Delay Dependent Stability Criteria and Stabilization for Discrete-Time Systems Via Three Terms approximation

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Abstract: This paper discusses the problem of delay-dependent stability and stabilization condition for discrete-time linear systems. By employing a three-term approximation for delayed state variables, a new model transformation is developed, which has a smaller approximation error than the two-term approach. By using scaled small gain theorem and an appropriate Lyapunov-Krasovskii functional, new delay-dependent stability conditions are proposed and formulated as linear matrix inequalities (LMIs). Before the end, a state feedback controller has investigated in the stabilization of discrete linear systems. Finally, numerical examples are presented to illustrate the effectiveness of the proposed method.

Keywords: Discrete time delay; Linear Matrix Inequality (LMI); Scaled Small Gain Theorem; Time-Delay system.

1. INTRODUCTION

Time delays are often an integral part of various physical systems like air-craft stabilization, communication systems, population dynamics, ship stabilization, electric power systems with lossless transmission lines and nuclear reactors, etc. The nature of these delays is time-varying. It is well known that the existence of time delay in various systems may provide poor performance and instability of dynamic systems, for more details see (Kim, 2011; Lakshmanan et al., 2011; Sun et al., 2010; Xu and Lam, 2008) and references therein.

Recently, the discrete time modelling has an essential role in many fields of science and engineering. Thus, most of systems are implemented with digital computers via the necessary input/output hardware. The digital computer uses the information in a way discrete. For the aforementioned considerations, much interest has been fixed to the analysis of discrete-time delay systems see for example (Chen et al., 2003; He et al., 2007; Lin et al., 2006; Park, 1999; Xu and Lam, 2005). Based on Lyapunov-krasovskii functional and on bounding techniques, delay-dependent stability of discrete-time systems has been investigated by (Fridman and Shaked, 2005; Gao and Chen, 2007; Gao et al., 2004; Jiang et al., 2005). On the other hand, many authors have been employed the input/output approach in the stability analysis of time-delay systems. This method is based on a specific transformation which aims to transform a pure system into two interconnected subsystems.

The input/output approach has been implemented in various works. Many works have proposed some results such as (Huang and Zhou, 2000; Park, 1999) for constant delays, and

it has been extended to time varying delay in (Fridman and Shaked, 2007; Kao and Rantzer, 2007). For time varying delay, the idea is to find an approximation of $x(k - d(k))$ for discrete-time case or $x(t - h(t))$ for continuous-time case, such that its approximation error is small as possible. (Fridman and Shaked, 2007) have adopted $x(k - d_a)$ as the approximation of $x(k - d(k))$ with $d_a = (d_1 + d_2)/2$, and $x(k - d_1)$ is used by (Kao and Lincoln, 2004). (Gu et al., 2011; Hmamed et al., 2015) have introduced the two term approximation $\left(x(t - h_1) + x(t - h_2) \right)/2$ as the approximation of $x(t - h(t))$ for continuous-time delayed case, and (Zhao et al., 2013) for T-S Fuzzy systems with time-varying delay. (Li and Gao, 2011) have also used the discrete two term approximation $\left(x(t - d_1) + x(t - d_2) \right)/2$ as the approximation of $x(k - d(k))$ for discrete delay systems. The same approximation has been considered in (Su et al., 2012) for filtering T-S fuzzy discrete-time systems with time-varying delay. It is pointed out that the approximation model of the delayed state with two terms is better than that based on only one term.

In this paper, Three terms approximation is proposed by using $\left(x(k - d_1) + x(k - d_2) + x(k - d_a) \right)/3$ as an approximation of $x(k - d(k))$ with d_1 and d_2 being the lower and the upper bounds of delay, within which the approximation error is smaller than the two-terms (Li and Gao, 2011). Then a new model transformation is formulated and will be analyzed and applied for the stability analysis and

stabilization of discrete delay systems. By using an appropriate Lyapunov-Krasovskii functional and the scaled small gain theorem (SSG) we present a new stability criterion of a subsystem. Before the end, the conditions which guarantee that discrete system is asymptotically stable under state feedback controller are given as LMIs by using the cone complementarity linearization algorithms. Finally, Numerical examples are given to illustrate the effective of the proposed method.

Notations: Notation P > 0(≥0) means that matrix P Is positive (semi) definite. The Superscript 'T' means the transpose. $G_1 \bullet G_2$ denotes the series connection of G_1 and G_2. **I** is an identity matrix with appropriate dimension. Denote "*" for the terms that can be deduced by symmetry In block matrices, we use diag{...} to express a block- diagonal matrix. $\|x\|_{l_2} \triangleq \sqrt[2]{\sum_{k=0}^{\infty} x^T(k)x(k)}$ denotes the l_2 norm of series x(k) and $\|\cdot\|_{\infty}$ represents the l_2-induced norm of a transfer function matrix or a general operator.

2. PROBLEM FORMULATION AND PRELIMINARIES

We consider the discrete time linear system with an interval time delay described by the following model.

$$\begin{cases} x(k+1) = Ax(k) + A_d x(k-d(k)) \\ x(k) = \phi(k), \ k = -d_2, -d_2+1,...,0 \end{cases} \quad (1)$$

Where $x(k) \in \mathbb{R}^n$ is the state vector, $A, A_d \in \mathbb{R}^{n \times n}$ are constant matrices, $\phi(k), \ k = -d_2, -d_2+1,...,0$ is the given initial condition sequence.

$d(k)$ is the time delay, time-varying satisfying.

$$1 \le d_1 \le d(k) \le d_2 \quad (2)$$

where d_1 and d_2 are known constants.

Before proceeding on, the following lemma is introduced which plays an important part in the development of our main results.

Lemma 1. (Huang and Feng, 2010) For any symmetric matrix $M \succ 0$, integer $l_1 \prec l_2$ and vector function $\omega : [l_1, l_1+1,...,l_2] \to \mathbb{R}^n$ such that the sums concerned are well defined, then

$$(l_2 - l_1 + 1)\sum_{i=l_1}^{l_2} \omega^T(i)M\omega(i) \ge \left(\sum_{i=l_1}^{l_2}\omega(i)\right)^T M \left(\sum_{i=l_1}^{l_2}\omega(i)\right)$$

The main objective of this work is to determine the stability condition for time delay system (1) using the Scaled Small Gain Theorem (SSG) (Li and Gao, 2011). To apply this theorem, we need to transform the original system (1) into the two following subsystems:

$$(S_1): z(t) = G\omega(t); \quad (S_2): \omega(t) = \Delta z(t) \quad (3)$$

Where the forward S_1 is a known linear time-invariant (LTI) system with operator G mapping $\omega(t)$ to z(t), and the feedback S_2 is an unknown linear time-varying one with operator $\Delta \in D \triangleq \{\Delta : \|\Delta\|_{\infty} \le 1\}$, $z(t) \in \mathbb{R}^z$ and $\omega(t) \in \mathbb{R}^{\omega}$. From the SSG Theorem in (Li and Gao, 2011), we derive sufficient condition for the robust asymptotic stability of the interconnected subsystems in (3). For this reason, we present the following Lemma.

Lemma 2. (Li and Gao, 2011) (SSG Theorem) Consider (3), and assume S_1 is internally stable. The closed-loop system formed by S_1 and S_2 is robustly asymptotically stable for all $\Delta \in D$ if there exist matrices $\{T_{\omega}, T_z\} \in T$ with $T \triangleq \{\{T_{\omega}, T_z\} \in \mathbb{R}^{\omega \times \omega} \times \mathbb{R}^{z \times z} : T_{\omega}, T_z \text{ nonsingular}, \|T_{\omega} \circ \Delta \circ T_z^{-1}\|_{\infty} \le 1\}$ such that the following SSG condition holds:

$$\|T_z \circ G \circ T_{\omega}^{-1}\|_{\infty} \prec 1 \quad (4)$$

3. MAIN RESULTS

In this section, we start with introducing the new model transformation method of system (1) and then we present the stability condition using SSG Theorem.

3.1 New Model Transformation

Inspired by the work in (Li and Gao, 2011), we propose a new approximation of the time-varying delay $d(k)$ using its lower, upper bounds d_1, d_2 and its average d_a. The estimation of $x(k - d(k))$ can be written as follows:

$$x(k-d(k)) = \frac{1}{3}\{x(k-d_1) + x(k-d_a) + x(k-d_2)\} + \frac{d_{12}}{3}\tilde{\omega}(k) \quad (5)$$

Where $\frac{1}{3}\{x(k-d_1) + x(k-d_a) + x(k-d_2)\}$ designed the approximation of $x(k-d(k))$, $\frac{d_{12}}{3}\tilde{\omega}(k)$ is the approximation error $d_{12} = d_2 - d_1, d_a = \frac{d_2+d_1}{2}$. From (5) system (1) can be written as:

$$\begin{cases} x(k+1) & = Ax(k) + \frac{1}{3}A_d x(k-d_1) + \frac{1}{3}A_d x(k-d_a) \\ & \quad + \frac{1}{3}A_d x(k-d_2) + \frac{d_{12}}{3}A_d \tilde{\omega}(k) \\ \omega(k) & = \Delta z(k) \text{ with } \tilde{\omega}(k) = \frac{3}{\sqrt{6}}\omega(k) \end{cases} \quad (6)$$

Remark 1. The equation $\tilde{\omega}(k) = \frac{3}{\sqrt{6}}\omega(k)$ is introduced to show that there is a relation between the feedback S_2 and the forward S_1, and to give a representation of subsystem S_1 in a compact form, similar to that in (Li and Gao, 2011).

From (5) and (6) the interconnection formulation of system (1) can be written as:

$$\begin{cases}(S_1): \begin{bmatrix} x(k+1) \\ z(k) \end{bmatrix} = \underbrace{\begin{bmatrix} \Phi_1 & \frac{d_{12}}{3} A_d \\ \Phi_1 & \frac{d_{12}}{3} A_d \end{bmatrix}}_{G} \begin{bmatrix} \xi(k) \\ \tilde{\omega}(k) \end{bmatrix} \\ (S_2): \omega(k) = \Delta z(k) \end{cases} \quad (7)$$

With $z(k) = x(k+1) - x(k)$

$$\Phi_1 = \begin{bmatrix} A & \frac{A_d}{3} & \frac{A_d}{3} & \frac{A_d}{3} \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} A-I & \frac{A_d}{3} & \frac{A_d}{3} & \frac{A_d}{3} \end{bmatrix}$$

$$\xi(k) = \begin{bmatrix} x(k) & x(k-d_1) & x(k-d_a) & x(k-d_2) \end{bmatrix}$$

From the position of $d(k)$ we obtain two scripture of $\frac{d_{12}}{3}\tilde{\omega}(k)$

Case 1: $d_1 \le d(k) \le d_a$

$$\frac{d_{12}}{3}\tilde{\omega}(k) = x(k-d(k)) - \frac{1}{3}\{x(k-d_1) + x(k-d_a) + x(k-d_2)\}$$

$$= \frac{1}{3}\left\{ \underbrace{-\sum_{i=k-d(k)}^{k-d_1-1} z(i)}_{\omega_1(k)} + 2\underbrace{\sum_{i=k-d_a}^{k-d(k)-1} z(i)}_{\omega_2(k)} + \underbrace{\sum_{i=k-d_2}^{k-d_2-1} z(i)}_{\omega_3(k)} \right\}$$

$$= \frac{1}{3}\{\omega_1(k) + \omega_2(k) + \omega_3(k)\}$$

Case 2: $d_a \le d(k) \le d_2$

$$\frac{d_{12}}{3}\tilde{\omega}(k) = x(k-d(k)) - \frac{1}{3}\{x(k-d_1) + x(k-d_a) + x(k-d_2)\}$$

$$= \frac{1}{3}\left\{ \underbrace{-\sum_{i=k-d_a}^{k-d_1-1} z(i)}_{\omega_1(k)} - 2\underbrace{\sum_{i=k-d(k)}^{k-d_a-1} z(i)}_{\omega_2(k)} + \underbrace{\sum_{i=k-d_2}^{k-d(k)-1} z(i)}_{\omega_3(k)} \right\}$$

$$= \frac{1}{3}\{\omega_1(k) + \omega_2(k) + \omega_3(k)\}$$

Before moving on, the following Lemma ensuring that the l_2-induced norm of Δ is bounded by one.

Lemma 3. The operator Δ denotes the mapping $z \to \omega$ and satisfies the condition $\|\Delta\|_\infty \le 1$.

Proof. For Case 1, we apply the Cauchy-Schwartz inequality (Fridman and Shaked, 2007)

$$\frac{d_{12}^2}{9}\|\tilde{\omega}(k)\|^2 = \frac{1}{9}\|\omega_1(k) + \omega_2(k) + \omega_3(k)\|^2$$
$$\le \frac{1}{9}\{\|\omega_1(k)\|^2 + \|\omega_2(k)\|^2 + \|\omega_3(k)\|^2\}$$

We continue the proof for each term separately. The function $j = p(k) = k - d(k)$ is strongly increasing. Hence, the inverse $k = p^{-1}(j) = q(j)$ is well-defined and

satisfies $|q(j) - (j+d_1)| \le \frac{d_{12}}{2}$. Then, summing $\omega_1(k)$ in k, changing the order of the summation and taking into account that $z(k) = 0, k \le 0$ we find that

$$\frac{d_{12}^2}{9}\|\tilde{\omega}(k)\|_{l_2}^2 \le \sum_{k=0}^{\infty}(d(k)-d_1)\sum_{j=k-d(k)}^{k-d_1-1}\|z(j)\|^2$$

$$= \left|\sum_{j=0}^{\infty}(d(q(j)-d_1)\sum_{k=q(j)}^{j+d_1-1}\|z(j)\|^2\right|$$

$$= \left|\sum_{j=0}^{\infty}(d(q(j)-d_1)(q(j)-(j+d_1))\|z(j)\|^2\right|$$

$$\le \sum_{j=0}^{\infty}(d_a-d_1)\frac{d_{12}}{2}\|z(j)\|^2$$

$$= \frac{d_{12}^2}{4}\|z(j)\|_{l_2}^2$$

For $\omega_2(k)$ and $\omega_3(k)$ we follow the same process, and we have

$$\|\omega_2(k)\|_{l_2}^2 \le 4\frac{d_{12}^2}{4}\|z(k)\|_{l_2}^2$$

$$\|\omega_3(k)\|_{l_2}^2 \le \frac{d_{12}^2}{4}\|z(k)\|_{l_2}^2$$

Then summing the three Terms together gives

$$\frac{d_{12}^2}{9}\|\tilde{\omega}(k)\|_{l_2}^2 \le \frac{1}{9}(\frac{d_{12}^2}{4} + 4\frac{d_{12}^2}{4} + \frac{d_{12}^2}{4})\|z(k)\|_{l_2}^2 = \frac{d_{12}^2}{6}\|z(k)\|_{l_2}^2$$

By substituting $\tilde{\omega}(k)$ by the relation given in equation (6), we obtain $\|\omega(k)\|_{l_2}^2 \le \|z(k)\|_{l_2}^2$. For case 2, using a proof process similar to that for case 1, we obtain the same results. This completes the proof.

Remark 2. The calculation of l_2-gain allows a comparative study. (Fridman and Shaked, 2007) have been approximate $x(k-d(k))$ by one term $x(k-d_a)$. (Li and Gao, 2011) have approximated $x(k-d(k))$ by $(x(k-d_1)+x(k-d_2))/2$ for two terms approximation. From a purely numerical standpoint, the evaluation of l_2-gain shows that the l_2-gain is smaller using three terms based on approximation model than obtained using one or two terms based models as will be shown in Table 1.

Table 1. l_2-gain of different approximation.

Methods	l_2-gain
(Fridman and Shaked, 2007)	$d_{12}/\sqrt{2}$
(Li and Gao, 2011)	$d_{12}/2$
Three-terms approximation	$d_{12}/\sqrt{6}$

Remark 3. Since the aforementioned considerations, we note that the three terms approximation is more general than that based on one or two terms. Thus

If $x(k - d_a) = \frac{1}{2}\{x(k - d_1) + x(k - d_2)\} + \frac{d_{12}}{2}\tilde{\omega}(k)$ (5) is reduced to

$$x(k - d(k)) = \frac{1}{2}\{x(k - d_1) + x(k - d_2)\} + \frac{d_{12}}{2}\tilde{\omega}(k)$$

which refers to the two terms approximation (Li and Gao, 2011)

If $x(k - d_1) + x(k - d_2) = 2\{x(k - d_a) + d_{12}\tilde{\omega}(k)\}$. (5) is reduced to

$$x(k - d(k)) = x(k - d_a) + \frac{d_{12}}{\sqrt{2}}\bar{\omega}(k)$$

Which refers to the one term approximation (Fridman and Shaked, 2007), with $\bar{\omega}(k) = \frac{3}{\sqrt{3}}\omega(k)$.

Remark 4. : Let $V(k)$ be a Lyapunov Krasovskii functional which guarantees the stability of subsystem S$_1$ and let

$$J \triangleq \sum_{k=0}^{\infty}\{z^T(k)Sz(k) - \omega^T(k)S\omega(k)\}$$

It is well known that the following condition along (S$_1$)

$$W \triangleq V(\infty) - V(0) + J = \sum_{k=0}^{\infty}\{j(k) + \Delta V(k)\} \prec 0 \quad (8)$$

guarantees that the H_∞ norm of (S$_1$) is less than 1. Therefore (8) is a sufficient condition for the bounded real lemma problem. In addition if (8) holds, according to Lemma 2 (SSG Theorem), we can conclude that system (S$_1$) is stable. Then, if J < 0, and letting $S = T^T T$ this means that $\left\| T \circ G \circ T^{-1} \right\|_\infty \prec 1$.

3.2 Stability Analysis

The forward subsystem S1 has three constant state delays So, the condition of the scaled small gain in Lemma 2 cannot be implemented to solve S1 directly by bounded real lemma. We can use an appropriate Lyapunov-Krasovskii functional to obtain sufficient LMIs conditions for a given $\gamma \succ 0$ ensuring $\left\| T \circ G \circ T^{-1} \right\|_\infty \prec \gamma$. Another possible way is applying the lifting method in (Xia et al., 2007) to convert S$_1$ to be delay-free. The following theorem presents two LMIs methods satisfying the SSG of S$_1$.

Theorem 1. Given scalars $d_2 \geq d_1 \geq 1$, and $\gamma \succ 0$, the forward subsystem S$_1$ is asymptotically and satisfies the SSG condition in **Lemma** 2 if one of the following two conditions holds:

    i)   if there exist matrices P and S such that

$$\begin{bmatrix} -P & P\tilde{A} & P\tilde{B} & 0 \\ * & -P & 0 & \tilde{C}^T S \\ * & * & -\frac{2}{3}\gamma^2 S & \tilde{D}^T S \\ * & * & * & -S \end{bmatrix} \prec 0 \quad (9)$$

Where

$$\tilde{A} = \begin{bmatrix} A & [0]_1 & \frac{1}{3}A_d & [0]_2 & \frac{1}{3}A_d & [0]_2 & \frac{1}{3}A_d \\ & I_{d_2 n} & & & & 0 \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} A - I_n & [0]_1 & \frac{1}{3}A_d & [0]_2 & \frac{1}{3}A_d & [0]_2 & \frac{1}{3}A_d \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} \frac{d_2}{3}A_d \\ 0_{d_2 n \times n} \end{bmatrix}, \quad \tilde{D} = \frac{d_2}{3}A_d, \quad [0]_1 = 0_{n \times (d_1 - 1)n}, \quad [0]_2 = 0_{n \times (\frac{d_{12}}{2} - 1)n}$$

    ii)   if there exist matrices,

$$P \succ 0, S \succ 0, Q_i \succ 0 \ (i = 1, 2, 3), \ R_j \succ 0 \ (j = 1, 2),$$

such that

$$\begin{bmatrix} \Sigma_1 & \begin{bmatrix} \Sigma_2^T P & d_1 \Sigma_3^T R_1 & d_2 \Sigma_3^T R_2 & \Sigma_3^T S \end{bmatrix} \\ * & \text{diag}\{-P, -R_1, -R_2, -S\} \end{bmatrix} \prec 0 \quad (10)$$

Where

$$\Sigma_1 = \begin{bmatrix} \Sigma_{11} & \begin{bmatrix} R_1 & 0 & R_2 & 0 \end{bmatrix} \\ * & diag\{\Sigma_{12}, -Q_2, \Sigma_{13}, -\frac{2}{3}\gamma^2 S\} \end{bmatrix}$$

$$\Sigma_{11} = Q_1 + Q_2 + Q_3 - P - R_1 - R_2$$

$$\Sigma_{12} = -Q_1 - R_1, \ \Sigma_{13} = -Q_3 - R_2$$

$$\Sigma_2 = \begin{bmatrix} \Phi_1 & \frac{d_{12}}{3}A_d \end{bmatrix}, \ \Sigma_3 = \begin{bmatrix} \Phi_2 & \frac{d_{12}}{3}A_d \end{bmatrix}$$

Proof. To prove (9) we use the scaled small gain theorem and the bounded real lemma. Define

$$\hat{x}(k) = col\{x(k), x(k-1), ..., x(k - d_2)\} \quad (11)$$

and using the lifting method in (Xia et al., 2007) to convert S$_1$ into delay-free of the following augmented state-space model:

$$\begin{bmatrix} \hat{x}(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ \tilde{\omega}(k) \end{bmatrix} \quad (12)$$

The operator G which is a mapping from $\omega(k)$ to z(k) guarantees the H_∞ norm of S$_1$ is less than γ. Then, the H_∞ norm can be written as $\left\| T \circ G \circ T^{-1} \right\|_\infty \prec \gamma$. According to **Lemma** 3.2 in (Apkarian and Gahinet, 1995) by setting P = X and S = L, then the condition (9) guarantees

$\left\|T \circ G \circ T^{-1}\right\|_{\infty} \prec \gamma$ so, it is clear that S_1 is asymptotically stable and satisfies $\left\|T \circ G \circ T^{-1}\right\|_{\infty} \prec \gamma$.

(**proof** of ii)). Let consider the discrete Lyapunov-Krasovskii functional for S_1 as

$$V(x(k)) = V_1(x(k)) + V_2(x(k)) + V_3(x(k)) \tag{13}$$

Where

$$V_1(x(k)) = x^T(k)Px(k)$$

$$V_2(x(k)) = \sum_{i=k-d_1}^{k-1} x^T(i)Q_1x(i) + \sum_{i=k-d_a}^{k-1} x^T(i)Q_2x(i)$$

$$+ \sum_{i=k-d_2}^{k-1} x^T(i)Q_3x(i)$$

$$V_3(x(k)) = \sum_{l=1}^{2} d_l \sum_{j=-d_l}^{-1} \sum_{i=k+j}^{k-1} z^T(i)R_lz(i)$$

And $z(i) = x(i+1) - x(i)$

The difference of V(k) can be calculated as

$$\begin{aligned}\Delta V(x(k)) = &\ x^T(k)\{Q_1 + Q_2 + Q_3 - P\} \\ &+ x^T(k+1)Px(k+1) \\ &- x^T(k-d_1)Q_1x(k-d_1) \\ &- x^T(k-d_a)Q_2x(k-d_a) \\ &- x^T(k-d_2)Q_3x(k-d_2) \\ &+ z^T(k)\left(d_1^2R_1 + d_2^2R_2\right)z(k) \\ &- d_1\sum_{i=k-d_1}^{k-1} z^T(i)R_1z(i) \\ &- d_2\sum_{i=k-d_2}^{k-1} z^T(i)R_2z(i)\end{aligned} \tag{14}$$

Applying **Lemma** 1 to deal with the cross-product items in (14), we obtain

$$-d_1\sum_{i=k-d_1}^{k-1} z^T(i)R_1z(i) \le -\xi_1^T(k)R_1\xi_1 \tag{15}$$

$$-d_2\sum_{i=k-d_2}^{k-1} z^T(i)R_2z(i) \le -\xi_2^T(k)R_2\xi_2 \tag{16}$$

With

$$\xi_1(k) = x(k) - x(k-d_1), \quad \xi_2(k) = x(k) - x(k-d_2).$$

Substituting the cross-product items in (14) by (15) and (16), we obtain

$$\Delta V(x(k)) = \xi^T(k)\left\{\overline{\Sigma}_1 + \Phi_1^T P\Phi_1 + \Phi_2^T\left(d_1^2R_1 + d_2^2R_2\right)\Phi_2\right\}\xi(k) \tag{17}$$

Where

$$\overline{\Sigma}_1 = \begin{bmatrix} \Sigma_{11} & \begin{bmatrix} R_1 & 0 & R_2 \end{bmatrix} \\ * & diag\{\Sigma_{12}, -Q_2, \Sigma_{13}\} \end{bmatrix}$$

Using Schur complement, (10) implies that $\Delta V(x(k)) \prec 0$, which means that S_1 is asymptotically stable.

Let $S > 0$

$$\begin{aligned}J &\triangleq \sum_{k=0}^{\infty} \underbrace{\left\{z^T(k)Sz(k) - \omega^T(k)S\omega(k)\right\}}_{j(k)} \\ &= \sum_{k=0}^{\infty}\left\{z^T(k)Sz(k) - \tilde{\omega}^T(k)\tfrac{2}{3}\gamma^2 S\omega(k)\right\}\end{aligned} \tag{18}$$

Taking $\Delta V(x(k))$ of S_1, we have under the zero initial condition

$$\begin{aligned}J &\le V(\infty) - V(0) + J \\ &= \sum_{k=0}^{\infty}\{j(k) + \Delta V(k)\} \\ &\le \sum_{k=0}^{\infty}\zeta^T(k)\left\{\Sigma_1 + \Sigma_2^T P\Sigma_2 + \Sigma_3^T\left(d_1^2R_1 + d_2^2R_2\right.\right. \\ &\quad\left.\left. +S\right)\Sigma_3\right\}\zeta(k)\end{aligned} \tag{19}$$

Where $\zeta^T(k) = \begin{bmatrix} \xi^T(k) & \tilde{\omega}^T(k) \end{bmatrix}$.

By using the Shur Complement, (19) implies (10). If letting $S = T^T T$, $J \prec 0$ means that $\left\|T \circ G \circ T^{-1}\right\|_{\infty} \prec \gamma$. This completes the proof.

Remark 5. : It has been observed that using $d_a = \frac{d_1+d_2}{2}$ in the constructed Lyapunov function can improve stability performance for many examples. Also it is seen from the approximation error that introducing $x(k-d_a)$ plays an important role to obtain an approximation error smaller than the existing ones.

Remark 6. : The number of decision variable in Theorem 1 has been reduced to $\frac{7}{2}n^2 + \frac{7}{2}n$, which is smaller than $9n^2 + 3n$ in (Zhang et al., 2008; Ramakrishnan and Ray, 2013). $\frac{11}{2}n^2 + \frac{9}{2}n$ in (Kwon et al., 2013), $4n^2 + 3n$ in (Liu ,and Zhang, 2012), $8n^2 + 3n$ in (Shao and Han, 2011) which means that the condition proposed is simple than the other conditions in literature.

3.3 Controller Design

This section is devoted to studying the state feedback controller design problem, whose goal is to guarantee the stability asymptotic of discrete-time delay system. The discrete-system controlled is represented as

$$x(k+1) = Ax(k) + Bu(k) \tag{20}$$

It should be noted that in the literature several author have been studying the stabilization of the system (20) and have chosen as control law $u(k) = Kx(k - d(k))$. In this section we are trying to stabilize our system with a control law Different than that used in (Gao and Chen, 2007; Kwon et al., 2013; Zhang et al., 2008). The state feedback controller is described by the following equation:

$$u(k) = K_1 x(k) + K_2 x(k - d(k)) \tag{21}$$

where K_1, K_2 are the controller gain to be determined and $d(k)$ is a time varying delay satisfying (2). Applying the controller law (21) to system (20) and using (5), the closed-loop system is obtained from (20) as

$$(\tilde{S}_1): \begin{cases} x(k+1) = (A + BK_1)x(k) + \frac{1}{3}A_d \{x(k - d_1) \\ \qquad + x(k - d_a) + x(k - d_2)\} + \frac{d_{12}}{3}A_d\tilde{\omega}(k) \end{cases}$$

Where $A_d = BK_2$.

The following Theorem presents the conditions should be satisfied $(\tilde{S}_1)$ to be asymptotically stable.

Theorem 2. Given scalars $d_2 \geq d_1 \geq 1$, the closed loop System $(\tilde{S}_1)$ is asymptotically stable if there exist matrices $P \succ 0, S \succ 0, X \succ 0, Z \succ 0, Q_i \succ 0 (i = 1,2,3)$ $R_j \succ 0, Y_j \succ 0 (j = 1,2), K_1$ and K_2 such that

$$\begin{bmatrix} \Sigma_1 & \begin{bmatrix} \tilde{\Sigma}_2^T & d_1\tilde{\Sigma}_3^T & d_2\tilde{\Sigma}_3^T & \tilde{\Sigma}_3^T \end{bmatrix} \\ * & diag\{-X, -Y_1, -Y_2, -Z\} \end{bmatrix} \prec 0 \tag{22}$$

With equality constraints

$$PX = I, \; R_1 Y_1 = I \; R_2 Y_2 = I, \; SZ = I$$

Where

$$\tilde{\Sigma}_2 = \begin{bmatrix} \tilde{\Phi}_1 & \frac{d_{12}}{3}A_d \end{bmatrix}, \tilde{\Sigma}_3 = \begin{bmatrix} \tilde{\Phi}_2 & \frac{d_{12}}{3}A_d \end{bmatrix}.$$
$$\tilde{\Phi}_1 = \begin{bmatrix} A + BK_1 & \frac{1}{3}A_d & \frac{1}{3}A_d & \frac{1}{3}A_d \end{bmatrix}$$
$$\tilde{\Phi}_2 = \begin{bmatrix} A + BK_1 - I & \frac{1}{3}A_d & \frac{1}{3}A_d & \frac{1}{3}A_d \end{bmatrix}.$$

Proof. replace X, Z, Y_1, Y_2 by $P^{-1}, S^{-1}, R_1^{-1}, R_2^{-1}$, respectively. By taking in consideration that $A = A + BK_1, A_d = BK_2$. Then we apply the Schur Complement lemma, we get the proposed conditions in Theorem 1. Therefore, by Theorem 1 the

desired result immediately follows. This completes the proof.

Remark 7. The resolution of the LMI in **Theorem** 1 using MATLAB toolbox is difficult then to put in evidence the problem, we need to transform it into minimization problem, such as LMIs are satisfied. By following the same procedure as that presented in (Zhang et al., 2007) then the resolution of our problem is easy to manipulate using the algorithm of cone complementarity linearization (CCL) algorithms (Ghaoui et al., 1997), which is adopted as **Algorithm** 1.

Algorithm 1. To maximize d_2:

Step 1: Choose a sufficiently small initial $d_2 \succ d_1 \geq 1$ and a tolerance ε (for example $\varepsilon = 10^{-6}$). Set $\{P_0, X_0, R_{10}, Y_{10}, R_{20}, Y_{20}, S_0, Z_0\} = \{I\}$ such that exists a feasible solution for the condition (22) and

$$\begin{bmatrix} P & I \\ I & X \end{bmatrix} \geq 0, \begin{bmatrix} R_1 & I \\ I & Y_1 \end{bmatrix} \geq 0, \begin{bmatrix} R_2 & I \\ I & Y_2 \end{bmatrix} \geq 0, \begin{bmatrix} S & I \\ I & Z \end{bmatrix} \geq 0 \tag{23}$$

Set $d_{2max} = d_2$; $k = 0$.

Step 2: Find a feasible solution of the following optimization problem for the variables $\{P, X, R_1, Y_1, R_2, Y_2, S, Z\}$

$$\begin{cases} \text{Minimize Trace} \; P_k X + X_k P + R_{1k}Y_1 + Y_{1k}R_1 \\ \qquad + R_{2k}Y_2 + Y_{2k}R_2 + + S_k Z + Z_k S \\ \text{subject to} \quad (22) \text{ and } (23) \end{cases}$$

Set

$$P_{k+1} = P, X_{k+1} = X, R_{1(k+1)} = R_1, Y_{1(k+1)} = Y_1$$
$$R_{2(k+1)} = R_2, Y_{2(k+1)} = Y_2, S_{k+1} = S, Z_{k+1} = Z$$

Step 3: if

$$\left\| P_{k+1} - X_{k+1}^{-1} \right\| \leq \varepsilon, \qquad \left\| R_{1(k+1)} - Y_{1(k+1)}^{-1} \right\| \leq \varepsilon$$
$$\left\| R_{2(k+1)} - Y_{2(k+1)}^{-1} \right\| \leq \varepsilon, \quad \left\| S_{k+1} - Z_{k+1}^{-1} \right\| \leq \varepsilon$$

are satisfied, then set $d_{2max} = d_2$, increase d_2, and return to **Step 2**. If it is not satisfied within a specified maximum number of iterations, then exit. Otherwise, set k = k + 1 and go to **Step 2**.

Table 2. The minimum γ for different methods for.

$8 \leq d(k) \leq 14$

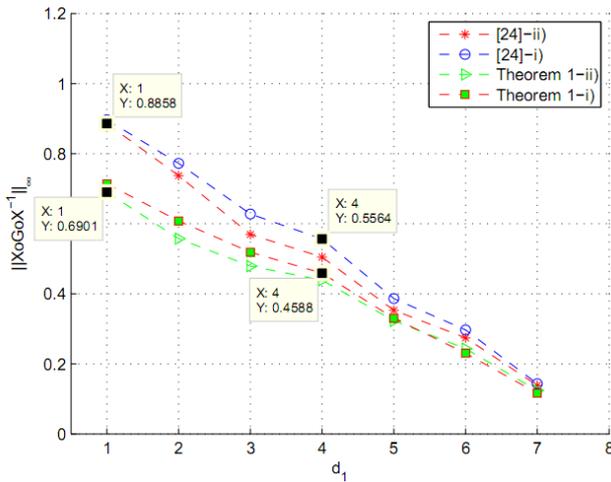| Methods	Num. of Var	γ
Lemma 3 for S_{h1} (Li and Gao, 2011)	174	1.00
Lemma 3 for S_{hm} (Li and Gao, 2011)	303	0.79
Corollary 1-(ii) (Li and Gao, 2011)	18	0.53
Corollary 1-(i) (Li and Gao, 2011)	468	0.49
Theorem 1-(ii)	21	0.44
Theorem 1-(i)	468	0.42

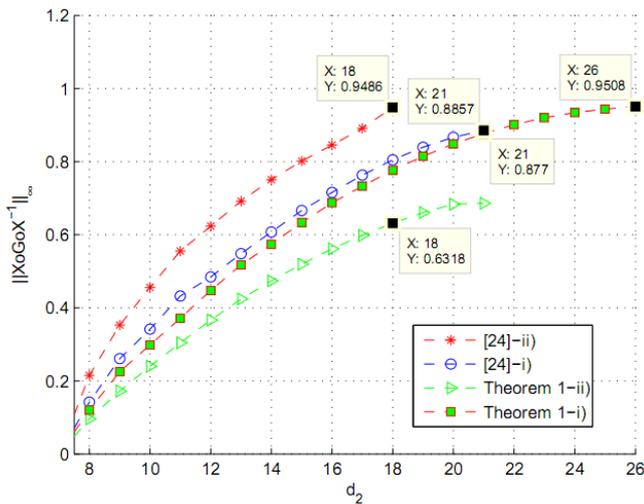Fig. 1. SSG condition according to d_1 when d_2=8 for example 2.

Fig. 2. SSG condition according to d_2 when d_1=7 for example 1.
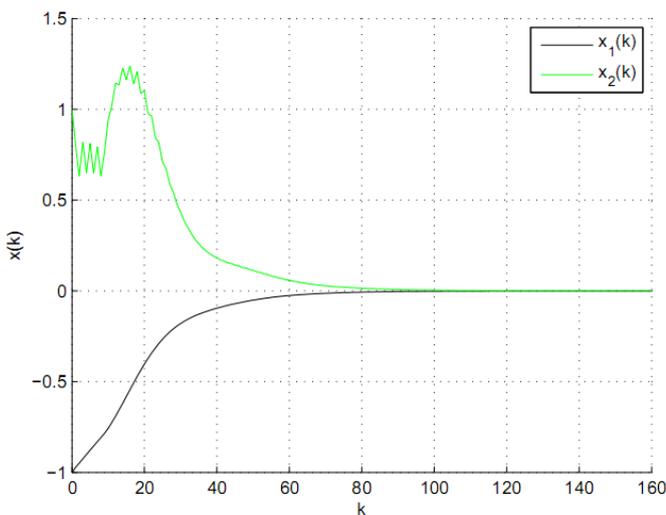

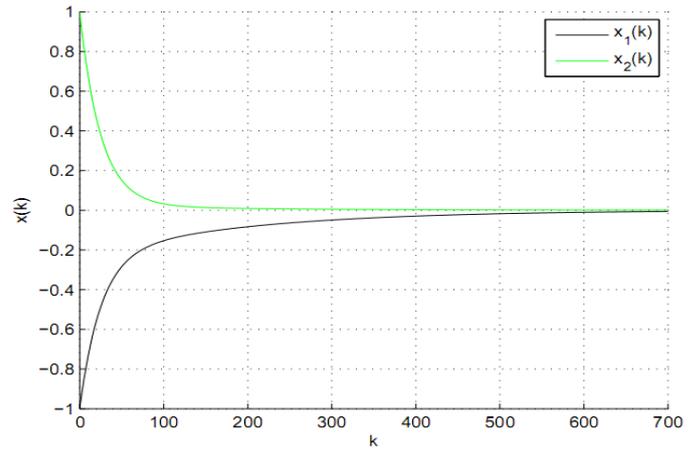Fig. 3. Closed-loop system $x_1(k)$ and $x_2(k)$ for $K_1=K_2$.

Fig. 4. The states trajectory $x_1(k)$ and $x_2(k)$ for $K_1 \neq K_2$.

4. NUMERICAL EXAMPLES

To illustrate the effectiveness of the proposed method, this section will provide three examples. It will be shown that the proposed results can provide less conservative results than recent ones proposed in literature.

Example 1. Consider the linear discrete-time delay systems

$$x(k+1) = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix} x(k) + \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix} x(k-d(k)) \qquad (24)$$

In order to test the advantages of the model transformation. (Li and Gao, 2011) is adopted the model transformation presented in other paper such as (Fridman and Shaked, 2007; Kao and Lincoln, 2004), and formulate its specific Lemmas. We compare our approach using **Theorem** 1 with results in (Li and Gao, 2011), and the results obtained by lemmas presented in (Li and Gao, 2011). Table 2 lists the Number of decision variable of different methods for $8 \leq d(k) \leq 14$, and the minimum of γ. One can see that the minimum γ obtained by our method is smaller than that given by other methods. From Table 2 we can see that Theorem 1 ii) need more decision variable and gives smaller γ than Corollary ii) in (Li and Gao, 2011) while that of Theorem 1 i) and with the same number of decision variables we obtains a smaller γ than Corollary i) in (Li and Gao, 2011) which means that the three term approximation gives less conservative results than two-term approximation.

For an appropriate choice of $\tilde{A}, \tilde{B}$ and $\tilde{C}$, Table 3 lists the upper delay bounds obtained by Theorem 1-i) and ii).

From Table 3, we can conclude that the proposed method yields less conservative results than the existing results in the literatures. Moreover, the relationship of SSG condition $\left\| X \circ G \circ X^{-1} \right\|_{\infty}$ according to d_2 when $d_1 = 7$ is figured in Figs.2. From Figs. 2, we observe that the three-term approximation has smaller SSG condition $\left\| X \circ G \circ X^{-1} \right\|_{\infty}$ than other approximation based model existing in the literature. For

example when $d_1 = 7$ the condition $\|X \circ G \circ X^{-1}\|_\infty$ obtained by Theorem 1-i) and ii) are 0.6318 for $d_2 = 18$ and 0.877 for $d_2 = 21$ respectively, while that of two-term approximation in (Li and Gao, 2011) by Corollary 1-i) and ii) are 0.9486 and 0.8857 respectively. In other side, we can observe from Fig 2 that Theorem 1-i) gives larger delay bounds $d_2 = 26$ than that obtained by Corollary 1-i) in (Li and Gao, 2011) $d_2 = 21$. This comparison shows that the proposed method is less conservative delay range.

Table 3. Maximum bounds d_2 for different value of d_1.

d_1	2	4	6	7	10	15	20	25
(Shao and Han,2011)	-	-	18	18	20	23	27	31
(Liu and Zhang, 2012)	-	-	18	18	20	23	27	31
(Kwon et al., 2013)	19	19	20	20	21	24	27	-
(Li and Gao, 2011)-(ii)	17	17	18	18	20	23	27	31
(Zhanga et al., 2015)	20	21	21	-	22	24	27	-
Theorem 1-(ii)	20	20	21	21	22	25	28	32
(Li and Gao, 2011)-(i)	17	19	201	22	25	30	35	40
Theorem 1-(i)	20	22	24	25	28	33	38	43

Example 2. Consider the linear discrete-delay systems

$$x(k+1) = \begin{bmatrix} 0.7 & 0.1 \\ 0.05 & 0.7 \end{bmatrix} x(k) + \begin{bmatrix} -0.1 & 0.1 \\ -0.1 & -0.2 \end{bmatrix} x(k - d(k)) \qquad (25)$$

For given $d_1 = \{2, 5, 6, 7, 10, 20\}$ the maximum upper bounds d_2 obtained by (Ramakrishnan and Ray, 2013) are $\{9, 11, 12, 13, 16, 26\}$, while that of Theorem 1-(ii) gives larger delay bounds $d_2 = \{12, 15, 16, 17, 20, 30\}$, which means that the proposed method is less conservative. Table 4 shows more results of the maximum bounds delay for different values of d_1.

Table 4. Maximum bounds d2 for different value of d1.

d_1	2	5	6	7	10	20	N.V
(Zhang et al., 2008)	7	9	10	11	14	24	42
(Huang and Feng, 2010)	8	10	11	12	15	25	18
(Liu and Zhang, 2012)	9	11	12	13	16	26	22
(Li and Gao, 2011)-(ii)	9	11	12	13	16	26	18
(Li and Gao, 2011)-(i)	9	13	14	15	17	27	-
Theorem 1-(ii)	10	13	14	15	18	28	21
Theorem 1-(i)	12	15	16	17	20	30	-

It is clear that the results obtained in this paper are better than the existing one in the literatures. From the last column of Table 4 (N.V), we observe that the proposed method needs more decision variables than (Huang and Feng, 2010) and (Li and Gao, 2011)-(ii) and smaller variables than other methods

which means that the proposed method still effective and gives less conservative results. In other hand, Fig.1 describes the relationship of SSG condition $\|X \circ G \circ X^{-1}\|_\infty$ according to d_1 when $d_2 = 8$. From Fig.1 we can be seen that the SSG condition obtained by our approach still smaller than other approximation based model used by other authors. This means that the proposed method is effective and less conservative.

Example 3. Consider the model of inverted pendulum system (Kwon et al., 2013), shown in Fig 5 with the following continuous description

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ \frac{3(M+m)g}{l(4M+m)} & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{3}{l(4M+m)} \end{bmatrix} u(t) \qquad (26)$$

When M = 8kg, m = 2.0kg, l = 0.5m, g = 9.8m/s^2 and choosing sampling time $T_s = 30$ms, then system (26) can be transformed to discrete-time system with the following parameters

$$A = \begin{bmatrix} 1.00078 & 0.0301 \\ 0.5202 & 1.0078 \end{bmatrix}, \quad B = \begin{bmatrix} -0.0001 \\ -0.0005 \end{bmatrix}$$
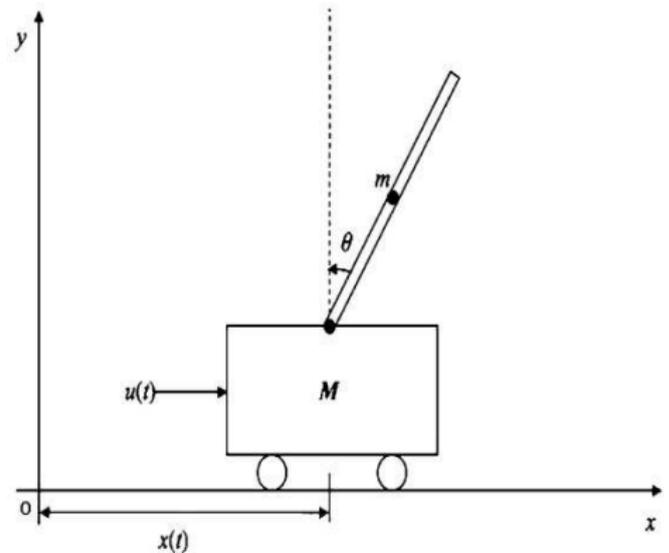

Fig. 5. Inverted pendulum system.

We consider this example to illustrate the advantages of the proposed method. When $d_1 = 1$ and by applying **Theorem 2** The maximum value of d_2 which guarantees the asymptotic stability of closed-loop system (20) is $d_2 = 8$ for $(K = K_1 = K_2)$ and $d_2 > 15$ for $K_1 \neq K_2$ while that of (Gao and Chen, 2007, Kwon et al., 2013; Zhang et al., 2008; Huang and Feng, 2010) are 3,4,5,6 respectively. This means that our approach gives larger delay bounds.

Table 5 summaries study devoted to stabilization of system (20) and lists the maximum delay bounds and the controller gain obtained by other methods. The last column in Table 5 lists the number of iteration (N.Iter) satisfied Theorem 2 to be feasible.

Table 5. Maximum bounds d₂ and controller gains K.

Methods	d_2	K		N.Iter
(Gao and Chen, 2007)	3	[102.9100	80.7916]	-
(Zhang et al., 2008)	4	[110.6827	34.6980]	-
(Huang and Feng, 2010)	5	[110.6827	34.6980]	-
(Kwon et al., 2013)	6	[110.6827	34.6980]	-
Theorem 2 $K_1 = K_2$	8	[85.9857	26.5128]	473
Theorem 2 $\begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$	>15	$\begin{bmatrix} 98.3007 & 7.0841 \\ -0.0005 & 0.0020 \end{bmatrix}$		59

Firstly, it should be noted that **Theorem** 2 is satisfied with a small controller gain than those in (Gao and Chen, 2007; Zhang et al., 2008; Huang and Feng, 2010, Kwon et al., 2013). Moreover, the number of iteration needed to obtain feasible solution for K = K_1 = K_2 is 473, while that of $K_1 \neq K_2$ is 59, which means that the proposed method with $K_1 \neq K_2$ is better than (K = K_1 = K_2). The controller gains in last line of Table 5 are obtained for given d_1 = 1 and d_2 = 15. Fig 3 and Fig 4 plots the closed loop system using the controller gain (K = K_1 = K_2) and $K_1 \neq K_2$ respectively. Fig 3 shows that the state responses converge to zero for small time k, while that of Fig 4 needs more time k to approach zero. From this example we conclude that our method can control practical system with a smaller controller gain better than the existing methods in literature. Fig 3 and Fig 4 emphasize the merit of the proposed method. In the simulation, the initial values of the states are x(0) = [1;1] and time-delay d(k) is assumed as $d(k) = 1 + 7\sin(k\pi / 2) \in \begin{bmatrix} 1 & 8 \end{bmatrix}$

5. CONCLUSION

In this paper, an improved delay-dependent stability for discrete-time linear systems has been developed. Based on a new model transformation performing a three-term approximation, stability criteria have been presented in term of a set of LMIs by using a direct Lyapunov-Krasovskii functional and SSG theorem. Thereafter, the problem of time-delayed controller design for discrete-time systems has been studied and a sufficient condition for the solvability of this problem has been given by using cone complementarity linearization (CCL) algorithms. It is better to mention that these results are extendable for filtering problem and for many types of systems. At the end, the proposed numerical examples have demonstrated the advantage of the method proposed to obtain less conservative results.

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