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THE MIT BAG MODEL AS AN INFINITE MASS LIMIT

NAIARA ARRIZABALAGA, LOÏC LE TREUST, ALBERT MAS, AND NICOLAS RAYMOND

ABSTRACT. The Dirac operator, acting in three dimensions, is considered. Assuming that a large mass $m > 0$ lies outside a smooth and bounded open set $\Omega \subset \mathbb{R}^3$, it is proved that its spectrum is approximated by the one of the Dirac operator on Ω with the MIT bag boundary condition. The approximation, which is developed up to an error of order $o(1/\sqrt{m})$, is carried out by introducing tubular coordinates in a neighborhood of $\partial\Omega$ and analyzing the corresponding one dimensional optimization problems in the normal direction.

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1. INTRODUCTION

1.1. Context. This paper is devoted to the spectral analysis of the Dirac operator with high scalar potential barrier in three dimensions. More precisely, we will assume that there is a large mass m outside a smooth and bounded open set Ω . From physical considerations, see [4, 5], it is expected that, when m becomes large, the eigenfunctions of low energy do not visit $\mathbb{R}^3 \setminus \Omega$ and tend to satisfy the so-called MIT bag condition on $\partial\Omega$. This boundary condition, that we will define in the next section, is usually chosen by the physicists [8, 5, 6], in order to get a vanishing normal flux at the bag surface. It was originally introduced by Bogolioubov in the late 60's [4] to describe the confinement of the quarks in the hadrons with the help of an infinite scalar potential barrier outside a fixed set Ω . In the mid 70's, this model has been revisited into a shape optimization problem named *MIT bag model* [8, 5, 6] in which the optimized energy takes the form

$$\Omega \mapsto \lambda_1(\Omega) + b|\Omega|,$$

where $\lambda_1(\Omega)$ is the first nonnegative eigenvalue of the Dirac operator with the boundary condition introduced by Bogolioubov, $|\Omega|$ is the volume of $\Omega \subset \mathbb{R}^3$ and $b > 0$. The interest of the bidimensional equivalent of this model has recently been renewed with the study of graphene where this condition is sometimes called “infinite mass condition”, see [1, 3]. The aim of this paper is to provide a mathematical justification of this terminology, and extend to dimension three the work [9].

1.2. The Dirac operator with large effective mass. In the whole paper, Ω denotes a fixed bounded domain of \mathbb{R}^3 with regular boundary. The Planck's constant and the velocity of light are assumed to be equal to 1.

Let us recall the definition of the Dirac operator associated with the energy of a relativistic particle of mass m_0 and spin $\frac{1}{2}$, see [10]. The Dirac operator is a first order differential operator $(H, \text{Dom}(H))$, acting on $L^2(\mathbb{R}^3; \mathbb{C}^4)$ in the sense of distributions, defined by

$$(1.1) \quad H = \alpha \cdot D + m_0\beta, \quad D = -i\nabla,$$

where $\text{Dom}(H) = H^1(\mathbb{R}^3; \mathbb{C}^4)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and β are the 4×4 Hermitian and unitary matrices given by

$$\beta = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \text{ for } k = 1, 2, 3.$$

Here, the Pauli matrices σ_1, σ_2 and σ_3 are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $\alpha \cdot X$ denotes $\sum_{j=1}^3 \alpha_j X_j$ for any $X = (X_1, X_2, X_3)$.

In this paper, we consider particles with large effective mass $m \gg m_0$ outside Ω . Their kinetic energy is associated with the self-adjoint operator $(H_m, \text{Dom}(H_m))$ defined by

$$H_m = \alpha \cdot D + (m_0 + m\chi_{\Omega'})\beta,$$

where Ω' is the complementary set of $\bar{\Omega}$, $\chi_{\Omega'}$ is the characteristic function of Ω' and $\text{Dom}(H_m) = H^1(\mathbb{R}^3; \mathbb{C}^4)$.

Notation 1.1. In the following, $\Gamma := \partial\Omega$ and for all $\mathbf{x} \in \Gamma$, $\mathbf{n}(\mathbf{x})$ is the outward-pointing unit normal vector to the boundary, $L(\mathbf{x}) = d\mathbf{n}_{\mathbf{x}}$ denotes the second fundamental form of the boundary and

$$\kappa(\mathbf{x}) = \text{Tr } L(\mathbf{x}) \quad \text{and} \quad K(\mathbf{x}) = \det L(\mathbf{x})$$

are the mean curvature and the Gauss curvature of Γ respectively.

Definition 1.2. The MIT bag Dirac operator $(H^\Omega, \text{Dom}(H^\Omega))$ is defined on the domain

$$\text{Dom}(H^\Omega) = \{\psi \in H^1(\Omega; \mathbb{C}^4) : \mathcal{B}\psi = \psi \text{ on } \Gamma\}, \quad \text{with} \quad \mathcal{B} = -i\beta(\alpha \cdot \mathbf{n}),$$

by $H^\Omega\psi = H\psi$ for all $\psi \in \text{Dom}(H^\Omega)$. Observe that the trace is well-defined by a classical trace theorem.

Notation 1.3. We denote by $\langle \cdot, \cdot \rangle$ the \mathbb{C}^4 scalar product (antilinear w.r.t. the left argument) and by $\langle \cdot, \cdot \rangle_U$ the L^2 scalar product on the set $U \subset \mathbb{R}^3$.

Notation 1.4. We define, for every $\mathbf{n} \in \mathbb{S}^2$, the orthogonal projections

$$(1.2) \quad \Xi^\pm = \frac{1_4 \pm \mathcal{B}}{2}$$

associated with the eigenvalues ± 1 of the matrix \mathcal{B} .

1.3. Squared operators, heuristics, and results. The aim of this paper is to relate the spectra of H_m and H^Ω in the limit $m \rightarrow +\infty$.

Notation 1.5. Let $(\lambda_k)_{k \in \mathbb{N}^*}$ and $(\lambda_{k,m})_{k \in \mathbb{N}^*}$ denote the increasing sequence of eigenvalues of the operator $|H^\Omega|$ and $|H_m|$, respectively, each one being repeated according to its multiplicity. By the min – max characterization, we have

$$\begin{aligned} \lambda_k &= \inf_{\substack{V \subset \text{Dom}(H^\Omega), \\ \dim V = k}} \sup_{\substack{\varphi \in V, \\ \|\varphi\|_{L^2(\Omega)} = 1}} \left\| H^\Omega \varphi \right\|_{L^2(\Omega)} \\ &= \sup_{\{\psi_1, \dots, \psi_{k-1}\} \subset \text{Dom}(H^\Omega)} \inf_{\substack{\varphi \in \text{span}(\psi_1, \dots, \psi_{k-1})^\perp, \\ \|\varphi\|_{L^2(\Omega)} = 1}} \left\| H^\Omega \varphi \right\|_{L^2(\Omega)}, \end{aligned}$$

and

$$\begin{aligned} \lambda_{k,m} &= \inf_{\substack{V \subset H^1(\mathbb{R}^3; \mathbb{C}^4), \\ \dim V = k}} \sup_{\substack{\varphi \in V, \\ \|\varphi\|_{L^2(\mathbb{R}^3)} = 1}} \left\| H_m \varphi \right\|_{L^2(\mathbb{R}^3)} \\ &= \sup_{\{\psi_1, \dots, \psi_{k-1}\} \subset H^1(\mathbb{R}^3; \mathbb{C}^4)} \inf_{\substack{\varphi \in \text{span}(\psi_1, \dots, \psi_{k-1})^\perp, \\ \|\varphi\|_{L^2(\mathbb{R}^3)} = 1}} \left\| H_m \varphi \right\|_{L^2(\mathbb{R}^3)}, \end{aligned}$$

for $k \in \mathbb{N}^*$ and $m > 0$. Here, $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

1.3.1. The quadratic forms. At first sight, it might seem surprising that λ_k and $\lambda_{k,m}$ are related, especially because of the boundary condition of H^Ω . It becomes less surprising when computing the squares of the operators. This is the purpose of the following lemma.

Lemma 1.6. *Let $\varphi \in \text{Dom}(H^\Omega)$ and $\psi \in H^1(\mathbb{R}^3; \mathbb{C}^4)$. Then*

$$(1.3) \quad \|H^\Omega \varphi\|_{L^2(\Omega)}^2 = \mathcal{Q}^{\text{int}}(\varphi) := \|\nabla \varphi\|_{L^2(\Omega)}^2 + \int_{\Gamma} \left(\frac{\kappa}{2} + m_0 \right) |\varphi|^2 d\Gamma + m_0^2 \|\varphi\|_{L^2(\Omega)}^2,$$

where κ is defined in Notation 1.1, and

$$(1.4) \quad \begin{aligned} \|H_m \psi\|_{L^2(\mathbb{R}^3)}^2 &= \|\nabla \psi\|_{L^2(\Omega)}^2 + \|\nabla \psi\|_{L^2(\Omega')}^2 + \|(m_0 + m\chi_{\Omega'})\psi\|_{L^2(\mathbb{R}^3)}^2 \\ &\quad - m \text{Re} \langle \mathcal{B}\psi, \psi \rangle_{\Gamma} \\ &= \|\nabla \psi\|_{L^2(\Omega)}^2 + \|\nabla \psi\|_{L^2(\Omega')}^2 + \|(m_0 + m\chi_{\Omega'})\psi\|_{L^2(\mathbb{R}^3)}^2 \\ &\quad + m \|\Xi^- \psi\|_{L^2(\Gamma)}^2 - m \|\Xi^+ \psi\|_{L^2(\Gamma)}^2. \end{aligned}$$

Proof. The identity (1.3) is proved for instance in [2, Section A.2]. Let $\psi \in H^1(\mathbb{R}^3; \mathbb{C}^4)$. Then, by integrations by parts,

$$\begin{aligned} \|H_m \psi\|_{L^2(\mathbb{R}^3)}^2 &= \|\alpha \cdot D\psi\|_{L^2(\mathbb{R}^3)}^2 + \|(m_0 + m\chi_{\Omega'})\psi\|_{L^2(\mathbb{R}^3)}^2 + 2m \text{Re} \langle \alpha \cdot D\psi, \beta\psi \rangle_{\Omega'} \\ &= \|\nabla \psi\|_{L^2(\mathbb{R}^3)}^2 + \|(m_0 + m\chi_{\Omega'})\psi\|_{L^2(\mathbb{R}^3)}^2 - m \text{Re} \langle \mathcal{B}\psi, \psi \rangle_{\Gamma}. \end{aligned}$$

Then, note that, for all $\psi \in H^1(\mathbb{R}^3; \mathbb{C}^4)$,

$$\text{Re} \langle \mathcal{B}\psi, \psi \rangle_{\Gamma} = \|\Xi^+ \psi\|_{L^2(\Gamma)}^2 - \|\Xi^- \psi\|_{L^2(\Gamma)}^2.$$

□

Considering (1.4) leads to the following minimization problem, for $v \in H^1(\Omega)$,

$$(1.5) \quad \Lambda_m(v) = \inf \{ \mathcal{Q}_m(u), u \in V_v \}, \quad \mathcal{Q}_m(u) = \|\nabla u\|_{L^2(\Omega')}^2 + m^2 \|u\|_{L^2(\Omega')}^2,$$

where

$$V_v = \{u \in H^1(\Omega', \mathbb{C}^4) \text{ s.t. } u = v \text{ on } \Gamma\}.$$

A classical extension theorem (see [7, Section 5.4]) ensures that V_v is non-empty.

1.3.2. Heuristics. In this paper, we will analyse the behavior of $\Lambda_m(v)$ and prove in particular (see Proposition 2.1) that there exists $C > 0$ such that for m large, and all $v \in H^1(\Omega; \mathbb{C}^4)$

$$(1.6) \quad o(1) \geq \Lambda_m(v) - \left(m \|v\|_{L^2(\Gamma)}^2 + \int_{\Gamma} \frac{\kappa}{2} |v|^2 d\Gamma \right) \geq -\frac{C}{m} \|v\|_{H^1(\Gamma)}^2.$$

Replacing m by $m_0 + m$ in (1.6), we get, for all $\psi \in H^1(\mathbb{R}^3; \mathbb{C}^4)$,

$$(1.7) \quad \begin{aligned} \|H_m \psi\|_{L^2(\mathbb{R}^3)}^2 &\geq \|\nabla \psi\|_{L^2(\Omega)}^2 + m_0^2 \|\psi\|_{L^2(\Omega)}^2 \\ &\quad + \int_{\Gamma} \left(\frac{\kappa}{2} + m_0 \right) |\psi|^2 d\Gamma + 2m \|\Xi^- \psi\|_{L^2(\Gamma)}^2 - \frac{C}{m} \|\psi\|_{L^2(\Gamma)}^2. \end{aligned}$$

Take any eigenfunction φ of H^Ω and consider a minimizer u_φ of (1.5) for $v = \varphi$ and m replaced by $m + m_0$. Then, letting $\psi = \mathbf{1}_\Omega \varphi + \mathbf{1}_{\Omega'} u_\varphi \in H^1(\mathbb{R}^3; \mathbb{C}^4)$, we get

$$\|H_m \psi\|_{L^2(\mathbb{R}^3)}^2 = \|\nabla \varphi\|_{L^2(\Omega)}^2 + m_0^2 \|\varphi\|_{L^2(\Omega)}^2 + \Lambda_{m+m_0}(\varphi) - m \|\Xi^+ \varphi\|_{L^2(\Gamma)}^2.$$

With (1.6) at hand, we deduce that, for all $j \in \mathbb{N}^*$,

$$\lambda_{j,m}^2 \leq \lambda_j^2 + o(1).$$

Conversely, if we are interested in the eigenvalues of $(H_m)^2$ that are of order 1 when $m \rightarrow +\infty$, we see from (1.7) that the corresponding normalized eigenfunctions must

satisfy $\Xi^- \psi = \mathcal{O}(m^{-1})$ and, in particular, $\mathcal{B}\psi = \psi + \mathcal{O}(m^{-1})$. Thus, we get formally, for all $j \in \mathbb{N}^*$,

$$\lambda_{j,m}^2 \geq \lambda_j^2 + o(1).$$

The aim of this paper is to make this heuristics rigorous. We can now state our main theorem.

Theorem 1.7. *The singular values of H_m can be estimated as follows:*

- (i) $\lim_{m \rightarrow +\infty} \lambda_{k,m} = \lambda_k$, for any $k \in \mathbb{N}^*$.
- (ii) Let $k_1 \in \mathbb{N}^*$ be the multiplicity of the first eigenvalue λ_1 of $|H^\Omega|$. For all $k \in \{1, \dots, k_1\}$, we have

$$\lambda_{k,m} = \left(\lambda_1^2 + \frac{\nu_k}{m} + o\left(\frac{1}{m}\right) \right)^{1/2},$$

where

$$(1.8) \quad \nu_k = \inf_{\substack{V \subset \ker(|H^\Omega| - \lambda_1), \\ \dim V = k}} \sup_{\substack{u \in V, \\ \|u\|_{L^2(\Omega)} = 1}} \eta(u),$$

with

$$\eta(u) = \int_{\Gamma} \left(\frac{|\nabla_s u|^2}{2} - \frac{|(\partial_n + \kappa/2 + m_0)u|^2}{2} + \left(\frac{K}{2} - \frac{\kappa^2}{8} - \frac{\lambda_1^2}{2} \right) |u|^2 \right) d\Gamma.$$

Here, $(\lambda_k)_{k \in \mathbb{N}^*}$ and $(\lambda_{k,m})_{k \in \mathbb{N}^*}$ are defined in Notation 1.5, κ and K are defined in Notation 1.1.

Remark 1.8. The max-min formula (1.8) makes sense since $\ker(|H^\Omega| - \lambda \text{Id}) \subset H^2(\Omega; \mathbb{C}^4)$ for any eigenvalue λ of $|H^\Omega|$.

Remark 1.9. H_m and H^Ω anticommute with the charge conjugation C defined for all $\psi \in \mathbb{C}^4$, by

$$C\psi = i\beta\alpha_2\bar{\psi},$$

where $\bar{\psi} \in \mathbb{C}^4$ is the vector obtained after complex conjugations of each of the components of ψ (see for instance [10, Section 1.4.6] and [2, Section A.1]). As a consequence, the spectrum of H_m and H^Ω are symmetric with respect to 0 and Theorem 1.7 may be rewritten as a result on the eigenvalues of H_m and H^Ω .

1.3.3. *A vectorial Laplacian with Robin-type boundary conditions.* Let us also mention an intermediate spectral problem whose study is needed in our proof of Theorem 1.7 and that may be of interest on its own. Let us consider the vectorial Laplacian associated with the quadratic form

$$(1.9) \quad \mathcal{Q}_m^{\text{int}}(u) = \|\nabla u\|_{L^2(\Omega)}^2 + m_0^2 \|u\|_{L^2(\Omega)}^2 + \int_{\Gamma} \left(\frac{\kappa}{2} + m_0 \right) |u|^2 d\Gamma + 2m \|\Xi^- u\|_{L^2(\Gamma)}^2$$

for $u \in \text{Dom}(\mathcal{Q}_m^{\text{int}}) = H^1(\Omega; \mathbb{C}^4)$ and $m > 0$ where Ξ^-, Ξ^+ are defined by (1.2). By a classical trace theorem, this form is bounded from below. More precisely, we have the following result.

Lemma 1.10. *The self-adjoint operator associated with $\mathcal{Q}_m^{\text{int}}$ is defined by*

$$(1.10) \quad \text{Dom}(L_m^{\text{int}}) = \left\{ u \in H^2(\Omega; \mathbb{C}^4) : \begin{array}{l} \Xi^- (\partial_{\mathbf{n}} + \kappa/2 + m_0 + 2m) u = 0 \text{ on } \Gamma, \\ \Xi^+ (\partial_{\mathbf{n}} + \kappa/2 + m_0) u = 0 \text{ on } \Gamma \end{array} \right\}$$

$$L_m^{\text{int}} u = (-\Delta + m_0^2) u \text{ for all } u \in \text{Dom}(L_m^{\text{int}}).$$

It has compact resolvent and its spectrum is discrete.

Notation 1.11. Let $(\lambda_{k,m}^{\text{int}})_{k \in \mathbb{N}^*}$ denote the sequence of eigenvalues, each one being repeated according to its multiplicity and such that

$$(1.11) \quad \lambda_{1,m}^{\text{int}} \leq \lambda_{2,m}^{\text{int}} \leq \dots$$

The asymptotic behavior of the eigenvalues of L_m^{int} is detailed in the following theorem.

Theorem 1.12. *The following holds:*

- (i) *For every $k \in \mathbb{N}^*$, $\lim_{m \rightarrow +\infty} \lambda_{k,m}^{\text{int}} = \lambda_k^2$.*
- (ii) *Let λ be an eigenvalue of $|H^\Omega|$ of multiplicity $k_1 \in \mathbb{N}^*$. Consider $k_0 \in \mathbb{N}$ the unique integer such that for all $k \in \{1, \dots, k_1\}$, $\lambda_{k_0+k} = \lambda$. Then, for all $k \in \{1, 2, \dots, k_1\}$, we have*

$$\lambda_{k_0+k,m}^{\text{int}} = \lambda^2 + \frac{\mu_{\lambda,k}}{m} + o\left(\frac{1}{m}\right)$$

where

$$(1.12) \quad \mu_{\lambda,k} := \inf_{\substack{V \subset \ker(|H^\Omega| - \lambda), \\ \dim V = k}} \sup_{\substack{v \in V, \\ \|v\|_{L^2(\Omega)} = 1}} -\frac{\|(\partial_{\mathbf{n}} + \kappa/2 + m_0)v\|_{L^2(\Gamma)}^2}{2}.$$

Here, $(\lambda_k)_{k \in \mathbb{N}^*}$ is defined in Notation 1.5, $(\lambda_{k,m}^{\text{int}})_{k \in \mathbb{N}^*}$ in Notation 1.11 and κ in Notation 1.1.

1.4. Organization of the paper. In Section 2, we discuss the asymptotic properties of the minimizers associated with the exterior optimization problem (1.5). In Section 3, we investigate the interior problem. In Section 4, we prove Theorem 1.7.

2. ABOUT THE EXTERIOR OPTIMIZATION PROBLEM

The aim of this section is to study the minimizers of (1.5) and their properties when m tends to $+\infty$. These properties are gathered in the following proposition.

Proposition 2.1. *For all $v \in H^1(\Omega)$, there exists a unique minimizer $u_m(v)$ associated with $\Lambda_m(v)$, and it satisfies, for all $u \in V_v$,*

$$\mathcal{Q}_m(u) = \Lambda_m(v) + \mathcal{Q}_m(u - u_m(v)).$$

There exists a constant $C > 0$ such that, for all $m \geq 1$,

- (i) *for all $v \in H^1(\Omega)$, we have*

$$o(1) \geq \Lambda_m(v) - \left(m \|v\|_{L^2(\Gamma)}^2 + \int_{\Gamma} \frac{\kappa}{2} |v|^2 d\Gamma \right) \geq -\frac{C}{m} \|v\|_{L^2(\Gamma)}^2,$$

and

$$Cm \|v\|_{H^1(\Omega)}^2 \geq \Lambda_m(v),$$

(ii) for all $v \in H^2(\Omega)$,

$$\left| \Lambda_m(v) - \tilde{\Lambda}_m(v) \right| \leq \frac{C}{m^{3/2}} \|v\|_{H^{3/2}(\Gamma)}^2,$$

(iii) for all $v \in H^2(\Omega)$,

$$\left| \|u_m(v)\|_{L^2(\Omega')}^2 - \frac{\|v\|_{L^2(\Gamma)}^2}{2m} \right| \leq \frac{C}{m^2} \|v\|_{H^{3/2}(\Gamma)}^2.$$

Here

$$\tilde{\Lambda}_m(v) = m \int_{\Gamma} |v|^2 d\Gamma + \int_{\Gamma} \frac{\kappa}{2} |v|^2 d\Gamma + m^{-1} \int_{\Gamma} \left\{ \frac{|\nabla_s v|^2}{2} + \left(\frac{K}{2} - \frac{\kappa^2}{8} \right) |v|^2 \right\} d\Gamma.$$

2.1. Existence, uniqueness and Euler-Lagrange equations. Let us discuss here the existence of the minimizers announced in Proposition 2.1 and their elementary properties. We will see later that, in the limit $m \rightarrow +\infty$, this minimization problem on Ω' is closely related to the same problem on a tubular neighborhood in Ω' of Γ . For $\delta > 0$, $m > 0$ and $v \in H^1(\Omega)$, we define

$$(2.1) \quad \Lambda_{m,\delta}(v) = \inf \{ \mathcal{Q}_m(u) , u \in V_{v,\delta} \} , \quad \mathcal{Q}_m(u) = \|\nabla u\|_{L^2(\mathcal{V}_\delta)}^2 + m^2 \|u\|_{L^2(\mathcal{V}_\delta)}^2 ,$$

where¹ \mathcal{Q}_m is defined in (1.5), $\mathcal{V}_\delta = \{ \mathbf{x} \in \Omega' : \text{dist}(\mathbf{x}, \Gamma) < \delta \}$ and

$$V_{v,\delta} = \{ u \in H^1(\mathcal{V}_\delta, \mathbb{C}^4) \text{ s.t. } u = v \text{ on } \Gamma \text{ and } u(\mathbf{x}) = 0 \text{ if } \text{dist}(\mathbf{x}, \Gamma) = \delta \}.$$

2.1.1. *Existence and uniqueness.*

Lemma 2.2. *The minimizers associated with (1.5) and (2.1) exist and are unique.*

Proof. Let (u_n) and $(u_{\delta,n})$ be minimizing sequences for $\Lambda_m(v)$ and $\Lambda_{m,\delta}(v)$ respectively. These two sequences are uniformly bounded in H^1 so that up to subsequences, they converge weakly to $u \in H^1(\Omega')$ and $v_\delta \in H^1(\mathcal{V}_\delta)$. By Rellich - Kondrachov compactness Theorem and the interpolation inequality, the sequences converges strongly in H_{loc}^s for any $s \in [0, 1)$. The trace theorem ensures then that the convergence also holds in $L_{\text{loc}}^2(\Gamma)$ and $L_{\text{loc}}^2(\partial\mathcal{V}_\delta)$ so that $u \in V_v$ and $u_\delta \in V_{v,\delta}$. Since

$$\Lambda_m(v) = \lim_{n \rightarrow +\infty} \mathcal{Q}_m(u_n) \geq \mathcal{Q}_m(u) \geq \Lambda_m(v)$$

and

$$\Lambda_{m,\delta}(v) = \lim_{n \rightarrow +\infty} \mathcal{Q}_m(u_{\delta,n}) \geq \mathcal{Q}_m(u_{\delta,n}) \geq \Lambda_{m,\delta}(v),$$

u and u_δ are minimizers.

Since V and V_δ are convex sets and the quadratic form \mathcal{Q}_m is a strictly convex function, the uniqueness follows. □

Notation 2.3. The unique minimizers associated with $\Lambda_m(v)$ and $\Lambda_{m,\delta}(v)$ are denoted by $u_m(v)$ and $u_{m,\delta}(v)$, respectively, or u_m and $u_{m,\delta}$ when the dependence on v is clear.

¹Note that, since Ω is a smooth set, there exists $\delta_0 > 0$ such that, for all $\delta \in (0, \delta_0)$, the set \mathcal{V}_δ has the same regularity as Ω .

2.1.2. *Euler-Lagrange equations.* The following lemma gathers some properties related to the Euler-Lagrange equations.

Lemma 2.4. *For all $\delta > 0$, $m > 0$ and $v \in H^1(\Omega)$, the following holds.*

- (i) $(-\Delta + m^2)u_m = 0$ and $(-\Delta + m^2)u_{m,\delta} = 0$,
- (ii) $\Lambda_m(v) = -\langle \partial_{\mathbf{n}} u_m, u_m \rangle_{\Gamma}$ and $\Lambda_{m,\delta}(v) = -\langle \partial_{\mathbf{n}} u_{m,\delta}, u_{m,\delta} \rangle_{\Gamma}$,
- (iii) $\mathcal{Q}_m(u) = \Lambda_m(v) + \mathcal{Q}_m(u - u_m)$ for all $u \in V_v$,
 $\mathcal{Q}_m(u) = \Lambda_{m,\delta}(v) + \mathcal{Q}_m(u - u_{m,\delta})$ for all $u \in V_{v,\delta}$,

where $\Lambda_m(v)$ and V_v are defined in (1.5).

Proof. Let $v \in H_0^1(\Omega')$. The function

$$\mathbb{R} \ni t \mapsto \mathcal{Q}_m(u_m + tv)$$

has a minimum at $t = 0$. Hence, the Euler-Lagrange equation is $(-\Delta + m^2)u_m = 0$. The same proof holds for $u_{m,\delta}$. The second point follows from integrations by parts. And for the last point, let $u \in V_v$. We have, by an integration by parts,

$$\begin{aligned} \mathcal{Q}_m(u - u_m) &= \mathcal{Q}_m(u) + \mathcal{Q}_m(u_m) - 2\operatorname{Re}\langle u, (-\Delta + m^2)u_m \rangle_{\Omega'} + 2\langle u_m, \partial_{\mathbf{n}} u_m \rangle_{\Gamma} \\ &= \mathcal{Q}_m(u) - \Lambda_m(v) \end{aligned}$$

and the result follows. The same proof works for $\Lambda_{m,\delta}(v)$. \square

2.2. **Agmon estimates.** This section is devoted to the decay properties of the minimizers in the regime $m \rightarrow +\infty$.

We will need the following localization formulas.

Lemma 2.5. *Let χ be any real bounded Lipschitz function on Ω' . Then,*

$$(2.2) \quad \mathcal{Q}_m(u_m \chi) = -\langle \partial_{\mathbf{n}} u_m, \chi^2 u_m \rangle_{\Gamma} + \|(\nabla \chi)u_m\|_{L^2(\Omega')}^2.$$

The same holds for $u_{m,\delta}$.

Proof. By definition, we have

$$\begin{aligned} \mathcal{Q}_m(u_m \chi) &= m^2 \|\chi u_m\|_{L^2(\Omega')}^2 + \|(\nabla \chi)u_m + \chi(\nabla u_m)\|_{L^2(\Omega')}^2 \\ &= m^2 \|\chi u_m\|_{L^2(\Omega')}^2 + \|(\nabla \chi)u_m\|_{L^2(\Omega')}^2 + \|\chi(\nabla u_m)\|_{L^2(\Omega')}^2 + 2\operatorname{Re}\langle u_m \chi, \nabla \chi \cdot \nabla u_m \rangle_{\Omega'}. \end{aligned}$$

Then, by an integration by parts,

$$\|\chi(\nabla u_m)\|_{L^2(\Omega')}^2 = -\langle \partial_{\mathbf{n}} u_m, \chi^2 u_m \rangle_{\Gamma} - 2\operatorname{Re}\langle u_m \chi, \nabla \chi \cdot \nabla u_m \rangle_{\Omega'} + \operatorname{Re}\langle -\Delta u_m, \chi^2 u_m \rangle_{\Omega'}.$$

It remains to use Lemma 2.4 to get

$$\mathcal{Q}_m(u_m \chi) = -\langle \partial_{\mathbf{n}} u_m, \chi^2 u_m \rangle_{\Gamma} + \|(\nabla \chi)u_m\|_{L^2(\Omega')}^2.$$

The conclusion follows. \square

We can now establish the following important proposition.

Proposition 2.6. *Let $\gamma \in (0, 1)$. There exist $\delta_0 > 0$, and $C_1, C_2 > 0$ such that, for all $\delta \in (0, \delta_0)$ and all $m > 0$,*

$$(2.3) \quad \|e^{m\gamma \operatorname{dist}(\cdot, \Gamma)} u_m\|_{L^2(\Omega')}^2 \leq C_1 \|u_m\|_{L^2(\Omega')}^2,$$

and, for all $v \in V_v$,

$$(2.4) \quad (1 - e^{-\gamma m^{1/2}} C_2 m^{-1}) \Lambda_{m, m^{-1/2}}(v) \leq \Lambda_m(v) \leq \Lambda_{m,\delta}(v).$$

Proof. Let us first prove (2.3). Given $\varepsilon > 0$, we define

$$\begin{aligned}\Phi &: \mathbf{x} \mapsto \min(\gamma \operatorname{dist}(\mathbf{x}, \Gamma), \varepsilon^{-1}), \\ \chi_m &: \mathbf{x} \mapsto e^{m\Phi(\mathbf{x})}.\end{aligned}$$

Let $c > 1$ and $R > 0$. Let $\chi_{1,m,R}, \chi_{2,m,R}$ be a smooth quadratic partition of the unity such that

$$\chi_{1,m,R}(\mathbf{x}) = \begin{cases} 1 & \text{if } \operatorname{dist}(\mathbf{x}, \Gamma) \leq R/2m \\ 0 & \text{if } \operatorname{dist}(\mathbf{x}, \Gamma) \geq R/m \end{cases}$$

and, for $k \in \{1, 2\}$,

$$\|\nabla \chi_{k,m,R}\|_{L^\infty(\Omega')} \leq \frac{2mc}{R}.$$

Since χ_m is a bounded, Lipschitz function and is equal to 1 on Γ , we get $u_m \chi_m \in V_v$. By definition and using (2.2), we get

$$\Lambda_m(v) = \mathcal{Q}_m(u_m) = -\langle \partial_{\mathbf{n}} u_m, u_m \rangle_\Gamma = \mathcal{Q}_m(u_m \chi_m) - \|(\nabla \chi_m) u_m\|_{L^2(\Omega')}^2.$$

Then, we use the fact that $\nabla(\chi_{1,m,R}^2 + \chi_{2,m,R}^2) = 0$ to get

$$\begin{aligned}\mathcal{Q}_m(u_m) &= \mathcal{Q}_m(u_m \chi_m \chi_{1,m,R}) + \mathcal{Q}_m(u_m \chi_m \chi_{2,m,R}) - \|(\nabla \chi_m) u_m\|_{L^2(\Omega')}^2 \\ &\quad - \|(\nabla \chi_{1,m,R}) \chi_m u_m\|_{L^2(\Omega')}^2 - \|(\nabla \chi_{2,m,R}) \chi_m u_m\|_{L^2(\Omega')}^2.\end{aligned}$$

Since $\mathcal{Q}_m(u_m \chi_m \chi_{1,m,R}) \geq \Lambda_m(v)$ and

$$\begin{aligned}\mathcal{Q}_m(u_m \chi_m \chi_{2,m,R}) &\geq m^2 \|u_m \chi_m \chi_{2,m,R}\|_{L^2(\Omega')}^2 \\ &= m^2 \|u_m \chi_m\|_{L^2(\Omega')}^2 - m^2 \|u_m \chi_m \chi_{1,m,R}\|_{L^2(\Omega')}^2,\end{aligned}$$

we get that

$$\begin{aligned}m^2 \left(1 - \gamma^2 - \frac{8c^2}{R^2}\right) \|u_m \chi_m\|_{L^2(\Omega')}^2 &\leq m^2 \|u_m \chi_m \chi_{1,m,R}\|_{L^2(\Omega')}^2 \\ &\leq m^2 e^{2m \min(\frac{\gamma R}{m}, \frac{1}{\varepsilon})} \|u_m\|_{L^2(\Omega')}^2 \leq m^2 e^{2\gamma R} \|u_m\|_{L^2(\Omega')}^2.\end{aligned}$$

Taking $R > 0$ big enough to get $1 - \gamma^2 - \frac{8c^2}{R^2} > 0$, we get that

$$\|u_m \chi_m\|_{L^2(\Omega')}^2 \leq C \|u_m\|_{L^2(\Omega')}^2$$

where C does not depend on ε . Taking the limit $\varepsilon \rightarrow 0$ and using the Fatou lemma we obtain (2.3).

Let us now prove (2.4). We have for any $\delta \in (0, \delta_0)$ that $V_{v,\delta} \subset V_v$ so that

$$\Lambda_m(v) \leq \Lambda_{m,\delta}(v).$$

Let us consider a Lipschitz function $\tilde{\chi}_m : \Omega' \rightarrow [0, 1]$ defined for all $\mathbf{x} \in \Omega'$ by

$$\tilde{\chi}_m(\mathbf{x}) = \begin{cases} 1 & \text{if } \operatorname{dist}(\mathbf{x}, \Gamma) \leq \frac{1}{2m^{1/2}} \\ 0 & \text{if } \operatorname{dist}(\mathbf{x}, \Gamma) \geq \frac{1}{m^{1/2}} \end{cases},$$

with $\|\nabla \tilde{\chi}_m\|_{L^\infty(\Omega')} \leq 2cm^{1/2}$. Thanks to (2.2), we find

$$(2.5) \quad \Lambda_{m,m^{-1/2}}(v) \leq \mathcal{Q}_m(u_m \tilde{\chi}_m) = \Lambda_m(v) + \|u_m \nabla \tilde{\chi}_m\|_{L^2(\Omega')}^2.$$

Then, by (2.3) we have

$$\|u_m \nabla \tilde{\chi}_m\|_{L^2(\Omega')}^2 \leq e^{-\gamma m^{1/2}} 4c^2 m \|e^{m\gamma \operatorname{dist}(\cdot, \Gamma)} u_m\|_{L^2(\Omega')}^2 \leq C_1 e^{-\gamma m^{1/2}} 4c^2 m \|u_m\|_{L^2(\Omega')}^2.$$

Observing that

$$m\|u_m\|_{L^2(\Omega')}^2 \leq m^{-1}\Lambda_m(v),$$

and using (2.5) we easily get (2.4). \square

2.3. Optimization problem in a tubular neighborhood. From Proposition 2.6, we see that, in order to estimate $\Lambda_m(v)$, it is sufficient to estimate $\Lambda_{m,m^{-1/2}}(v)$. For that purpose, we will use tubular coordinates.

2.3.1. Tubular coordinates. Let ι be the canonical embedding of Γ in \mathbb{R}^3 and g the induced metric on Γ . (Γ, g) is a C^4 Riemannian manifold, which we orientate according to the ambient space. Let us introduce the map $\Phi : \Gamma \times (0, \delta) \rightarrow \mathcal{V}_\delta$ defined by the formula

$$\Phi(s, t) = \iota(s) + t\mathbf{n}(s)$$

where \mathcal{V}_δ is defined in (2.1) below. The transformation Φ is a C^3 diffeomorphism for any $\delta \in (0, \delta_0)$ provided that δ_0 is sufficiently small. The induced metric on $\Gamma \times (0, \delta)$ is given by

$$G = g \circ (\text{Id} + tL(s))^2 + dt^2,$$

where $L(s) = d\mathbf{n}_s$ is the second fundamental form of the boundary at s . Let us now describe how our optimization problem is transformed under the change of coordinates. For all $u \in L^2(\mathcal{V}_\delta)$, we define the pull-back function

$$(2.6) \quad \tilde{u}(s, t) := u(\Phi(s, t)).$$

For all $u \in H^1(\mathcal{V}_\delta)$, we have

$$(2.7) \quad \int_{\mathcal{V}_\delta} |u|^2 d\mathbf{x} = \int_{\Gamma \times (0, \delta)} |\tilde{u}(s, t)|^2 \tilde{a} d\Gamma dt,$$

$$(2.8) \quad \int_{\mathcal{V}_\delta} |\nabla u|^2 d\mathbf{x} = \int_{\Gamma \times (0, \delta)} \left[\langle \nabla_s \tilde{u}, \tilde{g}^{-1} \nabla_s \tilde{u} \rangle + |\partial_t \tilde{u}|^2 \right] \tilde{a} d\Gamma dt.$$

where

$$\tilde{g} = (\text{Id} + tL(s))^2,$$

and $\tilde{a}(s, t) = |\tilde{g}(s, t)|^{\frac{1}{2}}$. Here $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product and ∇_s is the differential on Γ seen through the metric g . Since $L(s) \in \mathbb{R}^{2 \times 2}$, we have the exact formula

$$(2.9) \quad \tilde{a}(s, t) = 1 + t\kappa(s) + t^2K(s)$$

where κ and K are defined in Notation 1.1. In the following, we assume that

$$(2.10) \quad \delta = m^{-1/2}.$$

In particular, we will use (2.7) and (2.8) with this particular choice of δ .

2.3.2. *The rescaled transition optimization problem in boundary coordinates.* We introduce the rescaling

$$(s, \tau) = (s, mt),$$

and the new weights

$$(2.11) \quad \hat{a}_m(s, \tau) = \tilde{a}(s, m^{-1}\tau), \quad \hat{g}_m(s, \tau) = \tilde{g}(s, m^{-1}\tau).$$

Note that there exists $m_1 \geq 1$ such that for all $m \geq m_1$, $s \in \Gamma$ and $\tau \in [0, m^{1/2})$, $\hat{a}_m(s, \tau) \geq 1/2$. We set

$$(2.12) \quad \begin{aligned} \hat{\mathcal{V}}_m &= \Gamma \times (0, \sqrt{m}), \\ \hat{V}_m &= \{u \in H^1(\hat{\mathcal{V}}_m, \mathbb{C}^4; \hat{a}_m \, d\Gamma \, d\tau) : u(\cdot, \sqrt{m}) = 0\}, \\ \hat{\mathcal{Q}}_m(u) &= m^{-1} \int_{\hat{\mathcal{V}}_m} \left(\langle \nabla_s u, \hat{g}_m^{-1} \nabla_s u \rangle + m^2 |\partial_\tau u|^2 \right) \hat{a}_m \, d\Gamma \, d\tau \\ &\quad + m \int_{\hat{\mathcal{V}}_m} |u|^2 \hat{a}_m \, d\Gamma \, d\tau, \\ \hat{\mathcal{L}}_m &= -m^{-1} \hat{a}_m^{-1} \nabla_s (\hat{a}_m \hat{g}_m^{-1} \nabla_s) + m (-\hat{a}_m^{-1} \partial_\tau \hat{a}_m \partial_\tau + 1). \end{aligned}$$

Notation 2.7. Let $m \geq m_1$, $k, K \in \mathbb{R}$ and define

$$\begin{aligned} a_{m,\kappa,K} &: (0, \sqrt{m}) \longrightarrow \mathbb{R} \\ \tau &\longmapsto 1 + \frac{\tau\kappa}{m} + \frac{\tau^2 K}{m^2}. \end{aligned}$$

We let

$$(2.13) \quad A = \|\kappa\|_{L^\infty(\Gamma)} \text{ and } B = \|K\|_{L^\infty(\Gamma)}.$$

Remark 2.8. We can assume (up to taking a larger m_1) that for any

$$(m, \kappa, K) \in [m_1, +\infty) \times [-A, A] \times [-B, B],$$

we have $a_{m,\kappa,K}(\tau) \geq 1/2$ for all $\tau \in (0, \sqrt{m})$.

In the following, we assume that $m \geq m_1$.

2.4. One dimensional optimization problem with parameters. We denote by $\hat{\mathcal{Q}}_{m,\kappa,K}$ the ‘‘transversed’’ quadratic form defined for $u \in H^1((0, \sqrt{m}), a_{m,\kappa,K} \, d\tau)$ by

$$\hat{\mathcal{Q}}_{m,\kappa,K}(u) = \int_0^{\sqrt{m}} \left(|\partial_\tau u|^2 + |u|^2 \right) a_{m,\kappa,K} \, d\tau.$$

We let

$$(2.14) \quad \Lambda_{m,\kappa,K} = \inf \{ \hat{\mathcal{Q}}_{m,\kappa,K}(u) : u \in \hat{V}_{m,\kappa,K} \}$$

where

$$\hat{V}_{m,\kappa,K} = \{u \in H^1((0, \sqrt{m}), a_{m,\kappa,K} \, d\tau) : u(0) = 1, u(\sqrt{m}) = 0\}.$$

The following lemma follows from the same arguments as for Lemma 2.2.

Lemma 2.9. *There is a unique minimizer $u_{m,\kappa,K}$ for the optimization problem (2.14).*

Lemma 2.10. *Let $u, v \in H^1((0, \sqrt{m}), a_{m,\kappa,K} d\tau)$ be such that $u(\sqrt{m}) = v(\sqrt{m}) = 0$. We have*

$$(2.15) \quad \int_0^{\sqrt{m}} \langle \partial_\tau u, \partial_\tau v \rangle a_{m,\kappa,K} d\tau + \int_0^{\sqrt{m}} \langle u, v \rangle a_{m,\kappa,K} d\tau \\ = \int_0^{\sqrt{m}} \langle \hat{\mathcal{L}}_{m,\kappa,K} u, v \rangle a_{m,\kappa,K} d\tau - \langle \partial_\tau u(0), v(0) \rangle,$$

where

$$\hat{\mathcal{L}}_{m,\kappa,K} = -a_{m,\kappa,K}^{-1} \partial_\tau a_{m,\kappa,K} \partial_\tau + 1 = -\partial_\tau^2 - \frac{m^{-1}\kappa + m^{-2}2K\tau}{1 + m^{-1}\kappa\tau + m^{-2}K\tau^2} \partial_\tau + 1.$$

Proof. The lemma follows essentially by integration by parts and Notation 2.7. \square

Lemma 2.11. *We have that $u_{m,\kappa,K} \in \mathcal{C}^\infty([0, \sqrt{m}])$ and*

$$\hat{\mathcal{L}}_{m,\kappa,K} u_{m,\kappa,K} = 0, \quad \Lambda_{m,\kappa,K} = -\partial_\tau u_{m,\kappa,K}(0),$$

where $u_{m,\kappa,K}$ is defined in Lemma 2.9. Moreover, for all $u \in \hat{V}_{m,\kappa,K}$,

$$\hat{\mathcal{Q}}_{m,\kappa,K}(u) = \Lambda_{m,\kappa,K} + \hat{\mathcal{Q}}_{m,\kappa,K}(u - u_{m,\kappa,K}).$$

Proof. This follows from Lemma 2.10. \square

The aim of this section is to establish an accurate estimate of $\Lambda_{m,\kappa,K}$.

Proposition 2.12. *There exists a constant $C > 0$ such that for all*

$$(m, \kappa, K) \in [m_1, +\infty) \times [-A, A] \times [-B, B],$$

$$\left| \Lambda_{m,\kappa,K} - \left(1 + \frac{\kappa}{2m} + \frac{1}{m^2} \left(\frac{K}{2} - \frac{\kappa^2}{8} \right) \right) \right| \leq Cm^{-3},$$

and

$$\left| \int_0^{\sqrt{m}} |u_{m,\kappa,K}|^2 a_{m,\kappa,K} d\tau - \frac{1}{2} \right| \leq Cm^{-1}.$$

Proof. By Lemmas 2.9 and 2.11, the unique solution $u_{m,\kappa,K}$ of the problem satisfies

$$\left(-\partial_\tau^2 - \frac{m^{-1}\kappa + m^{-2}2K\tau}{1 + m^{-1}\kappa\tau + m^{-2}K\tau^2} \partial_\tau + 1 \right) u_{m,\kappa,K} = 0.$$

We expand formally $u_{m,\kappa,K}$ as $u_0 + m^{-1}u_1 + m^{-2}u_2 + \mathcal{O}(m^{-3})$:

(i) For the zero order term, we get

$$(-\partial_\tau^2 + 1)u_0 = 0 \text{ and } u_0(1) = 1, \lim_{\tau \rightarrow \infty} u_0(\tau) = 0,$$

so that $u_0(\tau) = e^{-\tau}$.

(ii) At the first order,

$$(-\partial_\tau^2 + 1)u_1 = \kappa \partial_\tau u_0 = -\kappa e^{-\tau} \text{ and } u_1(1) = 0, \lim_{\tau \rightarrow \infty} u_1(\tau) = 0,$$

so that $u_1(\tau) = -\frac{\kappa}{2}\tau e^{-\tau}$.

(iii) At the *second order*,

$$(-\partial_\tau^2 + 1)u_2 = \kappa \partial_\tau u_1 + (\kappa^2 - 2K)\tau \partial_\tau u_0 = -\frac{\kappa^2}{2}e^{-\tau} + \left(\frac{3\kappa^2}{2} - 2K\right)\tau e^{-\tau},$$

$$u_2(0) = 0 \text{ and } \lim_{\tau \rightarrow \infty} u_2(\tau) = 0,$$

$$\text{so that } u_2(\tau) = \left(\frac{\kappa^2}{8} - \frac{K}{2}\right)\tau e^{-\tau} + \left(\frac{3\kappa^2}{8} - \frac{K}{2}\right)\tau^2 e^{-\tau}.$$

This formal construction leads to define a possible approximation of $u_{m,\kappa,K}$. Consider

$$(2.16) \quad \begin{aligned} v_{m,\kappa,K}(\tau) &:= \chi_m(\tau) \left(u_0(\tau) + m^{-1}u_1(\tau) + m^{-2}u_2(\tau) \right), \\ \chi_m(\tau) &= \chi(\tau/\sqrt{m}), \end{aligned}$$

where $\chi : \mathbb{R}_+ \mapsto [0, 1]$ is a smooth function such that

$$\chi(\tau) = \begin{cases} 1 & \text{if } \tau \in [0, 1/2] \\ 0 & \text{if } \tau \geq 1 \end{cases}.$$

In the following, we denote $v_m = v_{m,\kappa,K}$ to shorten the notation. We immediately get that v_m belongs to $\widehat{V}_{m,\kappa,K}$. Note that

$$(2.17) \quad -\partial_\tau v_m(0) = 1 + \frac{\kappa}{2m} + m^{-2} \left(\frac{K}{2} - \frac{\kappa^2}{8} \right),$$

$$(2.18) \quad \|\widehat{\mathcal{L}}_{m,\kappa,K} v_m\|_{L^2((0,\sqrt{m}), a_{m,\kappa,K} d\tau)} = \mathcal{O}(m^{-3}).$$

Using Lemmas 2.10 and 2.11, we have

$$\Lambda_{m,\kappa,K} = \int_0^{\sqrt{m}} \langle \partial_\tau u_{m,\kappa,K}, \partial_\tau v_m \rangle a_{m,\kappa,K} d\tau + \int_0^{\sqrt{m}} \langle u_{m,\kappa,K}, v_m \rangle a_{m,\kappa,K} d\tau,$$

and

$$\Lambda_{m,\kappa,K} = \int_0^{\sqrt{m}} \langle \widehat{\mathcal{L}}_{m,\kappa,K} v_m, u_{m,\kappa,K} \rangle a_{m,\kappa,K} d\tau - \partial_\tau v_m(0).$$

By Lemma 2.10, the Cauchy-Schwarz inequality, (2.17), and (2.18),

$$\begin{aligned} & \left| \Lambda_{m,\kappa,K} - \left(1 + \frac{\kappa}{2m} + m^{-2} \left(\frac{K}{2} - \frac{\kappa^2}{8} \right) \right) \right| \\ &= \left| \int_0^{\sqrt{m}} \langle \widehat{\mathcal{L}}_{m,\kappa,K} v_m, u_{m,\kappa,K} \rangle a_{m,\kappa,K} d\tau \right| \\ &\leq \|\widehat{\mathcal{L}}_{m,\kappa,K} v_m\|_{L^2((0,\sqrt{m}), a_{m,\kappa,K} d\tau)} \|u_{m,\kappa,K}\|_{L^2((0,\sqrt{m}), a_{m,\kappa,K} d\tau)} \\ &\leq \Lambda_{m,\kappa,K}^{\frac{1}{2}} \|\widehat{\mathcal{L}}_{m,\kappa,K} v_m\|_{L^2((0,\sqrt{m}), a_{m,\kappa,K} d\tau)} \\ &\leq C m^{-3} \Lambda_{m,\kappa,K}^{\frac{1}{2}}. \end{aligned}$$

From this, it follows first that $\Lambda_{m,\kappa,K} = \mathcal{O}(1)$ uniformly in (κ, K) , and then the first estimate of the proposition is established. Using Lemmas 2.10 and 2.11, the fact

that $v_m(0) - u_{m,\kappa,K}(0) = 0$ and Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \widehat{\mathcal{Q}}_{m,\kappa,K}(v_m - u_{m,\kappa,K}) \\ & \leq \|\widehat{\mathcal{L}}_{m,\kappa,K}(v_m - u_{m,\kappa,K})\|_{L^2((0,\sqrt{m}),a_{m,\kappa,K} d\tau)} \|v_m - u_{m,\kappa,K}\|_{L^2((0,\sqrt{m}),a_{m,\kappa,K} d\tau)} \\ & \leq Cm^{-3} \|v_m - u_{m,\kappa,K}\|_{L^2((0,\sqrt{m}),a_{m,\kappa,K} d\tau)}. \end{aligned}$$

The second estimate follows since

$$\|v_m - u_{m,\kappa,K}\|_{L^2((0,\sqrt{m}),a_{m,\kappa,K} d\tau)}^2 \leq \widehat{\mathcal{Q}}_{m,\kappa,K}(v_m - u_{m,\kappa,K}),$$

$$\text{and } \|v_m\|_{L^2((0,\sqrt{m}),a_{m,\kappa,K} d\tau)}^2 = \frac{1}{2} + \mathcal{O}(m^{-1}).$$

□

2.5. Asymptotic study of $\Lambda_{m,m^{-1/2}}(v)$. From Proposition 2.12 and (2.12), we deduce the following lower bound.

Corollary 2.13. *There exists $C > 0$ such that for any $v \in H^1(\Omega)$,*

$$o(1) \geq \Lambda_{m,m^{-1/2}}(v) - \left(m\|v\|_{L^2(\Gamma)}^2 + \int_{\Gamma} \frac{\kappa}{2}|v|^2 d\Gamma \right) \geq -\frac{C}{m}\|v\|_{L^2(\Gamma)}^2,$$

and

$$Cm\|v\|_{H^1(\Omega)}^2 \geq \Lambda_{m,m^{-1/2}}(v).$$

Here, the term $o(1)$ depends on v (not only on the H^1 norm of v).

Proof. By Proposition 2.12, the lower bound follows. By the extension theorem for Sobolev functions (see for instance [7, Section 5.4.]), there exist a constant $C > 0$ and, for all $v \in H^1$, a function $Ev \in H^1(\mathbb{R}^3)$ that extends v and such that $\|Ev\|_{H^1(\mathbb{R}^3)} \leq C\|v\|_{H^1(\Omega)}$. Let us define the test function u_m by $u_m = v\tilde{u}_m$ where

$$\tilde{u}_m(\Phi(s, t)) = \begin{cases} v_{m,\kappa(s),K(s)}(mt) & \text{for all } (s, t) \in \Gamma \times [0, m^{-1/2}], \\ 0 & \text{for all } (s, t) \in \Gamma \times [m^{-1/2}, +\infty). \end{cases}$$

Here, the function v_m is defined in (2.16). With an integration by parts, Lemmas 2.10, 2.11 and Proposition 2.12, we get

$$\begin{aligned} \mathcal{Q}_m(u_m) &= \|\tilde{u}_m \nabla v + v \nabla \tilde{u}_m\|_{L^2(\Omega')}^2 + m^2 \|v \tilde{u}_m\|_{L^2(\Omega')}^2 \\ &= \|\tilde{u}_m \nabla v\|_{L^2(\Omega')}^2 + \|v \nabla \tilde{u}_m\|_{L^2(\Omega')}^2 + 2\text{Re} \langle \tilde{u}_m \nabla v, v \nabla \tilde{u}_m \rangle_{\Omega'} + m^2 \|v \tilde{u}_m\|_{L^2(\Omega')}^2 \\ &= \|\tilde{u}_m \nabla v\|_{L^2(\Omega')}^2 + \langle \tilde{u}_m v, v (-\Delta + m^2) \tilde{u}_m \rangle_{\Omega'} - \langle v \partial_{\mathbf{n}} \tilde{u}_m, v \tilde{u}_m \rangle_{\Gamma} \\ &\leq \|\tilde{u}_m \nabla v\|_{L^2(\Omega')}^2 + \langle \tilde{u}_m v, v (-\Delta + m^2) \tilde{u}_m \rangle_{\Omega'} \\ &\quad + m\|v\|_{L^2(\Gamma)}^2 + \int_{\Gamma} |v|^2 \kappa/2 d\Gamma + C\|v\|_{L^2(\Gamma)}^2/m. \end{aligned}$$

Since \tilde{u}_m is uniformly bounded in $W^{2,\infty}(\Omega)$ and pointwise converges to 0 with its derivatives in the tangential direction, Lebesgue's dominated convergence theorem ensures that $\|\tilde{u}_m \nabla v\|_{L^2(\Omega')}^2$ and $\langle \tilde{u}_m v, v (-\Delta + m^2) \tilde{u}_m \rangle_{\Omega'}$ tend to 0 as m goes to $+\infty$. We obtain that

$$\limsup_{m \rightarrow +\infty} \left(\Lambda_{m,m^{-1/2}} - m\|v\|_{L^2(\Gamma)}^2 - \int_{\Gamma} |v|^2 \kappa/2 d\Gamma \right) \leq 0.$$

□

With Proposition 2.6, this proves in particular (i) in Proposition 2.1. This section is devoted to the refinement of this lower bound and to the corresponding upper bound.

2.5.1. *Preliminary lemmas.* Let us state a few elementary lemmas that we will use later.

Lemma 2.14. *There exists $C > 0$ such that, for all $f, g \in H^{\frac{3}{2}}(\Gamma)$, we have*

$$\|fg\|_{H^{\frac{3}{2}}(\Gamma)} \leq C \|f\|_{H^{\frac{3}{2}}(\Gamma)} \|g\|_{H^{\frac{3}{2}}(\Gamma)}.$$

Proof. $H^{\frac{3}{2}}(\Gamma)$ is an algebra since $\frac{3}{2} > \frac{\dim \Gamma}{2} = 1$. \square

Lemma 2.15. *There exists $C > 0$ such that, for all $f \in H^{\frac{3}{2}}(\Gamma)$, we have*

$$\|f\|_{H^{\frac{1}{2}}(\Gamma)} \leq C \|f\|_{L^2(\Gamma)}^{\frac{1}{2}} \|f\|_{H^1(\Gamma)}^{\frac{1}{2}}.$$

Lemma 2.16. *There exists $C > 0$ such that, for all $f \in H^{\frac{1}{2}}(\Gamma, T\Gamma)$ and $g \in H^1(\Gamma, \mathbb{C})$, we have*

$$\left| \int_{\Gamma} f \cdot \nabla_s g \, d\Gamma \right| \leq C \|f\|_{H^{\frac{1}{2}}(\Gamma)} \|g\|_{H^{\frac{1}{2}}(\Gamma)}.$$

2.5.2. *Lower and upper bounds.*

Notation 2.17. In the following, we define

$$\begin{aligned} \hat{\Pi}_m : H^1(\Omega, \mathbb{C}^4) &\longrightarrow \hat{V}_m \\ v &\longmapsto [(s, \tau) \in \hat{V}_m \mapsto v(s)u_{m, \kappa(s), K(s)}(\tau) \in \mathbb{C}^4] \end{aligned}$$

where $u_{m, \kappa(s), K(s)}$ is defined by Proposition 2.12 with $\kappa = \kappa(s)$ and $K = K(s)$.

Lemma 2.18. *We have, uniformly in s ,*

$$\int_0^{\sqrt{m}} |\nabla_s u_{m, \kappa(\cdot), K(\cdot)}|^2 \, d\tau = \mathcal{O}(m^{-2}).$$

Proof. We have

$$\left(-a_{m, \kappa, K}^{-1} \partial_{\tau} a_{m, \kappa, K} \partial_{\tau} + 1 \right) u_{m, \kappa, K} = 0.$$

Let us take the derivative with respect to s :

$$\left(-a_{m, \kappa, K}^{-1} \partial_{\tau} a_{m, \kappa, K} \partial_{\tau} + 1 \right) \nabla_s u_{m, \kappa, K} = \left[\nabla_s, a_{m, \kappa, K}^{-1} \partial_{\tau} a_{m, \kappa, K} \partial_{\tau} \right] u_{m, \kappa, K}.$$

Taking the scalar product with $\nabla_s u_{m, \kappa, K}$ and integrating by parts by noticing that $\nabla_s u_{m, \kappa, K}(0) = 0$, we get

$$\begin{aligned} &\int_0^{\sqrt{m}} |\partial_{\tau} \nabla_s u_{m, \kappa, K}|^2 a_{m, \kappa, K} \, d\tau + \|\nabla_s u_{m, \kappa, K}\|_{L^2(a_{m, \kappa, K} \, d\tau)}^2 \\ &\leq \left| \left\langle \left[\nabla_s, a_{m, \kappa, K}^{-1} \partial_{\tau} a_{m, \kappa, K} \partial_{\tau} \right] u_{m, \kappa, K}, \nabla_s u_{m, \kappa, K} \right\rangle_{L^2(a_{m, \kappa, K} \, d\tau)} \right|. \end{aligned}$$

By an explicit computation and the Cauchy-Schwarz inequality, we find

$$\begin{aligned} & \left| \left\langle \left[\nabla_s, a_{m,\kappa,K}^{-1} \partial_\tau a_{m,\kappa,K} \partial_\tau \right] u_{m,\kappa,K}, \nabla_s u_{m,\kappa,K} \right\rangle_{L^2(a_{m,\kappa,K} d\tau)} \right| \\ & \leq C m^{-1} \|\partial_\tau u_{m,\kappa,K}\|_{L^2(a_{m,\kappa,K} d\tau)} \|\nabla_s u_{m,\kappa,K}\|_{L^2(a_{m,\kappa,K} d\tau)}. \end{aligned}$$

Since

$$\|\partial_\tau u_{m,\kappa,K}\|_{L^2(a_{m,\kappa,K} d\tau)} \leq \sqrt{\Lambda_{m,\kappa,K}},$$

we get

$$\int_0^{\sqrt{m}} |\partial_\tau \nabla_s u_{m,\kappa,K}|^2 a_{m,\kappa,K} d\tau + \|\nabla_s u_{m,\kappa,K}\|_{L^2(a_{m,\kappa,K} d\tau)}^2 \leq C m^{-2}.$$

□

Proposition 2.19. *There exist positive constants $C > 0$ and $m_1 > 0$ such that for all $m \geq m_1$, and all $v \in H^2(\Omega)$,*

$$|\Lambda_{m,m^{-1/2}}(v) - \tilde{\Lambda}_m(v)| \leq C m^{-3/2} \|v\|_{H^{3/2}(\Gamma)}^2,$$

where

$$\tilde{\Lambda}_m(v) = m \int_\Gamma |v|^2 d\Gamma + \int_\Gamma \frac{\kappa}{2} |v|^2 d\Gamma + m^{-1} \int_\Gamma \left(\frac{|\nabla_s v|^2}{2} + \left(\frac{K}{2} - \frac{\kappa^2}{8} \right) |v|^2 \right) d\Gamma.$$

More precisely, for all $u \in \hat{V}_m$ such that $u = v$ on Γ ,

$$\begin{aligned} \hat{\mathcal{Q}}_m(u) & \geq \tilde{\Lambda}_m(v) - \frac{C}{m^{3/2}} \|v\|_{H^{3/2}(\Gamma)}^2 + \frac{m}{2} \|u - \hat{\Pi}_m v\|_{L^2(\hat{V}_m, d\Gamma d\tau)}^2 \\ & \quad + \frac{1}{2m} \|\nabla_s (u - \hat{\Pi}_m v)\|_{L^2(\hat{V}_m, d\Gamma d\tau)}^2, \end{aligned}$$

and

$$\hat{\mathcal{Q}}_m(\hat{\Pi}_m(v)) \leq \tilde{\Lambda}_m(v) + C m^{-3/2} \left(\|v\|_{L^2(\Gamma)}^2 + \|\nabla_s v\|_{L^2(\Gamma)}^2 \right).$$

Proof. Let $v \in H^2(\Omega)$.

First, let us discuss the upper bound. For that purpose, we insert $\hat{\Pi}_m v$ in the quadratic form:

$$\hat{\mathcal{Q}}_m(\hat{\Pi}_m v) = m \int_\Gamma \hat{\mathcal{Q}}_{m,\kappa(\cdot),K(\cdot)}(\hat{\Pi}_m v) d\Gamma + m^{-1} \int_{\hat{V}_m} \langle \nabla_s \hat{\Pi}_m v, \hat{g}_m^{-1} \nabla_s \hat{\Pi}_m v \rangle \hat{a}_m d\Gamma d\tau.$$

We have

$$m \int_\Gamma \hat{\mathcal{Q}}_{m,\kappa(\cdot),K(\cdot)}(\hat{\Pi}_m v) d\Gamma = m \int_\Gamma |v|^2 \Lambda_{m,\kappa(\cdot),K(\cdot)} d\Gamma,$$

and

$$\int_{\hat{V}_m} \langle \nabla_s \hat{\Pi}_m v, \hat{g}_m^{-1} \nabla_s \hat{\Pi}_m v \rangle \hat{a}_m d\Gamma d\tau \leq (1 + C m^{-\frac{1}{2}}) \int_{\hat{V}_m} |\nabla_s \hat{\Pi}_m v|^2 d\Gamma d\tau.$$

Moreover, for all $\varepsilon > 0$,

$$\begin{aligned} \int_{\widehat{\mathcal{V}}_m} |\nabla_s \widehat{\Pi}_m v|^2 d\Gamma d\tau &\leq (1 + \varepsilon) \int_{\Gamma} |\nabla_s v|^2 \int_0^{\sqrt{m}} |u_{m,\kappa(\cdot),K(\cdot)}|^2 d\tau d\Gamma \\ &\quad + (1 + \varepsilon^{-1}) \int_{\Gamma} \|v\|^2 \int_0^{\sqrt{m}} |\nabla_s u_{m,\kappa(\cdot),K(\cdot)}|^2 d\tau d\Gamma. \end{aligned}$$

We recall Lemma 2.18. We choose $\varepsilon = m^{-1}$ and recall Proposition 2.12 to get

$$\int_{\widehat{\mathcal{V}}_m} |\nabla_s \widehat{\Pi}_m v|^2 d\Gamma d\tau \leq (1 + Cm^{-1}) \frac{1}{2} \int_{\Gamma} \|\nabla_s v\|^2 d\Gamma + Cm^{-1} \|v\|_{L^2(\Gamma)}^2.$$

Therefore,

$$\widehat{\mathcal{Q}}_m(\widehat{\Pi}_m v) \leq m \int_{\Gamma} |v|^2 \Lambda_{m,\kappa(\cdot),K(\cdot)} d\Gamma + m^{-1} \frac{1 + Cm^{-\frac{1}{2}}}{2} \int_{\Gamma} |\nabla_s v|^2 d\Gamma + Cm^{-2} \|v\|_{L^2(\Gamma)}^2.$$

It remains to use Proposition 2.12 to get the desired upper bound. Let us now discuss the lower bound. Let $u \in \widehat{\mathcal{V}}_m$ such that $u = v$ on Γ . By Lemma 2.11, we have

$$\begin{aligned} \widehat{\mathcal{Q}}_m(u) &= m \int_{\Gamma} \widehat{\mathcal{Q}}_{m,\kappa(\cdot),K(\cdot)}(u) d\Gamma + m^{-1} \int_{\widehat{\mathcal{V}}_m} \langle \nabla_s u, \widehat{g}_m^{-1} \nabla_s u \rangle \widehat{a}_m d\Gamma d\tau \\ &= m \int_{\Gamma} |v|^2 \Lambda_{m,\kappa(\cdot),K(\cdot)} d\Gamma + m \int_{\Gamma} \widehat{\mathcal{Q}}_{m,\kappa(\cdot),K(\cdot)}(u - \widehat{\Pi}_m v) d\Gamma \\ &\quad + m^{-1} \int_{\widehat{\mathcal{V}}_m} \langle \nabla_s u, \widehat{g}_m^{-1} \nabla_s u \rangle \widehat{a}_m d\Gamma d\tau. \end{aligned}$$

Thus,

$$\begin{aligned} \widehat{\mathcal{Q}}_m(u) &\geq m \int_{\Gamma} |v|^2 \Lambda_{m,\kappa(\cdot),K(\cdot)} d\Gamma + m \left(1 - Cm^{-\frac{1}{2}}\right) \|u - \widehat{\Pi}_m v\|_{L^2(\widehat{\mathcal{V}}_m, d\Gamma d\tau)}^2 \\ &\quad + m^{-1} \left(1 - Cm^{-\frac{1}{2}}\right) \int_{\widehat{\mathcal{V}}_m} |\nabla_s u|^2 d\Gamma d\tau. \end{aligned}$$

We have

$$(2.19) \quad \begin{aligned} \int_{\widehat{\mathcal{V}}_m} |\nabla_s u|^2 d\Gamma d\tau &= \int_{\widehat{\mathcal{V}}_m} |\nabla_s \widehat{\Pi}_m v|^2 d\Gamma d\tau + \int_{\widehat{\mathcal{V}}_m} \left| \nabla_s (u - \widehat{\Pi}_m v) \right|^2 d\Gamma d\tau \\ &\quad + 2\operatorname{Re} \int_{\widehat{\mathcal{V}}_m} \left\langle \nabla_s \widehat{\Pi}_m v, \nabla_s (u - \widehat{\Pi}_m v) \right\rangle d\Gamma d\tau. \end{aligned}$$

By Lemmas 2.16 and 2.14,

$$\left| 2\operatorname{Re} \int_{\widehat{\mathcal{V}}_m} \left\langle \nabla_s \widehat{\Pi}_m v, \nabla_s (u - \widehat{\Pi}_m v) \right\rangle d\Gamma d\tau \right| \leq C \|v\|_{H^{3/2}(\Gamma)} \|u - \widehat{\Pi}_m v\|_{H^{1/2}(\widehat{\mathcal{V}}_m, d\Gamma d\tau)}.$$

Then, with Lemma 2.15, we get, for all $\varepsilon_0 > 0$,

$$(2.20) \quad \begin{aligned} &\left| 2\operatorname{Re} \int_{\widehat{\mathcal{V}}_m} \left\langle \nabla_s \widehat{\Pi}_m v, \nabla_s (u - \widehat{\Pi}_m v) \right\rangle d\Gamma d\tau \right| \\ &\leq Cm^{-1} \varepsilon_0^{-1} \|v\|_{H^{3/2}(\Gamma)}^2 + m^2 \varepsilon_0 (1 + m^{-2}) \|u - \widehat{\Pi}_m v\|_{L^2(\widehat{\mathcal{V}}_m, d\Gamma d\tau)}^2 + \varepsilon_0 \left\| \nabla_s (u - \widehat{\Pi}_m v) \right\|_{L^2(\widehat{\mathcal{V}}_m, d\Gamma d\tau)}^2. \end{aligned}$$

Using (2.19) and (2.20), we get that

$$\begin{aligned} \widehat{\mathcal{Q}}_m(u) &\geq m \int_{\Gamma} |v|^2 \, d\Gamma + \int_{\Gamma} \frac{\kappa}{2} |v|^2 \, d\Gamma + m^{-1} \int_{\Gamma} \left(\frac{|\nabla_s v|^2}{2} + \left(\frac{K}{2} - \frac{\kappa^2}{8} \right) |v|^2 \right) \, d\Gamma \\ &- \frac{C}{m^{3/2}} (\varepsilon_0^{-1} + 1) \|v\|_{H^{3/2}(\Gamma)}^2 + m \left(1 - \varepsilon_0 - \frac{C}{m^{1/2}} \right) \|u - \widehat{\Pi}_m v\|_{L^2(\widehat{\mathcal{V}}_m, d\Gamma \, d\tau)}^2 \\ &+ m^{-1} \left(1 - \varepsilon_0 - \frac{C}{m^{1/2}} \right) \|\nabla_s (u - \widehat{\Pi}_m v)\|_{L^2(\widehat{\mathcal{V}}_m, d\Gamma \, d\tau)}^2. \end{aligned}$$

Taking $\varepsilon_0 = 3/4$ and m large enough, we get the result. \square

2.6. End of the proof of Proposition 2.1. Item (ii) of Proposition 2.1 follows from Propositions 2.19 and 2.6. It remains to prove (iii). Consider the minimizer u_m and a cut off function χ_m supported in a neighborhood of size $m^{-\frac{1}{2}}$ near the boundary. Then, we let

$$\check{u}_m(s, \tau) = (\chi_m u_m) \circ \Phi(s, m^{-1}\tau).$$

Let us use the lower bound in Proposition 2.19:

$$\widehat{\mathcal{Q}}_m(\check{u}_m) \geq \widetilde{\Lambda}_m(v) + \frac{m}{2} \|\check{u}_m - \widehat{\Pi}_m v\|_{L^2(\widehat{\mathcal{V}}_m, d\Gamma \, d\tau)}^2 - \frac{C}{m^{3/2}} \|v\|_{H^{3/2}(\Gamma)}^2.$$

As in the proof of Lemma 2.5 and recalling Item (ii) in Lemma 2.4, we get

$$\widehat{\mathcal{Q}}_m(\check{u}_m) = \mathcal{Q}_{m, m^{-\frac{1}{2}}}(\chi_m u_m) = \widetilde{\Lambda}_m(v) + \|(\nabla \chi_m) u_m\|^2 = (1 + \mathcal{O}(e^{-cm^{\frac{1}{2}}})) \widetilde{\Lambda}_m(v),$$

where we used (2.3).

We deduce that

$$\|\check{u}_m - \widehat{\Pi}_m v\|_{L^2(\widehat{\mathcal{V}}_m, d\Gamma \, d\tau)}^2 \leq \frac{C}{m^{5/2}} \|v\|_{H^{3/2}(\Gamma)}^2.$$

Thus

$$\left| \|\check{u}_m\|_{L^2(\widehat{\mathcal{V}}_m, d\Gamma \, d\tau)} - \|\widehat{\Pi}_m v\|_{L^2(\widehat{\mathcal{V}}_m, d\Gamma \, d\tau)} \right| \leq \frac{C}{m^{5/4}} \|v\|_{H^{3/2}(\Gamma)}.$$

Using Proposition 2.12, we get that

$$\left| \|\widehat{\Pi}_m v\|_{L^2(\widehat{\mathcal{V}}_m, d\Gamma \, d\tau)}^2 - \frac{\|v\|_{L^2(\Gamma)}^2}{2} \right| \leq C m^{-1} \|v\|_{L^2(\Gamma)}^2.$$

Therefore

$$\left| m \|\chi_m u_m\|_{L^2(\mathcal{V}_m, dx)}^2 - \frac{\|v\|_{L^2(\Gamma)}^2}{2} \right| \leq C m^{-1} \|v\|_{H^{3/2}(\Gamma)}^2.$$

We remove χ_m by using (2.3) and Item (iii) follows.

3. A VECTORIAL LAPLACIAN WITH ROBIN-TYPE BOUNDARY CONDITIONS

In this section, we study the vectorial Laplacian L_m^{int} associated with the quadratic form $\mathcal{Q}_m^{\text{int}}$ defined in section 1.3.3.

3.1. Preliminaries : proof of Lemma 1.10. We recall that the domain of L_m^{int} is the set of the $u \in H^1(\Omega; \mathbb{C}^4)$ such that the linear application

$$H^1(\Omega; \mathbb{C}^4) \ni v \mapsto \mathcal{Q}_m^{\text{int}}(v, u) \in \mathbb{C}$$

is continuous for the L^2 -norm. By using the Green-Riemann formula, we get that the domain is given by

$$\{u \in H^1(\Omega; \mathbb{C}^4) : -\Delta u \in L^2(\Omega; \mathbb{C}^4), \quad (\partial_{\mathbf{n}} + \kappa/2 + m_0 + 2m\Xi^-)u = 0 \text{ on } \Gamma\}.$$

By a classical regularity theorem, we deduce that the domain is included in $H^2(\Omega; \mathbb{C}^4)$. The compactness of the resolvent and the discreteness of the spectrum immediately follow.

3.2. Asymptotics of the eigenvalues. In this section, we describe the first terms in the asymptotic expansion of the eigenvalues of L_m^{int} . This is the aim of the following proposition.

Proposition 3.1. *The following properties hold.*

(i) For any $k \in \mathbb{N}^*$, $\lim_{m \rightarrow +\infty} \lambda_{k,m}^{\text{int}} = \lambda_k^2$.

Let λ be an eigenvalue of $|H^\Omega|$ of multiplicity $k_1 \in \mathbb{N}^*$. Consider $k_0 \in \mathbb{N}$ such that for all $k \in \{1, \dots, k_1\}$, $\lambda_{k_0+k} = \lambda$.

(ii) For all $k \in \{1, 2, \dots, k_1\}$, we have

$$\lambda_{k_0+k,m}^{\text{int}} = \lambda^2 + \frac{\mu_{\lambda,k}}{m} + o\left(\frac{1}{m}\right)$$

where

$$(3.1) \quad \mu_{\lambda,k} := \inf_{\substack{V \subset \ker(|H^\Omega| - \lambda), \\ \dim V = k}} \sup_{\substack{v \in V, \\ \|v\|_{L^2(\Omega)} = 1}} \frac{\|(\partial_{\mathbf{n}} + \kappa/2 + m_0)v\|_{L^2(\Gamma)}^2}{2}.$$

(iii) Let $(u_{k_0+1}, \dots, u_{k_0+k_1})$ be a H^1 -weak limit of a sequence

$$(u_{k_0+1,m}, \dots, u_{k_0+k_1,m})_{m>0}$$

of L^2 -orthonormal eigenvectors of L_m^{int} associated with the eigenvalues

$$(\lambda_{k_0+1,m}^{\text{int}}, \dots, \lambda_{k_0+k_1,m}^{\text{int}}).$$

Then, we have for all $v \in \ker(|H^\Omega| - \lambda)$ that,

$$-\frac{1}{2} \|(\partial_{\mathbf{n}} + \kappa/2 + m_0)v\|_{L^2(\Gamma)}^2 = \sum_{k=1}^{k_1} |\langle v, u_{k_0+k} \rangle_\Omega|^2 \mu_{\lambda,k}.$$

Here, $(\lambda_k)_{k \in \mathbb{N}^*}$ is defined in Notation 1.5 and $(\lambda_{k,m}^{\text{int}})_{k \in \mathbb{N}^*}$ in Notation 1.11.

3.3. Proof of Proposition 3.1. Since $\text{Dom}(H^\Omega) \subset \text{Dom}(\mathcal{Q}_m^{\text{int}})$, we have

$$(3.2) \quad \lambda_k^2 \geq \lambda_{k,m}^{\text{int}}$$

for all $k \in \mathbb{N}^*$ and all $m > 0$.

3.3.1. Lower bounds.

Lemma 3.2. *Let $k \in \mathbb{N}$. The following properties hold:*

- (i) *For all $j \in \{1, 2, \dots, k\}$, $\lim_{m \rightarrow +\infty} \lambda_{j,m}^{\text{int}} = \lambda_j^2$.*
- (ii) *For all subsequence $(m_n)_{n \in \mathbb{N}^*}$ going to $+\infty$ as $n \rightarrow +\infty$, all L^2 -orthonormal family of eigenvectors $(u_{1,m_n}, \dots, u_{k,m_n})$ of $L_{m_n}^{\text{int}}$ associated with $(\lambda_{1,m_n}^{\text{int}}, \dots, \lambda_{k,m_n}^{\text{int}})$ such that the sequence $(u_{1,m_n}, \dots, u_{k,m_n})_{n \in \mathbb{N}^*}$ converges weakly in H^1 , then the sequence $(u_{1,m_n}, \dots, u_{k,m_n})_{n \in \mathbb{N}^*}$ converges strongly in H^1 and*

$$(3.3) \quad \lim_{n \rightarrow +\infty} m_n \|\Xi^- u_{j,m_n}\|_{L^2(\Gamma)}^2 = 0$$

for all $j \in \{1, \dots, k\}$.

Proof. Let us prove (i) and (ii) by induction on $k \in \mathbb{N}^*$.

Case $k = 0$. There is nothing to prove.

Case $k > 0$. Assume that (i) and (ii) are valid for some $k \in \mathbb{N}$.

Let $(u_{1,m}, \dots, u_{k+1,m})$ be an L^2 -orthonormal family of eigenvectors of L_m^{int} associated with $(\lambda_{1,m}^{\text{int}}, \dots, \lambda_{k+1,m}^{\text{int}})$. By (3.2) and the trace Theorem [7, Section 5.5], the sequence $(u_{1,m}, \dots, u_{k+1,m})_{m>0}$ is bounded in $H^1(\Omega; \mathbb{C}^4)^{k+1}$, and

$$(3.4) \quad \lambda_{k+1}^2 \geq \limsup_{m \rightarrow +\infty} \lambda_{k+1,m}^{\text{int}} \geq \liminf_{m \rightarrow +\infty} \lambda_{k+1,m}^{\text{int}}.$$

Hence there exists a subsequence $(m_n)_{n \in \mathbb{N}^*}$ going to $+\infty$ as $n \rightarrow +\infty$ such that

$$\lim_{n \rightarrow +\infty} \lambda_{k+1,m_n}^{\text{int}} = \liminf_{m \rightarrow +\infty} \lambda_{k+1,m}^{\text{int}}$$

and $(u_{1,m_n}, \dots, u_{k+1,m_n})_{n \in \mathbb{N}^*}$ converges weakly in $H^1(\Omega; \mathbb{C}^4)$ to (u_1, \dots, u_{k+1}) .

Using the induction assumption, we get that $(u_{1,m_n}, \dots, u_{k,m_n})_{n \in \mathbb{N}^*}$ converges strongly in $H^1(\Omega; \mathbb{C}^4)$ to (u_1, \dots, u_k) , $\lim_{m \rightarrow +\infty} \lambda_{j,m}^{\text{int}} = \lambda_j^2$ and

$$\lim_{n \rightarrow +\infty} m_n \|\Xi^- u_{j,m_n}\|_{L^2(\Gamma)}^2 = 0$$

for all $j \in \{1, \dots, k\}$. By Rellich-Kondrachov Theorem [7, Section 5.7], the sequence (u_{k+1,m_n}) converges strongly in $L^2(\Omega; \mathbb{C}^4)$. This implies that (u_1, \dots, u_{k+1}) is an L^2 -orthonormal family. In addition, for all $j_1, j_2 \in \{1, \dots, k+1\}$, $j_1 \neq j_2$, and all $n \in \mathbb{N}^*$,

$$\begin{aligned} 0 &= \text{Re} \langle \nabla u_{j_1, m_n}, \nabla u_{j_2, m_n} \rangle_{\Omega} + m_0^2 \text{Re} \langle u_{j_1, m_n}, u_{j_2, m_n} \rangle_{\Omega} \\ &\quad + \text{Re} \langle (\kappa/2 + m_0) u_{j_1, m_n}, u_{j_2, m_n} \rangle_{\Gamma} + 2m_n \text{Re} \langle \Xi^- u_{j_1, m_n}, \Xi^- u_{j_2, m_n} \rangle_{\Gamma} \end{aligned}$$

and taking the limit $n \rightarrow +\infty$,

$$0 = \text{Re} \langle \nabla u_{j_1}, \nabla u_{j_2} \rangle_{\Omega} + m_0^2 \text{Re} \langle u_{j_1}, u_{j_2} \rangle_{\Omega} + \text{Re} \langle (\kappa/2 + m_0) u_{j_1}, u_{j_2} \rangle_{\Gamma}.$$

Since

$$\lim_{n \rightarrow +\infty} \mathcal{Q}_{m_n}^{\text{int}}(u_{j,m_n}) = \lambda_j^2 = \mathcal{Q}^{\text{int}}(u_j)$$

for all $j \in \{1, \dots, k\}$, where \mathcal{Q}^{int} is defined in (1.3), we deduce that the $(u_j)_{1 \leq j \leq k}$ are normalized eigenfunctions associated with $(\lambda_j^2)_{1 \leq j \leq k}$. By the min-max theorem, we get

$$\liminf_{n \rightarrow +\infty} \mathcal{Q}_{m_n}^{\text{int}}(u_{k+1, m_n}) \geq \mathcal{Q}^{\text{int}}(u_{k+1}) \geq \lambda_{k+1}^2.$$

We deduce that

$$\lim_{m \rightarrow +\infty} \lambda_{k+1, m}^{\text{int}} = \lambda_{k+1}^2.$$

We also get that

$$\lim_{n \rightarrow +\infty} \|\nabla u_{k+1, m_n}\|_{L^2(\Omega)} = \|\nabla u_{k+1}\|_{L^2(\Omega)}$$

and the strong convergence follows. Note that $\lim_{m \rightarrow +\infty} \lambda_{k+1, m}^{\text{int}} = \lambda_{k+1}^2$ implies that the previous arguments are valid for any weakly converging subsequence and Items (i) and (ii) follow for $k+1$. \square

3.3.2. A technical lemma. The following lemma is essential in the proof of Items (ii) and (iii).

Lemma 3.3. *Let $k \in \mathbb{N}^*$ and $m > 0$. Let u resp. $u_{k, m}$ be a L^2 -normalized eigenfunction of $|H^\Omega|$ resp. L_m^{int} associated with the eigenvalues λ resp. $\lambda_{k, m}^{\text{int}}$. Then*

$$(3.5) \quad m(\lambda_{k, m}^{\text{int}} - \lambda^2) \langle u_{k, m}, u \rangle_\Omega = -1/2 \langle (\partial_{\mathbf{n}} + \kappa/2 + m_0)u_{k, m}, (\partial_{\mathbf{n}} + \kappa/2 + m_0)u \rangle_\Gamma .$$

Proof. Since

$$\begin{aligned} \Xi^+(\partial_{\mathbf{n}} + \kappa/2 + m_0)u &= 0, & \Xi^-u &= 0, \\ \Xi^+(\partial_{\mathbf{n}} + \kappa/2 + m_0)u_{k, m} &= 0, & \Xi^-(\partial_{\mathbf{n}} + \kappa/2 + m_0 + 2m)u_{k, m} &= 0 \end{aligned} \quad \text{on } \Gamma ,$$

an integration by parts gives

$$\begin{aligned} (\lambda_{k, m}^{\text{int}} - \lambda^2) \langle u_{k, m}, u \rangle_\Omega &= \langle (-\Delta + m_0^2)u_{k, m}, u \rangle_\Omega - \langle u_{k, m}, (-\Delta + m_0^2)u \rangle_\Omega \\ &= -\langle \partial_{\mathbf{n}}u_{k, m}, u \rangle_\Gamma + \langle u_{k, m}, \partial_{\mathbf{n}}u \rangle_\Gamma \\ &= -\langle (\partial_{\mathbf{n}} + \kappa/2 + m_0)u_{k, m}, u \rangle_\Gamma + \langle u_{k, m}, (\partial_{\mathbf{n}} + \kappa/2 + m_0)u \rangle_\Gamma \\ &= \langle \Xi^-u_{k, m}, \Xi^-(\partial_{\mathbf{n}} + \kappa/2 + m_0)u \rangle_\Gamma \\ &= -1/2m \langle \Xi^-(\partial_{\mathbf{n}} + \kappa/2 + m_0)u_{k, m}, \Xi^-(\partial_{\mathbf{n}} + \kappa/2 + m_0)u \rangle_\Gamma . \end{aligned}$$

\square

3.3.3. Proof of Items (ii) and (iii). Let also $(u_{1, m_n}, \dots, u_{k_0+k_1, m_n})_{n \in \mathbb{N}^*}$ be a sequence of L^2 -orthonormal eigenvectors of $L_{m_n}^{\text{int}}$ that converges strongly in $H^1(\Omega; \mathbb{C}^4)^{k_0+k_1}$ to an L^2 -orthonormal family $(u_1, \dots, u_{k_0+k_1})$ of eigenvectors of $|H^\Omega|$. We have

$$\text{span}(u_{k_0+1}, \dots, u_{k_0+k_1}) = \ker(|H^\Omega| - \lambda).$$

By (3.5), we have for all $v = \sum_{k=1}^{k_1} a_k u_{k_0+k}$,

$$\begin{aligned} -1/2 \|\partial_{\mathbf{n}} + \kappa/2 + m_0 v\|_{L^2(\Gamma)}^2 &= \sum_{k, j=1}^{k_1} \bar{a}_k a_j \langle (\partial_{\mathbf{n}} + \kappa/2 + m_0)u_{k_0+k}, (\partial_{\mathbf{n}} + \kappa/2 + m_0)u_{k_0+j} \rangle_\Gamma \\ &= \lim_{n \rightarrow +\infty} \sum_{k, j=1}^{k_1} \bar{a}_k a_j \langle (\partial_{\mathbf{n}} + \kappa/2 + m_0)u_{k_0+k, m_n}, (\partial_{\mathbf{n}} + \kappa/2 + m_0)u_{k_0+j} \rangle_\Gamma \\ &= \lim_{n \rightarrow +\infty} \sum_{k, j=1}^{k_1} \bar{a}_k a_j m_n (\lambda_{k_0+k, m_n}^{\text{int}} - \lambda^2) \langle u_{k_0+k, m_n}, u_{k_0+j} \rangle_\Omega \\ &= \lim_{n \rightarrow +\infty} \sum_{k, j=1}^{k_1} \bar{a}_k a_j m_n (\lambda_{k_0+k, m_n}^{\text{int}} - \lambda^2) \langle u_{k_0+k}, u_{k_0+j} \rangle_\Omega \\ &= \lim_{n \rightarrow +\infty} \sum_{k=1}^{k_1} |a_k|^2 m_n (\lambda_{k_0+k, m_n}^{\text{int}} - \lambda^2). \end{aligned}$$

We deduce that for all $k \in \{1, \dots, k_1\}$,

$$\begin{aligned} \lim_{n \rightarrow +\infty} m_n(\lambda_{k_0+k, m_n}^{\text{int}} - \lambda^2) &= -\frac{\|(\partial_{\mathbf{n}} + \kappa/2 + m_0)u_{k_0+k}\|_{L^2(\Gamma)}^2}{2} \\ &= \inf_{\substack{V \subset \ker(|H^\Omega| - \lambda), \\ \dim V = k,}} \sup_{\substack{v \in V, \\ \|v\|_{L^2(\Omega)} = 1,}} -\frac{\|(\partial_{\mathbf{n}} + \kappa/2 + m_0)v\|_{L^2(\Gamma)}^2}{2} = \mu_{\lambda, k}, \end{aligned}$$

so that

$$\lim_{m \rightarrow +\infty} m(\lambda_{k_0+k, m}^{\text{int}} - \lambda^2) = \mu_{\lambda, k}.$$

The conclusion follows.

4. PROOF OF THE MAIN THEOREM

4.1. First term in the asymptotic. In this part, we work in the energy space without using any regularity result as Lemma 4.2.

4.1.1. Upper bound. Let $K \in \mathbb{N}^*$ and $(\varphi_1, \dots, \varphi_K)$ be an L^2 -orthonormal family of eigenvectors of $|H^\Omega|$ associated with the eigenvalues $(\lambda_1, \dots, \lambda_K)$. Using Proposition 2.1, we extend these functions outside Ω by

$$\tilde{u}_{j,m} = \begin{cases} u_j & \text{on } \Omega, \\ u_{m+m_0}(u_j) & \text{on } \Omega', \end{cases}$$

for $j \in \{1, \dots, K\}$. By Proposition 2.1, we get that

$$\|\tilde{u}_{j,m}\|_{L^2(\Omega')}^2 \leq (m + m_0)^{-2} \Lambda_{m+m_0}(u_j) \leq \frac{C}{m + m_0}.$$

so that $\tilde{u}_{1,m}, \dots, \tilde{u}_{K,m}$ are linearly independent vectors. Let $a_1, \dots, a_K \in \mathbb{C}$. Let us denote $\varphi_m^a := \sum_{j=1}^K a_j \tilde{u}_{j,m}$. By Lemma 1.6 and Proposition 2.1, we have

$$\begin{aligned} \|H_m \varphi_m^a\|_{L^2(\mathbb{R}^3)}^2 &= \|\nabla \varphi_m^a\|_{L^2(\Omega)}^2 + m_0^2 \|\varphi_m^a\|_{L^2(\Omega)}^2 - m \text{Re} \langle \mathcal{B}\psi, \psi \rangle_\Gamma + \Lambda_{m+m_0}(\varphi_m^a) \\ &\leq \mathcal{Q}^{\text{int}} \left(\sum_{j=1}^K a_j u_j \right) + o(1) = \sum_{j=1}^K |a_j|^2 \lambda_j^2 + o(1) \leq \lambda_K^2 \sum_{j=1}^K |a_j|^2 + o(1). \end{aligned}$$

We deduce that

$$(4.1) \quad \limsup_{m \rightarrow +\infty} \lambda_{K,m}^2 \leq \limsup_{m \rightarrow +\infty} \sup_{\substack{\varphi_m^a \in \text{span}(\tilde{u}_{1,m}, \dots, \tilde{u}_{K,m}) \\ \|\varphi_m^a\|_{L^2(\mathbb{R}^3)} = 1}} \|H_m \varphi_m^a\|_{L^2(\mathbb{R}^3)}^2 \leq \lambda_K^2.$$

4.1.2. Lower bound and convergence. Let $K \in \mathbb{N}^*$ and $(\varphi_{1,m}, \dots, \varphi_{K,m})$ be an L^2 -orthonormal family of eigenvectors of $|H_m|$ associated with the eigenvalues $(\lambda_{1,m}, \dots, \lambda_{K,m})$ for all $m \geq m_1$. By (4.1), there exists $C > 0$ such that

$$(4.2) \quad C \geq \sup_{\substack{k \in \{1, \dots, K\}, \\ m \geq m_1,}} \|H_m \varphi_{k,m}\|_{L^2(\mathbb{R}^3)}^2.$$

With (1.4) and Proposition 2.1, we get, for all $k \in \{1, \dots, K\}$ and all $m \geq m_1$, that

$$\begin{aligned}
(4.3) \quad \lambda_{k,m}^2 &= \|H_m \varphi_{k,m}\|_{L^2(\mathbb{R}^3)}^2 \\
&= \|\nabla \varphi_{k,m}\|_{L^2(\Omega)}^2 + m_0^2 \|\varphi_{k,m}\|_{L^2(\Omega)}^2 - m \langle \mathcal{B} \varphi_{k,m}, \varphi_{k,m} \rangle_\Gamma \\
&\quad + \Lambda_{m+m_0}(\varphi_{k,m}) + \mathcal{Q}_{m+m_0}(\varphi_{k,m} - u_{m+m_0}(\varphi_{k,m})) \\
&\geq \mathcal{Q}_m^{\text{int}}(\varphi_{k,m}) + (m + m_0)^2 \|\varphi_{k,m} - u_{m+m_0}(\varphi_{k,m})\|_{L^2(\Omega')}^2 - \frac{C}{m} \|\varphi_{k,m}\|_{L^2(\Gamma)}^2.
\end{aligned}$$

By the trace theorem, we deduce that there exists $C > 0$ such that

$$(4.4) \quad C \geq \sup_{\substack{k \in \{1, \dots, K\}, \\ m \geq m_1,}} \|\varphi_{k,m}\|_{H^1(\Omega)}.$$

Note also that by (4.3), (4.4) and the trace theorem, we get that

$$(4.5) \quad \left| \|\varphi_{k,m}\|_{L^2(\Omega')} - \|u_{m+m_0}(\varphi_{k,m})\|_{L^2(\Omega')} \right| \leq \|\varphi_{k,m} - u_{m+m_0}(\varphi_{k,m})\|_{L^2(\Omega')} \leq C/m.$$

Moreover, by Proposition 2.1, we obtain that

$$\|u_{m+m_0}(\varphi_{k,m})\|_{L^2(\Omega')}^2 \leq (m + m_0)^{-2} \Lambda_{m+m_0}(\varphi_{k,m}) \leq C(m + m_0)^{-1} \|\varphi_{k,m}\|_{H^1(\Omega)}^2,$$

and we deduce that

$$(4.6) \quad \|\varphi_{k,m}\|_{L^2(\Omega')} \leq C m^{-1}.$$

Combining (4.3), (4.4), (4.6), Proposition 3.1 with an induction procedure as in the proof of Lemma 3.2, we get the following result.

Lemma 4.1. *Let $K \in \mathbb{N}$. The following properties hold.*

- (i) For all $j \in \{1, 2, \dots, K\}$, $\lim_{m \rightarrow +\infty} \lambda_{j,m} = \lambda_j$.
- (ii) For all subsequence $(m_n)_{n \in \mathbb{N}}$ going to $+\infty$ as $n \rightarrow +\infty$, all L^2 -orthonormal family of eigenvectors $(\varphi_{1,m_n}, \dots, \varphi_{K,m_n})$ of $|H_m|$ associated with $(\lambda_{1,m_n}, \dots, \lambda_{K,m_n})$ such that the sequence $(\varphi_{1,m_n}, \dots, \varphi_{K,m_n})_{n \in \mathbb{N}}$ converges weakly in $H^1(\Omega)$, then the sequence $(\varphi_{1,m_n}, \dots, \varphi_{K,m_n})_{n \in \mathbb{N}}$ converges strongly in $H^1(\Omega)$ and

$$(4.7) \quad \lim_{n \rightarrow +\infty} m_n \|\Xi^- \varphi_{j,m_n}\|_{L^2(\Gamma)}^2 = 0$$

for all $j \in \{1, \dots, K\}$.

- (iii) Any weak limit $(\varphi_1, \dots, \varphi_K)$ of such a sequence is an L^2 -orthonormal family of eigenvectors of $|H^\Omega|$ associated with the eigenvalues $(\lambda_1, \dots, \lambda_K)$.

4.2. Second term in the asymptotic. In this section, we will freely use the following regularity result.

Lemma 4.2. *There exists a constant $C > 0$ such that for any $m \in \mathbb{R}$, any eigenfunction u of H_m associated with an eigenvalue $\lambda \in \mathbb{R}$, we have*

$$\|u\|_{H^2(\Omega)} \leq C(1 + |\lambda|) \|u\|_{L^2(\mathbb{R}^3)}.$$

We also have, for any eigenfunction u resp. v of H^Ω resp. L_m^{int} associated with an eigenvalue $\lambda \in \mathbb{R}$, resp. $\lambda^2 \in \mathbb{R}$ that

$$\|u\|_{H^2(\Omega)} \leq C(1 + |\lambda|) \|u\|_{L^2(\Omega)}$$

and

$$\|v\|_{H^2(\Omega)} \leq C(1 + |\lambda|) \|v\|_{L^2(\Omega)}$$

4.2.1. *Upper bound.* In this section, we prove the following lemma.

Lemma 4.3. *Let λ be an eigenvalue of $|H^\Omega|$ of multiplicity $k_1 \in \mathbb{N}^*$. Let $k_0 \in \mathbb{N}$ be the unique integer such that*

$$\lambda = \lambda_{k_0+1} = \dots = \lambda_{k_0+k_1}.$$

We have

$$(4.8) \quad \limsup_{m \rightarrow +\infty} m(\lambda_{k_0+k,m}^2 - \lambda^2) \leq \tilde{\nu}_{\lambda,k}.$$

where

$$(4.9) \quad \tilde{\nu}_{\lambda,k} := \inf_{\substack{V \subset \ker(|H^\Omega| - \lambda \text{Id}), \\ \dim V = k,}} \sup_{\substack{v \in V, \\ \|v\|_{L^2(\Omega)} = 1,}} \tilde{\eta}_\lambda(v),$$

and

$$\tilde{\eta}_\lambda(v) := \int_\Gamma \left(\frac{|\nabla_s v|^2}{2} - \frac{|(\partial_{\mathbf{n}} + \kappa/2 + m_0)v|^2}{2} + \left(\frac{K}{2} - \frac{\kappa^2}{8} - \frac{\lambda^2}{2} \right) |v|^2 \right) d\Gamma,$$

for $k \in \{1, \dots, k_1\}$.

Proof. Let $(u_{1,m}, \dots, u_{k_0+k_1,m})$ be an L^2 -orthonormal family of eigenvectors of L_m^{int} associated with the eigenvalues $(\lambda_{1,m}^{\text{int}}, \dots, \lambda_{k_0+k_1,m}^{\text{int}})$. Let $(m_n)_{n \in \mathbb{N}}$ be a subsequence which goes to $+\infty$ as n tends to $+\infty$ and which satisfies

- (i) $\limsup_{m \rightarrow +\infty} m(\lambda_{k_0+k,m}^2 - \lambda^2) = \lim_{n \rightarrow +\infty} m_n(\lambda_{k_0+k,m_n}^2 - \lambda^2)$,
- (ii) $(u_{1,m_n}, \dots, u_{k_0+k_1,m_n})$ converges in $L^2(\Omega)$ to $(u_1, \dots, u_{k_0+k_1})$,

where $(u_1, \dots, u_{k_0+k_1})$ is an L^2 -orthonormal family of eigenvectors of H^Ω associated with the eigenvalues $(\lambda_1, \dots, \lambda_{k_0+k_1})$. By Lemma 4.2, this sequence is uniformly bounded in $H^2(\Omega)$. By interpolation, the convergence also holds in $H^s(\Omega)$ for all $s \in [0, 2)$.

Since (4.9) is a finite dimensional spectral problem, there exists an L^2 -orthonormal basis $(w_{k_0+1}, \dots, w_{k_0+k_1})$ of $\ker(|H^\Omega| - \lambda \text{Id})$ such that

$$\tilde{\eta}_\lambda \left(\sum_{s=k_0+1}^{k_0+k_1} a_s w_s \right) = \sum_{s=k_0+1}^{k_0+k_1} |a_s|^2 \tilde{\eta}_\lambda(w_s) = \sum_{s=k_0+1}^{k_0+k_1} |a_s|^2 \tilde{\nu}_{\lambda,s-k_0},$$

for all $a_{k_0+1}, \dots, a_{k_0+k_1} \in \mathbb{C}$. Moreover, we have

$$\ker(|H^\Omega| - \lambda \text{Id}) = \text{span}(u_{k_0+1}, \dots, u_{k_0+k_1}) = \text{span}(w_{k_0+1}, \dots, w_{k_0+k_1}),$$

so that there exists a unitary matrix $B \in \mathbb{C}^{k_1 \times k_1}$ such that $Bu = w$ where $u = (u_{k_0+1}, \dots, u_{k_0+k_1})^T$ and $w = (w_{k_0+1}, \dots, w_{k_0+k_1})^T$. Using Proposition 2.1, we extend these functions outside Ω by

$$\tilde{u}_{j,m} = \begin{cases} u_{j,m} & \text{on } \Omega, \\ u_{m+m_0}(u_{j,m}) & \text{on } \Omega', \end{cases}$$

for $j \in \{1, \dots, k_0 + k_1\}$. We also define

$$\begin{aligned} u_m &:= (u_{k_0+1,m}, \dots, u_{k_0+k_1,m})^T \\ w_m &= (w_{k_0+1,m}, \dots, w_{k_0+k_1,m})^T := Bu_m \\ \tilde{w}_m &= (\tilde{w}_{k_0+1,m}, \dots, \tilde{w}_{k_0+k_1,m})^T := B(\tilde{u}_{k_0+1,m}, \dots, \tilde{u}_{k_0+k_1,m})^T, \end{aligned}$$

and

$$V_{k_0+k,m} = \text{span}(u_{1,m}, \dots, u_{k_0,m}, w_{k_0+1,m}, \dots, w_{k_0+k}),$$

$$\tilde{V}_{k_0+k,m} = \text{span}(\tilde{u}_{1,m}, \dots, \tilde{u}_{k_0,m}, \tilde{w}_{k_0+1,m}, \dots, \tilde{w}_{k_0+k}),$$

for all $k \in \{k_0 + 1, \dots, k_0 + k_1\}$ and all $m \geq m_1$. Let us remark that

$$\dim V_{k_0+k,m} = \dim \tilde{V}_{k_0+k,m} = k_0 + k$$

for all $k \in \{1, \dots, k_1\}$ (choosing if necessary a larger constant $m_1 > 0$). In the following, we consider test functions of the form

$$v_m = \sum_{j=1}^{k_0} a_j \tilde{u}_{j,m} + \sum_{j=k_0+1}^{k_0+k_1} a_j \tilde{w}_{j,m},$$

where $a_1, \dots, a_{k_0+k_1} \in \mathbb{C}$ satisfies $\sum_{j=1}^{k_0+k_1} |a_j|^2 = 1$ so that

$$\|v_m\|_{L^2(\Omega)}^2 = \sum_{j=1}^{k_0+k_1} |a_j|^2 = 1.$$

By Proposition 2.1, we have

$$(4.10) \quad \|v_m\|_{L^2(\mathbb{R}^3)}^2 = \|v_m\|_{L^2(\Omega)}^2 + \|v_m\|_{L^2(\Omega')}^2 = 1 + \frac{\|v_m\|_{L^2(\Gamma)}^2}{2m} + \mathcal{O}(m^{-2}),$$

and

$$(4.11) \quad \|H_m v_m\|_{L^2(\mathbb{R}^3)}^2 = \mathcal{Q}_m^{\text{int}}(v_m) + m^{-1} \int_{\Gamma} \left(\frac{|\nabla_s v_m|^2}{2} + \left(\frac{K}{2} - \frac{\kappa^2}{8} \right) |v_m|^2 \right) d\Gamma + \mathcal{O}(m^{-3/2}).$$

From (4.10) and (4.11), we deduce that

$$\begin{aligned} m \left(\frac{\|H_m v_m\|_{L^2(\mathbb{R}^3)}^2}{\|v_m\|_{L^2(\mathbb{R}^3)}^2} - \lambda^2 \right) &\leq m \left(\mathcal{Q}_m^{\text{int}}(v_m) - \lambda^2 \right) \\ &\quad + \int_{\Gamma} \left(\frac{|\nabla_s v_m|^2}{2} + \left(\frac{K}{2} - \frac{\kappa^2}{8} - \frac{\mathcal{Q}_m^{\text{int}}(v_m)}{2} \right) |v_m|^2 \right) d\Gamma + \mathcal{O}(m^{-1/2}). \end{aligned}$$

For $k \in \{1, \dots, k_1\}$, we get

$$(4.12) \quad \begin{aligned} &m \left(\lambda_{k_0+k,m}^2 - \lambda^2 \right) \\ &\leq \sup_{v_m \in \tilde{V}_{k_0+k,m} \setminus \{0\}} m \left(\frac{\|H_m v_m\|_{L^2(\mathbb{R}^3)}^2}{\|v_m\|_{L^2(\mathbb{R}^3)}^2} - \lambda^2 \right) \\ &\leq \sup_{\substack{v_m \in V_{k_0+k,m}, \\ \|v_m\|_{L^2(\Omega)} = 1}} m \left(\mathcal{Q}_m^{\text{int}}(v_m) - \lambda^2 \right) + \bar{\eta}_m(v_m) + \mathcal{O}(m^{-1/2}), \end{aligned}$$

where

$$\bar{\eta}_m(v) := \int_{\Gamma} \left(\frac{|\nabla_s v|^2}{2} + \left(\frac{K}{2} - \frac{\kappa^2}{8} - \frac{\mathcal{Q}_m^{\text{int}}(v)}{2} \right) |v|^2 \right) d\Gamma.$$

The remaining of the proof concerns the asymptotic behavior of

$$\mu_{k,m} := \sup_{\substack{v_m \in V_{k_0+k,m}, \\ \|v_m\|_{L^2(\Omega)} = 1}} m \left(\mathcal{Q}_m^{\text{int}}(v_m) - \lambda^2 \right) + \bar{\eta}_m(v_m),$$

for $k \in \{1, \dots, k_1\}$ when m goes to $+\infty$. Let us first remark that for any $v_m \in V_{k_0+k,m}$, we have

$$v_m = \sum_{j=1}^{k_0} a_j u_{j,m} + \sum_{j=k_0+1}^{k_0+k} a_j w_{j,m} = \sum_{j=1}^{k_0} a_j u_{j,m} + \sum_{s=k_0+1}^{k_0+k_1} \left(\sum_{j=k_0+1}^{k_0+k} a_j b_{j,s} \right) u_{s,m},$$

where $(b_{j,s})_{j,s \in \{k_0+1, \dots, k_0+k_1\}} = B$. With Proposition 3.1, we obtain

$$\begin{aligned} & m_n \left(\mathcal{Q}_{m_n}^{\text{int}}(v_{m_n}) - \lambda^2 \right) \\ (4.13) \quad &= \sum_{j=1}^{k_0} m_n (\lambda_{j,m_n}^{\text{int}} - \lambda^2) |a_j|^2 + \sum_{j=k_0+1}^{k_0+k_1} m_n (\lambda_{j,m_n}^{\text{int}} - \lambda^2) \left| \sum_{s=k_0+1}^{k_0+k} a_s b_{s,j} \right|^2 \\ &= \sum_{j=1}^{k_0} m_n (\lambda_{j,m_n}^{\text{int}} - \lambda^2) |a_j|^2 - \frac{\left\| (\partial_n + \kappa/2 + m_0) \sum_{j=k_0+1}^{k_0+k} a_j w_j \right\|_{L^2(\Gamma)}^2}{2} + o(1). \end{aligned}$$

Using (4.12) and (4.13) and taking $a_1 = \dots = a_{k_0+k-1} = 0$, $a_{k_0+k} = 1$, we deduce that

$$(4.14) \quad \liminf_{n \rightarrow +\infty} \mu_{k,m_n} \geq \tilde{\nu}_{\lambda,k}.$$

Let $(v^n)_{n \in \mathbb{N}}$ be a sequence of maximizer of μ_{k,m_n} . For all n , there exists a unitary vector $a^n = (a_{1,n}, \dots, a_{k_0+k,n}) \in \mathbb{C}^{k_0+k}$ such that

$$v^n = \sum_{j=1}^{k_0} a_{j,n} u_{j,m_n} + \sum_{j=k_0+1}^{k_0+k} a_{j,n} w_{j,m_n}.$$

Up to a subsequence, we can assume that (a^n) converges in \mathbb{C}^{k_0+k} to a unitary vector $a = (a_{k_0+1}, \dots, a_{k_0+k})$. Proposition 3.1, (4.13) and (4.14) ensure that

$$\lim_{n \rightarrow +\infty} \lambda_{j,m_n}^{\text{int}} - \lambda^2 \leq \lambda_j^2 - \lambda^2 < 0$$

for $j \in \{1, \dots, k_0\}$ so that there exists $c_0 > 0$ such that

$$m_n \sum_{j=1}^{k_0} |a_{j,n}|^2 \leq c_0$$

and

$$\limsup_{n \rightarrow +\infty} \mu_{k,m_n} \leq \tilde{\eta}_\lambda(v) \leq \tilde{\nu}_{\lambda,k}$$

where $v = \sum_{j=k_0+1}^{k_0+k} a_j w_j$. With (4.12), we conclude noticing that $\lim_{n \rightarrow +\infty} \mu_{k,m_n} = \tilde{\nu}_{\lambda,k}$ and

$$\limsup_{m \rightarrow +\infty} m (\lambda_{k_0+k,m}^2 - \lambda^2) \leq \tilde{\nu}_{\lambda,k}.$$

□

4.2.2. *Lower bound.* In the following, we look for the second term in the asymptotic expansions of the eigenvalues. More precisely, we will show the following lemma.

Lemma 4.4. *We have for all $k \in \{1, \dots, k_1\}$ that*

$$\liminf_{m \rightarrow +\infty} m(\lambda_{k,m}^2 - \lambda_1^2) \geq \tilde{\nu}_{\lambda_1, j},$$

where $\tilde{\nu}_{\lambda_1, j}$ is defined in (4.9).

Proof. Let λ be the first eigenvalue of $|H^\Omega|$ whose multiplicity is denoted $k_1 \in \mathbb{N}^*$:

$$\lambda = \lambda_1 = \dots = \lambda_{k_1}.$$

By Lemma 4.1 and Proposition 3.1, we have

$$\lim_{m \rightarrow +\infty} \lambda_{k,m}^2 = \lim_{m \rightarrow +\infty} \lambda_{k,m}^{\text{int}} = \lambda^2,$$

for all $k \in \{1, \dots, k_1\}$. Let $(\varphi_{1,m}, \dots, \varphi_{k_1,m})$ an L^2 -orthonormal family of eigenvectors of $|H_m|$ associated with the eigenvalues $(\lambda_{1,m}, \dots, \lambda_{k_1,m})$ for all $m \geq m_1$. By Lemma 4.2, there exists $C > 0$ such that

$$(4.15) \quad C \geq \sup_{\substack{m \geq m_1, \\ j \in \{1, \dots, k_1\}}} \|\varphi_{j,m}\|_{H^2(\Omega)}.$$

Let us remark that for all $k \in \{1, \dots, k_1\}$, and all $m \geq m_1$,

$$\lambda_{k,m}^2 = \|H_m \varphi_{k,m}\|_{L^2(\mathbb{R}^3)}^2 = \sup_{\substack{(a_1, \dots, a_k) \in \mathbb{C}^k, \\ \sum_{j=1}^k |a_j|^2 = 1}} \left\| H_m \left(\sum_{j=1}^k a_j \varphi_{j,m} \right) \right\|_{L^2(\mathbb{R}^3)}^2.$$

Let $a = (a_1, \dots, a_k) \in \mathbb{C}^k$ be such that $\sum_{j=1}^k |a_j|^2 = 1$. We define

$$\varphi_m^a = \sum_{j=1}^k a_j \varphi_{j,m}.$$

With (1.4), (4.15) and Proposition 2.1, we get

$$(4.16) \quad \begin{aligned} \lambda_{k,m}^2 &\geq \mathcal{Q}_m^{\text{int}}(\varphi_m^a) + m^{-1} \int_{\Gamma} \left(\frac{|\nabla_s \varphi_m^a|^2}{2} + \left(\frac{K}{2} - \frac{\kappa^2}{8} \right) |\varphi_m^a|^2 \right) d\Gamma \\ &\quad + (m + m_0)^2 \|\varphi_m^a - u_{m+m_0}(\varphi_m^a)\|_{L^2(\Omega')}^2 + \mathcal{O}(m^{-3/2}). \end{aligned}$$

By (4.5), we get

$$\begin{aligned} &\left| \|\varphi_m^a\|_{L^2(\Omega')}^2 - \|u_{m+m_0}(\varphi_m^a)\|_{L^2(\Omega')}^2 \right| \\ &\leq C/m \left(\|\varphi_m^a\|_{L^2(\Omega')} + \|u_{m+m_0}(\varphi_m^a)\|_{L^2(\Omega')} \right) \\ &\leq C/m \left(\|\varphi_m^a - u_{m+m_0}(\varphi_m^a)\|_{L^2(\Omega')} + 2\|u_{m+m_0}(\varphi_m^a)\|_{L^2(\Omega')} \right) \\ &\leq C/m \left(m^{-1} + 2\|u_{m+m_0}(\varphi_m^a)\|_{L^2(\Omega')} \right). \end{aligned}$$

Using Proposition 2.1 and (4.15), we deduce that

$$\left| \|u_{m+m_0}(\varphi_m^a)\|_{L^2(\Omega')}^2 - \frac{\|\varphi_m^a\|_{L^2(\Gamma)}^2}{2m} \right| \leq \frac{C}{m^{3/2}},$$

so that

$$(4.17) \quad \left| \|\varphi_m^a\|_{L^2(\Omega')}^2 - \frac{\|\varphi_m^a\|_{L^2(\Gamma)}^2}{2m} \right| \leq \frac{C}{m^{3/2}}.$$

With (4.16) and Proposition 3.1, we obtain

$$(4.18) \quad \begin{aligned} & m(\lambda_{k,m}^2 - \lambda^2) \\ & \geq m \left(\mathcal{Q}_m^{\text{int}}(\varphi_m^a) - \lambda^2 \|\varphi_m^a\|_{L^2(\Omega)}^2 \right) \\ & \quad + \int_{\Gamma} \left(\frac{|\nabla_s \varphi_m^a|^2}{2} + \left(\frac{K}{2} - \frac{\kappa^2}{8} - \frac{\lambda^2}{2} \right) |\varphi_m^a|^2 \right) d\Gamma + \mathcal{O}(m^{-1/2}). \end{aligned}$$

Let $(u_{j,m})_{j \in \mathbb{N}^*}$ be an L^2 -orthonormal basis of $L^2(\Omega; \mathbb{C}^4)$ whose elements are eigenvectors of L_m^{int} associated with the sequence of eigenvalues $(\lambda_{j,m}^{\text{int}})$. Since $\lambda_{j,m}^{\text{int}}$ converges to λ_j^2 as m goes to $+\infty$, we get that

$$\lambda_{j,m}^{\text{int}} - \lambda^2 \geq 0,$$

for all $j \geq k_1 + 1$ and all $m \geq m_1$ (choosing if necessary a larger constant $m_1 > 0$). We deduce that

$$(4.19) \quad \begin{aligned} & m \left(\mathcal{Q}_m^{\text{int}}(\varphi_m^a) - \lambda^2 \|\varphi_m^a\|_{L^2(\Omega)}^2 \right) = \sum_{s=1}^{+\infty} m \left(\lambda_{s,m}^{\text{int}} - \lambda^2 \right) |\langle \varphi_m^a, u_{s,m} \rangle_{\Omega}|^2 \\ & \geq \sum_{s=1}^{k_1} m \left(\lambda_{s,m}^{\text{int}} - \lambda^2 \right) |\langle \varphi_m^a, u_{s,m} \rangle_{\Omega}|^2. \end{aligned}$$

Let $(m_n)_{n \in \mathbb{N}^*}$ be a subsequence which goes to $+\infty$ as n tends to $+\infty$ and such that

- (i) $\liminf_{m \rightarrow +\infty} m(\lambda_{k,m}^2 - \lambda^2) = \lim_{n \rightarrow +\infty} m_n(\lambda_{k,m_n}^2 - \lambda^2)$,
- (ii) $(u_{1,m_n}, \dots, u_{k_1,m_n})$ converges in $H^1(\Omega)$ to (u_1, \dots, u_{k_1}) ,
- (iii) $(\varphi_{1,m_n}, \dots, \varphi_{k_1,m_n})$ converges in $H^1(\Omega)$ to $(\varphi_1, \dots, \varphi_{k_1})$,

where (u_1, \dots, u_{k_1}) and $(\varphi_1, \dots, \varphi_{k_1})$ are L^2 -orthonormal families of eigenvectors of H^{Ω} associated with the eigenvalue λ . By Proposition 3.1, we have that

$$(4.20) \quad \begin{aligned} & \lim_{n \rightarrow +\infty} \sum_{s=1}^{k_1} m \left(\lambda_{s,m_n}^{\text{int}} - \lambda^2 \right) |\langle \varphi_{m_n}^a, u_{s,m_n} \rangle_{\Omega}|^2 \\ & = \sum_{s=1}^{k_1} - \frac{\|(\partial_{\mathbf{n}} + \kappa/2 + m_0)u_s\|_{L^2(\Gamma)}^2}{2} |\langle \varphi^a, u_s \rangle_{\Omega}|^2 = - \frac{\|(\partial_{\mathbf{n}} + \kappa/2 + m_0)\varphi_a\|_{L^2(\Gamma)}^2}{2}, \end{aligned}$$

where $\varphi^a = \sum_{j=1}^k a_j \varphi_j$. We get from (4.18), (4.19), and (4.20) that

$$\liminf_{m \rightarrow +\infty} m(\lambda_{k,m}^2 - \lambda^2) \geq \tilde{\eta}_{\lambda}(\varphi^a),$$

and

$$\liminf_{m \rightarrow +\infty} m(\lambda_{k,m}^2 - \lambda^2) \geq \sup_{\substack{(a_1, \dots, a_k) \in \mathbb{C}^k, \\ \sum_{j=1}^k |a_j|^2 = 1,}} \tilde{\eta}_\lambda(\varphi^a) \geq \tilde{\nu}_{\lambda,j}.$$

The conclusion follows from the upper bound (4.8). \square

Remark 4.5. When considering a larger eigenvalue $\lambda > \lambda_1$, the proof above breaks down since

$$\sum_{s=1}^{k_0} m \left(\lambda_{s,m}^{\text{int}} - \lambda^2 \right) |\langle \varphi_m^a, u_{s,m} \rangle_\Omega|^2$$

is non positive and the non-wanted terms in (4.19) cannot be removed so easily anymore. Here k_0 denotes the unique integer such that

$$\lambda = \lambda_{k_0+1} = \dots = \lambda_{k_0+k_1}.$$

APPENDIX A. SKETCH OF THE PROOF OF LEMMA 4.2

The purpose of this appendix is to give the main ideas of the proof of Lemma 4.2. We do not intend to give a rigorous proof but rather to enlighten why the classical arguments give uniform bounds in m (see for instance [7, Section 6.3]). In particular, we restrict ourselves to the operator H_m for $\Omega : \mathbb{R}_+^3 = \{\mathbf{x} = (x_1, x_2, x_3) : x_3 > 0\}$ and consider the solution $u \in H^1(\mathbb{R}^3; \mathbb{C}^4)$ of

$$H_m u = (\alpha \cdot D + (m_0 + m\chi_{\mathbb{R}^3})\beta)u = f,$$

where $f \in H^1(\mathbb{R}^3; \mathbb{C}^4)$. By Lemma 1.6 and Proposition 2.12, we have

$$\begin{aligned} \|f\|_{L^2(\mathbb{R}^3)}^2 \geq & \left(\|\nabla u\|_{L^2(\Omega)}^2 + m_0^2 \|u\|_{L^2(\Omega)}^2 + m_0 \|u\|_{L^2(\Gamma)}^2 + \sum_{k=1}^2 \|\partial_k u\|_{L^2(\Omega')}^2 \right) \\ & + 2m \|\Xi^- u\|_{L^2(\Gamma)}^2 - C/m \|u\|_{L^2(\Gamma)}^2, \end{aligned}$$

so that by the trace theorem, there exists $C > 0$ such that

$$(A.1) \quad C \left(\|f\|_{L^2(\mathbb{R}^3)}^2 + \|u\|_{L^2(\Omega)}^2 \right) \geq \|\nabla u\|_{L^2(\Omega)}^2 + \sum_{k=1}^2 \|\partial_k u\|_{L^2(\Omega')}^2.$$

Using the notation of [7, Section 6.3], we introduce the difference quotients

$$D_k^h u(\mathbf{x}) = \frac{u(\mathbf{x} + h e_k) - u(\mathbf{x})}{h}, \quad h \in \mathbb{R}, h \neq 0, \mathbf{x} \in \mathbb{R}^3, k \in \{1, 2, 3\}.$$

For $j \in \{1, 2\}$, we get that

$$H_m D_j^h u = (\alpha \cdot D + (m_0 + m\chi_{\mathbb{R}^3})\beta) D_j^h u = D_j^h f,$$

so that using (A.1), we get

$$C \left(\|D_j^h f\|_{L^2(\mathbb{R}^3)}^2 + \|D_j^h u\|_{L^2(\Omega)}^2 \right) \geq \|\nabla D_j^h u\|_{L^2(\Omega)}^2 + \sum_{k=1}^2 \|\partial_k D_j^h u\|_{L^2(\Omega')}^2.$$

By [7, Section 5.8.2], we deduce that

$$(A.2) \quad \begin{aligned} & C \left(\|\partial_j f\|_{L^2(\mathbb{R}^3)}^2 + \|\partial_j u\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\mathbb{R}^3)}^2 + \|u\|_{L^2(\Omega)}^2 \right) \\ & \geq \|\nabla \partial_j u\|_{L^2(\Omega)}^2 + \sum_{k=1}^2 \|\partial_k \partial_j u\|_{L^2(\Omega')}^2 . \end{aligned}$$

We also have that on Ω ,

$$-\partial_3^2 u = H_m^2 u + \left(\sum_{k=1}^2 \partial_k^2 - m_0^2 \right) u = H_m f ,$$

so that

$$(A.3) \quad \|\partial_3^2 u\|_{L^2(\Omega)} \leq C \|f\|_{H^1(\Omega)} .$$

Using (A.1), (A.2) and (A.3), we get the result.

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