# Terminating Calculi and Countermodels for Constructive Modal Logics

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**Abstract.** We investigate terminating sequent calculi for constructive modal logics $\mathsf{CK}$ and $\mathsf{CCDL}$ in the style of Dyckhoff's calculi for intuitionistic logic. We first present strictly terminating calculi for these logics. Our calculi provide immediately a decision procedure for the respective logics and have good proof-theoretical properties, namely they allow for a syntactic proof of cut admissibility. We then present refutation calculi for non-provability in both logics. Their main feature is that they support direct countermodel extraction: each refutation directly defines a finite countermodel of the refuted formula in a natural neighbourhood semantics for these logics.

**Keywords:** Modal logic · Intuitionistic logic · Constructive modal logics · Sequent calculus · Refutation · Countermodels.

## 1 Introduction

Intuitionistic modal logic has a long history going back to the pioneering work by Fitch [8] in the late 40's and then by Prawitz [20] in the 60's. It is not possible to retrace here the whole history. It is now clear that there are two traditions leading to two distinct families of systems. The first one, called Intuitionistic modal logics have been introduced by Fischer Servi [7] and Plotkin and Stirling [19] and then systematised by Simpson [21] whose main goal is to define an analogous of classical modalities justified from an intuitionistic meta-theory. Simpson's basic systems is modal logic $\mathsf{IK}$, intended to be the intuitionistic counterpart of minimal normal modal logic $\mathsf{K}$. The second one, called Constructive modal logics, are mainly motivated by their applications to computer science, such as the type-theoretic interpretations (Curry-Howard correspondence, typed lambda calculi), verification and knowledge representation, together with their mathematical semantics. This second tradition has been developed independently,

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first by Wijesekera [23] who proposed the system CCDL (Constructive Concurrent Dynamic logic), and then by Bellin, De Paiva, and Ritter [2], among the others who proposed the logic CK (Constructive K) as the basic system for a constructive account of modality (see also the survey [22] and the references therein). Wijesekera's propositional CCDL was originally motivated as a logic of partial observations of concurrent actions, whereas CK can be also interpreted as a logic of contextual reasoning [16]. From an axiomatic point of view all systems (including Simpson's IK) share the same $\Box$-fragment, but they differ on the interpretation of diamond and interaction between the two modalities, in particular CCDL rejects diamond distribution over disjunction:

$$\Diamond(A \vee B) \to \Diamond A \vee \Diamond B$$

which is an axiom of IK, in addition CK further rejects its nullary version:

$$\neg\Diamond\bot$$

which is valid in CCDL.

The system CK has been extensively investigated from a proof-theoretical point of view: in addition to its Gentzen sequent calculus, a natural deduction system for it has been proposed [2], which leads to a type-theoretical interpretation of CK within an extended Lambda-calculus. Further proof systems for CK exist in the form of nested sequent calculus [1] and focused 2-sequent calculus [17], whereas a tableaux calculus for full CCDL is presented in [24].

From a semantical point of view, both CCDL and CK enjoy a Kripke semantics in terms of bi-relational Kripke models [23, 16], although in order to accommodate the failure of $\neg\Diamond\bot$ Kripke models for CK must be equipped with "inconsistent" worlds which force $\bot$. The failure of distribution of $\Diamond$ over disjunction makes $\Diamond$ a non-normal modality, so that it does not come as a surprise that the semantic tools for non-normal modal logics can be employed for analysing these logics. For CCDL Kojima [15] has proposed a semantics in terms of intuitionistic neighbourhood models (see also [11] for neighbourhood models of intuitionistic logics with only $\Box$). More recently an alternative semantics in terms of neighbourhood models has been provided in [3], in that semantics models are equipped with two neighbourhood functions for interpreting the two modalities, this semantics accounts uniformly both CCDL and CK without the need of "inconsistent" worlds. Moreover in both cases finite neighbourhood models can be transformed into relational models of the corresponding logics (but the obtained model may be much larger). This is the intended semantics for both CCDL and CK we consider in this work.

Despite the amount of research on proof systems, decision procedures based on proof systems have not been studied,[3] and there is no work on countermodel generation from failed derivations in sequent calculi neither for CK, nor for CCDL, which is the aim of this work. We are interested here in developing terminating

[3] Decidability for these logics follows from the finite model property established in Mendler and de Paiva [16] and Dalmonte et al. [3].

proof systems that can be used also to extract countermodels from failed proof search. Our starting point is the calculus $\mathsf{G4ip}'$ proposed by Dyckhoff [4]: his calculus has the form of a multiple-succedent sequent calculus comprising special decomposition rules; its main feature is that it is terminating in itself, without any control on proof-search. This calculus has been extended by Iemhoff [13] to intuitionistic/constructive modal logic, but only for the $\Box$-fragment (on which all systems, namely Simpson's $\mathsf{IK}$, $\mathsf{CCDL}$ and $\mathsf{CK}$ coincide). Extending Iemhoff's work, our first contribution is the proposition of terminating calculi for both $\mathsf{CK}$ and $\mathsf{CCDL}$ in their full language with both modalities. The two calculi provide then immediately a decision procedure for the respective logics. Moreover the calculi have good proof theoretical properties, first of all they allow a syntactic proof of cut-elimination.

Next we define a refutation calculus which allows for countermodel extraction. Our starting point is the refutation calculus $\mathsf{CRIP}$ for intuitionistic logic proposed by Pinto and Dyckhoff [18]: in this calculus a derivation, or better a refutation, directly provides a countermodel of the root-formula. In Pinto and Dyckhoff's view: Kripke countermodels are witnesses of refutations, as much as lambda terms are witnesses of proofs. We propose terminating refutation calculi for both $\mathsf{CK}$ and $\mathsf{CCDL}$. From one refutation in these calculi it can be defined *directly* a countermodel of the checked formula/sequent, namely a countermodel in the neighbourhood semantics mentioned above. In contrast we are not aware of any calculus for any of these two logics which allows for countermodel extraction within the original relational semantics.

The fact that a refutation corresponds directly to a neighbourhood countermodel confirms the significance of the neighbourhood semantics for these logics, thereby extending Pinto and Dyckhoff's views: neighbourhood countermodels are *the* natural witnesses of refutations for constructive modal logics.

## 2 Constructive modal logics and their semantics

In this section we present the constructive modal logics $\mathsf{CK}$ and $\mathsf{CCDL}$ in the form of axiomatic systems as well as their neighbourhood semantics. $\mathsf{CK}$ and $\mathsf{CCDL}$ are defined in a propositional modal language $\mathcal{L}$ based on a set $Atm = \{p_1, p_2, p_3, ...\}$ of countably many propositional variables; the *well-formed formulas* of $\mathcal{L}$ are generated by the following grammar, where $p_i$ is any element of $Atm$:

$$A ::= p_i \mid \bot \mid A \wedge A \mid A \vee A \mid A \supset A \mid \Box A \mid \Diamond A.$$

In the following, we call 'atomic formulas' the propositional variables and $\bot$, we call 'atomic implication' every implication whose antecedent is an atomic formula, finally we call '$\Box$-formula', resp. '$\Diamond$-formula', every formula whose outermost connective is $\Box$, resp. $\Diamond$. As usual we define $\neg A$ as $A \supset \bot$.

**Definition 1.** *The logic* $\mathsf{CK}$ *is defined by extending (any axiomatisation of) intuitionistic propositional logic, formulated in the modal language* $\mathcal{L}$*, with the following modal axioms and rules:*

$$Nec\ \frac{A}{\Box A} \qquad K_{\Box}\ \Box(A \supset B) \supset (\Box A \supset \Box B) \qquad K_{\Diamond}\ \Box(A \supset B) \supset (\Diamond A \supset \Diamond B).$$

*The logic* CCDL *is defined by extending* CK *with the additional axiom*

$$N_{\Diamond}\ \neg\Diamond\bot.$$

In the following we denote by $\mathsf{C}^*$ any of the two logics. CK and CCDL have both relational [23, 16] and neighbourhood semantics [15, 3]. Independently from its interest in itself, one of the advantages of the neighbourhood semantics is that, as we shall see, our refutation calculi directly build a neighbourhood countermodel of every refuted formula, whether the same does not seem to be the case with relational models. Here we consider a minor variation of the neighbourhood semantics of [3] (as explained below) which allows for a more immediate extraction of countermodels from the calculi.

**Definition 2.** *A* neighbourhood model *for* CK *is a tuple* $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_{\Box}, \mathcal{N}_{\Diamond}, \mathcal{V}\rangle$ *where:* $\mathcal{W}$ *is a non-empty set;* $\preceq$ *is a preorder over* $\mathcal{W}$*;* $\mathcal{V}$ *is a valuation function* $Atm \rightarrow \mathcal{P}(\mathcal{W})$ *satisfying the hereditary condition:*

$$\textit{if } w \in \mathcal{V}(p) \textit{ and } w \preceq v, \textit{ then } v \in \mathcal{V}(p);$$

*and* $\mathcal{N}_{\Box}$ *and* $\mathcal{N}_{\Diamond}$ *are two neighbourhood functions* $\mathcal{W} \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}))$ *satisfying the following conditions:*

| | |
|---|---|
| *if* $w \preceq v$*, then* $\mathcal{N}_{\Box}(w) \subseteq \mathcal{N}_{\Box}(v)$ *and* $\mathcal{N}_{\Diamond}(w) \subseteq \mathcal{N}_{\Diamond}(v)$ | ($\Box$*- and* $\Diamond$*-monotonicity*) |
| *if* $\alpha \in \mathcal{N}_{\Box}(w)$ *and* $\alpha \subseteq \beta$*, then* $\beta \in \mathcal{N}_{\Box}(w)$ | ($\Box$*-supplementation*) |
| *if* $\alpha \in \mathcal{N}_{\Diamond}(w)$ *and* $\alpha \subseteq \beta$*, then* $\beta \in \mathcal{N}_{\Diamond}(w)$ | ($\Diamond$*-supplementation*) |
| $\mathcal{W} \in \mathcal{N}_{\Box}(w)$ | ($\Box$*-containing the unit*) |
| *if* $\alpha, \beta \in \mathcal{N}_{\Box}(w)$*, then* $\alpha \cap \beta \in \mathcal{N}_{\Box}(w)$ | ($\Box$*-intersection closure*) |
| *if* $\alpha \in \mathcal{N}_{\Box}(w)$ *and* $\beta \in \mathcal{N}_{\Diamond}(w)$*, then* $\alpha \cap \beta \in \mathcal{N}_{\Diamond}(w)$ | ($\Box\Diamond$*-intersection closure*) |

*A neighbourhood model for* CCDL *is any neighbourhood model for* CK *where* $\mathcal{N}_{\Diamond}$ *satisfies the following additional condition:*

$$\emptyset \notin \mathcal{N}_{\Diamond}(w) \qquad (\Diamond\textit{-consistency}).$$

*The forcing relation* $\mathcal{M}, w \Vdash A$ *is defined as follows, where* $[\![B]\!]$ *denotes the set* $\{v \in \mathcal{W} \mid \mathcal{M}, v \Vdash B\}$ *of the worlds forcing* $B$ *in* $\mathcal{M}$*:*

| | | |
|---|---|---|
| $\mathcal{M}, w \Vdash p$ | *iff* | $w \in \mathcal{V}(p)$*;* |
| $\mathcal{M}, w \nVdash \bot$*;* | | |
| $\mathcal{M}, w \Vdash B \wedge C$ | *iff* | $\mathcal{M}, w \Vdash A$ *and* $\mathcal{M}, w \Vdash B$*;* |
| $\mathcal{M}, w \Vdash B \vee C$ | *iff* | $\mathcal{M}, w \Vdash A$ *or* $\mathcal{M}, w \Vdash B$*;* |
| $\mathcal{M}, w \Vdash B \supset C$ | *iff* | *for every* $v \succeq w$*,* $\mathcal{M}, v \Vdash B$ *implies* $\mathcal{M}, v \Vdash C$*;* |
| $\mathcal{M}, w \Vdash \Box B$ | *iff* | $[\![B]\!] \in \mathcal{N}_{\Box}(w)$*;* |
| $\mathcal{M}, w \Vdash \Diamond B$ | *iff* | $[\![B]\!] \in \mathcal{N}_{\Diamond}(w)$. |

In the following we simply write $w \Vdash A$ when $\mathcal{M}$ is clear from the context. It is easy to prove that neighbourhood models for CK and CCDL satisfy the *hereditary property* (cf. [3]):

$$\mathsf{init}\ \frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad \mathsf{L}\bot\ \frac{}{\Gamma, \bot \Rightarrow \Delta} \qquad \mathsf{R}\wedge\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

$$\mathsf{L}\wedge\ \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad \mathsf{R}\vee\ \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta} \qquad \mathsf{L}\vee\ \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}$$

$$\mathsf{R}\supset\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B, \Delta} \qquad \mathsf{L0}\supset\ \frac{\Gamma, p, B \Rightarrow \Delta}{\Gamma, p, p \supset B \Rightarrow \Delta} \qquad \mathsf{L}\wedge\supset\ \frac{\Gamma, C \supset (D \supset B) \Rightarrow \Delta}{\Gamma, (C \wedge D) \supset B \Rightarrow \Delta}$$

$$\mathsf{L}\vee\supset\ \frac{\Gamma, C \supset B, D \supset B \Rightarrow \Delta}{\Gamma, (C \vee D) \supset B \Rightarrow \Delta} \qquad \mathsf{L}\supset\supset\ \frac{\Gamma, C, D \supset B \Rightarrow D \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, (C \supset D) \supset B \Rightarrow \Delta}$$

**Fig. 1.** Rules of $\mathsf{G4ip}'$ [4, 5].

$$\text{for all } A \in \mathcal{L}, \text{ if } w \Vdash A \text{ and } w \preceq v, \text{ then } v \Vdash A.$$

Moreover, the equivalence of this semantics with the one of [3] can be easily shown with model transformations. Given a model $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ either as in Def. 2 or of the kind of [3], an equivalent model of the other kind can be obtained by taking the same $\mathcal{W}$, $\preceq$, $\mathcal{N}_\Box$ and $\mathcal{V}$, and defining $\mathcal{N}'_\Diamond(w) = \{\alpha \subseteq \mathcal{W} \mid \mathcal{W} \setminus \alpha \notin \mathcal{N}_\Diamond(w)\}$ for every $w \in \mathcal{W}$. By relying on the completeness result of [3] we then have:

**Theorem 1.** *The logics* $\mathsf{CK}$ *and* $\mathsf{CCDL}$ *are sound and complete with respect to the corresponding neighbourhood models.*

## 3 Sequent calculi

In this section we present G4-style sequent calculi for the logics $\mathsf{CK}$ and $\mathsf{CCDL}$. The calculi have the property that for every rule the complexity of the premiss(es) is strictly lower than the complexity of the conclusion (with respect to a suitable notion of complexity). From this it follows that bottom-up proof search always terminates. We show that the structural rules of weakening, contraction, and cut are admissible, and obtain thereby a proof of completeness of the calculi with respect to the axiomatic systems. As a consequence, bottom-up proof search in the calculi provides a decision procedure for the logics.

In the following, we denote by capital Greek letters $\Gamma, \Delta, \Sigma, \Pi$ possibly empty *multisets* of formulas of $\mathcal{L}$. If $\Gamma$ is the multiset $A_1, ..., A_n$, we respectively denote by $\Box\Gamma$ and $\Diamond\Gamma$ the multisets $\Box A_1, ..., \Box A_n$ and $\Diamond A_1, ..., \Diamond A_n$ (whence $\Box\Gamma$ and $\Diamond\Gamma$ only contain $\Box$-, resp. $\Diamond$-, formulas). We call *sequent* any pair $\Gamma \Rightarrow \Delta$ of multisets of formulas. As usual, sequents are interpreted in the language $\mathcal{L}$ as $\bigwedge \Gamma \supset \bigvee \Delta$ if $\Gamma$ is non-empty, and are interpreted as $\bigvee \Delta$ if $\Gamma$ is empty, where $\bigvee \emptyset$ is interpreted as $\bot$. We consider the following notions of weight of formulas and multiset ordering of sequents.

**Definition 3 (Weight of formulas and multiset ordering of sequents).** *For every formula $A$ of $\mathcal{L}$, its weight $wg(A)$ is defined as follows: $wg(\bot) = 0$;*

$$\mathsf{K}_\Box\ \frac{\Sigma \Rightarrow B}{\Gamma, \Box\Sigma \Rightarrow \Box B, \Delta} \qquad \mathsf{K}_\Diamond\ \frac{\Sigma, B \Rightarrow C}{\Gamma, \Box\Sigma, \Diamond B \Rightarrow \Diamond C, \Delta} \qquad \mathsf{N}_\Diamond\ \frac{\Sigma, B \Rightarrow}{\Gamma, \Box\Sigma, \Diamond B \Rightarrow \Delta}$$

$$\mathsf{L}\Box{\supset}\ \frac{\Sigma \Rightarrow C \qquad \Gamma, \Box\Sigma, B \Rightarrow \Delta}{\Gamma, \Box\Sigma, \Box C \supset B \Rightarrow \Delta} \qquad \mathsf{L}\Diamond{\supset}\ \frac{\Sigma, D \Rightarrow C \qquad \Gamma, \Box\Sigma, \Diamond D, B \Rightarrow \Delta}{\Gamma, \Box\Sigma, \Diamond D, \Diamond C \supset B \Rightarrow \Delta}$$

**Fig. 2.** Modal rules of G4.CK and G4.CCDL.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q, p \Rightarrow q}\ \mathsf{init}}{p \supset q, p \Rightarrow q}\ \mathsf{L0}{\supset}}{\Box(p \supset q), \Box p \Rightarrow \Box q}\ \mathsf{K}_\Box}{\Box(p \supset q) \Rightarrow \Box p \supset \Box q}\ \mathsf{R}{\supset}}{\Rightarrow \Box(p \supset q) \supset (\Box p \supset \Box q)}\ \mathsf{R}{\supset} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q, p \Rightarrow q}\ \mathsf{init}}{p \supset q, p \Rightarrow q}\ \mathsf{L0}{\supset}}{\Box(p \supset q), \Diamond p \Rightarrow \Diamond q}\ \mathsf{K}_\Diamond}{\Box(p \supset q) \Rightarrow \Diamond p \supset \Diamond q}\ \mathsf{R}{\supset}}{\Rightarrow \Box(p \supset q) \supset (\Diamond p \supset \Diamond q)}\ \mathsf{R}{\supset} \qquad \dfrac{\dfrac{\dfrac{}{\bot \Rightarrow}\ \mathsf{L}\bot}{\Diamond\bot \Rightarrow \bot}\ \mathsf{N}_\Diamond}{\Rightarrow \Diamond\bot \supset \bot}\ \mathsf{R}{\supset}$$

**Fig. 3.** Derivations of $K_\Box$ and $K_\Diamond$ in G4.C* and of $N_\Diamond$ in G4.CK.

$wg(p_i) = 1$ *for every* $p_i \in Atm$*;* $wg(A \supset B) = wg(A) + wg(B) + 1$*;* $wg(A \wedge B) = wg(A) + wg(B) + 2$*;* $wg(A \vee B) = wg(A) + wg(B) + 3$*; and* $wg(\Box A) = wg(\Diamond A) = wg(A) + 1$*. Then we define* $\Gamma \ll \Sigma$ *iff* $\Gamma$ *is the result of replacing one or more formulas in* $\Sigma$ *by zero or more formulas of lower weight; and* $\Gamma \Rightarrow \Delta \ll \Sigma \Rightarrow \Pi$ *iff* $\Gamma, \Delta \ll \Sigma, \Pi$*.*

In Fig. 1 it is displayed Dyckhoff's multi-succedent sequent calculus G4ip′ for intuitionistic logic [4], with the rule L⊃⊃ formulated as in [5]. The main peculiarity of Dyckhoff's calculus is that it terminates without need of loop-checking. This is obtained by considering four left implication rules rather than a single one, namely one rule for every possible outermost connective in the antecedent of the principal implication. As a consequence, the resulting calculus has the property that the premisses of every rule have a smaller complexity than the conclusion with respect to the multiset ordering of Def. 3.

By extending Dyckhoff's calculus with suitable rules for the modalities we now define the calculi G4.CK and G4.CCDL for constructive modal logics.

**Definition 4.** *The calculi* G4.CK *and* G4.CCDL *are defined by extending the calculus* G4ip′ *in Fig. 1 with the following sets of rules from Fig. 2:*

$$\begin{array}{l} \mathsf{G4.CK} := \mathsf{G4ip'} \cup \{\mathsf{K}_\Box, \mathsf{K}_\Diamond, \mathsf{L}\Box{\supset}, \mathsf{L}\Diamond{\supset}\}. \\ \mathsf{G4.CCDL} := \mathsf{G4.CK} \cup \{\mathsf{N}_\Diamond\}. \end{array}$$

The rules $\mathsf{K}_\Box$, $\mathsf{K}_\Diamond$, and $\mathsf{N}_\Diamond$ are the multi-conclusion formulation of the standard modal rules of sequent calculi for CK and CCDL (see e.g. [23]). In the spirit of G4ip′, the calculi G4.CK and G4.CCDL also contain two additional left implication rules, namely L□⊃ and L◊⊃, which take care of the □- or ◊-formulas occurring in the antecedent of an implication. The rule L□⊃ comes from [13] where a G4-stlyle calculus for the intuitionistic monomodal □-version of logic K is presented. Since this logic coincides with the ◊-free fragment of CK and CCDL the same rule is also adequate for our calculi. Moreover, the rule $\mathsf{L}\Diamond\supset$ reflects the different behaviour of the modality $\Diamond$, which is captured in the calculus by the rule $\mathsf{K}_\Diamond$, and requires the presence of a $\Diamond$-formula in addition to the principal implication. We point out that multi-succedent sequents are not necessary in order to define sequent calculi for CK, nor for CCDL: indeed analogous calculi could be formulated extending Dyckhoff's single-succedent calculus G4ip [4]. The reason for considering the multi-succedent version of the calculus is that it allows for a more immediate transformation into a refutation calculus, as we will see in the next section.

Some examples of derivation in the calculi $\mathsf{G4.C}^*$ are displayed in Fig. 3. It is easy to see that for every rule of $\mathsf{G4.C}^*$, the premisses have a smaller complexity than the conclusion with respect to the multiset ordering of Def. 3 (in particular the premisses of the modal rules only contain subformulas of formulas in the conclusion). Therefore it holds:

**Theorem 2.** *Backward proof search in* $\mathsf{G4.C}^*$ *always terminates after a finite number of steps.*

We now prove that the calculi $\mathsf{G4.C}^*$ are equivalent to the corresponding axiomatic systems. On the one hand, it is possible to show that all the rules of $\mathsf{G4.C}^*$ are derivable in $\mathsf{C}^*$. As an example, the derivation of the rule $\mathsf{L}\Diamond\supset$ in $\mathsf{C}^*$ is as follows:

| | | |
|---|---|---|
| 1. | $\bigwedge\Sigma \wedge D \supset C$ | (assumption) |
| 2. | $\bigwedge\Sigma \supset (D \supset C)$ | (1, IPL) |
| 3. | $\Box\bigwedge\Sigma \supset \Box(D \supset C)$ | (2, $Nec + K_\Box$) |
| 4. | $\bigwedge\Box\Sigma \supset \Box(D \supset C)$ | (3, $Nec + K_\Box$) |
| 5. | $\bigwedge\Box\Sigma \supset (\Diamond D \supset \Diamond C)$ | (4, $K_\Diamond$) |
| 6. | $\bigwedge\Box\Sigma \wedge \Diamond D \supset \Diamond C$ | (5, IPL) |
| 7. | $\bigwedge\Gamma \wedge \bigwedge\Box\Sigma \wedge \Diamond D \wedge (\Diamond C \supset B) \supset$ $\bigwedge\Gamma \wedge \bigwedge\Box\Sigma \wedge \Diamond D \wedge (\bigwedge\Box\Sigma \wedge \Diamond D \supset \Diamond C) \wedge (\Diamond C \supset B)$ | (6, IPL) |
| 8. | $\bigwedge\Gamma \wedge \bigwedge\Box\Sigma \wedge \Diamond D \wedge (\bigwedge\Box\Sigma \wedge \Diamond D \supset \Diamond C) \wedge (\Diamond C \supset B) \supset$ $\bigwedge\Gamma \wedge \bigwedge\Box\Sigma \wedge \Diamond D \wedge B$ | (IPL) |
| 9. | $\bigwedge\Gamma \wedge \bigwedge\Box\Sigma \wedge \Diamond D \wedge B \supset \bigvee\Delta$ | (assumption) |
| 10. | $\bigwedge\Gamma \wedge \bigwedge\Box\Sigma \wedge \Diamond D \wedge (\Diamond C \supset B) \supset \bigvee\Delta$ | (7,8,9, IPL) |

We now prove that $\mathsf{G4.C}^*$ is complete with respect to $\mathsf{C}^*$. We remark that Dyckhoff's original completeness proof of G4ip′ [4], as well as Iemhoff's completeness proof of intuitionistic monomodal calculi [13], are indirect as they rely on the completeness of G3-style calculi. An alternative proof of the completeness of G4ip′ with no reference to other kinds of calculi is provided in [5] by showing that the calculus in itself is syntactically complete with respect to the axiomatization: as usual the argument relies on a direct of proof of cut-admissibility within the calculus G4ip′. We follow here this latter approach as it can be modularly extended to our calculi.

As usual, we say that a rule is *admissible* in $\mathsf{G4.C}^*$ if whenever the premisses are derivable, the conclusion is also derivable, and that a single-premiss rule is *height-preserving admissible* (hp-admissible for short) if whenever the premiss is derivable, then the conclusion is derivable with a derivation of at most the same height. Moreover, we say that a rule $\dfrac{S_1 \quad ... \quad S_n}{S'}$ is *height-preserving invertible* (hp-invertible) *with respect to the premiss* $S_i$ if the rule $\dfrac{S'}{S_i}$ is hp-admissible, and that it is *height-preserving invertible* (tout court) if it is hp-invertible with respect to all its premisses. One can easily prove the following:

**Lemma 1.** *The rules* $\mathsf{L}\wedge$*,* $\mathsf{R}\wedge$*,* $\mathsf{L}\vee$*,* $\mathsf{R}\vee$*,* $\mathsf{L0}\supset$*,* $\mathsf{L}\wedge\supset$*,* $\mathsf{L}\vee\supset$ *are height-preserving invertible. The rules* $\mathsf{L}\supset\supset$*,* $\mathsf{L}\Box\supset$*, and* $\mathsf{L}\Diamond\supset$ *are height-preserving invertible with respect to the right premiss.*

We now prove admissibility of the structural rules in $\mathsf{G4.C}^*$.

**Proposition 1.** *The following weakening rules are height-preserving admissible in* $\mathsf{G4.C}^*$*, moreover, the following contraction rules are admissible in* $\mathsf{G4.C}^*$*:*

$$\mathsf{Lwk}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \mathsf{Rwk}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta} \qquad \mathsf{Lctr}\ \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \mathsf{Rctr}\ \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}$$

*Proof.* Hp-admissibility of weakening is straightforward. For contraction the proof extends the one of [5] for $\mathsf{G4ip}'$ and proceeds by induction on the height of the derivation of the premiss of contraction and case analysis. The proof is standard if the contracted formula is not principal in the last rule application in the derivation of the premiss of contraction. The cases where the contracted formula is principal and the last rule applied is a rule of $\mathsf{G4ip}'$ are covered in [5], in particular it is easy to see that the rule in Lemma 7.5 [5] is still admissible in $\mathsf{G4.C}^*$. Finally, for the modal rules we consider as an example the following application of contraction to the formula $\Diamond C \supset B$ which is obtained by $\mathsf{L}\Diamond\supset$ (on the left). The derivation is converted as follows (on the right) with an application of the hp-invertibilty of $\mathsf{L}\Diamond\supset$ with respect to the right premiss:

$$\dfrac{\dfrac{\Sigma, D \Rightarrow C \qquad \Gamma, \Box\Sigma, \Diamond D, B, \Diamond C \supset B \Rightarrow \Delta}{\Gamma, \Box\Sigma, \Diamond D, \Diamond C \supset B, \Diamond C \supset B \Rightarrow \Delta}\ \mathsf{L}\Diamond\supset}{\Gamma, \Box\Sigma, \Diamond D, \Diamond C \supset B \Rightarrow \Delta}\ \mathsf{Lctr} \quad \rightsquigarrow \quad \dfrac{\Sigma, D \Rightarrow C \qquad \dfrac{\dfrac{\Gamma, \Box\Sigma, \Diamond D, B, \Diamond C \supset B \Rightarrow \Delta}{\Gamma, \Box\Sigma, \Diamond D, B, B \Rightarrow \Delta}\ \mathsf{L}\Diamond\supset^{\mathrm{i}}}{\Gamma, \Box\Sigma, \Diamond D, B \Rightarrow \Delta}\ \mathsf{Lctr}}{\Gamma, \Box\Sigma, \Diamond D, \Diamond C \supset B \Rightarrow \Delta}\ \mathsf{L}\Diamond\supset$$

$\Box$

**Theorem 3 (Cut elimination).** *The following cut rule is admissible in* $\mathsf{G4.C}^*$*:*

$$\mathsf{cut}\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma', A \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

*Proof.* As usual we proceed by induction on the lexicographically ordered pairs $(c, h)$, where $c$ is the weight of the cut formulas (cf. Def. 3), and $h = h_1 + h_2$, called cut height, is the sum of the heights $h_1$ and $h_2$ of the derivations of the premisses of cut. As before, the proof extends the one in [5] for $\mathsf{G4ip}'$, and distinguishes some cases according to whether the cut formula is or not principal in the last rules applied in the derivation of the premisses of cut. We only show a few most relevant cases. (i) The cut formula is not principal in the last rule applied in the derivation of one premiss. As an example we consider:

$$\dfrac{\Gamma \Rightarrow A, \Delta \qquad \dfrac{\Sigma, B \Rightarrow C \qquad \Gamma', A, \Box\Sigma, \Diamond B, D \Rightarrow \Delta'}{\Gamma', A, \Box\Sigma, \Diamond B, \Diamond C \supset D \Rightarrow \Delta'}\ \mathsf{L}\Diamond\supset}{\Gamma, \Gamma', \Box\Sigma, \Diamond B, \Diamond C \supset D \Rightarrow \Delta, \Delta'}\ \mathsf{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\Sigma, B \Rightarrow C \qquad \dfrac{\Gamma \Rightarrow A, \Delta \qquad \Gamma', A, \Box\Sigma, \Diamond B, D \Rightarrow \Delta'}{\Gamma, \Gamma', \Box\Sigma, \Diamond B, D \Rightarrow \Delta, \Delta'}\ \mathsf{cut}}{\Gamma, \Gamma', \Box\Sigma, \Diamond B, \Diamond C \supset D \Rightarrow \Delta, \Delta'}\ \mathsf{L}\Diamond\supset$$

(ii) The cut formula is principal in the last rule applied in the derivation of both premisses. We consider the following two cases, where $R^*$ denotes multiple applications of the rule $R$. The other cases are similar and left to the reader.

$(\mathsf{R}\supset;\ \mathsf{L}\Diamond\supset)$

$$\dfrac{\mathsf{R}\supset\ \dfrac{\Gamma, \Diamond A \Rightarrow B}{\Gamma \Rightarrow \Diamond A \supset B, \Delta} \qquad \dfrac{\Sigma, C \Rightarrow A \qquad \Gamma', \Box\Sigma, \Diamond C, B \Rightarrow \Delta'}{\Gamma', \Box\Sigma, \Diamond C, \Diamond A \supset B \Rightarrow \Delta'}\ \mathsf{L}\Diamond\supset}{\Gamma, \Gamma', \Box\Sigma, \Diamond C \Rightarrow \Delta, \Delta'}\ \mathsf{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\mathsf{cut}\ \dfrac{\mathsf{K}_\Diamond\ \dfrac{\Sigma, C \Rightarrow A}{\Box\Sigma, \Diamond C \Rightarrow \Diamond A} \qquad \Gamma, \Diamond A \Rightarrow B}{\Gamma, \Box\Sigma, \Diamond C \Rightarrow B} \qquad \Gamma', \Box\Sigma, \Diamond C, B \Rightarrow \Delta'}{\Gamma, \Gamma', \Box\Sigma, \Box\Sigma, \Diamond C, \Diamond C \Rightarrow \Delta'}\ \mathsf{cut}}{}}{\Gamma, \Gamma', \Box\Sigma, \Diamond C \Rightarrow \Delta, \Delta'}\ \mathsf{Lctr}^* + \mathsf{Rwk}^*$$

$(\mathsf{K}_\Diamond;\ \mathsf{L}\Diamond\supset)$

$$\dfrac{\mathsf{K}_\Diamond\ \dfrac{\Sigma, A \Rightarrow B}{\Gamma, \Box\Sigma, \Diamond A \Rightarrow \Diamond B, \Delta} \qquad \dfrac{\Pi, B \Rightarrow C \qquad \Gamma', \Box\Pi, \Diamond B, D \Rightarrow \Delta'}{\Gamma', \Box\Pi, \Diamond B, \Diamond C \supset D \Rightarrow \Delta'}\ \mathsf{L}\Diamond\supset}{\Gamma, \Gamma', \Box\Sigma, \Box\Pi, \Diamond A, \Diamond C \supset D \Rightarrow \Delta, \Delta'}\ \mathsf{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\mathsf{cut}\ \dfrac{\Sigma, A \Rightarrow B \qquad \Pi, B \Rightarrow C}{\Sigma, \Pi, A \Rightarrow C} \qquad \dfrac{\Gamma, \Box\Sigma, \Diamond A \Rightarrow \Diamond B, \Delta \qquad \Gamma', \Box\Pi, \Diamond B, D \Rightarrow \Delta'}{\Gamma, \Gamma', \Box\Sigma, \Box\Pi, \Diamond A, D \Rightarrow \Delta, \Delta'}\ \mathsf{cut}}{\Gamma, \Gamma', \Box\Sigma, \Box\Pi, \Diamond A, \Diamond C \supset D \Rightarrow \Delta, \Delta'}\ \mathsf{L}\Diamond\supset$$

□

Given the admissibility of cut and the derivability in $\mathsf{G4.C}^*$ of the axioms and the modal rule of $\mathsf{C}^*$ we obtain the following result:

**Theorem 4 (Soundness and completeness).** *$\Gamma \Rightarrow \Delta$ is derivable in $\mathsf{G4.C}^*$ if and only if $\bigwedge \Gamma \supset \bigvee \Delta$ is derivable in $\mathsf{C}^*$.*

*Proof.* From right to left: For the intuitionistic axioms we refer to [5]. The derivations of specific instances of the modal axioms are displayed in Fig. 3. Since initial sequents can be generalised to arbitrary formulas, the same derivations can be applied to derive any instances of $K_\Box$ and $K_\Diamond$. Finally, the derivability of the rule $Nec$ follows immediately from the rule $\mathsf{K}_\Box$, whereas modus ponens is simulated by cut in the usual way. For the opposite direction: We have shown above the derivation of the rule $\mathsf{L}\Diamond\supset$ in $\mathsf{C}^*$. The derivation of $\mathsf{L}\Box\supset$ is similar, whereas the derivations of $\mathsf{K}_\Box$, $\mathsf{K}_\Diamond$, and $\mathsf{N}_\Diamond$ are standard and can be found in [23]. □

## 4 Refutation calculi and countermodel construction

We shall now present refutation calculi for constructive modal logics $\mathsf{CK}$ and $\mathsf{CCDL}$. These calculi can be seen as dual of the sequent calculi $\mathsf{G4.CK}$ and $\mathsf{G4.CCDL}$ of the previous section: instead of deriving all valid formulas, the refutation calculi allow one to *refute* all formulas which are *non-theorems* of the logics. We will further show that every refutation in these calculi explicitly constructs a neighbourhood countermodel of the refuted formula.

Refutation calculi handle so-called *anti-sequents*, which are pairs $\Gamma \not\Rightarrow \Delta$ of multiset of formulas of $\mathcal{L}$. Intuitively, the anti-sequent $\Gamma \not\Rightarrow \Delta$ expresses that $\bigvee \Delta$ does not follow from $\bigwedge \Gamma$, or equivalently that $\bigwedge \Gamma \supset \bigvee \Delta$ is not valid. The refutation calculi $\mathsf{Ref.CK}$ and $\mathsf{Ref.CCDL}$ of constructive modal logics extend the refutation calculus for intuitionistic logic by Pinto and Dyckhoff [18] in the following way.

**Definition 5.** *The refutation calculi* $\mathsf{Ref.CK}$ *and* $\mathsf{Ref.CCDL}$ *are defined by the following sets of rules from Fig. 4:*

$$\begin{aligned}\mathsf{Ref.CK} \;&:=\; \{\mathsf{init},\, \mathsf{init_{CK}},\, \mathsf{L}\wedge,\, \mathsf{R}\wedge_1,\, \mathsf{R}\wedge_2,\, \mathsf{L}\vee_1,\, \mathsf{L}\vee_2,\, \mathsf{R}\vee,\, \mathsf{L0}\supset,\, \mathsf{L}\wedge\supset,\, \mathsf{L}\vee\supset,\\ &\qquad \mathsf{L}\supset\supset,\, \mathsf{L}\Box\supset,\, \mathsf{L}\Diamond\supset,\, \mathsf{nip}\}.\\ \mathsf{Ref.CCDL} \;&:=\; \{\mathsf{init},\, \mathsf{L}\wedge,\, \mathsf{R}\wedge_1,\, \mathsf{R}\wedge_2,\, \mathsf{L}\vee_1,\, \mathsf{L}\vee_2,\, \mathsf{R}\vee,\, \mathsf{L0}\supset,\, \mathsf{L}\wedge\supset,\, \mathsf{L}\vee\supset,\, \mathsf{L}\supset\supset,\\ &\qquad \mathsf{L}\Box\supset,\, \mathsf{L}\Diamond\supset,\, \mathsf{nip},\, \mathsf{nip_{CCDL}}\}.\end{aligned}$$

Similarly to the refutation calculus in [18], the initial anti-sequents (or axioms) of $\mathsf{Ref.C}^*$ are all the pairs $\Gamma \not\Rightarrow \Delta$ such that the corresponding sequent $\Gamma \Rightarrow \Delta$ is neither an axiom of $\mathsf{G4.C}^*$, nor the conclusion of any rule of $\mathsf{G4.C}^*$. Concerning the other rules, every rule different from $\mathsf{nip}$ and $\mathsf{nip_{CCDL}}$ corresponds to an invertible premiss of some rule of $\mathsf{G4.C}^*$ (more precisely, to a rule $\frac{S'}{S_i}$ such that the $\mathsf{G4.C}^*$ rule $\frac{S_1 \quad ... \quad S_n}{S'}$ is invertible with respect to $S_i$), whereas $\mathsf{nip}$ and $\mathsf{nip_{CCDL}}$ deal at the same time with all the non-invertible premisses of the rules of $\mathsf{G4.C}^*$. Given their application conditions, the rules $\mathsf{nip}$ and $\mathsf{nip_{CCDL}}$ are (bottom-up) applicable only when no invertible rule of $\mathsf{G4.C}^*$ is applicable. Observe that the rules $\mathsf{nip}$ and $\mathsf{nip_{CCDL}}$ only differ with respect to the premisses where only $\Diamond$-formulas are principal. In particular, $\mathsf{nip_{CCDL}}$ allows one to reduce the anti-sequents where no $\Diamond$-formula occurs in the consequent, which is allowed by the logic $\mathsf{CCDL}$ but is not allowed by $\mathsf{CK}$. The idea is that in $\mathsf{Ref.CCDL}$ the rule $\mathsf{nip}$ is applied when $\Delta$ contains $\Diamond$-formulas, whereas $\mathsf{nip_{CCDL}}$ is applied when $\Delta$ does not contain $\Diamond$-formulas. Two examples of refutations in $\mathsf{Ref.C}^*$ of formulas which are valid in intuitionistic modal logics but are not valid in constructive ones are displayed in Fig. 5.

Note that similarly to $\mathsf{G4.C}^*$, for every rule of $\mathsf{Ref.C}^*$ the premisses have a smaller complexity than the conclusion with respect to the multiset ordering of Def. 3. Therefore we have:

**Theorem 5.** *Backward proof search in* $\mathsf{Ref.C}^*$ *is terminating.*

$$\mathsf{init}\ \frac{}{\Gamma, \Box\Gamma' \nRightarrow \Diamond\Delta, \Delta'} \qquad \mathsf{init_{CK}}\ \frac{}{\Gamma, \Box\Gamma', \Diamond\Gamma'' \nRightarrow \Delta}$$

$$\mathsf{L}\wedge\ \frac{\Gamma, A, B \nRightarrow \Delta}{\Gamma, A \wedge B \nRightarrow \Delta} \qquad \mathsf{R}\wedge_1\ \frac{\Gamma \nRightarrow A, \Delta}{\Gamma \nRightarrow A \wedge B, \Delta} \qquad \mathsf{R}\wedge_2\ \frac{\Gamma \nRightarrow B, \Delta}{\Gamma \nRightarrow A \wedge B, \Delta}$$

$$\mathsf{L}\vee_1\ \frac{\Gamma, A \nRightarrow \Delta}{\Gamma, A \vee B \nRightarrow \Delta} \qquad \mathsf{L}\vee_2\ \frac{\Gamma, B \nRightarrow \Delta}{\Gamma, A \vee B \nRightarrow \Delta} \qquad \mathsf{R}\vee\ \frac{\Gamma \nRightarrow A, B, \Delta}{\Gamma \nRightarrow A \vee B, \Delta}$$

$$\mathsf{L0}\supset\ \frac{\Gamma, p, B \nRightarrow \Delta}{\Gamma, p, p \supset B \nRightarrow \Delta} \qquad \mathsf{L}\wedge\supset\ \frac{\Gamma, C \supset (D \supset B) \nRightarrow \Delta}{\Gamma, (C \wedge D) \supset B \nRightarrow \Delta} \qquad \mathsf{L}\vee\supset\ \frac{\Gamma, C \supset B, D \supset B \nRightarrow \Delta}{\Gamma, (C \vee D) \supset B \nRightarrow \Delta}$$

$$\mathsf{L}\supset\supset\ \frac{\Gamma, B \nRightarrow \Delta}{\Gamma, (C \supset D) \supset B \nRightarrow \Delta} \qquad \mathsf{L}\Box\supset\ \frac{\Gamma, B \nRightarrow \Delta}{\Gamma, \Box C \supset B \nRightarrow \Delta} \qquad \mathsf{L}\Diamond\supset\ \frac{\Gamma, \Diamond D, B \nRightarrow \Delta}{\Gamma, \Diamond D, \Diamond C \supset B \nRightarrow \Delta}$$

$$\mathsf{nip}\ \frac{\begin{array}{ll} \{\Gamma^\Box \nRightarrow A \mid \Box A \supset B \in \Gamma\} & \{\Gamma^\Box \nRightarrow A \mid \Box A \in \Delta\} \\ \{\Gamma^\Box, C \nRightarrow A \mid \Diamond A \supset B, \Diamond C \in \Gamma\} & \{\Gamma^\Box, A \nRightarrow B \mid \Diamond A \in \Gamma, \Diamond B \in \Delta\} \\ \{\Gamma', D \supset B, C \nRightarrow D \mid (C \supset D) \supset B \in \Gamma\} & \{\Gamma, A \nRightarrow B \mid A \supset B \in \Delta\} \end{array}}{\Gamma \nRightarrow \Delta}$$

$$\mathsf{nip_{CCDL}}\ \frac{\begin{array}{ll} \{\Gamma^\Box \nRightarrow A \mid \Box A \supset B \in \Gamma\} & \{\Gamma^\Box \nRightarrow A \mid \Box A \in \Delta\} \\ \{\Gamma^\Box, C \nRightarrow A \mid \Diamond A \supset B, \Diamond C \in \Gamma\} & \{\Gamma^\Box, A \nRightarrow \ \mid \Diamond A \in \Gamma\} \\ \{\Gamma', D \supset B, C \nRightarrow D \mid (C \supset D) \supset B \in \Gamma\} & \{\Gamma, A \nRightarrow B \mid A \supset B \in \Delta\} \end{array}}{\Gamma \nRightarrow \Delta}$$

where • $\Gamma' = \Gamma \setminus \{(C \supset D) \supset B\}$, and
• if $\Box A_1, ..., \Box A_n$ are *all* the $\Box$-formulas of $\Gamma$, then $\Gamma^\Box = A_1, ..., A_n$.

**Application conditions**:

• init and $\mathsf{init_{CK}}$: (i) $\Gamma$ contains only propositional variables, atomic implications, and implications of the form $\Diamond A \supset B$; (ii) $\Delta$ contains only atomic formulas; (iii) if $p \supset A \in \Gamma$, then $p \notin \Gamma$; (iv) if $\Gamma$ contains an implication $\Diamond A \supset B$, then $\Diamond\Gamma'' = \emptyset$; (v) $\Gamma \cap \Delta = \emptyset$.

• nip: (i) $\Gamma$ does not contain $\bot$, conjunctions, disjunctions, and implications of the form $(C \wedge D) \supset B$ or $(C \vee D) \supset B$; (ii) $\Delta$ does not contain conjunctions and disjunctions; (iii) if $p \supset A \in \Gamma$, then $p \notin \Gamma$; (iv) if $p \in \Gamma$, then $p \notin \Delta$.

• $\mathsf{nip_{CCDL}}$: conditions of nip, plus (v) $\Delta$ does not contain $\Diamond$-formulas.

• nip and $\mathsf{nip_{CCDL}}$ must have at least one premiss.

**Fig. 4.** Rules of Ref.CK and Ref.CCDL.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \nRightarrow q}\ \mathsf{init}}{\nRightarrow p \supset q}\ \mathsf{nip}}{\Diamond p \supset \Box q \nRightarrow \Box(p \supset q)}\ \mathsf{nip}}{\nRightarrow (\Diamond p \supset \Box q) \supset \Box(p \supset q)}\ \mathsf{nip} \qquad \dfrac{\dfrac{\dfrac{\mathsf{L}\vee\ \dfrac{\mathsf{init}\ \dfrac{}{q \nRightarrow p}}{p \vee q \nRightarrow p} \qquad \dfrac{\dfrac{}{p \nRightarrow q}\ \mathsf{init}}{p \vee q \nRightarrow q}\ \mathsf{L}\vee}{\Diamond(p \vee q) \nRightarrow \Diamond p, \Diamond q}\ \mathsf{nip}}{\Diamond(p \vee q) \nRightarrow \Diamond p \vee \Diamond q}\ \mathsf{R}\vee}{\nRightarrow \Diamond(p \vee q) \supset \Diamond p \vee \Diamond q}\ \mathsf{nip}$$

**Fig. 5.** Examples of refutations in Ref.C*.

We can prove that the refutation calculi Ref.CK and Ref.CCDL are the dual of the sequent calculi G4.CK and G4.CCDL, in the sense that an anti-sequent $\Gamma \not\Rightarrow \Delta$ is derivable in a refutation calculus if and only if the sequent $\Gamma \Rightarrow \Delta$ is not derivable in the corresponding sequent calculus. It follows that the refutation calculi are complete with respect to the sets of non-valid formulas in the neighbourhood semantics for CK and CCDL.

**Theorem 6.** *$\Gamma \not\Rightarrow \Delta$ is derivable in* Ref.C$^*$ *if and only if $\Gamma \Rightarrow \Delta$ is not derivable in* G4.C$^*$*.*

We now show that every refutation of $\Gamma \not\Rightarrow \Delta$ provides a neighbourhood countermodel of $\Gamma \Rightarrow \Delta$. We thereby obtain a constructive proof of the completeness of the refutation calculi Ref.C$^*$ (and indirectly also of the calculi G4.C$^*$) with respect to the neighbourhood semantics of C$^*$. In order to define the countermodel construction, we enrich the anti-sequents occurring in a refutation with annotations that represent the worlds of a model in the following manner.

**Definition 6.** *An* annotation *is a finite sequence of natural numbers $n_1.n_2. \dots .n_k$. An* annotated anti-sequent *is an expression $\Gamma \not\Rightarrow^\sigma \Delta$, where $\Gamma \not\Rightarrow \Delta$ is an anti-sequent and $\sigma$ is an annotation. An* annotated refutation *is a refutation where all sequents are annotated according to the following prescriptions:*

- *The root anti-sequent $\Gamma \not\Rightarrow \Delta$ is annotated with the initial annotation* 1*.*
- *If the conclusion of any rule different from* nip *or* nip$_{\mathsf{CCDL}}$ *is annotated with $\sigma$, then its premiss has the same annotation $\sigma$.*
- *If the conclusion of* nip *or* nip$_{\mathsf{CCDL}}$ *is annotated with $\sigma$, then its premisses are annotated as follows:*
  - *The premisses obtained from formulas $(C \supset D) \supset B$ on the left of the conclusion, or formulas $A \supset B$ on the right, are annotated each with a different annotation $\sigma.n$ not already occurring in the refutation.*
  - *The premisses obtained from any other formulas are annotated each with a different annotation $k$ not already occurring in the refutation.*

As an example, the annotated versions of the refutations in Fig. 5 are displayed in Fig. 6. Note that every refutation in Ref.C$^*$ can be easily annotated according to Def. 6.

For any annotated refutation $\mathscr{R}$ of $\Gamma \not\Rightarrow^1 \Delta$ in Ref.C$^*$, we denote

$$\Gamma^\sigma = \bigcup\{\Gamma \mid \Gamma \not\Rightarrow^\sigma \Delta \in \mathscr{R}\} \text{ and } \Delta^\sigma = \bigcup\{\Delta \mid \Gamma \not\Rightarrow^\sigma \Delta \in \mathscr{R}\}.^4$$

We now show how to extract a countermodel from an annotated refutation of $\Gamma \not\Rightarrow^1 \Delta$. Intuitively, every annotation corresponds to a world of the model. The rules in which the premiss and conclusion have the same annotation (i.e., all the rules but nip and nip$_{\mathsf{CCDL}}$) are "local" as they deal with a single world. By contrast, bottom-up applications of nip and nip$_{\mathsf{CCDL}}$ create new worlds: the

[4] To be precise, the sets $\Gamma^\sigma$ and $\Delta^\sigma$ depend on the refutation $\mathscr{R}$. In order not to burden the notation we avoid explicit reference to $\mathscr{R}$ as it is clear from the context.

premisses annotated with $\sigma.n$ (i.e., those generated by non-modal $\supset$-formulas occurring in the conclusion) represent worlds related through $\preceq$ to the world $\sigma$ at the conclusion, whereas the other premisses represent worlds belonging to some neighbourhood of $\sigma$. The formal definition is as follows.

**Definition 7 (Countermodel extraction).** *Let $\mathscr{R}$ be an annotated refutation of $\Gamma \not\Rightarrow^1 \Delta$. The* countermodel *determined by $\mathscr{R}$ is defined as follows.*

- $\mathcal{W}$ *= the set of annotations occurring in $\mathscr{R}$.*
- $\sigma \preceq \rho$ *iff $\rho = \sigma.\pi$ for some possibly empty annotation $\pi$.*
- $\mathcal{V}(p) = \{\sigma \in \mathcal{W} \mid p \in \Gamma^\sigma\}$.
- *For every $\Box A, \Diamond A$ occurring in $\mathscr{R}$, $A^+ = \{\sigma \in \mathcal{W} \mid A \in \Gamma^\sigma\}$.*
- *For every $\sigma \in \mathcal{W}$, $\mathcal{N}_\Box(\sigma)$ and $\mathcal{N}_\Diamond(\sigma)$ are defined as follows:*
  - *If there are no $\Box$-formulas in $\Gamma^\sigma$, then:*
    - $\mathcal{N}_\Box(\sigma) = \{\mathcal{W}\}$.
    - $\mathcal{N}_\Diamond(\sigma) = \{\alpha \subseteq \mathcal{W} \mid \text{there is } \Diamond B \in \Gamma^\sigma \text{ s.t. } B^+ \subseteq \alpha\}$.
  - *Otherwise, if $\Box A_1, ..., \Box A_n$ are* all *the $\Box$-formulas in $\Gamma^\sigma$, then:*
    - $\mathcal{N}_\Box(\sigma) = \{\alpha \subseteq \mathcal{W} \mid A_1^+ \cap ... \cap A_n^+ \subseteq \alpha\}$.
    - $\mathcal{N}_\Diamond(\sigma) = \{\alpha \subseteq \mathcal{W} \mid \text{there is } \Diamond B \in \Gamma^\sigma \text{ s.t. } A_1^+ \cap ... \cap A_n^+ \cap B^+ \subseteq \alpha\}$.

Observe that $\mathcal{N}_\Diamond(\sigma) = \emptyset$ if there are no $\Diamond$-formulas in $\Gamma^\sigma$.

**Theorem 7.** *If $\mathscr{R}$ is an annotated refutation of $\Gamma \not\Rightarrow^1 \Delta$ in* Ref.C$^*$*, and $\mathcal{M}$ is the model extracted from $\mathscr{R}$ according to Def. 7, then $\mathcal{M}$ is a neighbourhood model for* C$^*$ *and it is a countermodel of $\Gamma \Rightarrow \Delta$.*

*Proof.* We first prove that $\mathcal{M}$ is a neighbourhood model for C$^*$. From the definition of $\mathcal{M}$ it immediately follows that $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ are supplemented, and $\mathcal{N}_\Box$ is closed under intersection and contains the unit. For Ref.CCDL we also have $\emptyset \notin \mathcal{N}_\Diamond(\sigma)$, since if $\Diamond B \in \Gamma^\sigma$, then by nip$_{\mathsf{CCDL}}$ and the annotation procedure there is $n \in \mathcal{W}$ such that $n \in \bigcap\{A^+ \mid \Box A \in \Gamma^\sigma\} \cap B^+$, thus for every $\alpha \in \mathcal{N}_\Diamond(\sigma)$, $\alpha \neq \emptyset$. Moreover, if $\alpha \in \mathcal{N}_\Box(\sigma)$ and $\beta \in \mathcal{N}_\Diamond(\sigma)$, then if $\Gamma^\sigma$ contains $\Box$-formulas we have $\bigcap\{A^+ \mid \Box A \in \Gamma^\sigma\} \subseteq \alpha$ and $\bigcap\{A^+ \mid \Box A \in \Gamma^\sigma\} \cap B^+ \subseteq \beta$ for some $\Diamond B \in \Gamma^\sigma$. Then $\bigcap\{A^+ \mid \Box A \in \Gamma^\sigma\} \cap B^+ \subseteq \alpha \cap \beta$, thus $\alpha \cap \beta \in \mathcal{N}_\Diamond(\sigma)$. Moreover, $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ are monotonic with respect to $\preceq$. For instance, if $\alpha \in \mathcal{N}_\Box(\sigma)$ and $\sigma.\pi \in \mathcal{W}$, then $\alpha = \mathcal{W}$ or $A_1^+ \cap ... \cap A_n^+ \subseteq \alpha$, where $\Box A_1, ..., \Box A_n$ are all the $\Box$-formulas in $\Gamma^\sigma$. In the first case, $\mathcal{W} \in \mathcal{N}_\Box(\sigma.\pi)$. In the second case, by nip and nip$_{\mathsf{CCDL}}$ $\Box A_1, ..., \Box A_n \in \Gamma^{\sigma.\pi}$. Then $\bigcap\{B^+ \mid \Box B \in \Gamma^{\sigma.\pi}\} \subseteq \bigcap\{A^+ \mid \Box A \in \Gamma^\sigma\} \subseteq \alpha$, thus $\alpha \in \mathcal{N}_\Box(\sigma.\pi)$. Finally $\mathcal{V}$ satisfies the hereditary condition: if $\sigma \Vdash p$, then $p \in \Gamma^\sigma$. By the rules and the annotation procedure it follows that $p \in \Gamma^{\sigma.\pi}$ for every $\sigma.\pi \in \mathcal{W}$, thus $\sigma.\pi \Vdash p$. Observe that since $\mathcal{M}$ is a neighbourhood model for C$^*$ it satisfies the ereditary property for every $A \in \mathcal{L}$.

Now we prove that for every formula $A$ and every annotation $\sigma$ occurring in $\mathscr{R}$, if $A \in \Gamma^\sigma$, then $\sigma \Vdash A$, and if $A \in \Delta^\sigma$, then $\sigma \nVdash A$. In order to carry on the proof we need the notion of "height of a label": we consider the forest of labels $F_\mathscr{R}$ generated by the labels $\sigma$ in $\mathscr{R}$ with their immediate successors $\sigma.1, ..., \sigma.n$ (the root of each tree is a unitary label); we then define the height of a label $\sigma$

as its height in $F_{\mathscr{R}}$. The two claims are proven simultaneously by induction on the pairs $(c, h)$, where $c$ is the weight of $A$ (Def. 3), and $h$ is the height of $\sigma$.

The basic case ($A \equiv p, \bot$) is trivial. If $\Gamma$ or $\Delta$ contains a conjuction or a disjunction, or $\Gamma$ contains an implication of the form $(C \wedge D) \supset B$ or $(C \vee D) \supset B$, then the claim easily follows from the i.h. and the structure of refutations. For instance, if $(C \wedge D) \supset B \in \Gamma^\sigma$, then $C \supset (D \supset B) \in \Gamma^\sigma$, and by i.h., $\sigma \Vdash C \supset (D \supset B)$, thus $\sigma \Vdash (C \wedge D) \supset B$.

If $B \supset C \in \Delta^\sigma$, then by the rule $\mathsf{nip}$ or $\mathsf{nip}_{\mathsf{CCDL}}$ and the annotation procedure there is $\sigma.n \in \mathcal{W}$ such that $B \in \Gamma^{\sigma.n}$ and $C \in \Delta^{\sigma.n}$, thus by i.h. $\sigma.n \Vdash B$ and $\sigma.n \nVdash C$, then since $\sigma \preceq \sigma.n$ it follows that $\sigma \nVdash B \supset C$.

If $p \supset B \in \Gamma^\sigma$, then for every chain of worlds starting from $\sigma$ either there is no world $\tau$ in the chain such that $p \in \Gamma^\tau$, or there is a $\preceq$-minimal world $\pi$ with $\sigma \preceq \pi$ such that $p \in \Gamma^\pi$. In the first case, by definition $p$ is false in every world of the chain. In the second case, $\rho \nVdash p$ for every $\rho \neq \pi$ such that $\sigma \preceq \rho \preceq \pi$, moreover there is $\Gamma \not\Rightarrow^\pi \Delta$ in $\mathscr{R}$ such that $p \in \Gamma$. Furthermore, by $\mathsf{nip}$ or $\mathsf{nip}_{\mathsf{CCDL}}$ $p \supset B \in \Gamma$, then by $\mathsf{L0}{\supset}$ and the application conditions there is $\Gamma' \not\Rightarrow^\pi \Delta'$ such that $B \in \Gamma'$, thus $B \in \Gamma^\pi$. By i.h. it follows $\pi \Vdash B$, and by the ereditary property we have $\omega \Vdash B$ for every $\omega$ such that $\pi \preceq \omega$. Therefore for every $\tau$ such that $\sigma \preceq \tau$, $\tau \nVdash p$ or $\tau \Vdash B$, thus $\sigma \Vdash p \supset B$.

If $(C \supset D) \supset B \in \Gamma^\sigma$, then if $B \in \Gamma^\sigma$, then by i.h. $\sigma \Vdash B$, and by the hereditary property $\rho \Vdash B$ for every $\rho$ such that $\rho \preceq \sigma$. If instead $B \notin \Gamma^\sigma$, then by $\mathsf{nip}$ or $\mathsf{nip}_{\mathsf{CCDL}}$ there is $\sigma.k \in \mathcal{W}$ such that $D \supset B, C \in \Gamma^{\sigma.k}$ and $D \in \Gamma^{\sigma.k}$, moreover for every other immediate successor $\sigma.m$ of $\sigma$, $(C \supset D) \supset B \in \Gamma^{\sigma.m}$. By i.h. $\sigma.m \Vdash (C \supset D) \supset B$, that is, for every $\pi$ such that $\sigma.m \preceq \pi$, $\pi \Vdash C \supset D$ implies $\pi \Vdash B$. Moreover, by i.h. $\sigma.n \Vdash D \supset B$, $\sigma.n \Vdash C$, and $\sigma.n \nVdash D$. Thus $\sigma \nVdash C \supset B$, and by the hereditary property, for every successor $\tau$ of $\sigma.n$, $\tau \Vdash C \wedge (D \supset B)$. Then if $\tau \Vdash C \supset D$ we have $\tau \Vdash D$, thus $\tau \Vdash B$. Therefore for every $\rho$ such that $\sigma \preceq \rho$, $\rho \Vdash C \supset D$ implies $\rho \Vdash B$. Then $\sigma \Vdash (C \supset D) \supset B$.

If $\Box C \supset B \in \Gamma^\sigma$, then if $B \in \Gamma^\sigma$, then by i.h. $\sigma \Vdash B$, and by the hereditary property $\rho \Vdash B$ for every $\rho$ such that $\rho \preceq \sigma$. If instead $B \notin \Gamma^\sigma$, then by $\mathsf{nip}$ or $\mathsf{nip}_{\mathsf{CCDL}}$ for every immediate successor $\sigma.k$ of $\sigma$, $\Box C \supset B \in \Gamma^{\sigma.k}$, then by i.h. $\sigma.k \Vdash \Box C \supset B$, moreover there is $n \in \mathcal{W}$ such that $C \in \Delta^n$ and for every $\Box D \in \Gamma^\sigma$, $D \in \Gamma^n$. Then by i.h. $\bigcap\{D^+ \mid \Box D \in \Gamma^\sigma\} \not\subseteq [\![C]\!]$, thus $[\![C]\!] \notin \mathcal{N}_\Box(\sigma)$, therefore $\sigma \nVdash \Box C$. Then for every $\rho$ such that $\sigma \preceq \rho$, $\rho \nVdash \Box C$ or $\rho \Vdash B$, therefore $\sigma \Vdash \Box C \supset B$.

If $\Diamond C \supset B \in \Gamma^\sigma$, then if $B \in \Gamma^\sigma$, then by i.h. $\sigma \Vdash B$, and by the hereditary property $\rho \Vdash B$ for every $\rho$ such that $\rho \preceq \sigma$. If instead $B \notin \Gamma^\sigma$, then by $\mathsf{nip}$ or $\mathsf{nip}_{\mathsf{CCDL}}$ for every immediate successor $\sigma.k$ of $\sigma$, $\Box D \supset B \in \Gamma^{\sigma.k}$, then by i.h. $\sigma.k \Vdash \Box D \supset B$. Moreover, if there is no $\Diamond$-formula in $\Gamma^\sigma$, then $\mathcal{N}_\Diamond(\sigma) = \emptyset$, whence $\sigma \nVdash \Diamond C$. Otherwise for every $\Diamond D \in \Gamma^\sigma$, by $\mathsf{nip}$ or $\mathsf{nip}_{\mathsf{CCDL}}$ there is $n \in \mathcal{W}$ such that $D \in \Gamma^n$, $C \in \Delta^n$, and $E \in \Gamma^n$ for every $\Box E \in \Gamma^\sigma$. Then by i.h. $\bigcap\{E^+ \mid \Box E \in \Gamma^\sigma\} \cap D^+ \not\subseteq [\![C]\!]$, thus $[\![C]\!] \notin \mathcal{N}_\Diamond(\sigma)$, therefore $\sigma \nVdash \Diamond C$. Then for every $\rho$ such that $\sigma \preceq \rho$, $\rho \nVdash \Diamond C$ or $\rho \Vdash B$, therefore $\sigma \Vdash \Diamond C \supset B$.

If $\Box B \in \Gamma^\sigma$ (resp. $\Diamond B \in \Gamma^\sigma$), then by i.h. $B^+ \subseteq [\![B]\!]$, and by definition $[\![B]\!] \in \mathcal{N}_\Box(\sigma)$ (resp. $\mathcal{N}_\Diamond(\sigma)$), so $\sigma \Vdash \Box B$ (resp. $\sigma \Vdash \Diamond B$).

**1. Annotated refutation and countermodel for** $(\Diamond p \supset \Box q) \supset \Box(p \supset q)$**:**

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \not\Rightarrow^{2.1} q}\,\mathsf{init}}{\not\Rightarrow^{2} p \supset q}\,\mathsf{nip}}{\Diamond p \supset \Box q \not\Rightarrow^{1.1} \Box(p \supset q)}\,\mathsf{nip}}{\not\Rightarrow^{1} (\Diamond p \supset \Box q) \supset \Box(p \supset q)}\,\mathsf{nip}$$

$\mathcal{W} = \{1, 1.1, 2, 2.1\}$. $1 \preceq 1.1$. $2 \preceq 2.1$.
$\mathcal{V}(p) = \{2.1\}$. $\mathcal{V}(q) = \emptyset$.
$\mathcal{N}_{\Box}(w) = \{\mathcal{W}\}$ for every $w \in \mathcal{W}$.
$\mathcal{N}_{\Diamond}(w) = \emptyset$ for every $w \in \mathcal{W}$.

**2. Annotated refutation and countermodel for** $\Diamond(p \vee q) \supset \Diamond p \vee \Diamond q$**:**

$$\dfrac{\dfrac{\dfrac{\mathsf{L}\vee\,\dfrac{\mathsf{init}\,\dfrac{}{q \not\Rightarrow^{2} p}}{p \vee q \not\Rightarrow^{2} p} \qquad \dfrac{\dfrac{}{p \not\Rightarrow^{3} q}\,\mathsf{init}}{p \vee q \not\Rightarrow^{3} q}\,\mathsf{L}\vee}{\Diamond(p \vee q) \not\Rightarrow^{1.1} \Diamond p, \Diamond q}\,\mathsf{nip}}{\Diamond(p \vee q) \not\Rightarrow^{1.1} \Diamond p \vee \Diamond q}\,\mathsf{R}\vee}{\not\Rightarrow^{1} \Diamond(p \vee q) \supset \Diamond p \vee \Diamond q}\,\mathsf{nip}$$

$\mathcal{W} = \{1, 1.1, 2, 3\}$. $1 \preceq 1.1$.
$\mathcal{V}(p) = \{3\}$. $\mathcal{V}(q) = \{2\}$.
$\mathcal{N}_{\Box}(w) = \{\mathcal{W}\}$ for every $w \in \mathcal{W}$.
$\mathcal{N}_{\Diamond}(w) = \emptyset$ for every $w \in \mathcal{W}$, $w \neq 1.1$.
$\mathcal{N}_{\Diamond}(1.1) = \{\alpha \mid (p \vee q)^{+} \subseteq \alpha\} =$
$\{\{2,3\}, \{2,3,1\}, \{2,3,1.1\}, \{2,3,1,1.1\}\}$.

**3. Annotated refutation and countermodel for** $\Diamond\bot \supset \bot$ **in** $\mathsf{Ref.CK}$**:**

$$\dfrac{\dfrac{}{\Diamond\bot \not\Rightarrow^{1.1} \bot}\,\mathsf{init}_{\mathsf{CK}}}{\not\Rightarrow^{1} \Diamond\bot \supset \bot}\,\mathsf{nip}$$

$\mathcal{W} = \{1, 1.1\}$. $1 \preceq 1.1$.
$\mathcal{N}_{\Box}(1) = \{\mathcal{W}\}$. $\mathcal{N}_{\Box}(1.1) = \{\mathcal{W}\}$.
$\mathcal{N}_{\Diamond}(1) = \emptyset$. $\mathcal{N}_{\Diamond}(1.1) = \mathcal{P}(\mathcal{W})$.

**Fig. 6.** Annotated refutations and countermodels.

If $\Box B \in \Delta^{\sigma}$, then by the rule nip or $\mathsf{nip}_{\mathsf{CCDL}}$ there is $n \in \mathcal{W}$ such that $B \in \Delta^{n}$ and for every $\Box C \in \Gamma^{\sigma}$, $C \in \Gamma^{n}$. Then by i.h. $\bigcap\{C^{+} \mid \Box C \in \Gamma^{\sigma}\} \not\subseteq [\![B]\!]$, thus $[\![B]\!] \notin \mathcal{N}_{\Box}(\sigma)$, therefore $\sigma \not\Vdash \Box B$.

If $\Diamond B \in \Delta^{\sigma}$, then if there is no $\Diamond C \in \Gamma^{\sigma}$, then $\mathcal{N}_{\Diamond}(\sigma) = \emptyset$, thus $\sigma \not\Vdash \Diamond B$. If instead there is $\Diamond C \in \Gamma^{\sigma}$, then by the rule nip, for every $\Diamond C \in \Gamma^{\sigma}$ there is $n \in \mathcal{W}$ such that $B \in \Delta^{n}$, $C \in \Gamma^{n}$, and $D \in \Gamma^{n}$ for every $\Box D \in \Gamma^{\sigma}$. Then by i.h. $\bigcap\{D^{+} \mid \Box C \in \Gamma^{\sigma}\} \cap C^{+} \not\subseteq [\![B]\!]$, thus $[\![B]\!] \notin \mathcal{N}_{\Diamond}(\sigma)$, therefore $\sigma \not\Vdash \Diamond B$. $\Box$

Some relevant examples of refutations of non-valid formulas and corresponding countermodels are displayed in Fig. 6.

As shown in [3], every neighbourhood model for CK or CCDL can be transformed into an equivalent relational model.[5] For instance, by applying the transformation to the last model in Fig. 6 we obtain a relational model $\langle \mathcal{W}', \preceq', \mathcal{R}, \mathcal{V}' \rangle$ for CK, where $\mathcal{W}' = \{(1, \{1, 1.1\}), (1.1, \{1, 1.1, f\}), (f, \{f\})\}$; $(1, \{1, 1.1\}) \preceq' (1.1, \{1, 1.1, f\})$; $(1.1, \{1, 1.1, f\})\mathcal{R}(f, \{f\})$; and $(f, \{f\}) \Vdash \bot$. Moreover, a simplified transformation is possible for the models where $\mathcal{N}_{\Diamond}$ is empty, whence in particular for neighbourhood models for the $\Box$-fragment of the logics. The simplified transformation generates relational models of the same size as the original neighbourhood ones. By contrast, the general transformation can produce relational models that are exponentially larger than the original neighbourhood ones. It follows that the 1-1 correspondence between the premisses of the non-

[5] The transformation in [3] must be slightly modified given the alternative formulation of the neighbourhood semantics.

invertible rules in a refutation and the worlds of the extracted countermodel is not preserved in the relational semantics. For this reason, while it is possible to directly extract relational models from refutations for the $\Box$-fragment of the two logics,[6] the same does not seem possible for CK and CCDL with both $\Box$ and $\Diamond$. In this sense neighbourhood models are the natural semantics of our refutation calculi.

## 5 Conclusion and future work

In this paper we have proposed terminating sequent calculi for constructive modal logics CK and CCDL. First we have presented the calculi G4.CK and G4.CCDL which extend both Dyckhoff's calculus for intuitionistic logic and Iemhoff's one for the $\Box$-fragment of IK. Our calculi provide a decision procedure for the respective logics. They have also good proof-theoretical properties, as they allow for a syntactic proof of cut admissibility. Then we have proposed dual refutation calculi for non-provability. The dual calculi are likewise terminating. Their main interest is that they support direct countermodel extraction: each refutation uniquely determines a finite neighbourhood countermodel of the refuted formula in the semantics defined in [3].

There are a number of issues that we intend to explore in future work. We have already mentioned the issue of transforming a neighbourhood countermodel into a "small" relational countermodel. There are also some computational issues: although the exact complexity of CK and CCDL has not been explicitly stated, we strongly conjecture that both are in PSPACE, in this hypothesis, the calculi G4.CK and G4.CCDL would not be optimal, since a derivation may have an exponential size, the same happens within Dyckhoff's G4ip′; this naturally leads to the issue of studying refinements of our calculi, following the line of [6] which would match (and establish) the PSPACE upper bound. Moreover, we believe that our terminating calculi are very suitable for implementation: a theorem prover based on them would expand the realm of intuitionistic modal theorem proving, in addition to the recent prover presented in [10]. Following Iemhoff [14] we also intend to use our terminating calculi to prove constructively the uniform interpolation property for both CK and CCDL.

Finally, we plan to extend our calculi to other (non-normal) intuitionistic modal logics in two directions: on the one hand to subsystems of CK and CCDL defined in [3], and on the other hand their extensions with axioms of the standard modal cube. To this regard, nested sequents for the standard cube extensions of CK have been proposed in [1], but terminating calculi of the kind considered here have not been investigated yet for them. A further direction could be to study a constructive version of Bi-Intuitionistic Logic with tense modalities [12]. The investigation of refutation calculi for these logics, along the lines of this work, would of course presuppose the extension of the neighbourhood semantics itself to these logics, a non-trivial task which may have an independent interest.

[6] As an example, an extraction of relational countermodels from failed proofs in a G4-calculus for Intuitionistic Strong Löb Logic with only $\Box$ is presented in [9].

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