that possess their own ridges. Then, if the ridges are not too close from each other in the (b, a) half-plane (in which case the approximation of oscillatory integrals fails), one can associate a skeleton to every ridge, and then compute the canonical pairs of all components. We refer to [De.E.Gu.KM.Tc.To] for a detailed presentation of that method.

We have then seen in this section that one-dimensional wavelets can be used to give a precise meaning to the concept of local frequency, i.e. as a position-frequency analysis method. As such, it has to be compared with the windowed Fourier analysis, or Gabor analysis, defined as follows. Let $g \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ be a window function, i.e. a function possessing good decay at infinity both in the x-space and the Fourier domain. To any $f \in L^2(\mathbf{R})$ is associated its windowed Fourier transform G_f:

$$G_f(b, \omega) = \int_{\mathbf{R}} f(x) e^{-i\omega.(x-b)} g(x-b)^* dx \qquad (III.12)$$

Simple arguments show that the map

$$f \in L^2(\mathbf{R}) \to G_f \in L^2(\mathbf{R}^2) \qquad (III.13)$$

is actually an isometry (up to a constant factor), so that f can be expressed as

$$f(x) = \frac{1}{2\pi ||g||^2} \int_{\mathbf{R}^2} G_f(b, \omega) e^{i\omega.(x-b)} g(x-b) dx \qquad (III.14)$$

(the R.H.S. converging weakly to f). The associated wavelets are the so-called Gabor functions[(3)]:

$$g_{(b,\omega)}(x) = e^{i\omega.(x-b)} g(x-b), \quad b, \omega \in \mathbf{R} \qquad (III.15)$$

The Gabor functions are then obtained from the $g(x)$-window by simple transformations, namely modulations and translations. We will use this fact in the next section.

[(3)] Notice that $G_f(b, \omega)$ is the scalar product of f and $g_{(b,\omega)}$.

IV. PHASE SPACE AND ADAPTED WAVELET DECOMPOSITIONS:

IV.1: Affine wavelets and phase space:

In the previous sections, we have described two one-dimensional position-frequency decompositions, corresponding essentially to two geometries of the position-frequency plane. The generalizations to arbitrary dimensions are quite different. First of all the generalization of Gabor analysis is straightforward. Let $g \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ be the mother window, and associate with it the following Gabor functions

$$g_{(\underline{b},\underline{\omega})}(\underline{x}) := e^{i\underline{\omega}.(\underline{x}-\underline{b})} g(\underline{x}-\underline{b}), \quad \underline{b},\underline{\omega} \in \mathbf{R}^n \qquad (IV.1)$$

Then any $f \in L^2(\mathbf{R}^n)$ can be decomposed as follows

$$f(\underline{x}) = \frac{1}{(2\pi)^n ||g||^2} \int_{\mathbf{R}^{2n}} G_f(\underline{b},\underline{\omega}) e^{i\underline{\omega}.(\underline{x}-\underline{b})} g(\underline{x}-\underline{b}) d\underline{x} \qquad (IV.2)$$

with $G_f(\underline{b},\underline{\omega}) :=< f, g_{(\underline{b},\underline{\omega})} >$. G_f is then a function in $L^2(\mathbf{R}^{2n})$, i.e. a function on the phase space of $\mathbf{R}^n$.

The generalization of affine wavelet transform to arbitrary dimensions as a mapping of $L^2(\mathbf{R}^n)$ into the L^2-space of the phase space of $\mathbf{R}^n$ is a little bit more complex. In other words, it is not obvious to generalize Calderón's one-dimensional formula to the n-dimensional case and to keep this notion of wavelets indexed by a position and a frequency. This is of course necessary if one wants to be able to generalize to arbitrary dimensions the algorithms described above. First of all, one can't use the "radial wavelets" version described in section II, since the corresponding wavelet transform is not a function on the phase space. An alternative one, proposed by R. Murenzi [Mu] is based on an extension of the set of translations and dilations of $\mathbf{R}^n$ by rotations matrices (we recall here that the rotations of $\mathbf{R}^n$ form a group denoted by $SO(n)$). To the mother wavelet $\psi \in L^1(\mathbf{R}^n)$ are then associated the following wavelets

$$\psi_{(\underline{b},a,\underline{\underline{r}})}(\underline{x}) := \frac{1}{a^{n/2}} \psi\left(\underline{\underline{r}}^{-1}.\frac{\underline{x}-\underline{b}}{a}\right) \qquad (IV.3)$$

where $\underline{b} \in \mathbf{R}, a \in \mathbf{R}_+^*, \underline{\underline{r}} \in SO(n)$. The admissibility condition then reads

$$0 < k_\psi := Vol(SO(n-1)) \int_{\mathbf{R}^n} |\hat{\psi}(\underline{\xi})|^2 \frac{d\underline{\xi}}{||\underline{\xi}||^n} < \infty \qquad (IV.4)$$

(We recall here that as a compact group, $SO(n)$ has finite volume). In such a context, any $f \in L^2(\mathbf{R}^n)$ can be decomposed as

$$f(\underline{x}) = \frac{1}{k_\psi} \int_{\mathbf{R}_+^* \times \mathbf{R}^n \times SO(n)} T_f(\underline{b},a,\underline{\underline{r}})\, \psi_{(\underline{b},a,\underline{\underline{r}})}(\underline{x}) \frac{da}{a} \frac{d\underline{b}}{a^n} dm(\underline{\underline{r}}) \qquad (IV.5)$$

where $dm(\underline{\underline{r}})$ is a Haar measure on $SO(n)$ (which is known to be unique up to a multiplicative factor), and the wavelet transform is now the function of $L^2(\mathbf{R}_+^* \times \mathbf{R}^n \times SO(n))$ defined by

$$T_f(\underline{b}, a, \underline{\underline{r}}) :=< f, \psi_{(\underline{b},a,\underline{\underline{r}})} > \qquad (IV.6)$$

The wavelet transform is then in that case a function of $n(n+1)/2+1$ variables, and still can't be considered as a function on the phase space. Nevertheless, one can observe in eq. $(IV.4)$ that the admissibility constant k_ψ contains as a factor the volume of the rotation group in $\mathbf{R}^{n-1}$, $SO(n-1)$. This suggests that some $SO(n-1)$-subgroup of $SO(n)$ does not contribute to the reconstruction.

To see that, it is necessary to introduce some more technical tools, namely the Euler angles parametrization of $SO(n)$. Let $e_1, e_2, ...e_n \in \mathbf{R}^n$ be a fixed orthonormal frame. Then the stabilizer of e_n is isomorphic to an $SO(n-1)$ subgroup of $SO(n)$, and the decomposition $SO(n) = S^{n-1} \times SO(n-1)$ (where $S^{n-1} \cong SO(n)/SO(n-1)$ is the $(n-1)$-dimensional sphere embedded into $\mathbf{R}^n$) yields the following decomposition of elements of $\underline{\underline{r}} \in SO(n)$:

$$\underline{\underline{r}} = \underline{\underline{r}}^{(n-1)}.\underline{\underline{r}}^{(n-2)}...\underline{\underline{r}}^{(1)} \qquad (IV.7)$$

where

$$\underline{\underline{r}}^{(k)} := \underline{\underline{r}}_1(\theta_1^k)...\underline{\underline{r}}_k(\theta_k^k) \qquad (IV.8)$$

and $\underline{\underline{r}}_k(\theta)$ is the rotation of angle θ in the oriented plane defined by e_k and e_{k+1} if $k = 1, ...n-1$ and e_n and e_1 if $k = n$. Here, θ_j^k runs over $[0, \pi]$ when $j \neq k$ and over $[0, 2\pi]$ when $j = k$. Such a decomposition makes explicit the factorization of $SO(n)$ into the product of $SO(n-1)$ and the sphere S^{n-1}. If $\underline{\underline{r}}$ is a generic element of $SO(n)$, we will write $\underline{\underline{r}} = \underline{\underline{r}}_0.\underline{\underline{s}}$ its corresponding factorization ($\underline{\underline{r}}_0 \in SO(n-1)$). The corresponding form for the Haar measure on $SO(n)$ reads

$$dm(\underline{\underline{r}}) = dm(\theta_1^1, ...\theta_{n-1}^{n-1}) = A(n)^{-1} \prod_{k=1}^{n-1} \prod_{j=1}^{k} \left[\sin^{j-1}\left(\theta_j^k\right)\, d\theta_j^k\right] \qquad (IV.9)$$

for some constant $A(n)$ only depending on the dimension (see [Vil]). $dm(\underline{\underline{r}})$ then factorizes into a product of the Haar measure $dm'(\underline{\underline{s}})$ by an invariant measure $d\nu(\underline{\underline{r}}_0)$ on the sphere S^{n-1}.

We now come back to the affine wavelets. Let $f(\underline{x}) \in L^2(\mathbf{R}^n)$. Parametrizing $SO(n)$ by the associated Euler angles, and setting to zero the Euler angles corresponding to the factor $SO(n-1)$, it is still possible to reconstruct $f(\underline{x})$ from the restricted wavelet transform, as follows:

$$f(\underline{x}) = \frac{1}{k'_\psi} \int_{\mathbf{R}_+^* \times \mathbf{R}^n \times SO(n)/SO(n-1)} T_f(\underline{b}, a, \underline{\underline{r}}_0)\, \psi_{(\underline{b},a,\underline{\underline{r}}_0)}(\underline{x})\, \frac{da}{a} \frac{d\underline{b}}{a^n}\, d\nu(\underline{\underline{r}}_0) \qquad (IV.10)$$

The reconstruction formula is the same as in the $SO(n)$ case, except that the integral is now taken over the quotient space $\mathbf{R}_+^* \times \mathbf{R}^n \times SO(n)/SO(n-1) \cong \mathbf{R}^n \times \mathbf{R}_+^* \times S^{n-1}$, and that the admissibility constant k_ψ has now to be replaced by $k'_\psi := k_\psi / Vol(SO(n-1))$.

IV.2: Square-integrable group representations:

In the previous section, we have seen that to generalize wavelets (as position-frequency objects) to arbitrary dimensions, it was natural to introduce a rotation group. Actually, it must also be remarked that the set of simple geometrical transformations used to generate the wavelets from a single function $\psi(\underline{x})$ possesses a group structure. More precisely, the set of translations, dilations and rotations of $\mathbf{R}^n$ is a Lie group, denoted by $IG(n)$, introduced first by R. Murenzi in the wavelet community [Mu].

$$IG(n) := \mathbf{R}^n \times \mathbf{R}_+^* \times SO(n) \qquad (IV.11)$$

with group operation

$$(\underline{b}, a, \underline{\underline{r}}).(\underline{b}', a', \underline{\underline{r}}') = (\underline{b} + a\underline{\underline{r}}.\underline{b}', aa', \underline{\underline{rr}}'), \quad (\underline{b}, a, \underline{\underline{r}}), (\underline{b}', a', \underline{\underline{r}}') \in IG(n) \qquad (IV.12)$$

In fact, the n-dimensional version of wavelet analysis, and more precisely eq. $(IV.5)$, can be viewed as a particular case of the general theory of square-integrable group representations.

The connexion between position-frequency analysis methods and square-integrable group representations was realized by A. Grossmann, J. Morlet and T. Paul in [Gr.Mo.Pa]. We start by briefly describing their construction. Let $\mathbf{G}$ be a separable locally compact Lie group, let μ be its left-invariant measure and let π be a unitary strongly continuous representation of $\mathbf{G}$ on the Hilbert space $\mathcal{H}$. π is said to be *square-integrable* (or to belong to the *discrete series* of $\mathbf{G}$) if

- π is irreducible.
- There exists at least a vector $v \in \mathcal{H}$ such that

$$0 \ < \ \int_{\mathbf{G}} | < \pi(g).v, v > |^2 \, d\mu(g) \ < \ \infty \qquad (IV.13)$$

Such a vector is said to be *admissible*.

Square-integrable group representations have been extensively studied in the literature, in particular for compact groups by Bargmann, locally compact unimodular groups by Godement and non-unimodular locally compact groups, in particular by Duflo and Moore [Du.Moo]. One of the main results concerns the so-called orthogonality relations for Schur coefficients, that can be expressed as follows.

Theorem:
Let π be a square-integrable strongly continuous unitary representation of the locally compact group $\mathbf{G}$ on $\mathcal{H}$. Then there exists a positive self-adjoint operator C such that for any admissible vectors $v_1, v_2 \in \mathcal{H}$ and for any $u_1, u_2 \in \mathcal{H}$ such that

$$\int_{\mathbf{G}} < u_1, \pi(g).v_1 >< \pi(g).v_2, u_2 > \ d\mu(g) =< C^{1/2}.v_2, C^{1/2}.v_1 > < u_1, u_2 > \qquad (IV.14)$$

Moreover, the set of admissible vectors coincides with the domain of C.$\circ$

Let us denote by λ the left-regular representation of $\mathbf{G}$ [(4)]. As a simple consequence of the previous theorem, a representation π of $\mathbf{G}$ is square integrable if and only if is is unitarily equivalent to a subrepresentation of the left-regular representation λ. The corresponding intertwining operators can be realized as follows. If v is an admissible vector in $\mathcal{H}$, and $v' \in \mathcal{H}$, one can then introduce the corresponding Schur coefficients, i.e the matrix coefficients of elements of $\mathbf{G}$:

$$c_{v,v'}(g) := < v', \pi(g).v >, \; g \in \mathbf{G} \qquad (IV.15)$$

Denote by T the map which assigns to any $u \in \mathcal{H}$ the family of coefficients $c_{v,u}(g)$ $g \in \mathbf{G}$

$$T : u \in \mathcal{H} \to T_u = c_{v,u}(.) \in L^2(\mathbf{G}) \qquad (IV.16)$$

We will call T a generalized wavelet transform. T realizes the intertwinning between π and λ as follows.

$$T \circ \pi = \lambda \circ T \qquad (IV.17)$$

The idea of Grossmann, Morlet and Paul was to use $(IV.16)$ and $(IV.17)$ for the analysis of functions, in the case where $\mathcal{H}$ is a function space. This was the starting point of many applications, especially in a signal analysis context. The T transform is used to obtain another representation of functions, and $(IV.17)$ expresses the covariance of the transform.

Consider now the $IG(n)$ group, and let π be the following representation of $IG(n)$ on $L^2(\mathbf{R}^n)$:

$$\left[\pi(\underline{b}, a, \underline{\underline{r}}).f\right](\underline{x}) := a^{-n/2} f\left(\underline{\underline{r}}^{-1}.\frac{\underline{x}-\underline{b}}{a}\right) \qquad (IV.18)$$

Let $\psi \in L^1(\mathbf{R}^n)$ be such that $(IV.4)$ holds. Then the wavelets $\psi_{(\underline{b},a,\underline{\underline{r}})}$ defined by $(IV.3)$ are nothing but

$$\psi_{(\underline{b},a,\underline{\underline{r}})} = \left[\pi(\underline{b}, a, \underline{\underline{r}}).\psi\right] \qquad (IV.19)$$

and the resolution of the identity $(IV.5)$ is a rewriting of $(IV.14)$

IV.3: Group representations square-integrable modulo a subgroup:

Consider again the example of wavelets associated with representations of $IG(n)$. We have seen at the end of section IV.1 that the corresponding wavelet transform does not map $L^2(\mathbf{R}^n)$ into the L^2-space of the corresponding phase space. Otherwise stated, it can't be considered as a position-frequency analysis. Nevertheless, we have also seen that restricting to a quotient group, namely $IG(n)/SO(n-1)$, it is still possible to get a resolution of the identity, in a position-frequency representation context.

Another interesting example is that of the (say one-dimensional for simplicity) Gabor functions. According to the theory of canonical coherent states (see [Pe],[Kl.Sk] for a review), they are built from the (polarized) Weyl-Heisenberg group (see [Sch] for a review):

$$\mathbf{G}_{WH} := \mathbf{R}^2 \times S^1 \qquad (IV.20)$$

---

(4) Let us recall here that λ is a strongly continuous unitary representation of $\mathbf{G}$ on $L^2(\mathbf{G}, \mu)$ defined by $[\lambda(g).f](h) = f(g.h)$.

with group law

$$(q,p,\varphi).(q',p',\varphi') := (q+q',p+p',\varphi+\varphi'+pq' \ [mod 2\pi]) \qquad (IV.21)$$

and the following representation on $L^2(\mathbf{R})$:

$$[\pi(p,q,\varphi).f](x) := e^{i\varphi}e^{ip(x-q)}f(x-q) \qquad (IV.22)$$

Nevertheless, it is worth mentioning that the S^1 factor does not play any role in the construction, and that the Gabor functions are associated with the quotient $\mathbf{G}_{WH}/S^1$ more than with the $\mathbf{G}_{WH}$ group itself.

From such examples arises the notion of square-integrability modulo a subgroup. Indeed, in both cases one is interested to drop the extra factor (here S^1 or $SO(n-1)$), to reduce the number of variables of the representation. Given a representation π of a group $\mathbf{G}$ on a Hilbert space $\mathcal{H}$ such that all integrals of the type $(IV.13)$ diverge, one may wonder whether such integrals might converge or diverge when restricted to an appropriate homogeneous space $\mathbf{G}/\mathbf{H}$ for some closed subgroup $\mathbf{H} \subset \mathbf{G}$. Of course π is not defined directly on $\mathbf{G}/\mathbf{H}$, and it is necessary to first embed $\mathbf{G}/\mathbf{H}$ in $\mathbf{G}$. This is realized by using the canonical fiber bundle structure of $\mathbf{G}$.

$$\Pi : \ \mathbf{G} \to \mathbf{G}/\mathbf{H}$$

Let σ be a Borel section of this fiber bundle (it is well known that such sections always exist), and introduce

$$\pi_\sigma := \pi \circ \sigma \qquad (IV.23)$$

Let μ be some quasi-invariant measure on $\mathbf{G}/\mathbf{H}$. It then makes sense to study the operator

$$\mathcal{A} : \ u \in \mathcal{H} \to \int_{\mathbf{G}/\mathbf{H}} < u, \pi_\sigma(x) > \pi_\sigma(x)\, d\mu(x) \qquad (IV.24)$$

Depending on the properties of the $\mathcal{A}$ operator, it may be possible to associate with it an isometry

$$T : \ \mathcal{H} \to L^2(\mathbf{G}/\mathbf{H})$$

similar to wavelet transforms for $\mathcal{H} \cong L^2(\mathbf{R})$. Such a program was carried out in particular cases, namely for special groups: in particular the Poincaré group [Al.An.Ga.1-2] or the affine Weyl-Heisenberg group [To.1-2], [Ka.To].

Depending on the choice of the σ section, the $\mathcal{A}$ operator can enjoy quite different properties. The following notions were introduced in [Ka.To]:

Definition:

The section σ is said to be admissible if there exists a bounded positive invertible operator $\mathcal{A}$, with bounded inverse, and a function $\psi \in L^2(\mathbf{R}^n)$ such that for all $f \in L^2(\mathbf{R}^n)$

$$\int_X |< \pi_\sigma(x).\psi, \psi >|^2 \, d\mu(x) \ = < f, \mathcal{A}.f > \qquad (IV.25)$$

σ is said to be strictly admissible if there exists a function $\psi \in L^2(\mathbf{R}^n)$ such that for all $f \in L^2(\mathbf{R}^n)$

$$\int_X |<\pi_\sigma(x).\psi, \psi>|^2 \, d\mu(x) \;=\; K||f||^2 \qquad (IV.26)$$

for some positive constant K.
σ is said to be weakly admissible if there exists a continuous field of operators $\mathcal{T}(x), x \in X$, and a function $\psi \in L^2(\mathbf{R}^n)$ such that for all $f \in L^2(\mathbf{R}^n)$

$$\int_X |<\mathcal{T}(x).\pi_\sigma(x).\psi, \psi>|^2 \, d\mu(x) \;=\; K||f||^2 \qquad (IV.27)$$

for some positive constant K. $\circ$

Otherwise stated, the section σ is strictly admissible if and only if σ is admissible and the $\mathcal{A}$ operator is a multiple of the identity. Admissible sections generate a continuous frame in the terminology of [Al.An.Ga.1-2]. In any case, we will denote by ψ_x the function $\pi_\sigma(x).\psi$ when σ is strictly admissible, and the function $\mathcal{A}.\pi_\sigma(x).\psi$ (resp. $\mathcal{T}(x).\pi_\sigma(x).\psi$) in the admissible (resp. weakly admissible) case. Given a weakly admissible section, there is then an associated resolution of the identity if and only if the orbit of X through ψ is total in $L^2(\mathbf{R}^n)$. In such cases, one can then construct an associated wavelet transform.

IV.4: more general wavelets:

Up to now, we have only described families of functions (wavelets) coming either from n-dimensional versions of the affine group or the Weyl-Heisenberg group. We will now consider other geometries of the phase space. For simplicity, consider again the one-dimensional case. The main property of Gabor functions is that they are of constant absolute bandwidth, i.e. their Fourier transform is of constant width. On the other hand, since $\widehat{\psi_{(b,a)}}(\xi) = e^{-i\xi b}\hat{\psi}(a\xi)$, it is easy to see that the affine wavelets are of constant relative bandwidth, i.e. the width of their Fourier transform is proportional to the value of the frequency they are centered on. We will say that these two families of functions correspond to different geometries of the phase space. We will now describe families of wavelets associated with geometries of phase spaces different than the previous ones, namely descriptions that lie "in between" the Gabor and the affine wavelet ones. Such intermediate wavelets have been found very useful in a speech analysis context [d'A.Be]. To perform such a generalization, we proceed as follows. First consider a larger group, containing the affine group and the Weyl-Heisenberg group as subgroups. Second pick a specific representation of this larger group. Such a representation is in general not square-integrable. Finally, restrict the representation to appropriate coset spaces, and look for a resolution of the identity.

We will focus on the n-dimensional affine Weyl-Heisenberg group $\mathbf{G}_{aWH}$, that is the group generated by translations, modulations, dilations and rotations in $\mathbf{R}^n$, studied in [Ka.To]. It is the n-dimensional generalization of the one-dimensional affine Weyl-Heisenberg group considered in [To.1].

The affine Weyl-Heisenberg group is topologically isomorphic to

$$\mathbf{G}_{aWH} \cong \mathbf{R}^{2n+1} \times \mathbf{R}_+^* \times SO(n) \qquad (IV.28)$$

and has a structure of semi-direct product of the n-dimensional Weyl-Heisenberg group by $\mathbf{R}_+^* \times SO(n)$.

The corresponding generic element is of the form

$$g = (\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi),\ \underline{q}, \underline{p} \in \mathbf{R}^n, a \in \mathbf{R}_+^*, \varphi \in \mathbf{R}, \underline{\underline{r}} \in SO(n) \qquad (IV.29)$$

with group operation

$$(\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi)(\underline{q}', \underline{p}', a', \underline{\underline{r}}', \varphi') = (\underline{q} + a\underline{\underline{r}}.\underline{q}', \underline{p} + a^{-1}\underline{\underline{r}}.\underline{p}', aa', \underline{\underline{r}}.\underline{\underline{r}}', \varphi + \varphi' + \underline{p}.(a\underline{\underline{r}}.\underline{q}')) \quad (IV.30)$$

It is easy to see that $\mathbf{G}_{aWH}$ is unimodular, that is that the following measure is both left and right invariant:

$$d\mu(\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi) = d\underline{q}\, d\underline{p}\, \frac{da}{a}\, dm(\underline{\underline{r}})\, d\varphi \qquad (IV.31)$$

where $dm(\underline{\underline{r}})$ is the Haar measure on $SO(n)$, normalized so that $m(SO(n)) = 1$. The rotation group $SO(n)$ is conveniently described by means of the corresponding Euler angles. The n-dimensional Weyl-Heisenberg group can be realized as a matrix group as follows. The generic element $g = (\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi) \in \mathbf{G}_{aWH}$ is realized as the matrix:

$$g = \begin{pmatrix} 1 & {}^t\left[a\underline{\underline{r}}^{-1}.\underline{p}\right] & \varphi \\ 0 & \left[a\underline{\underline{r}}\right] & \left[\underline{q}\right] \\ 0 & 0 & 1 \end{pmatrix} \qquad (IV.32)$$

the group law being represented by matrix multiplications. Here, ${}^t\left[a\underline{\underline{r}}^{-1}.\underline{p}\right]$ is the transpose of the vector $a\underline{\underline{r}}^{-1}.\underline{p}$. A detailed study of the representation theory of the affine Weyl-Heisenberg group can be found in [Ka.To]. Let $\mathcal{H} = L^2(\mathbf{R}^n)$, and consider in particular following continuous unitary representation $\pi : \mathbf{G}_{aWH} \to \mathcal{U}(\mathcal{H})$:

$$\left[\pi(\underline{q}, \underline{p}, a, \underline{\underline{r}}, 0).f\right](\underline{x}) = a^{-n/2} e^{2i\pi \underline{p}.(\underline{x}-\underline{q})} f\left(\underline{\underline{r}}^{-1}.\frac{\underline{x}-\underline{q}}{a}\right) \qquad (IV.33)$$

It is easy to see that π is irreducible, but not square-integrable. Nevertheless, it can be made square-integrable when restricted to appropriate coset spaces. Consider for example the following subgroup (a more detailed analysis, together with other examples can be found in [Ka.To]):

$$\Gamma = \{(0, 0, a, 1, \varphi) \in \mathbf{G}_{aWH}\} \cong \mathbf{R}_+^* \times \mathbf{R} \qquad (IV.34)$$

Let

$$X = \mathbf{G}_{aWH}/\Gamma \qquad (IV.35)$$

X can be parametrized by elements of the form $(\underline{q}, \underline{p}, \underline{\underline{r}}) \in \mathbf{R}^{2n} \times SO(n)$. and is provided with the following left and right invariant measure

$$d\mu(\underline{q}, \underline{p}, \underline{\underline{r}}) = d\underline{q}\, d\underline{p}\, dm(\underline{\underline{r}}) \qquad (IV.36)$$

Let σ_0 be the following flat section of the fiber bundle $\Pi : \mathbf{G}_{aWH} \to X$

$$\sigma_0(\underline{q}, \underline{p}, \underline{\underline{r}}) = (\underline{q}, \underline{p}, 1, \underline{\underline{r}}, 0) \in \mathbf{G}_{aWH} \tag{IV.37}$$

and let σ_β be the Borel section defined by

$$\sigma_\beta(\underline{q}, \underline{p}, \underline{\underline{r}}) = (\underline{q}, \underline{p}, \beta(\underline{q}, \underline{p}, \underline{\underline{r}}), \underline{\underline{r}}, 0) \tag{IV.38}$$

where β is a piecewise differentiable Borel mapping of X into Γ.

Let $\psi \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$, and set

$$\psi_{(\underline{p},\underline{q},\underline{\underline{r}})}(\underline{x}) = |\beta(\underline{p}, \underline{q}, \underline{\underline{r}})|^{-n/2} e^{i\underline{p}.(\underline{x}-\underline{q})} \psi\left(\beta(\underline{p}, \underline{q}, \underline{\underline{r}})^{-1} \underline{\underline{r}}^{-1}.(\underline{x} - \underline{q})\right) \tag{IV.39}$$

Let $f \in L^2(\mathbf{R}^n)$; associate with it its transform

$$T_f(\underline{p}, \underline{q}, \underline{\underline{r}}) = < f, \psi_{(\underline{p},\underline{q},\underline{\underline{r}})} > \tag{IV.40}$$

T_f is bounded, and by Plancherel equality

$$T_f(\underline{p}, \underline{q}, \underline{\underline{r}}) = (2\pi)^{-n} |\beta(\underline{p}, \underline{q}, \underline{\underline{r}})|^{n/2} \int_{\mathbf{R}^n} \hat{f}(\underline{\xi}) e^{i\underline{\xi}.\underline{q}}\ \hat{\psi}\left[\underline{k}_\xi(\underline{p}, \underline{q}, \underline{\underline{r}})\right]^*\ d\underline{\xi} \tag{IV.41}$$

where

$$\underline{k}_\xi(\underline{p}, \underline{q}, \underline{\underline{r}}) = \beta(\underline{p}, \underline{q}, \underline{\underline{r}})\ \underline{\underline{r}}^{-1}.(\underline{\xi} - \underline{p}) \tag{IV.42}$$

Let $\underline{\Xi}$ be a constant $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ tensor, and set

$$\beta(\underline{q}, \underline{p}, \underline{\underline{r}}) = \frac{1}{< \underline{\Xi}, \underline{p} > + f(\underline{\underline{r}})} \tag{IV.43}$$

where f is some (smooth) function of $\underline{\underline{r}}$. It then follows from simple algebra that the corresponding σ_β section is strictly admissible. Indeed, one has that

$$\hat{F}(\underline{\xi}) = \hat{f}(\underline{\xi}) \int_{SO(n)} dm(\underline{\underline{r}}) \int_{\mathbf{R}^n} |\hat{\psi}(\underline{k})|^2 \frac{d\underline{k}}{|1- < \underline{\Xi}, \underline{\underline{r}}.\underline{k} > |} = C_\psi \hat{f}(\underline{\xi}) \tag{IV.44}$$

so that if the C_ψ constant is finite and nonzero, we are in the case described by $(IV.26)$, and directly get a resolution of the identity: for any $f \in L^2(\mathbf{R}^n)$:

$$f = \frac{1}{C_\psi} \int_{\mathbf{R}^{2n} \times SO(n)} < f, \psi_{(\underline{p},\underline{q},\underline{\underline{r}})} > \psi_{(\underline{p},\underline{q},\underline{\underline{r}})} d\underline{p}\, d\underline{q}\, dm(\underline{\underline{r}}) \tag{IV.45}$$

As in the affine wavelet case, the finiteness of C_ψ is obtained as soon as $\hat{\psi}$ has sufficient decay at infinity, and vanishes on the sphere $\underline{k} = \underline{\underline{r}}^{-1}.{}^t\underline{\Xi}, \underline{\underline{r}} \in SO(n)$. Notice that $\underline{\Xi} = 0$ (i.e.

constant β) corresponds to N-dimensional Gabor analysis, and the admissibility condition reduces to $\psi \in L^2(\mathbf{R}^n)$, which we have by assumption.

It is in general much easier to get admissible or weakly admissible sections. Some explicit examples are given in [Ka.To], and in [To.1-2]. Let us just quote the following construction, taken from [To.2]. To the wavelet $\psi_{(\underline{p},\underline{q},\underline{r})}(\underline{x})$ asociate the function $\Psi_{(\underline{p},\underline{q},\underline{r})}(\underline{x})$, defined as follows by its Fourier transform

$$\widehat{\Psi_{(\underline{p},\underline{q},\underline{r})}}(\underline{\xi}) = \widehat{\psi_{(\underline{p},\underline{q},\underline{r})}}(\underline{\xi}) . \sqrt{\frac{J_\xi}{\beta(\underline{p},\underline{r})}} \qquad (IV.46)$$

where J_ξ is the following Jacobian

$$J_\xi = |Det\left[\underline{\nabla}_p . \underline{k}_\xi\right]| \qquad (IV.47)$$

Using the same arguments as before, it is not difficult to see that if the section $\beta(\underline{p},\underline{r})$ is such that

$$\chi(\underline{\xi}) = \int_{\underline{k}_\xi(\mathbf{R}^n)} |\hat{\psi}(\underline{k})|^2 d\underline{k} \qquad (IV.48)$$

is strictly positive and bounded almost everywhere as a function of $\underline{\xi}$ (i.e. is the multiplyer of an invertible convolution operator $\mathcal{C}_\chi$), then eq. $(IV.25)$ holds, with $\mathcal{A} = \mathcal{C}_\chi$.

The interesting point in such a procedure is that since ψ is square-integrable by assumption, the admissibility of the section now only depends on β through the set $\underline{k}_\xi(\mathbf{R}^n)$, and can be analyzed in a quite simple way.

Nevertheless, our knowledge in such a context is quite poor, and we do not know for the moment which are the most convenient sections. It is of course possible to check directly the admissibility of a given section. But an important problem is to find the *best-adapted* section for a given problem, and then to define a criterion of adaptness. This remains a very appealing open problem.

V. CONCLUSIONS:

In this paper, we have described and illustrated by various examples two complementary aspects of wavelet analysis, namely the position-scale aspect and the position-frequency aspect. While the first point of view is quite convenient for the analysis of global and local regularity properties of functions, the second one appears to provide a local version of Fourier analysis. Applications to many specific problems of signal analysis can be mentioned.

It turns out that the interpretation of wavelets as position-frequency tools can be formulated in a quite simple geometric language. (generalized) wavelets can be introduced as generated by a unitary irreducible group representation, under the assumption that the representation is square-integrable with respect to the whole group or some quotient space of the group.

The case of square-integrability with respect to the whole group is well known, and was described in [Gr.Mo.Pa]. In such a case, the geometrical interpretation of the transform is peculiarly simple. The representation is an irreducible subrepresentation of the left-regular representation of the group, and the wavelet transform is an intertwining operator, i.e. realizes the embedding into the regular representation. In the coset space case, the problem of the existence of associated families of wavelets (or coherent states) is related to the choice of an appropriate section, mapping the coset space into the group. The notion of position-frequency representation is recovered in the case where the representation belongs to some coadjoint orbit of the group, and the coset space is isomorphic to the associated phase space. This is in particular the case of the usual wavelet and Gabor analysis.

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