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Robust Ekeland variational principles. Application to the formation and stability of partnerships

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ABSTRACT

This paper has two parts. The mathematical part provides generalized versions of the robust Ekeland variational principle in terms of set-valued EVP with variable preferences, uncertain parameters and changing weights given to vectorial perturbation functions. The behavioural part that motivates our findings models the formation and stability of a partnership in a changing, uncertain and complex environment in the context of the variational rationality approach of stop, continue and go human dynamics. Our generalizations allow us to consider two very important psychological effects relative to ego depletion and goal gradient hypothesis.

KEYWORDS

Robust Ekeland variational principle; variational rationality; variable preferences; ego depletion; goal gradient hypothesis

1. Introduction

Ekeland's variational principle [1] (briefly, denoted by EVP) is a well-known theorem that has several versions, extensions and applications in nonlinear analysis, optimization, psychology and the behavioural sciences; see [2–17] and the references therein. Since EVP provides the existence of a strict minimum of a perturbed lower semicontinuous function on complete metric spaces and it has several applications in mathematics and behavioural sciences, many authors have extended EVP to vector-valued functions, and, more recently, to set-valued maps and bimap. We refer the reader to books [8,11,13] and several recent articles [2–7,10,12,14–16,18–28] for a good reporting of the results and various approaches. Moreover, inspired by set-valued optimization, there are two approaches to extend EVP to set-valued maps, the vector approach, see, for example, [2,5–7,19,20,24] and the set approach, see, for example, [3,17,18,21–23,26,28]. The vector criterion provides the existence of an approximate minimal (efficient) point of the range set of a set-valued map. In the set approach, it is necessary to introduce an ordering for sets and it provides the existence of an optimum of a perturbed set optimization problem. Since these two

types criteria of set-valued optimization are not comparable [26,29], there is no direct comparison between the two types of EVP extensions to set-valued maps. On the other hand, the equilibrium problem is a unified method for solving several problems in nonlinear analysis; see [30]. In order to study the equilibrium problem and the vector equilibrium problem in the setting of nonconvex sets, many authors have been interested in obtaining the equilibrium versions of EVP, see [31–37] and the references therein. Recently, some authors have obtained the equilibrium versions of EVP for set-valued bimaps and given their applications in the set-valued equilibrium problems and the behavioural sciences, see, for example, [14,38–41].

Our interest in studying EVP for set-valued bimaps with variable ordering in the setting of quasi-metric spaces was motivated mainly by modelling the dynamic of the formation and stability of a partnership in a changing, uncertain and complex environment within the context of the variational rationality approach. Further extensions of EVP for set-valued bimaps, where the perturbations need not satisfy the triangle inequality, are necessary for their own sake and for later applications. We now turn to the two parts mentioned above.

Mathematical part. The mathematical part provides generalized versions of the robust Ekeland variational principle in terms of set-valued formulations of EVP with variable preferences, uncertain parameters and changing weights given to vectorial perturbation functions. This sequence of challenges works as follows. We start with:

- an optimization (minimizing) problem $\min\{f(y) : y \in X\}$ where $f : X \rightarrow \mathbb{R}$ is a real valued function defined on a metric space (X, d) . The maximizing counterpart being $\max\{g(y), y \in X\}$ with $g = -f$;
- a perturbed optimization problem $\min\{f(y) + \varepsilon d(x, y) : y \in X\}$ where, starting from some $x \in X$, the perturbation function is $Q(\cdot, x) = f(\cdot) + \varepsilon d(x, \cdot)$. The size of the weight $\varepsilon \in \mathbb{R}^{++}$, where $\mathbb{R}^{++} =]0, +\infty[$, models the importance of the perturbation. In this perturbed setting, the EVP shows that, even if the initial optimization problem has no solution, then, for each $x \in X$, the perturbed optimization problem will have an optimal solution:
- a vectorial EVP problem, suppose that the image of f is a subset of a linear space Y ;
- a vectorial robust optimization problem $\min\{h(y, \eta) : y \in X, \eta \in H\}$ models a situation where, for each $y \in X$, the vector of objectives $h(y, \eta) \in Y$ depends on uncertain parameters. The problem being that the decision relative to the choice of y must be made before the realization of any of the uncertainties $\eta \in H$ [42]. It was originally used to protect optimization problems from infeasibility caused by uncertainties in the model parameters. In this setting, the objective is an uncertainty set of vectors $f(y) = \{h(y, \eta) \in \mathbb{R}, \eta \in H\} \subseteq Y$, the problem is a set-valued optimization problem;

- a vectorial robust EVP optimization problem considers a perturbation of a robust optimization problem [14].
- a variable preference formulation of an EVP optimization problem models variable preferences as changing cones $C(x)$ [4];
- an adaptive EVP optimization problem considers changing weights given to perturbation functions.

A behavioural formulation of the EVP requires, in the context of the variational rationality approach, the introduction of an asymmetric generalized metric space like a quasi-metric space. This model the fact that costs and inconveniences of moving from x to y are different from the opposite [14].

If we introduce all these challenges in the same formulation, the reason is that we have in mind a nice application in behavioural sciences. In contrast, the problem with the vast majority of generalized EVP formulations is that generalizations seem to have been done just for the sake of generalizations. Most of the time, there are no applications, relative to mathematics, physics, or behavioural sciences. Hopefully, recently, the (VR) variational rationality approach of stop, continue and go human dynamics [43–48] has many applications to most generalized variational principles and optimizing algorithms as a way to provide a general and (for the first time) formalized theory of motivation and emotion in the behavioural sciences.

Behavioural part. The behavioural part that motivates our generalizations of the EVP models the formation and stability of a partnership in a changing, uncertain and complex environment within the context of the VR approach [46,47]. Our generalization allows us to consider two very important psychological effects relative to ego depletion and goal gradient hypothesis.

The article is structured as follows. In Section 2, some notions, notations and basic results that will be used in the following are given. Section 3 is devoted to the mathematical part. Following Fakhar et al. [9] and Qiu et al. [14], and driven by behavioural applications, we give new formulations of the robust Ekeland variational principle in terms of set-valued EVP with variable preferences, uncertain parameters and changing weights given to vectorial perturbation functions. Section 4 is devoted to modelling the formation and stability of a partnership in a changing, uncertain and complex environment in the context of the variational rationality approach of stop, continue and go human dynamics.

2. Preliminaries

In this section, we introduce some notions and notations which will be used in the sequel. A quasi-metric q on a nonempty set X is a bifunction $q : X \times X \rightarrow \mathbb{R}^+$, where \mathbb{R}^+ denotes the set of non-negative real numbers, such that

- (i) $q(x, y) = 0$ if and only if $x = y$,

(ii) $q(x, y) \leq q(x, z) + q(z, y)$, for all $x, y, z \in X$.

A set X equipped with a quasi-metric q is said to be a quasi-metric space and it is denoted by (X, q) . Let (X, q) be a quasi-metric space. A sequence $\{x_n\}$ in X is said to be left convergent to $x \in X$, if $\lim_{n \rightarrow \infty} q(x_n, x) = 0$. The sequence $\{x_n\}$ is said to be left Cauchy if for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $q(x_n, x_m) < \varepsilon$ for all $m \geq n \geq N_\varepsilon$. A quasi-metric space (X, q) is said to be left-complete if every left Cauchy sequence is left convergent.

For any nonempty subset A of a linear space Y , the vector closure of A is defined as follows:

$$vcl(A) = \{y \in Y : \exists v \in Y, \exists \lambda_n \geq 0, \lambda_n \rightarrow 0 \text{ such that } y + \lambda_n v \in A, \forall n \in \mathbb{N}\}.$$

For any $k_0 \in Y$, we define the k_0 -vector closure of A as follows:

$$vcl_{k_0}(A) = \{y \in Y : \exists \lambda_n \geq 0, \lambda_n \rightarrow 0 \text{ such that } y + \lambda_n k_0 \in A, \forall n \in \mathbb{N}\}.$$

The set A is called k_0 -closed if and only if $A = vcl_{k_0}(A)$, for more details, see [15,49,50]. Let K be a nonempty subset of a linear space Y and $k_0 \in Y \setminus \{0\}$. The nonlinear scalarization function $\xi_{k_0} : Y \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is defined as follows:

$$\xi_{k_0}(y) := \begin{cases} +\infty & \text{if } y \notin \mathbb{R}k_0 - K, \\ \inf\{t \in \mathbb{R} : y \in tk_0 - K\} & \text{otherwise.} \end{cases}$$

The function ξ_{k_0} is also called Gerstewitz's function generated by K and k_0 . For its main properties one can refer to [8,11,12,14,15,51] and the references therein. Also, according to Lemma 2.6 in [15], $\xi_{k_0}(y) > -\infty$, for each $y \in Y$ if and only if $k_0 \notin -vcl(K)$.

Suppose that C is a nonempty subset of Y , we say that ξ_{k_0} is C -nondecreasing, if $\xi_{k_0}(y_1) \leq \xi_{k_0}(y_2)$ for all $y_1, y_2 \in Y, y_2 - y_1 \in C$. If K is a closed convex cone of a topological vector space Y and $k_0 \in K \setminus (-K)$, then ξ_{k_0} is subadditive, positively homogeneous, lower semicontinuous, K -nondecreasing and $\xi_{k_0}(y) > -\infty$ for all $y \in Y$; see [8,15]. Let A be a nonempty set of a linear space Y , we say that A is free disposal with respect to a convex cone $D \subseteq Y$ if $A + D = A$. This notion was introduced by Debreu [52] and it has been used in mathematical economics and optimization. Recall that the cone generated by a nonempty set $B \subseteq Y$ is the set $cone(B) := \{ty : t \in \mathbb{R}^+, y \in B\}$.

Proposition 2.1 (see [12,50,51]): *Let K, C be two nonempty subsets of a linear space Y and $k_0 \in Y \setminus \{0\}$. Then the Gerstewitz function ξ_{k_0} has the following properties:*

- (i) $\xi_{k_0}(y) < +\infty$ if and only if $y \in \mathbb{R}k_0 - vcl_{k_0}(K)$;
- (ii) ξ_{k_0} is C -nondecreasing if and only if $K + C \subseteq cone(\{k_0\}) + vcl_{k_0}(K)$;
- (iii) for any real number r , $\xi_{k_0}(y + rk_0) = \xi_{k_0}(y) + r$.

Definition 2.2: Let Y be a linear space and Ω be a nonempty set.

- (i) A nonempty set S of Y is said to be a domination set in Y , if $S + S \subseteq S$.
- (ii) A set-valued map $D : \Omega \rightrightarrows Y$ is called a domination structure in Y , if $D(x)$ is a domination set, for each $x \in \Omega$.

The notion of domination structure was introduced by Yu [53], in the case where $D : Y \rightrightarrows Y$ and $D(y)$ is a convex cone, for every $y \in Y$. Notice that every convex cone subset of a linear space is a domination set but the converse is not necessarily true, for example, the integer numbers set \mathbb{Z} in \mathbb{R} is a domination set but it is neither convex nor cone.

In order to obtain our main results, we need to state a pre-order principle provided by Qiu [16]. A binary relation \preceq on X is called a pre-order if the transitivity property is satisfied. Let (X, \preceq) be a pre-order set. An extended real-valued function $\eta : (X, \preceq) \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is called monotone with respect to \preceq if and only if for any $x_1, x_2 \in X$,

$$x_1 \preceq x_2 \Rightarrow \eta(x_1) \leq \eta(x_2).$$

For any given $x_0 \in X$, denotes $S(x_0)$ the set $\{x \in X : x \preceq x_0\}$.

Theorem 2.3 ([16]): Let (X, \preceq) be a pre-order set, $x_0 \in X$ such that $S(x_0) \neq \emptyset$ and $\eta : (X, \preceq) \rightarrow \mathbb{R} \cup \{\pm\infty\}$ be an extended real-valued function which is monotone with respect to \preceq . Suppose that the following conditions are satisfied:

- (A) $-\infty < \inf\{\eta(x) : x \in S(x_0)\} < +\infty$;
- (B) for any $x \in S(x_0)$ with $-\infty < \eta(x) < +\infty$ and $x' \in S(x) \setminus \{x\}$, one has $\eta(x) > \eta(x')$;
- (C) for any sequence $\{x_n\} \subseteq S(x_0)$ with $x_n \in S(x_{n-1}), \forall n \in \mathbb{N}$, such that $\eta(x_n) - \inf\{\eta(x) : x \in S(x_{n-1})\} \rightarrow 0, (n \rightarrow \infty)$, there exists $\bar{x} \in X$ such that $\bar{x} \in S(x_n)$, for all $n \in \mathbb{N}$.

Then there exists $u \in X$ such that

- (a) $u \in S(x_0)$;
- (b) $S(u) \subseteq \{u\}$.

3. Robust Ekeland variational principle

Inspired by [9,14], we are going to obtain general forms of the robust Ekeland variational principle in terms of set-valued EVP with variable preferences, uncertain parameters and changing weights given to vectorial perturbation functions.

Motivated by the notion of sequentially lower monotonicity for vector-valued and set-valued maps; see [10,14,15,50], we extend this notion for set-valued bimap with variable ordering.

Definition 3.1: Let Y be a linear space, $C : X \rightrightarrows Y$ be a set-valued map, (X, q) be a quasi-metric space, $\psi : X \times X \rightarrow \mathbb{R}^{++}$ be a bifunction, and $F : X \times X \rightrightarrows Y$ be a set-valued bimap with nonempty values.

- (i) F is said to be left- $C(\cdot)$ -sequentially lower monotone (briefly, denoted left- $C(\cdot)$ -slm) if for any sequence $\{x_n\}$ that left converges to an element \bar{x} such that $F(x_n, x_{n+1}) \subseteq -C(x_n)$, for any $n \in \mathbb{N}$, we have $F(x_n, \bar{x}) \subseteq -C(x_n)$, for all $n \in \mathbb{N}$.
- (ii) ψ is called C - F -decreasing in the first argument (resp., C - F -increasing in the second argument) if and only if for all $x, x' \in X$ with $F(x, x') \subseteq -C(x)$ we have $\psi(x', y) \geq \psi(x, y)$ (resp., $\psi(y, x') \leq \psi(y, x)$), for all $y \in X$.

Example 3.2: Let $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^3$, $C : \mathbb{R} \rightrightarrows \mathbb{R}^3$, and $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^{++}$ be defined as follows:

$$F(x, y) =] - \infty, 1 - x[\times]y[\times] - \infty, -x + y[\quad \forall x, y \in \mathbb{R},$$

$$C(x) = \mathbb{R}_+^3 \quad \forall x \in \mathbb{R},$$

and

$$\psi(x, y) = e^{-x}(e^y + 1) \quad \forall x, y \in \mathbb{R}.$$

Now, if $F(x, x') \subseteq -C(x)$, then $x' < x$, and so ψ is both C - F -decreasing in the first argument and C - F -increasing in the second argument.

In what follows, we give some elementary results which will be used in the sequel.

Proposition 3.3: Let F be left- $C(\cdot)$ -slm and $\psi : X \rightarrow \mathbb{R}^{++}$ be C - F -increasing in the second argument. Let $\{x_n\} \subseteq X$ be left convergent to $\bar{x} \in X$, and $F(x_n, x_{n+1}) \subseteq -C(x_n)$, for all $n \in \mathbb{N}$. Then $\psi(x, \bar{x}) \leq \psi(x, x_n)$, for all $n \in \mathbb{N}$ and for each $x \in X$.

Proof: Assume on the contrary that there exist $n_0 \in \mathbb{N}$ and $x \in X$ such that $\psi(x, x_{n_0}) < \psi(x, \bar{x})$. Since ψ is C - F -increasing in the second argument, then $F(x_{n_0}, \bar{x}) \not\subseteq -C(x_{n_0})$ which contradicts the left- $C(\cdot)$ -sequentially lower monotonicity of F . ■

The following example shows that a C - F -increasing function ψ is not necessarily lower semicontinuous or sequentially lower monotone.

Example 3.4: Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^{++}$ be defined as follows:

$$F(x, x') = x' - x \quad \forall (x, x') \in \mathbb{R}^2$$

and

$$\psi(y, x) = \begin{cases} 1 & \text{if } x > 0, \\ 2 & \text{if } x \leq 0. \end{cases}$$

For every $x \in \mathbb{R}$, we consider $C(x) = \mathbb{R}^+$. It is obvious that F is left- $C(\cdot)$ -slm, and ψ is C - F -increasing in the second argument but ψ is not lower semicontinuous and it is not even sequentially lower monotone in the second argument. Suppose $x_n = \frac{1}{n}$, for every $n \in \mathbb{N}$. Then, we have $F(x_n, x_{n+1}) \subseteq -C(x_n)$, for all $n \in \mathbb{N}$ and $x_n \rightarrow 0$, but $2 = \psi(y, 0) \not\leq \psi(y, \frac{1}{n}) = 1$.

Proposition 3.5: Let Y be a linear space, $k_0 \in Y \setminus \{0\}$ and $S \subseteq Y$ be free disposal with respect to $\text{cone}(\{k_0\})$. Suppose that $t_1, t_2 \in \mathbb{R}$ and $t_1 \leq t_2$, then $t_1 k_0 - S \subseteq t_2 k_0 - S$.

Proof: Since $t_2 - t_1 \geq 0$ and S is free disposal with respect to $\text{cone}(\{k_0\})$, then

$$t_1 k_0 - S = t_2 k_0 + (t_1 - t_2)k_0 - S \subseteq t_2 k_0 - S.$$

■

In the following, we give our main result.

Theorem 3.6: Let (X, q) be a left-complete quasi-metric space, Y be a real linear space, $k_0 \in Y \setminus \{0\}$ and K be a nonempty subset of Y . Suppose that $\text{vcl}_{k_0}(K)$ is free disposal with respect to $\text{cone}(\{k_0\})$. Let $C : X \rightrightarrows Y$ be a set-valued domination structure with nonempty values. Suppose that for every $x \in X$, $C(x)$ is k_0 -vectorially closed, free disposal with respect to $\text{cone}(\{k_0\})$ and $C(x) + K \subseteq \text{vcl}_{k_0}(K)$. Suppose that $F : X \times X \rightrightarrows Y$ be a set-valued bimap with nonempty values satisfy the following conditions:

- (i) there exists $x_0 \in X$ such that $F(x_0, x_0) \subseteq -C(x_0)$;
- (ii) there exist $\alpha \in \mathbb{R}$ and $s_0 \in \mathbb{R}k_0 - K$ such that $F(x_0, X) \cap (-s_0 + \alpha k_0 - C(x_0)) = \emptyset$;
- (iii) $F(x, z) \subseteq F(x, y) + F(y, z) - C(x)$, for all $x, y, z \in X$;
- (iv) F is left $C(\cdot)$ -slm;
- (v) $F(x, y) \subseteq -C(x)$ implies $C(x) + C(y) \subset C(x)$, for all $x, y \in X$.

Assume that $\psi : X \times X \rightarrow \mathbb{R}^{++}$ is C - F -decreasing in the first argument and C - F -increasing in the second argument, then there exists $\hat{x} \in X$ such that

- (a) $F(x_0, \hat{x}) + \psi(x_0, \hat{x})q(x_0, \hat{x})k_0 \subseteq -C(x_0)$;

(b) $F(\hat{x}, x) + \psi(\hat{x}, x)q(\hat{x}, x)k_0 \not\subseteq -C(\hat{x})$, for each $x \in X \setminus \{\hat{x}\}$.

Proof: We prove this result by applying Theorem 2.3. For this purpose, we define a relation \preceq on X as follows: for any $x, x' \in X$,

$$x' \preceq x \iff F(x, x') + \psi(x, x')q(x, x')k_0 \subseteq -C(x).$$

In the first step, we show that \preceq is a pre-order. Suppose that $x' \preceq x, x \preceq y$, then

$$F(x, x') + \psi(x, x')q(x, x')k_0 \subseteq -C(x), \quad (1)$$

$$F(y, x) + \psi(y, x)q(y, x)k_0 \subseteq -C(y). \quad (2)$$

Since $C(x)$ and $C(y)$ are free disposal with respect to $\text{cone}(\{k_0\})$, then $F(x, x') \subseteq -C(x)$ and $F(y, x) \subseteq -C(y)$. Therefore, since ψ is C - F -decreasing in the first argument and C - F -increasing in the second argument we have

$$\psi(y, x') \leq \psi(x, x') \quad \text{and} \quad \psi(y, x') \leq \psi(y, x). \quad (3)$$

On the other hand, by condition (iii), the triangle property for q and Proposition 3.5, we have

$$\begin{aligned} F(y, x') + \psi(y, x')q(y, x')k_0 &\subseteq F(y, x) + F(x, x') - C(y) + \psi(y, x')q(y, x')k_0 \\ &\subseteq F(x, x') + F(y, x) - C(y) + \psi(y, x')(q(y, x) + q(x, x'))k_0. \end{aligned} \quad (4)$$

Furthermore, from (3) and Proposition 3.5, we get

$$\begin{aligned} F(x, x') + F(y, x) - C(y) + \psi(y, x')(q(y, x) + q(x, x'))k_0 \\ &\subseteq F(x, x') + \psi(x, x')q(x, x')k_0 + F(y, x) + \psi(y, x')q(y, x)k_0 - C(y) \\ &\subseteq F(x, x') + \psi(x, x')q(x, x')k_0 + F(y, x) + \psi(y, x)q(y, x)k_0 - C(y). \end{aligned} \quad (5)$$

Also, by (1) and (2), we have

$$\begin{aligned} F(x, x') + \psi(x, x')q(x, x')k_0 + F(y, x) + \psi(y, x)q(y, x)k_0 \\ &\subseteq -C(x) - C(y) - C(y). \end{aligned} \quad (6)$$

Since $F(y, x) \subseteq -C(y)$, then by condition (v) and since C is a domination structure, we get

$$-C(x) - C(y) - C(y) \subseteq -C(y). \quad (7)$$

Therefore, from (4), (5), (6) and (7), we have

$$F(y, x') + \psi(y, x')q(y, x')k_0 \subseteq -C(y).$$

This shows that $x' \preceq y$, and so \preceq has the transitive property. Hence, the relation \preceq is a pre-order.

Now, we set

$$S(x) := \{x' \in X : x' \preceq x\} \quad \forall x \in X,$$

and we define $\eta : (X, \preceq) \rightarrow \mathbb{R} \cup \{\pm\infty\}$ as follows:

$$\eta(x) := \sup\{\xi_{k_0}(y + s_0) : y \in F(x_0, x)\} \quad \forall x \in X.$$

By condition (i), $x_0 \in S(x_0)$, and so $S(x_0) \neq \emptyset$. We now show that η is monotone with respect to \preceq . Let $x, x' \in X$ and $x' \preceq x$, then $F(x, x') + \psi(x, x')q(x, x')k_0 \subseteq -C(x)$, and so $F(x, x') \subseteq -\psi(x, x')q(x, x')k_0 - C(x)$. Therefore, by condition (iii), we have

$$\begin{aligned} F(x_0, x') &\subseteq F(x_0, x) + F(x, x') - C(x_0) \\ &\subseteq F(x_0, x) - \psi(x, x')q(x, x')k_0 - C(x_0) - C(x). \end{aligned}$$

Hence, for every $y' \in F(x_0, x')$ there exists $y \in F(x_0, x)$ such that

$$y' \in y - \psi(x, x')q(x, x')k_0 - C(x_0) - C(x).$$

Therefore,

$$y' + s_0 \in y + s_0 - \psi(x, x')q(x, x')k_0 - C(x_0) - C(x). \quad (8)$$

Since $C(x_0) + C(x) + vcl_{k_0}(K) \subseteq C(x) + vcl_{k_0}(K) \subseteq vcl_{k_0}(K)$, then by (8) and parts (ii), (iii) of Proposition 2.1, we have

$$\begin{aligned} \xi_{k_0}(y' + s_0) &\leq \xi_{k_0}(y + s_0) - \psi(x, x')q(x, x') \\ &\leq \sup\{\xi_{k_0}(z + s_0) : z \in F(x_0, x)\} - \psi(x, x')q(x, x') \\ &= \eta(x) - \psi(x, x')q(x, x'). \end{aligned}$$

Thus,

$$\eta(x') = \sup\{\xi_{k_0}(y' + s_0) : y' \in F(x_0, x')\} \leq \eta(x) - \psi(x, x')q(x, x').$$

This shows that η is monotone with respect to \preceq . Moreover, if $x' \preceq x$, $x \neq x'$ and $\eta(x) < \infty$, then

$$\eta(x') \leq \eta(x) - \psi(x, x')q(x, x') < \eta(x). \quad (9)$$

Now, it is enough to show that conditions (A), (B) and (C) in Theorem 2.3 hold.

Checking condition A. Since $s_0 \in \mathbb{R}k_0 - K$, then $\xi_{k_0}(s_0) < +\infty$. Hence, by condition (i) and Proposition 2.1(ii), we have

$$\xi_{k_0}(y + s_0) \leq \xi_{k_0}(s_0) < +\infty \quad \forall y \in F(x_0, x_0).$$

Therefore,

$$\eta(x_0) = \sup\{\xi_{k_0}(y + s_0) : y \in F(x_0, x_0)\} \leq \xi_{k_0}(s_0) < +\infty.$$

On the other hand, from condition (ii) $y \notin -s_0 + \alpha k_0 - K$, for each $x \in X$ and $y \in F(x_0, x)$. Hence, from definition of $\xi_{k_0}(\cdot)$, we have $\xi_{k_0}(y + s_0) \geq \alpha$, and so

$$\eta(x) = \sup\{\xi_{k_0}(y + s_0) : y \in F(x_0, x)\} \geq \alpha \quad \forall x \in X.$$

Thus,

$$-\infty < \alpha \leq \inf\{\eta(x) : x \in S(x_0)\} \leq \eta(x_0) < +\infty.$$

Hence, condition **(A)** holds.

Checking condition B. Consider $x \in S(x_0)$ with $-\infty < \eta(x) < +\infty$, and let $x' \in S(x) \setminus \{x\}$. By relation (9), we have $\eta(x') < \eta(x)$ and so condition **B** holds.

Checking condition C. Let $\{x_n\}$ be an arbitrary sequence in $S(x_0)$ such that $x_n \in S(x_{n-1})$, for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} F(x_0, x_1) + \psi(x_0, x_1)q(x_0, x_1)k_0 &\subseteq -C(x_0), \\ F(x_1, x_2) + \psi(x_1, x_2)q(x_1, x_2)k_0 &\subseteq -C(x_1), \\ &\vdots \\ F(x_{n-1}, x_n) + \psi(x_{n-1}, x_n)q(x_{n-1}, x_n)k_0 &\subseteq -C(x_{n-1}). \end{aligned}$$

Adding the current n inclusions up, we obtain

$$\sum_{i=1}^n F(x_{i-1}, x_i) + \sum_{i=1}^n \psi(x_{i-1}, x_i)q(x_{i-1}, x_i)k_0 \subseteq \sum_{i=1}^n -C(x_{i-1}) \subseteq -C(x_0). \quad (10)$$

The last inclusion holds, because $F(x_{n-1}, x_n) + \psi(x_{n-1}, x_n)q(x_{n-1}, x_n)k_0 \subseteq -C(x_{n-1})$, thus we have $F(x_{n-1}, x_n) \subseteq -C(x_{n-1})$. Therefore, by condition (v), we have $C(x_n) \subseteq C(x_{n-1})$, for each $n \in \mathbb{N}$, and so $\sum_{i=1}^n -C(x_{i-1}) \subseteq -C(x_0)$.

On the other hand, by condition (iii), we have

$$F(x_0, x_n) \subseteq \sum_{i=1}^n F(x_{i-1}, x_i) - C(x_0). \quad (11)$$

Since $F(x_i, x_{i+1}) \subseteq -C(x_i)$, ψ is C - F -decreasing in the first argument and C - F -increasing in the second argument, we deduce that

$$\psi(x_0, x_n) \leq \psi(x_0, x_{n-1}) \leq \psi(x_0, x_{n-2}) \leq \cdots \leq \psi(x_0, x_1)$$

and

$$\psi(x_0, x_k) \leq \psi(x_1, x_k) \leq \psi(x_2, x_k) \leq \cdots \leq \psi(x_{k-1}, x_k) \quad \forall k \geq 2.$$

Hence,

$$\psi(x_0, x_n) \leq \psi(x_{i-1}, x_i), \quad 1 \leq i \leq n, \forall n \in \mathbb{N}. \quad (12)$$

Now, from (12) and Proposition 3.5 we deduce that

$$\psi(x_0, x_n) \sum_{i=1}^n q(x_{i-1}, x_i)k_0 - C(x_0) \subseteq \sum_{i=1}^n \psi(x_{i-1}, x_i)q(x_{i-1}, x_i)k_0 - C(x_0). \quad (13)$$

Combining (10), (11), (13) and since $C(x_0)$ is a domination set, we have

$$\begin{aligned} & F(x_0, x_n) + \psi(x_0, x_n) \sum_{i=1}^n q(x_{i-1}, x_i)k_0 \\ & \subseteq \sum_{i=1}^n F(x_{i-1}, x_i) - C(x_0) + \psi(x_0, x_n) \sum_{i=1}^n q(x_{i-1}, x_i)k_0 \\ & \subseteq \sum_{i=1}^n F(x_{i-1}, x_i) + \sum_{i=1}^n \psi(x_{i-1}, x_i)q(x_{i-1}, x_i)k_0 - C(x_0) \\ & \subseteq -C(x_0) - C(x_0) \subseteq -C(x_0). \end{aligned}$$

Therefore,

$$F(x_0, x_n) + s_0 + \psi(x_0, x_n) \sum_{i=1}^n q(x_{i-1}, x_i)k_0 \subseteq s_0 - C(x_0). \quad (14)$$

If $y_n \in F(x_0, x_n)$, then by condition (ii), $y_n \notin -s_0 + \alpha k_0 - K$, and so

$$\xi_{k_0}(y_n + s_0) \geq \alpha. \quad (15)$$

Hence, by (14) and parts (ii), (iii) of Proposition 2.1 we have

$$\xi_{k_0}(y_n + s_0) + \psi(x_0, x_n) \sum_{i=1}^n q(x_{i-1}, x_i) < \xi_{k_0}(s_0) < +\infty. \quad (16)$$

Therefore, from (15) and (16), we get

$$\begin{aligned}\sum_{i=1}^n q(x_{i-1}, x_i) &\leq \frac{1}{\psi(x_0, x_n)} (\xi_{k_0}(s_0) - \xi_{k_0}(y_n + s_0)) \\ &\leq \frac{1}{\psi(x_0, x_n)} (\xi_{k_0}(s_0) - \alpha) < +\infty \quad \forall n \in \mathbb{N}.\end{aligned}$$

Thus,

$$\sum_{i=1}^{\infty} q(x_{i-1}, x_i) \leq \frac{1}{\inf_{x \in X} \psi(x_0, x)} (\xi_{k_0}(s_0) - \alpha) < +\infty.$$

So, for every $m > n$,

$$q(x_n, x_m) \leq \sum_{i=n}^{m-1} q(x_i, x_{i+1}) \rightarrow 0 (m > n \rightarrow \infty).$$

This means that the sequence $\{x_n\}$ is a left-Cauchy sequence in (X, q) and since (X, q) is left-complete, there exists $\bar{x} \in X$, such that $q(x_n, \bar{x}) \rightarrow 0 (n \rightarrow \infty)$. Since $F(x_{n-1}, x_n) \subseteq -C(x_{n-1})$, condition (iv) implies

$$F(x_{n-1}, \bar{x}) \subseteq -C(x_{n-1}) \quad \forall n \in \mathbb{N}. \quad (17)$$

Now, for fixed n and $m > n$, we have $x_m \in S(x_n)$, and so

$$F(x_n, x_m) + \psi(x_n, x_m)q(x_n, x_m)k_0 \subseteq -C(x_n). \quad (18)$$

Hence, by condition (iii), relations (17), (18), Propositions 3.3, 3.5 and since $q(x_n, \bar{x}) - q(x_n, x_m) \leq q(x_m, \bar{x})$, we have

$$\begin{aligned}&F(x_n, \bar{x}) + \psi(x_n, \bar{x})q(x_n, \bar{x})k_0 \\ &= F(x_n, \bar{x}) + \psi(x_n, \bar{x})q(x_n, x_m)k_0 + \psi(x_n, \bar{x})(q(x_n, \bar{x}) - q(x_n, x_m))k_0 \\ &\subseteq F(x_n, x_m) + F(x_m, \bar{x}) - C(x_n) + \psi(x_n, \bar{x})q(x_n, x_m)k_0 \\ &\quad + \psi(x_n, \bar{x})(q(x_n, \bar{x}) - q(x_n, x_m))k_0 \\ &\subseteq F(x_n, x_m) + F(x_m, \bar{x}) - C(x_n) + \psi(x_n, x_m)q(x_n, x_m)k_0 \\ &\quad + \psi(x_n, \bar{x})(q(x_n, \bar{x}) - q(x_n, x_m))k_0 \\ &\subseteq -C(x_m) - C(x_n) - C(x_n) + \psi(x_n, \bar{x})(q(x_n, \bar{x}) - q(x_n, x_m))k_0 \\ &\subseteq -C(x_n) + \psi(x_n, \bar{x})q(x_m, \bar{x})k_0.\end{aligned}$$

Since $q(x_m, \bar{x}) \rightarrow 0 (m \rightarrow \infty)$, and $C(x_n)$ is a k_0 -closed, we obtain

$$F(x_n, \bar{x}) + \psi(x_n, \bar{x})q(x_n, \bar{x})k_0 \subseteq -C(x_n).$$

Therefore, $\bar{x} \leq x_n$, for each $n \in \mathbb{N}$, and so $\bar{x} \in S(x_n)$, for each $n \in \mathbb{N}$. Then, condition C is satisfied. Hence, by Theorem 2.3, conditions (a) and (b) hold, and the proof is completed. \blacksquare

Let us show an example to illustrate Theorem 3.6.

Example 3.7: Let $F : \mathbb{R}^+ \times \mathbb{R}^+ \rightrightarrows \mathbb{R}^2$, $C : \mathbb{R}^+ \rightrightarrows \mathbb{R}^2$, and $\psi : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^{++}$ be defined as follows:

$$F(x, y) = \text{conv}(\{(-e^y, -1), (0, -x), (0, 0)\}),$$

$$C(x) = \text{cone}(\text{conv}(\{(e^x, 1), (-1, 0), (0, 0)\})),$$

and $\psi(x, y) = e^{-x}(e^y + 1)$. We show that all conditions of Theorem 3.6 are satisfied. Assume that $K :=] - \infty, +\infty[\times]0, \infty[$, $k_0 = (-1, 1)$, and $(x_0, y_0) = (0, 0)$. It is clear that $F(0, 0) = \text{conv}(\{(-1, -1), (0, 0)\}) \subseteq \text{cone}(\text{conv}(\{(-1, -1), (1, 0), (0, 0)\})) = -C(0)$, so condition (i) holds. Since $F(0, y) = \text{conv}(\{(-e^y, -1), (0, 0)\})$, for every $y \in \mathbb{R}^+$, so for $\alpha = -3, s_0 = -k_0 = (1, -1)$, we have $(-s_0 + \alpha k_0 - K) = (2, -2) - C(0)$. Hence, $F(0, y) \cap (-s_0 + \alpha k_0 - C(0)) = \emptyset$, and so condition (ii) holds too. Also, it is clear that $F(x, z) \subseteq F(x, y) + F(y, z) - C(x)$. Conditions (iv) and (v) can be obtained by the following fact:

$$F(x, y) \subseteq -C(x) \iff y \leq x \iff C(y) \subseteq C(x).$$

Remark 3.1: Note that Theorem 3.6 reduces to Theorem 4.1 of [39], in the case where (X, q) is a complete metric space, Y is a topological vector space, K is a closed convex cone in Y , $C(x) = K$ for all $x \in X$, $k_0 \in K \setminus (-K)$ and $\psi(x, y) = \lambda$ for all $x, y \in X$, where λ is a positive real number. Hence, Theorem 3.6 extends Theorem 4.1 in [39] to set-valued bimaps defined on quasi-metric spaces with values in linear spaces ordered by variable preferences with the perturbation function as the form $\lambda d(\cdot, \cdot)k_0$ which satisfies the triangle inequality. Furthermore, by Remark 4.5 of [39], conditions (ii) and (iv) in Theorem 3.6 are strictly weaker than conditions (ii) and (iii) in Theorem 3.1 of [38] and conditions (ii) and (iv) in Theorem 3.1 of [41], respectively.

As a consequence of the above result, we can obtain the following consequence which is a moving cone version of Theorem 3.1 of [14].

Theorem 3.8: Let (X, q) , Y , K and $C : X \rightrightarrows Y$ be the same as in Theorem 3.6. Let $F : X \times X \rightrightarrows Y$ be a set-valued bimap with nonempty values that satisfies conditions (i)–(v). Let $\delta : X \rightarrow \mathbb{R}^{++}$ be a C - F -decreasing function, in the sense that $F(x, x') \subseteq -C(x) \implies \delta(x) \leq \delta(x')$. Then, there exists $\hat{x} \in X$ such that

- (a) $F(x_0, \hat{x}) + \delta(x_0)q(x_0, \hat{x})k_0 \subseteq -C(x_0)$;
- (b) $F(\hat{x}, x) + \delta(\hat{x})q(\hat{x}, x)k_0 \not\subseteq -C(\hat{x})$, for each $x \in X \setminus \{\hat{x}\}$.

Proof: It is enough to consider $\psi : X \times X \rightarrow \mathbb{R}^{++}$ as $\psi(x, y) = \delta(x)$ for each $x, y \in X$ and apply Theorem 3.6. ■

As another consequence of Theorem 3.6, we have the following result.

Theorem 3.9: Let (X, q) , Y , K and $C : X \rightrightarrows Y$ be the same as in Theorem 3.6. Let $F : X \times X \rightrightarrows Y$ be a set-valued bimap with nonempty values that satisfies conditions (i)–(v). Let $\delta : X \rightarrow \mathbb{R}^{++}$ be a C - F -increasing function, in the sense that $F(x, x') \subseteq -C(x) \Rightarrow \delta(x') \leq \delta(x)$, and let $\inf_{x \in X} \delta(x) > 0$. Then, there exists $\hat{x} \in X$ such that

- (a) $F(x_0, \hat{x}) + \delta(\hat{x})q(x_0, \hat{x})k_0 \subseteq -C(x_0)$;
- (b) $F(\hat{x}, x) + \delta(x)q(\hat{x}, x)k_0 \not\subseteq -C(\hat{x})$, for each $x \in X \setminus \{\hat{x}\}$.

Proof: It is enough to define $\psi : X \times X \rightarrow \mathbb{R}^{++}$ as $\psi(x, y) = \delta(y)$ for all $x, y \in X$ and apply Theorem 3.6. ■

In the sequel, we are going to obtain another version of Theorem 3.4 in [6] from Theorem 3.6. For this purpose, we need the following notions.

Definition 3.10 (Limiting monotonicity condition [6]): Let (X, q) be a quasi-metric space and Y be a topological vector space. Given $G : X \rightrightarrows Y$ and $\bar{x} \in X$, we say that G satisfies the limiting monotonicity condition at \bar{x} if for any sequence of pairs $\{(x_n, y_n)\} \subset \text{gph}G = \{(x, y) \in X \times Y : y \in G(x)\}$ with $q(x_n, \bar{x}) \rightarrow 0$ as $n \rightarrow \infty$ one has the implication

$$[y_{n+1} \leq y_n \text{ for all } n \in \mathbb{N}] \Rightarrow [\text{there is a } \bar{y} \in \text{Min}G(\bar{x}) \text{ with } \bar{y} \leq y_n, n \in \mathbb{N}].$$

Theorem 3.11: Let (X, q) be a left-complete quasi-metric space and Y be a topological vector space. Let $G : X \rightrightarrows Y$ be a set-valued map with compact values, where Y is partially ordered by a proper closed convex cone $C \subset Y$ with $C \setminus (-C) \neq \emptyset$. Assume that G satisfies the limiting monotonicity condition on X , $\varepsilon > 0$, $\lambda > 0$, $\kappa \in C \setminus (-C)$, $(x_0, y_0) \in \text{gph}G$, $G(X) \cap (y_0 + \varepsilon\kappa - C) = \emptyset$ and $G(x) \cap (y_0 + \mathbb{R}\kappa - C) \neq \emptyset$, for all $x \in X$. Then there exists $(\bar{x}, \bar{y}) \in \text{gph}G$ satisfying

$$\bar{y} - y_0 + \frac{\varepsilon}{\lambda}q(x_0, \bar{x})\kappa \leq 0, \quad \bar{y} \in \text{Min}G(\bar{x}), \quad \text{and} \quad (19)$$

$$y - \bar{y} + \frac{\varepsilon}{\lambda}q(\bar{x}, x)\kappa \not\leq 0 \quad \text{for all } (x, y) \in \text{gph}G \text{ with } x \neq \bar{x}. \quad (20)$$

Proof: Let $\xi_\kappa(\cdot)$ be the Gerstewitz function generated by C and κ . Since $G(x) \cap (y_0 + \mathbb{R}\kappa - C) \neq \emptyset$ and $\kappa \in C \setminus -C$, then $-\infty < \xi_\kappa(y - y_0) < +\infty$ for all $y \in G(x)$. Suppose that $F : X \times X \rightarrow \mathbb{R}$ and $\psi : X \times X \rightarrow \mathbb{R}^+$ are defined as follows:

$$F(x, z) := \min\{\xi_\kappa(u - y_0) : u \in G(z)\} - \min\{\xi_\kappa(v - y_0) : v \in G(x)\},$$

and
$$\psi(x, z) := \frac{\varepsilon}{\lambda}, \quad \forall x, z \in X.$$

Not that, since G is compact valued, the bifunction F is well defined. By considering $K = C(x) = \mathbb{R}^+$ for all $x \in X$ and $k_0 = 1$, we show that all conditions

of Theorem 3.6 hold for bifunctions F and ψ . Since $F(x_0, x_0) = 0$, condition (i) of Theorem 3.6 is satisfied. It is easy to see that conditions (iii) and (iv) of Theorem 3.6 hold, and ψ is C - F -decreasing in the first argument and C - F -increasing in the second argument. Now, we show that F is left $C(\cdot)$ -slm. Let $\{x_n\}$ be a left convergent sequence to an element \bar{x} and $F(x_n, x_{n+1}) \leq 0$, for every $n \in \mathbb{N}$. Since G is compact valued, then for every $n \in \mathbb{N}$ there exists $y_n \in G(x_n)$ such that

$$\xi_\kappa(y_{n+1} - y_0) \leq \xi_\kappa(y_n - y_0).$$

Therefore, $y_{n+1} - y_n \in -C$. Since G satisfies the limiting monotonicity condition, there is $\bar{y} \in \text{Min}G(\bar{x})$ with $\bar{y} - y_n \in -C, n \in \mathbb{N}$. Hence,

$$F(x_n, \bar{x}) \leq \xi_\kappa(\bar{y} - y_0) - \xi_\kappa(y_n - y_0) \leq 0.$$

Also, since $G(X) \cap (y_0 + \varepsilon\kappa - C) = \emptyset$, then $\xi_\kappa(v - y_0) \geq \xi_\kappa(\varepsilon\kappa) = \varepsilon$ for every $v \in G(x)$ and $x \in X$. Therefore, $\inf_{x \in X} F(x_0, x) \geq \varepsilon$, and so $F(x_0, X) \cap (-\frac{\varepsilon}{3} - \frac{\varepsilon}{3} - K) = \emptyset$, and therefore condition (ii) of Theorem 3.6 holds. Hence, there exists $\bar{x} \in X$ such that

- (a1) $F(x_0, \bar{x}) + \frac{\varepsilon}{\lambda}q(x_0, \bar{x}) \leq 0$;
- (a2) $F(\bar{x}, x) + \frac{\varepsilon}{\lambda}(\bar{x}, x)q(\hat{x}, x) \not\leq 0$, for each $x \in X \setminus \{\bar{x}\}$.

On the other hand, by Lemma 3.1 [54] there is $\bar{y} \in G(\bar{x})$ such that $\bar{y} \in \text{Min}G(\bar{x})$ and $\min\{\xi_\kappa(u - y_0) : u \in G(\bar{x})\} = \xi_\kappa(\bar{y} - y_0)$. So, by condition (a1), we have $\xi_\kappa(\bar{y} - y_0) + \frac{\varepsilon}{\lambda}q(x_0, \bar{x}) \leq 0$. Hence, $\bar{y} - y_0 + \frac{\varepsilon}{\lambda}q(x_0, \bar{x})\kappa \in -C$. Also, condition (a2) implies $y - \bar{y} + \frac{\varepsilon}{\lambda}q(\bar{x}, x)\kappa \notin -C$ for all $x \neq \bar{x}$. ■

Remark 3.2: Notice that condition $G(X) \cap (y_0 - \varepsilon\kappa - K) = \emptyset$ is strictly weaker than quasiboundedness from below condition, see Proposition 3.6 and Example 3.9 in [39]. Moreover, in Theorem 3.4 of [6] the set-valued map G is level-closed, but in Theorem 3.11 we assume that G is compact valued. On the other hand, the comparison of condition (20) with known results of this type for set-valued mappings is discussed in Remark 3.4 of [6] and for further discussion the reader may refer to Chapter 9 of Modukhovich's book [13].

4. Application to the formation and stability of a partnership

4.1. The formation of partnerships: a variational rationality approach

A partnership is an arrangement in which two or more parties, known as partners, agree to cooperate to advance their mutual interests. The partners may be individuals, businesses, interest-based organizations, schools, governments or combinations (see [55,56], for a survey). We are interested in the dynamical formation of a partnership: 'Coming together is a beginning, staying together is

progress, and working together is success', famously said Henry Ford. Very often, a partnership is not so easy to form and unstable (rather easy to break). The formation and stability of a partnership is collective action problem. It occurs when a number of people work together to achieve some common objective. Because taking part in a collective action can be costly, and if individuals believe that the collective act will occur without their individual contributions, then they may try to free ride. Then, this free-riding problem comes from disincentives that tend to discourage joint action by individuals in the pursuit of a common goal.

4.1.1. A simple variational rational model of a partnership

To save space, let us consider a simplified version of a model given in [48] to explain more transparently the formation and stability of a partnership in the context of the (VR) variational rationality approach [43–48].

Collective action within a partnership. Consider two partners $j \in J = \{1, 2\}$ and two tasks $i \in \mathcal{T} = \{a, b\}$ that they must do jointly. A collective action (task) refers to how much of these two tasks $x = (x^a, x^b) \in X^a \times X^b$ the two partners jointly do. This means that each partner can do a part of each of these two tasks. For example, each partner performs one of these two tasks with some help from the other partner. Suppose that each time they perform a collective task $x = (x^a, x^b) \in X$ they get a joint surplus $\mathbb{S}(x) = G(x) - \Gamma(x) \in \mathbb{R}^+$, where, for example, the joint benefit of the partnership is $G(x) = \theta x^a x^b$ and joint costs of production are $\Gamma(x) \in \mathbb{R}^+$ if $X^a = X^b = \mathbb{R}^+$. In this setting, for simplification, doing a task (action) refers to spending some effort in order to provide a joint surplus.

Utilities to participate in a partnership. For each partner $j \in J$, his individual utility $g^j(y)$ to participate in the partnership in the current period must increase with the surplus. A separable formulation gives $g^j(y) = s^j(y)\mathbb{S}(y)$, $j \in J$, with $s^j = s^j(y) \in \mathbb{R}^{++}$. For a joint venture, $s^1 + s^2 = 1$. This formulation is linear if weights s_j are constant. The vectorial utility of the partnership is $g(y) = (g^1(y), g^2(y)) \in \mathbb{R}^2$.

Moving: more or less engagements in the partnership. Consider three periods: the previous, the current and the future periods. Suppose that partners want to move from $x = (x^a, x^b) \in X$, i.e., having done the collective action $x \in X$ in the previous period to $y = (y^a, y^b) \in X$, i.e., doing the collective action $y \in X$ in the current period. What occurs in the future period will be examined later. This means that partners make the collective move $x = (x^a, x^b) \curvearrowright y = (y^a, y^b)$. Then, $y^j - x^j > 0$ ($=, < 0$) is an engagement (continuation or disengagement) of the two partners in the collective task $i \in \mathcal{T}$. To save space, we do not model the concomitant disengagements and engagements in activities of each partner outside the partnership, when resources are limited; see [48]. A collective change is the move $m = x \curvearrowright y$, $y \neq x$ and a collective stay is the move $\sigma = x \curvearrowright x$.

Advantages to change rather than to stay. When partners consider changing, that is, moving from having done x in the previous period to do $y \neq x$ in the current period, they compare, in the current period,

- (i) the expected utilities $g(x) = (g^1(x), g^2(x))$ of doing x for the second time if they stay with,
- (ii) the expected utilities $h(y) = (h^1(y), h^2(y))$ to do $y \neq x$ for the first time if they change. Then, the expected vectorial advantages to move $A(x, y) = (A^1(x, y), A^2(x, y))$ are $A(x, y) = h(y) - g(x)$ if $y \neq x$ with $A(x, x) = g(x) - g(x) = 0$ if $y = x$.

Fair advantages to move. Let $K \subseteq \mathbb{R}^2$ be a convex pointed cone. Then, $m = x \curvearrowright y$ is a fair advantageous move if $A(x, y) \in K$. For example, consider the cone $K = \mathbb{R}^+ \times \mathbb{R}^+$ or an acute cone $K = \left\{ \begin{array}{l} (A^1, A^2) \geq 0, \\ \text{with } \underline{\mu}A^1 \leq A^2 \leq \bar{\mu}A^1 \end{array} \right\}$, $0 < \underline{\mu} \leq \bar{\mu} < +\infty$. This models the fact that advantages to move must be not too different for the two partners.

Inconveniences to moves as quasi-distances. A collective move $x = (x^a, x^b) \curvearrowright y = (y^a, y^b)$ requires that the two partners perform more or less of the two activities than previously. That is, they jointly spend more (less) collective efforts and resources in task $i \in T = \{a, b\}$ if $y^i - x^i > 0$ ($x^i - y^i > 0$). These joint engagements and disengagements generate joint costs of moving $q^i(x^i, y^i) = c_+^i(y^i - x^i)$ if $y^i - x^i \geq 0$ and $q^i(x^i, y^i) = c_-^i(x^i - y^i)$ if $x^i - y^i > 0$. Hence, expected joint costs of moving are, relative to each task $i \in T = \{a, b\}$, $q^i(x^i, y^i) = \begin{cases} c_+^i(y^i - x^i), & \text{if } y^i - x^i \geq 0 \\ c_-^i(x^i - y^i), & \text{if } y^i - x^i < 0 \end{cases}$ where $c_+^i, c_-^i > 0$ model unit costs of engagement and disengagement.

Expected joint costs of moving of the partnership are $q(x, y) = q^a(x^a, y^a) + q^b(x^b, y^b)$.

The two partners $j \in J = \{1, 2\}$ share a part $k^j > 0, k^1 + k^2 = 1, j = 1, 2$, of the joint costs of moving $q(x, y)$. That is, for each partner j , his expected inconvenience to move is $I^j(x, y) = k^j q(x, y)$. Let $I(x, y) = (I^1(x, y), I^2(x, y))$. Then, we have $I(x, y) = kq(x, y)$.

Furthermore, $q(\cdot, \cdot) : (x, y) \in X \times X \mapsto q(x, y) \in \mathbb{R}^+$ is a quasi-distance, given that each $q^i(\cdot, \cdot)$, $i = a, b$ is a quasi-distance. For simplification, we can suppose that inconveniences to move are known.

Worthwhile balances. Let $B(x, y) = (B^1(x, y), B^2(x, y))$ be the vector of worthwhile balances $B^j(x, y) = A^j(x, y) - \xi I^j(x, y)$, $j = 1, 2$, with $A(x, y) = (A^1(x, y), A^2(x, y))$ and $I(x, y) = (I^1(x, y), I^2(x, y))$. The scalar $\xi > 0$ weights equally (for simplification) each inconvenience to move.

Then, given what has been said before, $B(x, y) = [h(y) - g(x)] - \xi kq(x, y)$.

Let us emphasize that, in this model, a worthwhile balance depends on two kinds of sharing rules. That is, it depends of,

- (i) the shares $s = (s^1, s^2) \in \mathbb{R}^+ \times \mathbb{R}^+$, given that $g^j(x) = s^j(x)\mathbb{S}(x)$ and $g^j(y) = s^j(y)\mathbb{S}(y)$. These shares define how much is it beneficial for each partner j to participate in the joint project;
- (ii) the shares $k = (k^1, k^2) \in \mathbb{R}^+ \times \mathbb{R}^+$ of the total inconveniences to move $q(x, y)$. They define how much it is costly to participate in the joint project.

The fair and worthwhile formation of a partnership. Fair collective worthwhile moves satisfy $B(x, y) \in K$. For example, $K = \{(B^1, B^2) \geq 0, \underline{\mu}B^1 \leq B^2 \leq \bar{\mu}B^1\}$, with $0 < \underline{\mu} \leq \bar{\mu} < +\infty$. This formulation means that worthwhile balances are close enough.

4.1.2. A rational aspect of the status quo bias: when the repetition of collective action (stay) is rewarding

Let us highlight the following simple, but important behavioural result given in [48]. We give again the proof, because this will help us to generalize it later, when uncertainty matters.

Result. Expected advantages to move are superadditive when doing something for the second time is better than doing it for the first time.

Proof: Advantages to move are superadditive when, for all $x, y, z \in X$, it exists $\eta \in K$ such that $A(x, z) = A(x, y) + A(y, z) + \eta$. This is true if $g(y) - h(y) \in K$, i.e. if $g(y) = h(y) + \eta$, for some $\eta \in K$ for all $y \in X$. Let us consider three cases.

- (i) $z \neq x$ and $y \notin \{x, z\} \implies A(x, z) = A(x, y) + A(y, z) + \eta$, $\eta \in K$ if and only if, $h(z) - g(x) = h(y) - g(x) + h(z) - g(y) + \eta(y)$, i.e. if and only if $g(y) = h(y) + \eta \in K$ for all $y \in X$. This means that the expected utility $g(y) \geq_K h(y)$ of doing an activity for the second time is high enough relative to the expected utility $h(y)$ of doing the same activity for the first time.
- (ii) $z \neq x$ and $y = x \implies A(x, z) = A(x, x) + A(x, z) = A(x, z)$ because $A(x, x) = 0$.
- (iii) $z \neq x$ and $y = z \implies A(x, z) = A(x, z) + A(z, z) = A(x, z)$ because $A(z, z) = 0$. ■

The inequality $h(y) = g(y) - \eta(y)$ where $\eta = \eta(y) \in K$ occurs when, in a partnership, partners have optimistic expectations relative to an expanding market, or higher productivities (learning by doing). In this way, the utility of doing something new is penalized with respect to the utility of doing something for the second time (from increased benefits due to repetition). This effect has to do with the famous exploration–exploitation trade-off [57]. We can also see this effect as a rational aspect of the famous status quo bias ([58]).

In this context, given that $A(x, x) = 0$ and $A(x, y) = h(y) - g(x)$ if $y \neq x$, we get two formulations of expected advantages to move,

- (i) $A(x, y) = [g(y) - \eta(y)] - g(x) = g(y) - g(x) - \eta(y) = v \in K$; or,
- (ii) $A(x, y) = h(y) - [h(x) + \eta(x)] = h(y) - h(x) - \eta(x) = v \in K$.

That is, if $y \neq x$, $A(x, y) \in K$ if and only if $g(y) - g(x) \geq_K \eta(y)$ or $h(y) - h(x) \geq_K \eta(x)$.

Then, a worthwhile balance can be $B(x, y) = [g(y) - g(x) - \eta(y)] - \xi kq(x, y)$, or $B(x, y) = [h(y) - h(x) - \eta(x)] - \xi kq(x, y)$.

4.1.3. A formulation of a worthwhile move in terms of frustration feelings

Aspiration levels and frustration feelings. Let $\bar{h}^j = \sup\{h^j(x), x \in X\} < +\infty$ be the aspiration level of partner $j \in J = \{1, 2\}$ when, for example, the two partners perform the collective action x for the first time (the case of doing the collective action for the second time being similar). Then, $\bar{h} = (\bar{h}^1, \bar{h}^2)$ is the aspiration (ideal) point of the partnership. Let $f^j(x) = \bar{h}^j - h^j(x) \in \mathbb{R}^+$ be the frustration feeling of partner j of not doing his best for the first time. Thus, the vector $f(x) = (f^1(x), f^2(x))$ defines the unsatisfaction levels of the two partners when they do x for the first time. We have $f(x) = \bar{h} - h(x) \in \mathbb{R}^+ \times \mathbb{R}^+$. Then, when moving from x to $y \neq x$, $f(x) - f(y) = h(y) - h(x)$ while, as seen before, advantages to move are $A(x, y) = h(y) - h(x) - \eta(x)$. As a consequence, advantages to move are the difference between frustration feelings $f(x)$ and $f(y)$, i.e. $A(x, y) = f(x) - f(y) - \eta(x)$.

Loss to move. They are the opposite of advantages to move, i.e. $F(x, y) = -A(x, y) = f(y) - f(x) + \eta(x)$.

Worthwhile moves in term of frustration feelings. The opposite of a worthwhile balance is $L(x, y) = -B(x, y) = -A(x, y) + \xi kq(x, y) = F(x, y) + \xi kq(x, y)$.

Then, a worthwhile move $m = x \curvearrowright y$ is such that $B(x, y) = A(x, y) - \xi kq(x, y) \in K$, i.e. such that $L(x, y) = F(x, y) + \xi kq(x, y) \in -K$.

4.2. First result of this paper: the formation of a robust partnership in a changing and complex environment

In this paper, taking advantage of the previous example relative to the formation of a partnership, we give a new application of our generalized and robust variational principles in the context of the variational rationality approach. This application is relative to the formation and stability of a partnership that generalizes Qiu et al. [14] in different contexts. We,

- (i) give a new and explicit set-valued formulation of the subadditive inclusion $F(x, z) \subseteq F(x, y) + F(y, z) - K$;
- (ii) provide with a new set-valued formulation of a worthwhile move $F(x, y) + \psi kq(x, y) \subseteq -K$ that models the worthwhile formation of a partnership.

- (iii) consider changing weights on resistance to move leading to two different formulations of a robust worthwhile move, $F(x, y) + \psi(x, y)kq(x, y) \subseteq -K$. This formulation is new;
- (iv) introduce changing preferences (moving cones $C(x)$). This formulation is new in the context of robust variational principles.

4.2.1. A new set-valued formulation of the superadditivity of advantages to move

Let us give a behavioural interpretation of the superadditivity of a set-valued loss to move function, i.e. $F(x, z) \subseteq F(x, y) + F(y, z) - K$.

Suppose now that partners do not know how much it is rewarding to repeat a collective action x before doing it again. That is, they do not know $g(x)$, i.e. they do not know the difference $g(x) - h(x) = \eta(x) \in K$, given that $h(x)$ is known. Then, set $g(x) = h(x) + \eta$, with $\eta \in K$, η unknown. In this setting advantages to move $A(x, y) = h(y) - g(x)$ from x to y can be written as $A(x, y) = h(y) - [h(x) + \eta] = h(y) - h(x) - \eta = \mathbb{A}(x, y, \eta)$.

Then, the superadditive condition $A(x, z) = A(x, y) + A(y, z) + v$, $v \in K$ means that

$$\mathbb{A}(x, z, \eta) = \mathbb{A}(x, y, \eta') + \mathbb{A}(y, z, \eta'') + v (**), \text{ where,}$$

$$\mathbb{A}(x, z, \eta) = h(z) - h(x) - \eta, \eta \in K,$$

$$\mathbb{A}(x, y, \eta') = h(y) - h(x) - \eta', \eta' \in K, \text{ and}$$

$$\mathbb{A}(y, z, \eta'') = h(z) - h(y) - \eta'', \eta'' \in K.$$

This shows that condition $(**)$ is satisfied if and only if it exists $\eta, \eta', \eta'' \in K$ such that

$$-\eta = -\eta' - \eta'' + v, \text{ i.e. if } v = \eta' + \eta'' - \eta \in K. \text{ This is true if } \eta' = \eta.$$

Let $\mathbf{A}(x, z) = \{\mathbb{A}(x, z, \eta), \eta \in K\}$, $\mathbf{A}(x, y) = \{\mathbb{A}(x, y, \eta'), \eta' \in K\}$ and $\mathbf{A}(y, z) = \{\mathbb{A}(y, z, \eta''), \eta'' \in K\}$, then, $\mathbf{A}(x, z) \subseteq \mathbf{A}(x, y) + \mathbf{A}(y, z) + K$ means that for all $\eta \in K$, it exist $\eta', \eta'' \in K$ and $v \in K$ such that $\mathbb{A}(x, z, \eta) = \mathbb{A}(x, y, \eta') + \mathbb{A}(y, z, \eta'') + v$.

This shows that $\mathbf{A}(x, z) \subseteq \mathbf{A}(x, y) + \mathbf{A}(y, z) + K$. Then, given that $F(x, y) = -A(x, y)$, $F(x, y) = \mathbb{F}(x, y, \eta)$ and $A(x, y) = \mathbb{A}(x, y, \eta)$, we have $\mathbb{F}(x, y, \eta) = -\mathbb{A}(x, y, \eta)$. This shows that $\mathbb{F}(x, z, \eta) = \mathbb{F}(x, y, \eta') + \mathbb{F}(y, z, \eta'') - v$.

Let $\mathbf{F}(x, z) = \{\mathbb{F}(x, z, \eta), \eta \in K\}$, $\mathbf{F}(x, y) = \{\mathbb{F}(x, y, \eta'), \eta' \in K\}$ and $\mathbf{F}(y, z) = \{\mathbb{F}(y, z, \eta''), \eta'' \in K\}$. Then, the inclusion $\mathbf{A}(x, z) \subseteq \mathbf{A}(x, y) + \mathbf{A}(y, z) + K$ is equivalent to the inclusion $\mathbf{F}(x, z) \subseteq \mathbf{F}(x, y) + \mathbf{F}(y, z) - K$.

4.2.2. The formation of a robust and worthwhile partnership

Starting from having done the collective action x , a partnership evolves when it can make a collective worthwhile move $x \curvearrowright y$. That is, when $B(x, y) = A(x, y) - \xi kq(x, y) \in K$, i.e. when $L(x, y) = -B(x, y) = F(x, y) + \xi kq(x, y) \in -K$. Let $A(x, y) = h(y) - g(x)$ and $g(x) - h(x) = \eta \in K$ as before. Then, $A(x, y) = \mathbb{A}(x, y, \eta) = h(y) - h(x) - \eta$. This implies $F(x, y) = -A(x, y) = h(x) - h(y) +$

$\eta = \mathbb{F}(x, y, \eta) = -\mathbb{A}(x, y, \eta)$. Then, $\mathbb{B}(x, y, \eta) = \mathbb{A}(x, y, \eta) - \xi kq(x, y)$ and $\mathbb{L}(x, y, \eta) = -\mathbb{B}(x, y, \eta) = \mathbb{F}(x, y, \eta) + \xi kq(x, y)$.

Conclusion. Given that $\mathbf{A}(x, y) = \{\mathbb{A}(x, y, \eta), \eta \in K\}$ and $\mathbf{F}(x, y) = \{\mathbb{F}(x, y, \eta), \eta \in K\}$, a move $x \rightsquigarrow y$ is worthwhile if $\mathbf{B}(x, y) = \mathbf{A}(x, y) - \xi kq(x, y) \subseteq K$. That is, if $\mathbf{L}(x, y) = -\mathbf{B}(x, y) = \mathbf{F}(x, y) + \xi kq(x, y) \subseteq -K$.

4.2.3. Changing preferences

In this paper, preferences over worthwhile balances change with the status quo x when the cone $C = C(x)$ depends of x . For example, the acute cone $C(x) = \{0, 0\} \cup \left\{ \text{with } \begin{array}{l} (B^1, B^2) \geq 0, \\ \underline{\mu}(x)B^1 \leq B^2 \leq \bar{\mu}(x)B^1 \end{array} \right\}$, with $0 < \underline{\mu}(x) \leq \bar{\mu}(x) < +\infty$, is a moving cone. This formulation means that (i) the ratio between worthwhile balances is bounded (a fair constraint between partners) and (ii) bounds are changing (the fair constraint changes with the status quo). In this paper, $K \subseteq \bigcup_{x \in X} C(x)$.

4.2.4. Variable weights given to resistance to move

We consider now two different situations where, given a move $m = (x, y)$, and the worthwhile balance $\mathbb{B}(x, y, \eta) = \mathbb{A}(x, y, \eta) - \xi I(x, y)$, the weight given to resistance to move $\xi = \psi(x, y) > 0$ depends both of the status quo x and of the desired end y . Consider, for an easier illustration, the two specific cases given in Theorems 3.8 and 3.9.

First case: $\xi = \psi(x, y) = \delta(x) > 0$. In this situation, we will suppose that the weight $\psi(\cdot)$ is F -decreasing. This means that, if the given move $m = (x, y)$ is advantageous, i.e. if $\mathbb{A}(x, y, \eta) = -\mathbb{F}(x, y, \eta) \in K$, then the weight assigned to resistance to the move increases when moving from x to y , i.e. $\delta(y) \geq \delta(x)$. This model the famous ego depletion effect coming from physiological, physical and mental fatigue after having done a succession of tasks [59].

Second case: $\xi = \psi(x, y) = \delta(y) > 0$. In this situation, we will suppose that the weight $\delta(\cdot)$ is F -increasing. This means that if the given move $m = (x, y)$ is advantageous, i.e. if $\mathbb{A}(x, y, \eta) = -\mathbb{F}(x, y, \eta) \in K$, then the weight assigned to resistance to move decreases when moving from x to y , i.e. $\delta(y) \leq \delta(x)$. This model the well-known goal gradient effect ([60]). The goal-gradient hypothesis states that the tendency to approach a goal increases with proximity to the goal. This is the case if an individual gives less importance to the resistance to move.

4.3. Second result of this paper: robust variational principles and the stability of partnership

4.3.1. Robust EVP principles and the stability of a partnership: robust traps

We are now in a good position to show how the mathematical part of our paper provides the existence of two periods go and stop worthwhile dynamic. That is, in the context of our example, how it proves the formation (go) and the stability (stay) of a partnership in a changing and complex environment, when,

- (i) vectorial advantages to change depend on an uncertain status quo bias η ;
- (ii) preferences are changing: they are defined by a moving cone $C(x), x \in X$;
- (iii) the importance of resistance to moving depends on the starting or endpoint of a move.

In this way, our set-valued Ekeland equilibrium theorem gives sufficient conditions for the existence of robust traps in a partly unknown and changing environment.

Behavioural interpretations of our robust EVP theorems. Given what we have shown before, the behavioural signification of our set-valued Ekeland equilibrium theorems, for example Theorem 3.6, is the following. It tells us that, starting from any initial position x_0 , there exists a robust trap $\hat{x} \in X$ such that,

- (a) it is worthwhile to directly move from x_0 to \hat{x} . That is,

$$F(x_0, \hat{x}) + \psi(x_0, \hat{x})q(x_0, \hat{x})k_0 \subseteq -C(x_0).$$

- (b) it is not worthwhile to move from \hat{x} to any $x \neq \hat{x}$. That is,

$$F(\hat{x}, x) + \psi(\hat{x}, x)q(\hat{x}, x)k_0 \not\subseteq -C(\hat{x}), \forall x \neq \hat{x}.$$

To save space, we will leave it to the reader to interpret the technical hypothesis of Theorem 3.6 in terms of the VR approach ([46,47]). This is easy.

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