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TIME-FREQUENCY ANALYSIS, FROM GEOMETRY TO SIGNAL PROCESSING

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We describe some aspects of time-frequency analysis, involving mainly two arguments: time-frequency localization, and symmetry constraints. We show how such arguments appear in the classical time-frequency transforms (Wigner transforms, wavelet and Gabor transforms). Then we discuss deformed versions of one of these transforms, namely the reassigned Gabor transform, and show that it may be a promising alternative to Wigner function.

1 Introduction

Time-Frequency analysis aims at providing representations for signals, involving simultaneously time and frequency descriptions. Classical signal analysis is generally based upon a representation of functions, for example the usual one (hereafter termed the “time representation”), or the Fourier representation (“frequency representation”): one studies

$$\hat{f}(\omega) = \int f(t)e^{-i\omega t} dt ,$$

whenever such an expression makes sense. In fact, most representations may be regarded as representations of functions by coefficients with respect to some “basis” (in some generalized sense if needed).

Among the representations, time-frequency representations have become quite popular in the recent years. The first contributions seem to be due to the engineer J. Ville on one hand, who proposed to use the Wigner distribution as a “time-frequency density”, and the physicist D. Gabor on the other hand, whose approach was based upon decompositions into “time-frequency atoms”, generated as time and frequency translates of a Gaussian function. Since then, Ville’s approach has led to the theory of quadratic time-frequency representations, very popular in the signal analysis community. Gabor-type expansions are also very popular, and have more recently enjoyed a renewed interest, since the development of wavelet theory in the early eighties. For completeness, let us also quote the contribution of M. Priestley ³⁷, who was

the first to formulate the time-frequency representation problem in a random situation.

A. Grossmann and J. Morlet realized in 1983 that any square-integrable function may be expanded into wavelets of constant shape, shifted and scaled copies of a single function (the mother wavelet), provided that the latter possess some mild oscillation properties. This simple result, and the remark that the corresponding expansions are in fact very close to Gabor's expansions, showed that wavelet expansions had an important potential as a tool for signal analysis. Grossmann and his collaborators also provided a beautiful interpretation of wavelet and Gabor expansions in terms of square-integrable group representations, emphasizing the importance of symmetry groups in the construction.

More recently, interesting connections between wavelet and Gabor transforms on one hand, and Wigner-Ville transforms and generalizations on the other hand have been discovered, the former being some "blurred" versions of the latter. The problem of restoring the time-frequency resolution from such "blurred" transforms has then attracted some attention, in particular with the systematic study of the so-called "reassignment methods" by P. Flandrin¹⁸ and his collaborators, after the pioneering work of Kodera et al.f²⁶. The geometrical status of such methods also deserves some interest¹⁵.

The present contribution is a description of the relationship between the above mentioned tools. We briefly recall the basics of time-frequency analysis, Wigner, Gabor and wavelet transforms. Then, we turn to a description of the reassignment methods, in their geometrical version, and conclude with some prospective remarks.

2 Time-frequency analysis

Continuous wavelet analysis was introduced^a in a seminal article by A. Grossmann and J. Morlet²³. The continuous wavelet transform is a prototype of *linear time frequency representations*, which provide decompositions of functions as superpositions of elementary waveforms. Those linear time-frequency representations are often compared with the *quadratic time-frequency representations*, a family of tools which are in some respect, more powerful, but also lack of robustness in complex practical situations. We first give a short account of the quadratic time-frequency representations.

^aIn fact, several earlier contributions could be quoted, both in the signal processing and mathematical literature, but wavelet transform was considered as such onmy since that work.

2.1 Wigner functions and quadratic time-frequency transforms

The Wigner-Ville distribution. The first instance of time-frequency transform is the *ambiguity function*, familiar to radar specialists⁹. The (cross) ambiguity function^b $\mathcal{A}_{f,g}$ of a pair of functions f and g may be introduced as a method for solving a simple estimation problem. Suppose that $g \in L^2(\mathbb{R})$ is a reference signal, and that f be an observation, time and frequency shifted copy of g (possibly perturbed): $f(t) = Ag(t - \tau)e^{i\xi t}$. Here A is a constant and τ and ξ are the parameters to be estimated. It follows from the Cauchy-Schwarz inequality that the parameters τ and ξ may be obtained by maximizing with respect to b and ω the square-modulus of the following quantity:

$$\int f(t)\overline{g}(t - b)e^{-i\omega t}dt .$$

The latter is essentially a time-frequency cross-correlation of f and g , obtained by considering scalar products of f with time and frequency shifted copies of g of the form $e^{i\omega t}g(t - b)$ (with $b, \omega \in \mathbb{R}$). Therefore, it measures how close f is to time-frequency shifted copies of g . The same problem may be formulated in a context of random signals. The definition of ambiguity function (including the case of random time series) is given below.

DEFINITION 2.1 1. Let $f \in L^2(\mathbb{R})$. Its ambiguity function is defined by

$$\mathcal{A}_f(\tau, \xi) = \int f(t + \tau/2)\overline{f}(t - \tau/2)e^{-i\xi t}dt . \quad (1)$$

2. Let $\{X_t, t \in \mathbb{R}\}$ a second order random time series. Then its ambiguity function is defined by

$$\mathcal{A}_X(\tau, \xi) = \mathbb{E} \left\{ \int X_{t+\tau/2}\overline{X}_{t-\tau/2}e^{i\xi t}dt \right\} . \quad (2)$$

REMARK 2.1 Clearly, if $f \in L^2(\mathbb{R})$, then \mathcal{A}_f is a bounded function (with $\|\mathcal{A}_f\|_\infty \leq \|f\|^2$). In addition, a direct calculation shows that if $f \in L^2(\mathbb{R})$, then $\mathcal{A}_f \in L^2(\mathbb{R}^2)$, and that $\|\mathcal{A}_f\|_2^2 = 2\pi\|f\|^4$. In fact, a result due to E. Lieb²⁸ shows that $\mathcal{A} \in L^p(\mathbb{R}^2)$ for all $p \in [1, \infty]$.

The definition of ambiguity function may be extended to cover the case of distributions, and it may be shown that the ‘‘ambiguity distribution’’ of a distribution $\phi \in \mathcal{S}'(\mathbb{R})$ is a distribution $\mathcal{A}_\phi \in \mathcal{S}'(\mathbb{R}^2)$.

REMARK 2.2 The non-deterministic version may be given a similar interpretation. Given a second order random time series $\{X_t, t \in \mathbb{R}\}$, the properties

^bIt is in fact the *narrow-band* ambiguity function.

of \mathcal{A}_X depend on the properties of the covariance operator \mathcal{C} of the process, defined by its matrix elements: for all $f, g \in \mathcal{D}(\mathbb{R})$,

$$\langle \mathcal{C}f, g \rangle = \mathbb{E} \left\{ \langle X, g \rangle \overline{\langle X, f \rangle} \right\}. \quad (3)$$

For example, if \mathcal{C} extends to a Hilbert-Schmidt operator, which we denote by $\mathcal{C} \in \mathcal{L}^2$, then $\mathcal{A} \in L^2(\mathbb{R}^2)$.

The ambiguity function $\mathcal{A}_f(\tau, \xi)$, as a time-frequency autocorrelation, provides estimates for the *spreading* of the function f in the joint time-frequency plane. The Wigner-Ville distribution (WV), defined below, provides informations on its localization in that space.

DEFINITION 2.2 1. Let $f \in L^2(\mathbb{R})$. Its Wigner-Ville distribution is the function of two real variables \mathcal{W}_f defined by

$$\mathcal{W}_f(b, \omega) = \int f(b + \tau/2) \overline{f(b - \tau/2)} e^{-i\omega\tau} d\tau \quad (4)$$

2. Let $\{X_t, t \in \mathbb{R}\}$ a second order time series. Then its Wigner-Ville distribution (or Wigner spectrum) is defined by

$$\mathcal{E}_X(b, \omega) = \mathbb{E} \left\{ \int X_{b+\tau/2} \overline{X_{b-\tau/2}} e^{-i\omega\tau} d\tau \right\} \quad (5)$$

More generally, one also defines the *cross Wigner-Ville function* of $f, g \in L^2(\mathbb{R})$ by

$$\mathcal{W}_{f,g}(b, \omega) = \int f(b + \tau/2) \overline{g(b - \tau/2)} e^{-i\omega\tau} d\tau \quad (6)$$

REMARK 2.3 The Wigner-Ville function and the ambiguity function are related via a symplectic Fourier transform

$$\mathcal{A}(\tau, \xi) = \frac{1}{2\pi} \iint \mathcal{W}(b, \omega) e^{-i(\xi b - \omega\tau)} db d\omega, \quad (7)$$

$$\mathcal{W}(b, \omega) = \frac{1}{2\pi} \iint \mathcal{A}(\tau, \xi) e^{i(\xi b - \omega\tau)} d\tau d\xi. \quad (8)$$

The same holds true in the non-deterministic context. Therefore, the Wigner function is square-integrable as soon as the ambiguity function is so, i.e. as soon as $f \in L^2(\mathbb{R})$. In fact, Lieb's estimates show that when $f \in L^2(\mathbb{R})$, $\mathcal{W}_f \in L^p(\mathbb{R}^2)$ for all $p \in [1, \infty]$.

REMARK 2.4 Like the ambiguity function, the Wigner-Ville distribution may also be defined when f is a distribution. Indeed, if $\phi \in \mathcal{S}'(\mathbb{R})$, it may be shown²⁰ that $\mathcal{W}_\phi \in \mathcal{S}'(\mathbb{R}^2)$. In fact, the Wigner-Ville distribution is nothing

but the Weyl symbol of the operator of orthogonal projection onto f , which may be written:

$$P_f g(t) = \frac{1}{2\pi\|f\|^2} \int \mathcal{W}_f\left(\frac{t+b}{2}, \omega\right) e^{i\omega(t-b)} g(b) db d\omega .$$

Properties of the Wigner-Ville distribution. The WV distributions enjoy a set of quite remarkable properties.

1. *Orthogonality relations*^c: let $f, f', g, g' \in L^2(\mathbb{R})$. Then $\mathcal{W}_{f,g}, \mathcal{W}_{f',g'} \in L^2(\mathbb{R}^2)$ and

$$\langle \mathcal{W}_{f,g}, \mathcal{W}_{f',g'} \rangle = 2\pi \langle f, f' \rangle \langle g', g \rangle . \quad (9)$$

The orthogonality relations of the Wigner-Ville coefficients are a direct consequence of the corresponding relations for ambiguity functions, and the symplectic Fourier transform formulas (7) and (8).

2. *Covariance*: The WV distribution is covariant with respect to a certain number of simple transformations. Namely:

- *Translations*: if $g(t) = f(t - \tau)$, $\mathcal{W}_g(b, \omega) = \mathcal{W}_f(b - \tau, \omega)$.
- *Modulations*: if $g(t) = e^{i\lambda t} f(t)$, then $\mathcal{W}_g(b, \omega) = \mathcal{W}_f(b, \omega - \lambda)$.
- *Rescalings*: if $g(t) = \frac{1}{\sqrt{a}} f\left(\frac{t}{a}\right)$, then $\mathcal{W}_g(b, \omega) = \mathcal{W}_f\left(\frac{b}{a}, a\omega\right)$.
- *Time-frequency rotations*: Let θ be such that $\cos \theta \neq 0$, and set

$$g_\theta(t) = \frac{1}{\sqrt{\cos \theta}} \frac{1}{2\pi} \int e^{i \tan \theta (t^2 + \xi^2) / 2} e^{it\xi / \cos \theta} \hat{f}(\xi) d\xi .$$

Then

$$\mathcal{W}_{g_\theta}(b, \omega) = \mathcal{W}_f(b \cos \theta + \omega \sin \theta, -b \sin \theta + \omega \cos \theta) .$$

The Wigner transform is actually covariant under a general group of transformations, called the *metaplectic group*. We refer to the textbook of Folland ²⁰ for more details.

3. *Time-frequency localization*: Let us start with simple examples, namely the case of the *pure oscillations*, i.e. the distribution of the form $f(t) = e^{i\lambda t}$. The WV transform of such an $f(t)$ has to be defined as a two-dimensional distribution, and one easily shows that $\mathcal{W}_f(b, \omega) = 2\pi\delta(\omega - \lambda)$, which has *sharp localization* in the time-frequency space.

^calso known as *Moyal's formula*.

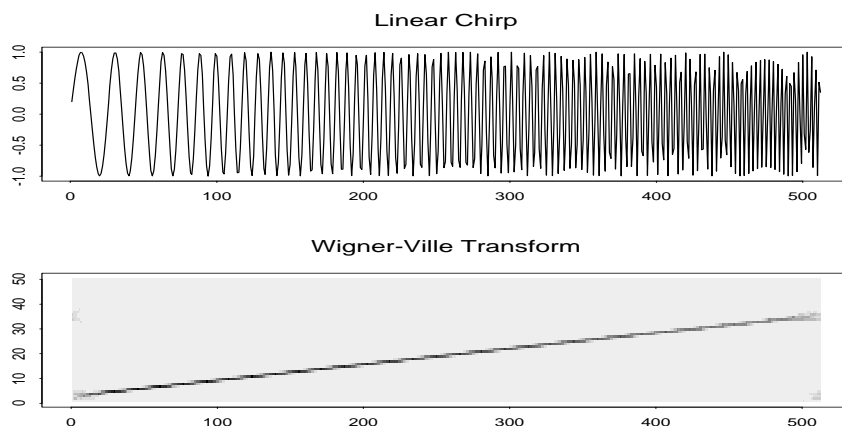


Figure 1. Example of a WV transform: the case of a linear chirp (coded with gray levels).

Such optimal localization properties are preserved by the simple transformations alluded to in the previous section. While the effect of translations, modulations and rescalings are easy to visualize, let us pay special attention to the consequences of the time-frequency rotations covariance. It is easily shown that the so-called *linear chirps*, i.e. the distributions with a linearly time-varying frequency $e^{i\lambda t + \alpha t^2/2}$ may be obtained by appropriate translation, modulation, rescaling and time-frequency rotation of a pure oscillation. Hence, its WV distribution inherits the perfect localization properties from those of the pure oscillations, and one obtains $\mathcal{W}_f(b, \omega) = 2\pi\delta(\omega - (\lambda + \alpha b))$. An example of such a behavior is provided in Fig. 1, which represents the WV distribution of a linear chirp.

Interferences. As a quadratic functional of the function $f(t)$, the WV distribution yields interference terms. Namely, let $f_1, f_2 \in L^2(\mathbb{R})$, and let $f = f_1 + f_2$. Then one immediately sees that

$$\mathcal{W}_f(b, \omega) = \mathcal{W}_{f_1}(b, \omega) + \mathcal{W}_{f_2}(b, \omega) + 2\Re(\mathcal{W}_{f_1, f_2}(b, \omega)) .$$

Even in the case where both $\mathcal{W}_{f_1}(b, \omega)$ and $\mathcal{W}_{f_2}(b, \omega)$ are sharply localized in the (b, ω) plane, the “cross term” $2\Re\mathcal{W}_{f_1, f_2}(b, \omega)$ introduces an extra component in the WV transform of f , which degrades the resolution.

The treatment of such interference terms has received a considerable attention in the signal processing literature during the past 10 years. Interferences terms are generally attenuated by appropriate smoothings of the WV

distribution. Examples are provided by the following classes of time-frequency distributions:

1. *Usual smoothings*: given a kernel $\Pi \in L^1(\mathbb{R}^2)$, one may associate with it the following time-frequency distribution

$$\rho_f(b, \omega) = \int \Pi(\tau, \xi) \mathcal{W}(b - \tau, \omega - \xi) d\tau d\xi . \quad (10)$$

$\rho_f(b, \omega)$ is smoother than the Wigner-Ville distribution. In addition, the interference terms of the Wigner distribution are generally much smaller in ρ_f . In addition, since ρ_f is obtained from a WV distribution via a two-dimensional convolution (a translation invariant operation), it inherits from $\mathcal{W}(b, \omega)$ the time-frequency translations covariance. The family of ρ_f generated in that way is called the *Cohen's class*.

2. *Affine smoothings*: Given a function of two variables $\Pi(b, \omega)$, introduce

$$\rho_f(b, a) = \int \Pi(\tau, \xi) \mathcal{W}\left(\frac{b - \tau}{a}, a\xi\right) d\tau d\xi . \quad (11)$$

$\rho_f(b, a)$ is a *time-scale representation* of f . With the same arguments as before, it may be shown that it inherits from the WV distributions its properties of covariance with respect to translations and rescalings. The family of representations generated in that way is called the *affine class*.

Generalized Wigner functions. The covariance properties are an important ingredient in the construction of Wigner functions. In fact, it may be shown that the Wigner function is the only quadratic representation possessing the covariance properties described above. This suggests that imposing different covariance conditions would yield generalized Wigner functions. This program has been achieved in a number of situations, in particular in the work of Bertrand and Bertrand ⁸, who constructed generalized Wigner functions, covariant with respect to 3-dimensional extensions of the affine group (see P. Flandrin's book ¹⁸ for more details on signal processing applications). Let us also mention the more abstract recent contribution of Ali et al. ², who extend the concept to general type one Lie groups.

2.2 Continuous Gabor transform (CGT)

In his seminal paper, in the mid forties, Gabor proposed to develop signals with respect to a family of Gaussian, translated and modulated (with discrete translations and modulations). The concept has evolved further, in several directions, yielding among other generalizations the *continuous Gabor transform*

(CGT for short). The CGT essentially first localizes the signal by multiplying it by a smooth and localized window, and then performs a Fourier transform. More precisely, start from a function $g \in L^2(\mathbb{R})$, such that $\|g\| \neq 0$, and associate with it the following family of *Gaborlets*

$$g_{(b,\omega)}(t) = e^{i\omega(t-b)}g(t-b) . \quad (12)$$

The corresponding Gabor transform is defined by

DEFINITION 2.3 Let $g \in L^2(\mathbb{R})$, $\|g\| \neq 0$, be a window. The continuous Gabor transform of a finite-energy signal $f \in L^2(\mathbb{R})$ is defined by the integral transform

$$G_f(b,\omega) = \langle f, g_{(b,\omega)} \rangle = \int f(t) \overline{g(t-b)} e^{-i\omega(t-b)} dt . \quad (13)$$

Gaborlets yield decomposition formulas for functions in $L^2(\mathbb{R})$, as follows.

THEOREM 2.1 Let $g \in L^2(\mathbb{R})$ be a non trivial window (i.e. $\|g\| \neq 0$.) Then every $f \in L^2(\mathbb{R})$ admits the decomposition

$$f(t) = \frac{1}{2\pi\|g\|^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_f(b,\omega) g_{(b,\omega)}(t) db d\omega , \quad (14)$$

where equality holds in the weak $L^2(\mathbb{R})$ sense.

In other words, the mapping $L^2(\mathbb{R}) \ni f \mapsto \frac{1}{\|g\|\sqrt{2\pi}} G_f \in L^2(\mathbb{R}^2)$ is an isometry between $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^2)$, and the inverse mapping is provided by the adjoint mapping.

In the case of random time series, the CGT is defined similarly.

DEFINITION 2.4 Let $\{X_t, t \in \mathbb{R}\}$ be a mean zero second order random time series, and let $g \in L^2(\mathbb{R})$ be a window. The CGT of X_t is the random time-frequency series defined by

$$G_X(b,\omega) = \langle X, g_{(b,\omega)} \rangle \quad (15)$$

By definition, the covariance operator \mathcal{C} of the time series is defined by its matrix elements $\langle \mathcal{C}f, g \rangle = \mathbb{E} \{ \langle X, g \rangle \langle f, X \rangle \}$. Obviously,

$$\mathbb{E} \{ G_f(b,\omega) \overline{G_f(b',\omega')} \} = \langle \mathcal{C}g_{(b',\omega')}, g_{(b,\omega)} \rangle \quad (16)$$

REMARK 2.5 The case where the time series $\{X_t, t \in \mathbb{R}\}$ under consideration is (second order) stationary is particularly interesting. By definition, the covariance operator is in such a case a convolution operator, and one immediately sees that

$$\mathbb{E} \{ G_f(b,\omega) \overline{G_f(b',\omega')} \} = \frac{1}{2\pi} \int e^{i\eta(b-b')} |\hat{g}(\eta-\omega)|^2 d\nu(\eta) ,$$

where $d\nu(\eta)$ stands for the spectral measure of the time series.

However, the main interest of the CGT lies in its potential for handling non stationary situations (in the stationary case, the covariance operator is a convolution operator, which is perfectly handled by Fourier methods). Particularly interesting is the case of the so-called *locally stationary time series*, which are basically random time series whose covariance operator is “almost diagonal” in an appropriate Gabor representation. Such a situation has been discussed by various authors in various contexts ^{27,31,11}.

examples. The CGT and similar tools have been quite popular in the speech processing literature, because of its capability of handling the so-called *locally harmonic signals*, namely signals which may be modeled in the form

$$f(t) = \sum_{k=1}^K A_k(t) e^{i\phi_k(t)} , \quad (17)$$

where the local amplitudes $a_k(t)$ and the local frequencies $\omega_k(t) = \phi'_k(t)$ are assumed to be slowly varying. Such signals are called locally harmonic when, in addition, the local frequencies are close to be integer multiples of a fundamental frequency (the *pitch frequency*: $\omega_k(t) \approx k\omega_1(t)$).

The CGT of such signals have been studied by several authors. Let us consider a simpler signal, of the form

$$f(t) = A(t) e^{i\phi(t)} ,$$

where $A, \phi \in C^2(\mathbb{R})$, and A and ϕ' are supposed slowly varying. Then, a simple application of Taylor's formula yields ^{30,16,38,11})

$$G_f(b, \omega) = A(b) e^{i\phi(b)} \overline{\hat{g}}(\phi'(b) - \omega) [1 + R(b, \omega)] , \quad (18)$$

where the remainder $R(b, \omega)$ is bounded as

$$|R(b, \omega)| \leq K_1 \frac{|A'(b)|}{|A(b)|} + K_2 \frac{\sup_u |A''(u)|}{|A(b)|} + K_3 \frac{\sup_u |\phi''(u)|}{|A(b)|} .$$

Therefore, as soon as the amplitudes $A(t)$ and frequencies $\phi'(t)$ are slowly varying enough, one may keep only the first term, and

$$G_f(b, \omega) \approx A(b) e^{i\phi(b)} \overline{\hat{g}}(\phi'(b) - \omega) .$$

Assuming that $g(t)$ is a smooth function, located near the origin $t = 0$ (typically a Gaussian function.) Then for each value of the time variable b , the modulus of the CGT attains its maximum on a curve (the so-called *ridge*) of equation $\omega = \phi'(b)$, i.e. describing the instantaneous frequency of the function $f(t)$.

a large number of numerical illustrations may be found in the book ¹¹. We limit our illustrations to an example of speech signal, in which several

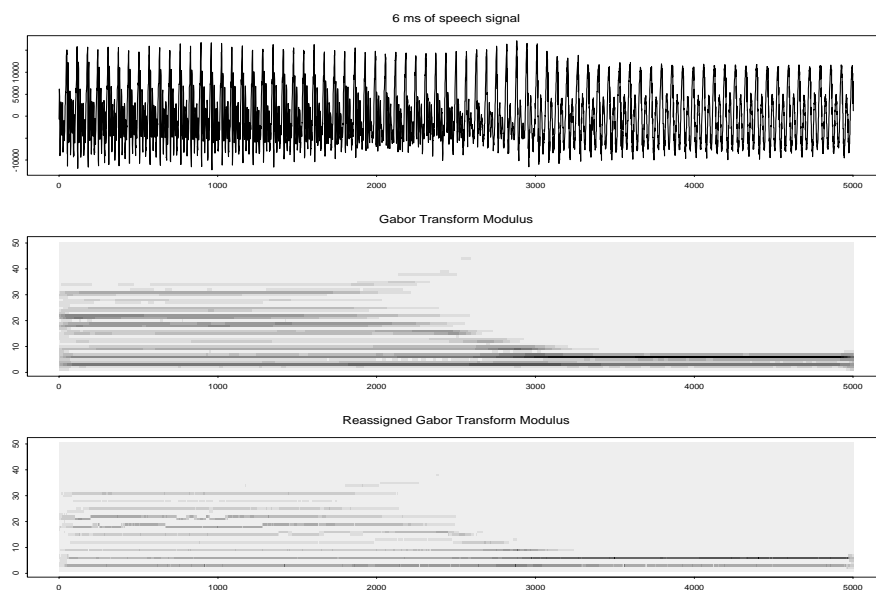


Figure 2. CGT (middle) and reassigned CGT (bottom) of a few milliseconds of speech signal (top).

such components are present. Thanks to the linearity of the CGT, the CGT of a signal of the type (17) is of the form

$$G_f(b, \omega) \approx \sum_{k=1}^K A_k(b) e^{i\phi_k(b)} \bar{g}(\phi'_k(b) - \omega) ,$$

and as soon as g is such that the various $\phi'_k(b) - \phi'_\ell(b)$ for $\ell \neq k$ are small enough, the CGT of such a signal localizes itself near K different ridges. An example of such a behavior is given in Fig. 2, where we display 625 milliseconds of speech signal: (top) and the modulus of the CGT (middle). Observe the localisation near the ridges, which are more easily estimated thanks to the reassigned CGT (bottom), to be discussed below.

2.3 Continuous Wavelet Transform (CWT)

Continuous wavelet transform (CWT) may be presented as an alternative to CGT. In the CGT, one improves²³ the time resolution of Gabor transform,

by changing the rule for generating the “basis functions”. The modulation is replaced with a scaling operation.

More precisely, let $\psi \in L^2(\mathbb{R})$ be a fixed function, called the *analyzing wavelet*. The corresponding family of wavelets is the family $\{\psi_{(b,a)}; b \in \mathbb{R}, a \in \mathbb{R}_+^*\}$ of shifted and scaled copies of ψ defined as follows. If $b \in \mathbb{R}$ and $a \in \mathbb{R}_+^*$ we set:

$$\psi_{(b,a)}(t) = \frac{1}{a} \psi\left(\frac{t-b}{a}\right), \quad t \in \mathbb{R}. \quad (19)$$

The wavelet $\psi_{(b,a)}$ can be viewed as a copy of the original wavelet ψ rescaled by a and centered around the “time” b . Given an analyzing wavelet ψ , the associated continuous wavelet transform is defined as follows (we give the definition in the case of $L^2(\mathbb{R})$ signals. The CWT of random time series is defined similarly to their CGT).

DEFINITION 2.5 *Let $\psi \in L^2(\mathbb{R})$ be an analyzing wavelet. The continuous wavelet transform (CWT for short) of a finite-energy signal $f(t)$ is defined by the integral:*

$$T_f(b,a) = \langle f, \psi_{(b,a)} \rangle = \frac{1}{a} \int f(t) \overline{\psi}\left(\frac{t-b}{a}\right) dt. \quad (20)$$

Like Gaborlets, wavelets may form complete sets of functions in $L^2(\mathbb{R})$, and we have in particular

THEOREM 2.2 *Let $\psi \in L^2(\mathbb{R})$, be such that the number c_ψ defined by:*

$$c_\psi = \int_0^\infty |\hat{\psi}(a\xi)|^2 \frac{da}{a} \quad (21)$$

is finite, nonzero and independent of $\xi \in \mathbb{R}$. Then every $f \in L^2(\mathbb{R})$ admits the decomposition

$$f(t) = \frac{1}{c_\psi} \int_{-\infty}^\infty \int_0^\infty T_f(b,a) \psi_{(b,a)}(t) \frac{da}{a} db, \quad (22)$$

in the sense of strong $L^2(\mathbb{R})$ -convergence.

In particular, we also have “energy conservation”: if $f \in L^2(\mathbb{R})$, then $T_f \in L^2(\mathbb{R} \times \mathbb{R}_+^*, db \frac{da}{a})$, and $\|T_f\|^2 = c_\psi \|f\|^2$. Notice that in (21), the constant c_ψ can only depend on the sign of $\xi \in \mathbb{R}$, therefore assuming independence wrt ξ is a simple symmetry assumption. The fact that $0 < c_\psi < \infty$ implies that $\hat{\psi}(0) = 0$, so that the wavelet $\psi(t)$ has to oscillate enough to be of zero mean.

The CWT has a behavior similar to that of the CGT in many respects. The main difference lies in the fact that wavelets are extremely precise at small scales (where they lose frequency resolution), and more frequency localized at

large scales (where time resolution is lost). A main application of this fact is the analysis of regularity (see below). A visualization of this effect may be found in Fig. 2 (bottom), where we display the modulus of the CWT of the speech signal shown at the top of the figure. As may be seen, at large scales, the wavelets have a sufficient frequency resolution to analyze carefully the first harmonics (namely, the pitch frequency and the first harmonic). For smaller scales, frequency resolution is lost, and the same wavelet is unable to “separate” several harmonic components. This results in interferences between the harmonic components, which yield the oscillations of the modulus in the b direction that appear on the image. We refer to ¹¹ for a more detailed analysis of such applications.

2.4 Higher dimensional generalizations

The constructions of wavelet and Gabor transforms are easy to generalize in arbitrary dimensions. The n -dimensional CGT is a mere generalization of the one-dimensional version. The case of wavelets is more interesting, as it allows one to introduce generalized dilations. Let us mention the construction of Murenzi’s wavelets in two dimensions. Start from $\psi \in L^2(\mathbb{R}^2)$, and consider the family of wavelets $\psi_{(b,a,\theta)}$, $b \in \mathbb{R}^2$, $a \in \mathbb{R}^+$, $\theta \in [0, 2\pi[$:

$$\psi_{(b,a,\theta)}(x) = \frac{1}{a} \psi \left(r_{-\theta} \frac{x-b}{a} \right) . \quad (23)$$

Then, if ψ is such that

$$0 < c_\psi = \int |\hat{\psi}(k)|^2 \frac{dk}{|k|^2} < \infty ,$$

any $f \in L^2(\mathbb{R})$ may be decomposed as

$$f = \frac{1}{c_\psi} \int \langle f, \psi_{(b,a,\theta)} \rangle \psi_{(b,a,\theta)} \frac{da db d\theta}{a^2} . \quad (24)$$

Such transforms have found many applications in image processing. We give below an application to the estimation of local frequencies in images presenting “regular textures”. In such a case, the local frequency of the texture may be estimated, and the corresponding component may be estimated as well, and resynthesized separately. The techniques used for such a work are fairly close to the ones described above in the one-dimensional CGT of speech signal. More details may be found in ²².

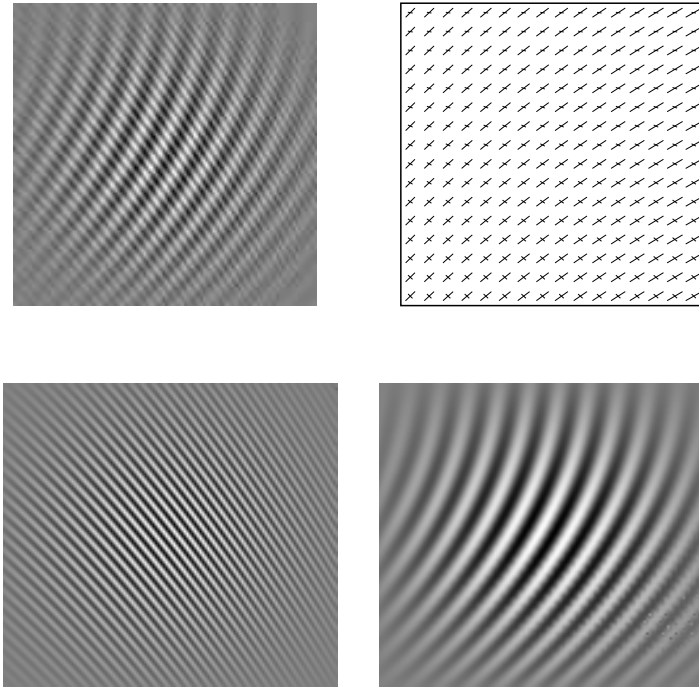


Figure 3. Top: Textured image with regular texture, and estimation of the local frequencies (right). The two components and displayed in the bottom figures.

2.5 Linear transforms as approximations

Interestingly enough, there is a strong connection between the Wigner-Ville distributions and the linear decompositions we just reviewed (or more precisely their squared modulus). We express this in the following

PROPOSITION 2.1 *Let $f \in L^2(\mathbb{R})$, and let T_f, G_f and W_f denote respectively its CWT, CGT and WV transforms. Then the following two properties are true:*

1. *The squared modulus $|G_f(b, \omega)|^2$ of the CGT is a smoothed version of the*

WV distribution \mathcal{W}_f of f :

$$|G_f(b, \omega)|^2 = \frac{1}{2\pi} \int \mathcal{W}_f(\tau, \xi) \overline{\mathcal{W}_g(\tau - b, \xi - \omega)} d\tau d\xi \quad (25)$$

2. The squared modulus $|T_f(b, a)|^2$ of the CWT is an affine smoothing of the WV distribution \mathcal{W}_f of f :

$$|T_f(b, a)|^2 = \frac{1}{2\pi} \int \mathcal{W}_f(\tau, \xi) \overline{\mathcal{W}_\psi\left(\frac{\tau - b}{a}, a\xi\right)} d\tau d\xi \quad (26)$$

These results follow from an easy calculation. See e.g. ¹⁸ for more details. This makes the connection between the spectrogram and the scalogram in one hand, and the Cohen's class and the affine class in the other hand. This also provides an alternative interpretation of the unperfect time-frequency localization properties of these transforms: a blurring, resulting from the convolution products appearing in the above proposition. We shall come back to that point later on, when discussing reassignment methods.

3 Time-frequency transforms from group theory

We already emphasized the fact that the continuous (linear or quadratic) time-frequency transforms possess important built-in symmetry properties. Such covariance properties in fact characterize the transforms. This is a consequence of the geometrical origins of the transforms, which may be described in terms of group-theoretical methods. As remarked by several authors ^{24,34,35} the set of simple transformations used to generate the wavelets from a single one in general inherits the structure of a group G (as is the case for instance for translations, modulations or dilations). The convenient language is the theory of square-integrable representations.

3.1 Square-integrable group representations

Accounts of the theory may be found in many textbooks as well as review articles on wavelets. We only briefly sketch the construction here.

DEFINITION 3.1 *Let G be a separable locally compact Lie group, and let π be a unitary strongly continuous representation of G on the Hilbert space \mathcal{H} . π is said to be square-integrable if π is irreducible, and if there exists at least a vector $v \in \mathcal{H}$ such that*

$$0 < \int_G |\langle \pi(g)v, v \rangle|^2 d\mu(g) < \infty. \quad (27)$$

Such a vector is said to be admissible.

Square-integrable group representations have been extensively studied in the literature, in particular for compact groups, locally compact unimodular groups and non-unimodular locally compact groups. The results may be summarized in the following theorem, due to Duflo and Moore, and Carey independently:

THEOREM 3.1 *Let π be a square-integrable strongly continuous unitary representation of the locally compact group G on \mathcal{H} . Then there exists a positive self-adjoint operator C such that for any admissible vectors $v_1, v_2 \in \mathcal{H}$ and for any $u_1, u_2 \in \mathcal{H}$*

$$\int_G \langle u_1, \pi(g).v_1 \rangle \langle \pi(g).v_2, u_2 \rangle d\mu(g) = \langle C^{1/2}v_2, C^{1/2}v_1 \rangle \langle u_1, u_2 \rangle \quad (28)$$

Moreover, the set of admissible vectors coincides with the domain of C .

Let $\lambda : G \rightarrow \mathcal{U}(L^2(G))$ be the left-regular representation of G : if $f \in L^2(G)$,

$$(\lambda(h)f)(g) = f(h^{-1}g) . \quad (29)$$

Theorem 3.1 shows that a representation π of G is square integrable if and only if it is unitarily equivalent to a subrepresentation of the left-regular representation λ . The corresponding intertwiners can be realized as follows. If v is an admissible vector in \mathcal{H} , and $v' \in \mathcal{H}$, introduce the Schur coefficients, i.e. the matrix coefficients of elements of G :

$$c_{v,v'}(g) = \langle v', \pi(g)v \rangle, \quad g \in G \quad (30)$$

Let T be the left transform ²⁴, i.e. the map defined by

$$T : u \in \mathcal{H} \rightarrow T_u = c_{v,u}(\cdot) \in L^2(G) \quad (31)$$

T intertwines π and λ :

$$T \circ \pi = \lambda \circ T \quad (32)$$

Grossmann, Morlet and Paul suggested to use (28) and (31) for the analysis of functions, in the case where \mathcal{H} is a function space. This was the starting point of many applications, especially in a signal analysis context. The left transform T is used to obtain another representation of functions, and (32) expresses the covariance of the transform.

REMARK 3.1 The Duflo-Moore theorem has been applied to several groups (including discrete groups), yielding many different generalizations of wavelets and Gabor transforms. Let us quote for the record the contributions of Bernier and Taylor ⁶, Führ ²¹ and Aniello et al ³, who studied semi-direct products of the form $H \times \mathbb{R}^n$, where H is a closed subgroup of $GL(n, \mathbb{R})$, with group law $(h, \underline{b})(h', \underline{b}') = (hh', \underline{b} + h\underline{b}')$. In such cases, the question of square-integrability

is formulated in a simple way, and the Dufflo-Moore operator \mathcal{C} may be written explicitly.

REMARK 3.2 For non square integrable representations, it is still possible to modify the construction so as to obtain wavelet type decompositions. The price to pay is a loss of the full covariance with respect to the group action. These aspects are described in great details in ¹.

We discuss below in more details the case of the Weyl-Heisenberg group.

3.2 The case of the Gabor transform

The CGT revisited. We consider the one-dimensional Weyl-Heisenberg group $G_{WH} = \mathbb{R}^2 \times S^1$, with group operation

$$(b, \omega, \varphi) \cdot (b', \omega', \varphi') = (b + b', \omega + \omega', \varphi + \varphi' + \omega b' \text{ [mod } 2\pi]) \quad (33)$$

The Stone-Von-Neumann theorem states that any irreducible unitary representations of G_{WH} is unitarily equivalent to one of the following form (see ^{29,39}). If $f \in L^2(\mathbb{R}^n)$

$$[\pi(b, \omega, \varphi) \cdot f](t) = e^{i\mu(\varphi - \omega \cdot (t-b))} f(t - b)$$

for some $\mu \in \mathbb{Z}^*$. For the sake of simplicity, we assume $\mu = 1$. The representation is square-integrable, and the corresponding left transform is the Gabor transform described in Definition 2.3, up to a phase factor $e^{i\varphi}$. More precisely, the Gaborlets read

$$g_{(b,\omega)} = \pi(b, \omega, 0)g .$$

More on the Weyl-Heisenberg group. As stated by the general theory, the Gabor transform intertwines the representation π with the left-regular representation λ of G_{WH} on $L^2(G_{WH})$, defined as follows: if $F \in L^2(G_{WH})$,

$$\lambda(g)F(h) = F(g^{-1}h) . \quad (34)$$

In explicit form, we then have

$$\lambda(b', \omega', \varphi')F(b, \omega, \varphi) = F(b - b', \omega - \omega', \varphi - \varphi' + \omega'(b' - b)) . \quad (35)$$

λ is a unitary representation, and is highly reducible.

The infinitesimal generators of λ_n are also of interest: introduce the differential operator $\nabla = (\check{\partial}_b, \check{\partial}_\omega)^t$, where for any smooth enough function F , we set

$$\check{\partial}_b F(b, \omega) = \partial_b F(b, \omega) \quad (36)$$

$$\check{\partial}_\omega F(b, \omega) = \partial_\omega F(b, \omega) - ibF(b, \omega) . \quad (37)$$

Then we have that for all $F \in L^2(\mathbb{R}^2)$,

$$[\lambda(v_b, v_\nu)F](b, \omega) = e^{-v_b \hat{\partial}_b} e^{-v_\omega \hat{\partial}_\omega} F(b, \omega) . \quad (38)$$

The right regular representation ρ is introduced similarly:

$$\rho(g)F(h) = F(hg) . \quad (39)$$

One may also define, on $L^2(\mathbb{R}^2)$:

$$\rho(b', \omega')F(b, \omega) = e^{-i\omega b'} F(b + b', \omega + \omega') \quad (40)$$

Again, the infinitesimal generators of the right regular representation are of interest. As before, we introduce the modified derivative $\widehat{\nabla} = (\hat{\partial}_b, \hat{\partial}_\omega)^t$, where for any smooth enough function F , we set

$$\hat{\partial}_b F(b, \omega) = \partial_b F(b, \omega) - i\omega F(b, \omega) \quad (41)$$

$$\hat{\partial}_\omega F(b, \omega) = \partial_\omega F(b, \omega) . \quad (42)$$

Then we have that for all $F \in L^2(\mathbb{R}^2)$,

$$[\rho(v_b, v_\omega)F](b, \omega) = e^{v_b \hat{\partial}_b} e^{v_\omega \hat{\partial}_\omega} F(b, \omega) . \quad (43)$$

4 Time-frequency localization and reassignment

We have seen earlier that the spectrogram and the scalogram may be interpreted as blurred versions of the Wigner transform. This lack of resolution however goes together with an improvement of other aspects of the transform, which become almost interference free.

4.1 Generalities on reassignment

In a seminal paper ²⁶, Kodera and his coworkers propose a procedure for “de-blurring” the transforms. They propose an extremely original idea, which essentially amounts to “move” CGT coefficients in the time-frequency plane, so as to “put them back where they should be”. To give a more precise meaning to such a statement, they propose a prescription for computing the “new location” of a coefficient $G_f(b, \omega)$:

$$\begin{pmatrix} \tilde{b}(b, \omega) \\ \tilde{\omega}(b, \omega) \end{pmatrix} = \begin{pmatrix} b \\ \omega \end{pmatrix} - \underline{v}(b, \omega) , \quad (44)$$

where

$$\underline{v}(b, \omega) = \begin{pmatrix} \partial_\omega \Omega(b, \omega) \\ \omega - \partial_b \Omega(b, \omega) \end{pmatrix} \quad (45)$$

where $\Omega(b, \omega)$ is the argument of the CGT:

$$\Omega(b, \omega) = \arg G_f(b, \omega) . \quad (46)$$

Remarkably enough, such expressions may also be obtained from the Wigner function:

$$\begin{cases} \tilde{b}(b, \omega) = \frac{1}{2\pi|G_f(b, \omega)|^2} \int \tau \mathcal{W}_f(\tau, \eta) \overline{\mathcal{W}_g}(\tau - b, \eta - \omega) d\tau d\eta , \\ \tilde{\omega}(b, \omega) = \frac{1}{2\pi|G_f(b, \omega)|^2} \int \eta \mathcal{W}_f(\tau, \eta) \overline{\mathcal{W}_g}(\tau - b, \eta - \omega) d\tau d\eta . \end{cases}$$

Given such a prescription, the reassigned transform is defined, formally, as follows:

$$M_r(b, \omega) = \sum_{(b', \omega'):(b, \bar{\omega})=(b, \omega)} |G_f(b, \omega)|^2 . \quad (47)$$

REMARK 4.1 While the significance of such an expression is clear in a discrete context, some care is needed for defining it in for continuous transforms.

REMARK 4.2 As proposed by Kodera et al., we give in (47) the expression for reassigning the spectrogram. An alternative amounts to reassign the CGT itself.

EXAMPLE 4.1 As an illustration, we give in FIG. 4 an example of reassigned CGT. The signal is a simple linear chirp, with a Gaussian envelope, and we display the square modulus of its CGT and its reassigned CGT. Notice the improvement of time-frequency localization near the instantaneous frequency line.

However, even though such a formulation yields remarkable numerical results, its mathematical interpretation is somewhat difficult. We describe below a formalism which allows us to interpret it from first principles.

Before going to the geometric interpretation, let us first give a short account of the *differential reassignment* method, introduced by E. Chassande-Mottin and coworkers¹². The idea of differential reassignment is to make use of the vector field \underline{v} obtained in (45) to construct a dynamical system on the time-frequency space. The spectrogram (or CGT) coefficients are then “displaced” following that vector field. Such a procedure has found several applications in signal processing. We shall show below interesting interpretations in terms of the action of the Weyl-Heisenberg group on the time-frequency plane.

4.2 Constant Phase Deformations

Let us now return to the problem of reassigning, or deforming, a CGT. The results of this section have been published in¹⁵. Our goal is to provide a

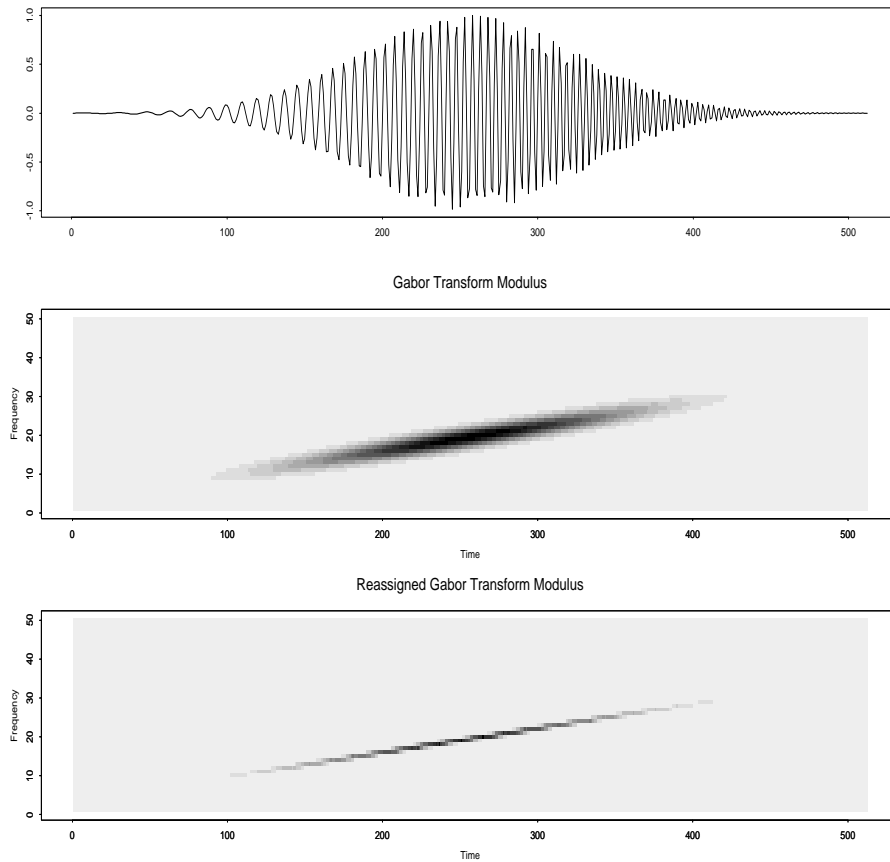


Figure 4. Example of CGT (middle) and reassigned CGT (bottom): the case of a linear chirp (top) .

geometrical interpretation of the reassignment maps proposed and studied in ²⁶ and ⁵ for the CGT. Let us first set the problem. We are interested in finding a vector field

$$\underline{v} : (b, \omega) \rightarrow \underline{v}(b, \omega) = (v_b(b, \omega), v_\omega(b, \omega))^t \in \mathbb{R}^2$$

defined on the phase space, such that the argument of the CGT remains unchanged under the infinitesimal action of the Weyl-Heisenberg group. More

precisely, we ask for invariance with respect to the right regular action $\rho_n(\epsilon \underline{v})$, at first order in $\epsilon > 0$.

Let F be a smooth function defined on the plane. For $\epsilon > 0$ and $\underline{v} \in \mathbb{R}^2$, let $\tau_{\underline{v}}(\epsilon) : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$ denote the operator defined by

$$\tau_{\underline{v}}(\epsilon)F(b, \omega) = \rho_n(-\epsilon \underline{v}(b, \omega)) F(b, \omega) . \quad (48)$$

At first order in ϵ , we have

$$\tau_{\underline{v}}(\epsilon)F(b, \omega) = F(b, \omega) - \epsilon \underline{v}(b, \omega) \cdot \widehat{\nabla} F(b, \omega) + O(\epsilon^2) , \quad (49)$$

where the symbol “ \cdot ” stands for the scalar product in \mathbb{C}^2 .

Let now $f \in L^2(\mathbb{R})$, and let G_f denote its CGT. Let us write

$$G_f(b, \omega) = |G_f(b, \omega)| e^{i\Omega(b, \omega)} , \quad (50)$$

where $\Omega(b, \omega) = \arg(G_f(b, \omega))$. One immediately sees that, again at first order in ϵ ,

$$\tau_{\underline{v}}(\epsilon)G_f(b, \omega) = G_f(b, \omega) - \epsilon G_f(b, \omega) \left(\underline{v} \cdot \frac{|\nabla G_f(b, \omega)|}{|G_f(b, \omega)|} + i \underline{v} \cdot \nabla^m \Omega(b, \omega) \right) + O(\epsilon^2) , \quad (51)$$

where we have introduced the *modified gradient* ∇^m , defined by

$$\nabla^m \Omega(b, \omega) = \begin{pmatrix} \partial_b^m \Omega(b, \omega) \\ \partial_\omega^m \Omega(b, \omega) \end{pmatrix} = \begin{pmatrix} \partial_b \Omega(b, \omega) - \omega \\ \partial_\omega \Omega(b, \omega) \end{pmatrix} \quad (52)$$

Therefore, imposing that the argument of $G_f(b, \omega)$ be unchanged imposes that the vector field \underline{v} be of the form

$$\underline{v}(b, \omega) = \sigma \begin{pmatrix} \partial_\omega^m \Omega(b, \omega) \\ -\partial_b^m \Omega(b, \omega) \end{pmatrix} = \sigma \begin{pmatrix} \partial_\omega \Omega(b, \omega) \\ \omega - \partial_b \Omega(b, \omega) \end{pmatrix} , \quad (53)$$

where $\sigma \in \mathbb{R}$ is a real number. We notice that this expression coincides (for $\sigma = 1$) with the prescription given in (44) and (45).

4.3 Back to the reassignment problem

We have described above a formulation (developed in ¹⁵) which derives the reassignment field as a Hamiltonian vector field, obtained from the argument of the CGT. The use of such a vector field for the reassignment of the CGT itself is illustrated in FIG. 4 (for an academic signal), and in FIG. 2 (in the case of a real signal). As may be seen, reassigning the CGT itself (and not only the spectrogram), by handling carefully its argument, improves the time-frequency localization of the representation.

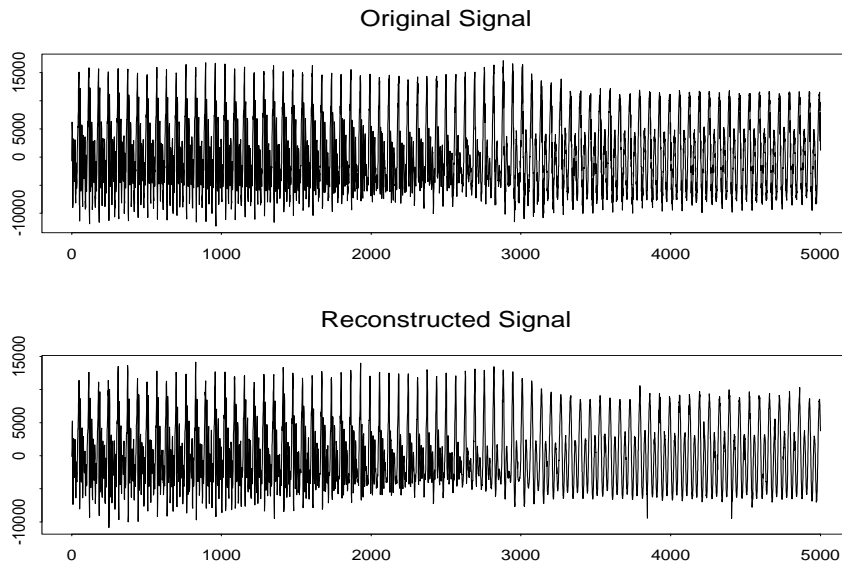


Figure 5. Reconstruction from a reassigned CGT: the case of a speech signal.

An important issue is that of extending such an analysis to different time-frequency transforms, such as wavelet transforms in 1D or higher dimensions. Prescriptions for reassignment fields have been proposed in the 1D wavelet case, none of which seems to fit into the formalism we developed.

Another important question regards the completeness of the representation provided by the reassigned CGT. In other words, does there exist reconstruction formulas for signals from reassigned transforms. To our knowledge, there is no general formulation for that problem. However, there are examples for which reconstruction formulas may be derived (even though the reassignment prescriptions they are derived for do not fit into the formalism we just developed), for example in the case of the *synchrosqueezed wavelet transform*¹⁴. As an illustration, we display in FIG. 5 a reconstruction of the speech signal of FIG. 2 from its reassigned CGT, following the lines of¹⁴. As may be seen, the numerical results are extremely promising.

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