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A new regularization of equilibrium problems on Hadamard manifolds: applications to theories of desires

G. C. Bento[1] · J. X. Cruz Neto[2] · P. A. Soares Jr.[3] · A. Soubeyran[4]

Abstract
In this paper, we introduce a new proximal algorithm for equilibrium problems on a genuine Hadamard manifold, using a new regularization term. We first extend recent existence results by considering pseudomonotone bifunctions and a weaker sufficient condition than the coer-civity assumption. Then, we consider the convergence of this proximal-like algorithm which can be applied to genuinely Hadamard manifolds and not only to specific ones, as in the recent literature. A striking point is that our new regularization term have a clear interpretation in a recent "variational rationality" approach of human behavior. It represents the resistance to change aspects of such human dynamics driven by motivation to change aspects. This allows us to give an application to the theories of desires, showing how an agent must escape to a succession of temporary traps to be able to reach, at the end, his desires.

Keywords Proximal algorithms · Equilibrium problem · Hadamard manifold · Desires · Trap · Worthwhile changes

Mathematics Subject Classification 65K05 · 47J25 · 90C33 · 91E10

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1 Introduction

The equilibrium problem EP has been widely studied and it is a very active field of research. One of the motivations is that various problems may be formulated as an equilibrium problem, for instance, optimization problems, Nash equilibria pro-blems, complementarity problems, fixed point problems and variational inequality problems. An extensive development can be found in Blum and Oettli (1994), Bianchi and Schaible (1996) and references therein.

An important issue is under what conditions there exists a solution to EP. In the linear setting, several authors have provided results answering this question; see, for instance, Iusem and Sosa (2003) and Iusem et al. (2009). For an approach on this issue in Hadamard manifolds see, for example, Colao et al. (2012), where the authors considered the case where EP is associated to a monotone bifunction satisfying a certain coercivity condition. Cruz Neto et al. (2018b), the authors establish existence of solutions to equilibrium problems in Hadamard manifolds by perturbing the equilibrium bifunction. An approach on the existence of solutions in the case where the equilibrium problem is of the type vectorial with variable order can be found in Batista et al. (2016). Li et al. (2019) addressed the issue of the existence of solutions for EP in general Riemannian manifolds in the case where the equilibrium problem is equivalent to suitable VIP (Variational Inequality Problem) and, consequently, the technique considered was mainly focused in applying the corresponding results in Colao et al. (2012) for the variational inequality problem obtained. See also Pany et al. (2018) where the authors use a similar strategy to address this issue in the framework of finitely continuous space. Our first contribution in this paper is to establish this important result in the case where the bifunction involved is not necessarily monotone. More precisely, following the ideas presented in Iusem et al. (2009), in the presente paper we have extended the existence result in Colao et al. (2012) (for EP) by considering pseudomonotone bifunctions and a weaker sufficient condition than the coercivity assumption used there.

Another issue of interest is the development of an algorithmic analysis to deal with EP; see for example Bulavsky and Kalashnikov (1998), Moudafi (1999), Konnov (2003), Facchinei and Kanzow (2010), Cruz Neto et al. (2016) and Pany et al. (2018). From an algorithmic point of view, in Colao et al. (2012) the authors presented an iterative process to approximate a solution of the equilibrium problem on a Hadamard manifold, which is associated to the proximal iteration studied, for example, by Moudafi (1999), Konnov (2003) and Iusem and Sosa (2010), all in the linear setting. Although natural, the iterative scheme considered in Colao et al. (2012) is ensured to be well defined only on Hadamard manifolds with null sectional curvature; see Remark 7. The approach via variational inequality considered in Li et al. (2019) allowed the authors to ensure the well-definedness of the iterative scheme presented in Colao et al. (2012) to a genuine Hadamard manifold. It is known that although the variational inequality theory provides us a tool for formulating a variety of equilibrium problems, even in the linear setting the generalization given by EP formulation with respect to VIP (variational inequality problem) is genuine, in the sense that there are EP which do not fit the format of VIP, see Remark 2. In this direction, our contribution in this paper has been to introduce a a new proximal algorithm for EP by using a new regularization term that, unlike the classical model, does not come as a natural extension of the optimality condition of a minimization problem. Consequently, we provide a convergence analysis that can be applied to genuinely Hadamard manifolds.

Several researchers have studied the proximal point method on Hadamard manifolds for particular instances of the EP; see, for example, Ferreira and Oliveira (2002), Li et al. (2009) and Tang et al. (2013). In recent years, extensions to Riemannian manifolds of concepts and

techniques which fit in Euclidean spaces are natural; see, for instance, Ledyaev and Zhu (2007), Ferreira et al. (2005), Li et al. (2011b), Li et al. (2011a), Kristály (2014), Yang et al. (2013), Cruz Neto et al. (2013), Quiroz (2013) and the references therein. One reason for the success in the extension of techniques from the linear setting to the Riemannian context, is the possibility to transform nonconvex problems in convex problems by introducing a suitable Riemannian metric; see Rapcsák (1997), Cruz Neto et al. (2002), Bento and Melo (2012) and Colao et al. (2012). On the application side, a striking point is that our new regularization term have a clear interpretation in a recent (VR) "variational rationality" approach of human behavior, see Soubeyran (2009, 2010, 2016, 2019a, b). It represents resistance to change aspects of such human dynamics driven by motivation to change aspects. This allows us to give an application to the theories of desires, showing how an agent must escape to a succession of temporary traps to be able to reach, at the end, his desires. In a dynamical context, a Riemannian manifold greatly helps to model human dynamics, because it represents what is possible, relative to what is not, that is, which bundles of situated activities an individual can do, each period, in this setting and which move he can do. The main interest to extend the VR approach in the Riemannian context is that this setting allows to formalize in a better way different aspects of the famous field theory in psychology; see, for example, Lewin (1935, 1936, 1938, 1951). In other words, a Riemannian manifold helps to model constraints that Lewin (1936) call "what is not possible" in the life space of free movement. A constraint is a fundamental concept in the Lewin approach as well as in the VR approach, because it represents an important aspect of resistance to move (opportunity costs). For example, when, given a time constraint, a time consuming activity forbids to have enough time to do an other activity within a period. In dimension three, the hyperbolic paraboloid

$$S = \left\{(x, y, z) \in X = \mathbb{R}^3_+ : z = xy\right\},$$

is a surface with negative Gauss curvature as well as negative sectional curvature; see, for example, Haesen et al. (2003) and Ng (1995). This constraint represents a classical Cobb Douglas utility or a production function with complementary and substitutability effects between the quantities of inputs (either consumption goods, or factors of production like labour and capital),

(i) geodesics model directions and distinguished paths which are central to Lewin's approach in term of psychological forces;
(ii) the link between curvature and resistance to move remains open for us.

The organization of our paper is as follows. In Sect. 2, we give some elementary facts on Riemannian manifolds and convexity needed for reading this paper. In Sect. 3, we present a sufficient condition for existence of a solution for the equilibrium problem on Hadamard manifolds under conditions similar to the linear case. In Sect. 4, the proximal point algorithm for equilibrium problems on Hadamard manifolds is presented and convergence analysis is derived. In Sect. 5, we give a behavioral application to the existence of temporary traps and the existence/reachability of desires, in the context of the recent and unifying approach of stability and change dynamics, (see Soubeyran 2009, 2010, 2016, 2019a, b). This approach focuses the attention on four main concepts:

(i) worthwhile single changes, where, for an isolated agent or several interrelated agents, their motivation to change from the current position to a new position is higher than some adaptive and satisficing worthwhile to change ratio, time their resistance to change. Motivation to change refers to the utility of advantages to change, while resistance to change refers to the disutility of inconveniences to change. Resistance to change includes inertia,

frictions, obstacles, difficulties to change, costs to be able to change and inconveniences to change as well as different resource constraints;

(ii) worthwhile transitions, i.e, succession of worthwhile single stays and changes;

(iii) traps, which can be stationary or variational. A trap is stationary when, starting from it, no feasible change is worthwhile. An equilibrium appears to be a very particular stationary trap, in a world with no resistance to change, when only motivation to change matters. In this case the agent has no motivation to change (no advantage to change, i.e only losses to change), and zero resistance to change. A trap is variational with respect to a subset of initial positions, when it is stationary, and, starting from any of these initial positions, agents can find a succession of worthwhile single changes and temporary stays which converge to this stationary trap. Then, a variational trap is rather easy to reach and difficult to leave in a worthwhile way. Furthermore, traps can be weak or strong, depending of large or strict inequalities;

(iv) desires, which represent, both to have what you want and to want what you have. The idea is that to be able to reach your desires, you must escape to several temporary traps.

This last section, devoted to applications, focuses the attention on a succession of worthwhile changes and stays, moving from a weak stationary trap to a new one, given that the agent can change, each step, his satisficing worthwhile to change ratio. The algorithm given in Sect. 4 represents a nice instance of such a worthwhile stability and change dynamic. The result of this paper shows that this dynamic converges to an equilibrium which represents a desired situation or desire. This worthwhile stability and change dynamic is a very important benchmark case of the more general stability and change dynamic (see Soubeyran 2009, 2010, 2016, 2019a, b), where a succession of worthwhile changes move from a position to a new one (which are not supposed to be stationary traps) and converges to an end point, which is shown to be a variational trap. Finally, Sect. 6 contains concluding discussions of the main results obtained in the paper.

2 Preliminary

A complete, simply connected Riemannian manifold with nonpositive sectional curvature is called a Hadamard manifold. Throughout this paper, every manifold M denotes a finite dimensional Hadamard manifold and, when we refer to a genuine Hadamard manifold we are referring to a Hadamard manifold whose sectional curvature is not identically zero. The notations, results, and concepts used in this paper can be found in Ferreira and Oliveira (2002).

A set $\Omega \subset M$ is said to be *convex* iff any geodesic segment with end points in Ω is contained in Ω, that is, iff $\gamma : [a, b] \to M$ is a geodesic such that $x = \gamma(a) \in \Omega$ and $y = \gamma(b) \in \Omega$, then $\gamma((1-t)a + tb) \in \Omega$ for all $t \in [0, 1]$. Given $\mathcal{B} \subset M$, we denote by $\mathrm{conv}(\mathcal{B})$ the convex *hull* of $\mathcal{B}$, that is, the smallest convex subset of M containing $\mathcal{B}$. Let $\Omega \subset M$ be a convex set. A function $f : \Omega \to \mathbb{R}$ is said to be *convex* iff for any geodesic segment $\gamma : [a, b] \to \Omega$ the composition $f \circ \gamma : [a, b] \to \mathbb{R}$ is convex. Take $p \in \Omega$. A vector $s \in T_pM$ is said to be a *subgradient* of f at p iff

$$f(q) \geq f(p) + \langle s, \exp_p^{-1} q \rangle, \quad q \in \Omega.$$

The set of all subgradients of f at p, denoted by $\partial f(p)$, is called the *subdifferential* of f at p. It is known that if f is convex and M is a Hadamard manifold, then $\partial f(p)$ is a nonempty set, for each $p \in \Omega$; see Udriste (1994, Theorem 4.5, p. 74).

Let $\mathcal{B} \subset M$ be a non-empty, convex and closed set. The distance function associated with $\mathcal{B}$ is given by

$$M \ni x \longmapsto d_{\mathcal{B}}(x) := \inf\{d(y, x) : y \in \mathcal{B}\} \in \mathbb{R}_+.$$

It is well-known (see Ferreira and Oliveira 2002, Corollary 3.1) that for each $x \in M$ there exists a unique element $\tilde{x} \in \mathcal{B}$ such that

$$\langle \exp_{\tilde{x}}^{-1} x, \exp_{\tilde{x}}^{-1} y \rangle \leq 0, \quad y \in \mathcal{B}.$$

In this case, $\tilde{x}$ is the projection of x onto the set $\mathcal{B}$ which we will denote by $P_{\mathcal{B}}(x)$.

Remark 1 It is important to mention that for every $y \in M$, $x \mapsto d(x, y)$ is a continuous and convex function; see Sakai (1996, Proposition 4.3, p. 222).

3 Equilibrium problem

In this section, following the ideas given in Iusem et al. (2009), we present a sufficient condition for the existence of solution of equilibrium problems on Hadamard manifolds. We chose to present a proof only for the main result. With the exception to the proof of Proposition 1, the proof of the other results can be extended, from those presented in linear environments (see Iusem et al. 2009; Iusem and Sosa 2003), with minor adjustments to the nonlinear context of this paper.

From now on, $\Omega \subset M$ will denote a nonempty closed convex set, unless explicitly stated otherwise. Given a bifunction $F : \Omega \times \Omega \to \mathbb{R}$ satisfying the property $F(x, x) = 0$, for all $x \in \Omega$, the *equilibrium problem* in the Riemannian context (denoted by EP) consists in:

$$\text{Find } x^* \in \Omega : F(x^*, y) \geq 0, \quad \forall\, y \in \Omega. \tag{3.1}$$

In this case, the bifunction F is called an *equilibrium bifunction*. As far as we know, this problem was considered firstly, in this context, in Colao et al. (2012) where the authors pointed out important problems, which are retrieved by (3.1). Particularly, given $V \in \mathcal{X}(M)$, if

$$F(x, y) = \langle V(x), \exp_x^{-1} y \rangle, \quad \forall\, x, y \in \Omega,$$

Equation (3.1) reduces to the variational inequality problem which, as far as we know, was first introduced in Németh (2003), for single-valued vector fields on Hadamard manifolds, and afterward extended for single-valued vector fields and multivalued vector fields on general Riemannian manifolds in Li et al. (2009) and Li and Yao (2012) respectively.

Remark 2 Although the Variational Inequality Theory provides us a tool for formulating a variety of equilibrium problems, Iusem and Sosa (2003, Proposition 2.6) showed that the generalization given by EP formulation with respect to VIP (variational inequality problem) is genuine, in the sense that there are EP problems which do not fit the format of VIP. We affirm that it is possible to guarantee the genuineness of the EP formulation compared to VIP, by considering the important class of quasi-convex optimization problems that appears, for instance, in many micro-economical models devoted to maximize utilities. Indeed, the absence of convexity allows us to obtain situations where this important class of problems can not be considered as a VIP in the sense that their possible representation in this format would lead us to a problem, whose solution set contains points that do not necessarily belong to the solution set of the original optimization problem. On the other hand even in the absence of convexity, this class of problems can be placed in the EP format.

Definition 1 Let $F : \Omega \times \Omega \to \mathbb{R}$ be a bifunction. F is said to be

(1) *monotone* iff $F(x, y) + F(y, x) \leq 0$, for all $(x, y) \in \Omega \times \Omega$;
(2) *pseudomonotone* iff, for each $(x, y) \in \Omega \times \Omega$, $F(x, y) \geq 0$ implies $F(y, x) \leq 0$.

Remark 3

(i) Clearly, monotonicity implies pseudomonotonicity, but the converse does not hold even in a linear context, see, for instance, Iusem and Sosa (2003).
(ii) If F is pseudomonotone, then for $\tilde{x}, \tilde{y} \in \Omega$, $F(\tilde{x}, \tilde{y}) > 0$ implies $F(\tilde{y}, \tilde{x}) < 0$. Indeed, let us suppose, for contradiction, that $F(\tilde{y}, \tilde{x}) = 0$ (in particular $F(\tilde{y}, \tilde{x}) \geq 0$). From the pseudomonotonicity of F it follows that $F(\tilde{x}, \tilde{y}) \leq 0$, which is an absurd, and the affirmation is proved.

Next result was presented by Colao et al. (2012) and it is fundamental to establish our existence result for the EP.

Proposition 1 *Let $\mathcal{B} \subset M$ be a closed convex subset and $H : \mathcal{B} \to 2^{\mathcal{B}}$ be a mapping such that, for each $y \in \mathcal{B}$, $H(y)$ is closed. Suppose that*

(i) *there exists $y_0 \in \mathcal{B}$ such that $H(y_0)$ is compact;*
(ii) $\forall y_1, \ldots, y_m \in \mathcal{B}$, $\operatorname{conv}(\{y_1, \ldots, y_m\}) \subset \bigcup_{i=1}^{m} H(y_i)$.

Then,

$$\bigcap_{y \in \mathcal{B}} H(y) \neq \varnothing.$$

Proof See Colao et al. (2012). □

Unless stated to the contrary, in the remainder of this paper we assume that $F : \Omega \times \Omega \to \mathbb{R}$ is an *equilibrium bifunction* satisfying the following assumptions:

$\mathcal{H}1$) For every $x \in \Omega$, $y \mapsto F(x, y)$ is convex and lower semicontinuous;
$\mathcal{H}2$) For every $y \in \Omega$, $x \mapsto F(x, y)$ is upper semicontinuous.

For each $y \in \Omega$, let us define:

$$L_F(y) := \{x \in \Omega : F(y, x) \leq 0\}.$$

From this set, we can consider the following *convex feasibility problem* (denoted by CFP):

$$\text{Find } x^* \in \bigcap_{y \in \Omega} L_F(y).$$

As far as we know, this problem was first studied, in the Riemannian context, by Bento and Melo (2012), in the particular case where the domain of F is given $M \times \{1, \ldots, m\}$. In this case, $y \in \{1, \ldots, m\}$ and Ω is the whole M.

Next result establishes a relationship between CFP and EP.

Lemma 1 *The solution set of* CFP *is contained in the solution set of* EP.

Remark 4 Note that, as it is in the Euclidean context, the equality between the two sets in the previous lemma in general does not happen, see Iusem and Sosa (2003). However, in the particular case where F is pseudomonotone, the equality is immediately verified.

Take $z_0 \in M$ fixed. For each $k \in \mathbb{N}$ consider the following set:

$$\Omega_k := \{x \in \Omega : d(x, z_0) \leq k\}.$$

Note that Ω_k is a nonempty set, for $k \in \mathbb{N}$ sufficiently large. For simplicity, we can suppose, without loss of generality, that Ω_k is a nonempty set for all $k \in \mathbb{N}$. Moreover, as Ω_k is contained in the closed ball $B(z_0, k) := \{x \in M : \ d(z_0, x) \leq k\}$, it is a bounded set. On the other hand, since $d(\cdot, z_0)$ is a continuous and convex function (this follows from Remark 1), Ω_k is a convex and closed set and, hence, compact; see Ropf–Rinow's Theorem. We denote, by Ω_k^0, the following set:

$$\Omega_k^0 := \{x \in \Omega : d(x, z_0) < k\}.$$

For each $y \in \Omega$, let us define:

$$L_F(k, y) := \{x \in \Omega_k : F(y, x) \leq 0\}.$$

Lemma 2 *Let $k \in \mathbb{N}$, $\bar{x} \in \bigcap_{y \in \Omega_k} L_F(k, y)$ and assume that there exists $\bar{y} \in \Omega_k^0$ such that $F(\bar{x}, \bar{y}) \leq 0$. Then, $F(\bar{x}, y) \geq 0$, for all $y \in \Omega$, i.e., $\bar{x}$ is a solution for* (3.1).

Assumption 1 *Given $k \in \mathbb{N}$, for all finite set $\{y_1, \ldots, y_m\} \subset \Omega_k$, one has*

$$\operatorname{conv}(\{y_1, \ldots, y_m\}) \subset \bigcup_{i=1}^{m} L_F(k, y_i).$$

Remark 5 Note that, in the particular case where F is pseudomonotone, the property described by the previous assumption is naturally verified. Indeed, let us consider $y_1, \ldots, y_m \in \Omega_k$, take $\bar{y} \in \operatorname{conv}(\{y_1, \ldots, y_n\})$ and let us suppose, for contradiction, that $\bar{y} \notin \bigcup_{i=1}^{m} L_F(k, y_i)$. Then,

$$F(y_i, \bar{y}) > 0, \quad i \in \{1, \ldots, m\}. \tag{3.2}$$

Now, define the following set $B := \{x \in \Omega_k : F(\bar{y}, x) < 0\}$. In the particular case where F is pseudomonotone, using (3.2) and taking into account that B is convex (this follows from $\mathcal{H}1$), we conclude that $\bar{y} \in B$ (see item ii) of Remark 3). But this contradicts that $F(x, x) = 0$ and the affirmation is proved.

Assumption 2 *Given $z_0 \in M$ fixed, consider a sequence $\{z^k\} \subset \Omega$ such that $\{d(z^k, z_0)\}$ converges to infinity as k goes to infinity. Then, there exists $x^* \in \Omega$ and $k_0 \in \mathbb{N}$ such that*

$$F(z^k, x^*) \leq 0, \quad k \geq k_0.$$

It is worth noting that this last assumption has been presented by Iusem et al. (2009), in a space with a linear structure. It is a sufficient condition for the existence of solutions of the equilibrium problem EP.

Next result (see Iusem and Sosa 2003 for similar results, in the linear setting) assures us that Assumption 2 is a weaker sufficient condition than the coercivity assumption used by Colao et al. (2012), for the existence of solutions of EP.

Proposition 2 *Let $\mathcal{B} \subset M$ be a compact set and $y_0 \in \mathcal{B} \cap \Omega$ a point such that $F(x, y_0) < 0$, for all $x \in \Omega \backslash \mathcal{B}$. Then, F satisfies* Assumption 2.

Next, we state and prove the main result of this section.

Theorem 1 *Under* Assumptions 1 *and* 2*, EP admits a solution.*

Proof Recall that Ω_k is a convex and compact set for each $k \in \mathbb{N}$. Now, given $k \in \mathbb{N}$ and $y \in \Omega$, note that $L_F(k, y)$ is a compact set. Indeed, this fact follows from the definition of $L_F(k, y)$ combined with assumption $\mathcal{H}1$ ($F(y, \cdot)$ is a lower semicontinuous function on Ω) and compactness of Ω_k. Now, since Assumption 1 holds true, using Proposition 1 with $\mathcal{B} = \Omega_k$ and $H(y) = L_F(k, y)$, we conclude that, for each $k \in \mathbb{N}$,

$$\bigcap_{y \in \Omega_k} L_F(k, y) \neq \emptyset.$$

For each k, choose $z^k \in \bigcap_{y \in \Omega_k} L_F(k, y)$ and take $z_0 \in M$ fixed. If there exists $k \in \mathbb{N}$ such that $d(z^k, z_0) < k$, then $z^k \in \Omega_k^0$ and, from Lemma 2, it follows that z^k solves EP. On the other hand, if $d(z^k, z_0) = k$, from Assumption 2, there exist, $x^* \in \Omega$ and $k_0 \in \mathbb{N}$ such that $F(z^k, x^*) \leq 0$, for all $k \geq k_0$. Taking $k' > k_0$ such that $d(x^*, z_0) < k'$, we have $F(z^{k'}, x^*) \leq 0$ and $x^* \in \Omega_{k'}^0$. Therefore, using again Lemma 2, we conclude that $z^{k'}$ solves EP, and the proof is complete. □

Next example was inspired by Colao et al. (2012, Example 3.4). It illustrates the usefulness of our previous result, in the sense that it can be applied to some situations not covered in the linear setting. For other papers that highlight such advantage, in regard to the linear setting, see Cruz Neto et al. (2006) and Bento and Melo (2012).

Example 1 Let $\Omega = \{(x, y, z) : 0 \leq x \leq 1, y^2 - z^2 = -1, y \geq 0, z \geq 1\} \subset \mathbb{R} \times \mathbb{H}^1$ and consider the following bifunction $F : \Omega \times \Omega \to \mathbb{R}$, given by:

$$F((x_1, y_1, z_1), (x_2, y_2, z_2)) := (2 - x_1)\big(\big(y_2^2 + z_2^2\big(- \big(y_1^2 + z_1^2\big(\big(.$$

Note that Ω is indeed a not convex set in $\mathbb{R}^3$. So, an equilibrium problem defined on Ω cannot be solved by using the classical results known in the linear context. Let $(\mathbb{H}^n, \langle\, , \,\rangle)$ be the Riemannian manifold, where

$$\mathbb{H}^n := \{x = (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0 \text{ and } \langle x, x\rangle = -1\}$$

(hyperbolic n space), and $\langle \cdot, \cdot \rangle$ is the Riemannian metric:
$\langle x, y\rangle := x_1 y_1 + x_2 y_2 + \ldots + x_n y_n - x_{n+1} y_{n+1}$ (Lorentz metric). As noted in Colao et al. (2012), $(\mathbb{H}^n, \langle \cdot, \cdot \rangle)$ is a Hadamard manifold with sectional curvature -1 and, given initial conditions $x \in \mathbb{H}^n$, $v \in T_x\mathbb{H}^n$ ($\|v\| = 1$), the normalized geodesic $\gamma : \mathbb{R} \to \mathbb{H}^n$, is given by:

$$\gamma(t) = (\cosh t)x + (\sinh t)v, \quad t \in \mathbb{R}.$$

Hence, we obtain the following expression for the Riemannian distance d:

$$d(x, y) = \operatorname{arccosh}(-\langle x, y\rangle), \quad x, y \in \mathbb{H}^n.$$

Again, as observed in Colao et al. (2012), Ω is a convex set which is immersed in the Hadamard manifold $M := \mathbb{R} \times \mathbb{H}^1$. Using the expression of the geodesic curves, it can be deduced that F is a convex function in the second variable. Moreover, from the definition of F, it is easy to see that $F(x, x) = 0$, assumptions $\mathcal{H}1$ and $\mathcal{H}2$ are satisfied, and F is a pseudomonotone bifunction. In particular, from Remark 5, it follows that *Assumption* 1 holds. Now, take $w^0 \in M$ fixed and a sequence $\{w^k\} \subset \Omega$, $w^k := (x_k, y_k, z_k)$, such that $d(w^k, w^0) \to +\infty$. There exists a point $x^* := (1, 0, 1) \in \Omega$ such that $F(w^k, x^*) \leq 0$, for all $k \in \mathbb{N}$, i.e., *Assumption* 2 holds. Therefore, *Theorem* 1 implies the existence of an equilibrium point for F.

4 Proximal point for equilibrium problem

In this section, we propose a new proximal point algorithm for equilibrium problems on Hadamard manifolds where the convergence result is obtained for monotone bifunctions. As far as we know, a proximal point algorithm for EP was first introduced in Colao et al. (2012). In comparison with the algorithm given in Colao et al. (2012), the novelty here is the new regularization term, whose convexity does not impose any restrictive hypothesis on the Hadamard manifold.

Let us denote the equilibrium point set of F by $\mathrm{EP}(F, \Omega)$ and, for $\lambda > 0$ and $z \in \Omega$ fixed, consider the regularized bifunction

$$F_{\lambda,z}(x, y) := F(x, y) + \lambda \left[d^2(y, z) - d^2(x, z)\right], \quad x, y \in \Omega. \tag{4.3}$$

Now, we describe a proximal algorithm to solve the equilibrium problem (3.1).

Algorithm 1 *Take $\{\lambda_k\}$ a bounded sequence of positive real numbers.*

INITIALIZATION. *Choose an initial point $x^0 \in \Omega$;*
STOPPING CRITERION. *Given x^k, if $x^{k+1} = x^k$ STOP. Otherwise;*
ITERATIVE STEP. *Given x^k, take as the next iterate any $x^{k+1} \in \Omega$ such that:*

$$x^{k+1} \in \mathrm{EP}(F_k, \Omega), \quad F_k := F_{\lambda_k, x^k}. \tag{4.4}$$

Remark 6 Note that if $x^{k+1} = x^k$, then $x^k \in \mathrm{EP}(F,\Omega)$. It is worth noting that the iterative process (4.4) retrieves the proximal point method for minimization problems on Hadamard manifolds; see Ferreira and Oliveira (2002).

Remark 7 Our algorithm uses a new type of regularization which is convex on genuine Hadamard manifolds, namely, $y \mapsto \left[d^2(y, z) - d^2(x, z)\right]$. Note that our regularization is different from one presented in Colao et al. (2012) ($y \mapsto \langle \exp_x^{-1} z, \exp_x^{-1} y\rangle$), whose convexity is ensured only on Hadamard manifolds with null sectional curvature, as shown by Kristály et al. (2016).

Next results are useful to ensure the well-definition of Algorithm 1. In the remainder of the paper we assume that F is monotone, λ is a positive real number and $z \in \Omega$.

Lemma 3 *Let F be a monotone bifunction. Then,*

(i) *$F_{\lambda,z}$ is monotone;*
(ii) *$F_{\lambda,z}$ satisfies assumption $\mathcal{H}1$.*

Proof Item (i) follows immediately from the monotony of F and item (ii) for considering that F and $d^2(\cdot, \cdot)$ satisfy $\mathcal{H}1$. □

Lemma 4 *Let* EP*(F, Ω) be a nonempty set. If F satisfies* Assumption 2*, then $F_{\lambda,z}$ also satisfies this assumption.*

Proof First of all, given $z_0 \in M$, consider a sequence $\{z^k\} \subset \Omega$ such that $\{d(z^k, z_0)\}$ converges to infinity as k goes to infinity. Take $\tilde{x} \in \mathrm{EP}(F, \Omega)$. Using (4.3) with $x = z^k$ and $y = \tilde{x}$, we get

$$F_{\lambda,z}(z^k, \tilde{x}) = F(z^k, \tilde{x}) + \lambda \left[d^2(\tilde{x}, z) - d^2(z^k, z)\right]. \tag{4.5}$$

Since $\{d(z^k, z_0)\}$ converges to infinity as k goes to infinity, in particular, $\{d(z^k, z)\}$ also converges to infinity. Moreover, as $\{z^k\} \subset \Omega$ and $\tilde{x} \in \mathrm{EP}(F, \Omega)$, monotonicity of F implies that $F(z^k, \tilde{x}) \leq 0$. Hence, the desired result follows immediately from (4.5), which concludes the proof. □

Theorem 2 *Assume that* Assumption 2 *holds and* EP*(F,Ω) is a nonempty set. Then, there exists* $x^* \in \Omega$ *such that*

$$F_{\lambda,z}(x^*, y) \geq 0, \quad y \in \Omega.$$

Proof From item (i) of Lemma 3 it follows that $F_{\lambda,z}$ is monotone and, in particular, pseudomotone (this follows from Remark 3). Moreover, Remark 5 implies that $F_{\lambda,z}$ satisfies Assumption 1 and Lemma 3 (resp. Lemma 4) tells us that $F_{\lambda,z}$ satisfies $\mathcal{H}1$ (resp. Assumption 2). Hence, the desired result follows from Theorem 1 and the proof is concluded. □

Corollary 1 *Assume that* Assumption 2 *holds and* EP*(F, Ω) is a nonempty set. Then, Algorithm 1 is well-defined.*

Proof It follows immediately from Theorem 2. □

In the remainder of this paper we assume that the assumption of the previous corollary hold and $\{x^k\}$ is a sequence generated from Algorithm 1. Taking into account that if Algorithm 1 terminates after a finite number of iterations, it terminates at an equilibrium point of F, from now on, we assume also that $\{x^k\}$ is an infinite sequence.

4.1 Convergence analysis

In this section we present the convergence of the sequence $\{x^k\}$. Let us start it presenting the concept of Fejér convergence and a known result on this subject.

Definition 2 A sequence $\{y^k\}$ in the complete metric space (M, d) is said to be *Fejér convergent* to a nonempty set $\mathcal{S} \subset M$ iff for every $\bar{y} \in \mathcal{S}$,

$$d(y^{k+1}, \bar{y}) \leq d(y^k, \bar{y}) \quad k = 0, 1, \ldots.$$

The following result is well known and its proof is elementary.

Proposition 3 *Let* $\{y^k\}$ *be a sequence in the complete metric space* (M, d). *If* $\{y^k\}$ *is Fejér convergent to a nonempty set* $\mathcal{S} \subset M$, *then* $\{y^k\}$ *is bounded. If, furthermore, an accumulation point* $\bar{y}$ *of* $\{y^k\}$ *belongs to* $\mathcal{S}$, *then* $\lim_{k\to\infty} y^k = \bar{y}$.

The next result is useful for establishing Fejér convergence of $\{x^k\}$.

Proposition 4 *For each* $x^* \in EP(F, \Omega)$ *it holds*

$$\langle \exp^{-1}_{x_{k+1}} x^k, \exp^{-1}_{x_{k+1}} x^* \rangle \leq 0. \tag{4.6}$$

Proof Indeed, from the definition of the iterate x^{k+1} and F_k in (4.4) combined with (4.3), we obtain

$$F_k(x^{k+1}, y) = F(x^{k+1}, y) + \lambda_k \left[d^2(y, x^k) - d^2(x^{k+1}, x^k) \right] \geq 0, \quad y \in \Omega.$$

Since $F_k(x^{k+1}, x^{k+1}) = 0$, last inequality tells us that

$$x^{k+1} = \arg\min_{y \in \Omega} F_k(x^{k+1}, y)$$

or, equivalently, that there exists

$$w^{k+1} \in \partial F_k(x^{k+1}, x^{k+1}) = \partial F(x^{k+1}, x^{k+1}) - \lambda_k \exp^{-1}_{x^{k+1}} x^k$$

such that

$$\langle w^{k+1}, \exp^{-1}_{x^{k+1}} y \rangle \geq 0, \quad y \in \Omega.$$

Note that $w^{k+1} = u^{k+1} - \lambda_k \exp^{-1}_{x^{k+1}} x^k$, where $u^{k+1} \in \partial F(x^{k+1}, x^{k+1})$. So, last inequality yields

$$\langle u^{k+1}, \exp^{-1}_{x^{k+1}} y \rangle - \lambda_k \langle \exp^{-1}_{x^{k+1}} x^k, \exp^{-1}_{x^{k+1}} y \rangle \geq 0, \quad y \in \Omega. \tag{4.7}$$

On the other hand, convexity of $F(x^{k+1}, \cdot)$ (this follows from $\mathcal{H}1$) implies that, for all $y \in \Omega$, we have

$$F(x^{k+1}, y) \geq F(x^{k+1}, x^{k+1}) + \langle u^{k+1}, \exp^{-1}_{x^{k+1}} y \rangle = \langle u^{k+1}, \exp^{-1}_{x^{k+1}} y \rangle. \tag{4.8}$$

Take $\bar{x} \in \mathrm{EP}(F, \Omega)$. As $\mathrm{EP}(F, \Omega) \subset \Omega$, from the last inequality, we obtain

$$\langle u^{k+1}, \exp^{-1}_{x^{k+1}} \bar{x} \rangle \leq F(x^{k+1}, \bar{x}) \leq 0, \tag{4.9}$$

where last inequality is a consequence of the monotony of F, since $F(\bar{x}, x^{k+1}) \geq 0$. Therefore, because $\lambda_k > 0$ for all k, the result of the claim immediately follows by combining inequalities (4.7) and (4.9). □

Now, we present our main convergence result.

Theorem 3 *The sequence $\{x^k\}$ converges to a point in $EP(F, \Omega)$.*

Proof Take $x^* \in \mathrm{EP}(F, \Omega)$. Using Sakai (1996, Theorem 4.2, p. 161) (law of cosines) with $x_i = x^*$, $x_{i+1} = x^k$ and $x_{i+2} = x^{k+1}$, we obtain

$$d^2(x^{k+1}, x^*) + d^2(x^{k+1}, x^k) - 2\langle \exp^{-1}_{x^{k+1}} x^k, \exp^{-1}_{x^{k+1}} x^* \rangle \leq d^2(x^k, x^*), \quad k = 0, 1, \ldots. \tag{4.10}$$

Since $x^* \in \mathrm{EP}(F, \Omega)$, combining inequality (4.6) with the last inequality and taking into account that $d^2(x^{k+1}, x^k) > 0$, it follows that $\{x^k\}$ is Fejér convergent to the set $\mathrm{EP}(F, \Omega)$. Moreover, from (4.10) combined with inequality (4.6), we have that $\{d(x^{k+1}, x^k)\}$ converges to zero as k goes to infinity. Applying Proposition 3 with $y^k = x^k$, $k \in \mathbb{N}$, and $\mathcal{S} = \mathrm{EP}(F, \Omega)$, we have that $\{x^k\}$ is a bounded sequence. In particular, from Hopf-Rinow Theorem, there exists a subsequence $\{x^{k_j}\}$ of $\{x^k\}$ converging to some point $\hat{x}$. Note that $\{d(x^{k_j+1}, x^{k_j})\}$, $\{d(x^{k_j+1}, \hat{x})\}$ converge to zero and, in particular, $\{x^{k_j+1}\}$ goes to $\hat{x}$ as j goes to infinity. For combining (4.7), (4.8) and (4.10), we obtain

$$F(x^{k_j+1}, y) \geq \frac{\lambda_{k_j}}{2} \left[d^2(y, x^{k_j+1}) + d^2(x^{k_j+1}, x^{k_j}) - d^2(y, x^{k_j}) \right].$$

Using simple upper limit properties applied to last inequality, since $d^2(y, x^{k_j+1}), d^2(x^{k_j+1}, x^{k_j})$, $d^2(y, x^{k_j})$ converge to zero and $\{\lambda_k\}$ is bounded, it follows from $\mathcal{H}2$ that $0 \leq \limsup_{j \to +\infty} F(x^{k_j}, y) \leq F(\hat{x}, y)$, for all $y \in \Omega$. But this tells us that $\hat{x} \in \mathrm{EP}(F, \Omega)$ and the desired result it follows by using Proposition 3 with $y^k = x^k$, $k \in \mathbb{N}$, and $\mathcal{S} = \mathrm{EP}(F, \Omega)$.

5 Application: the problem of traps and the reachability of desires in behavioral sciences

In this last section we want to show how the present paper offers, in a specific context, a nice solution to the behavioral problem of how an agent can escape to a succession of traps to be able to reach his desires, using the recent variational rationality modelization of desires and temporary/permanent traps all along stay and change worthwhile transitions; see Soubeyran (2009, 2010, 2016, 2019a, b).

This paper considers two related problems, in a given space of alternatives Ω (possible choices of actions, states, positions, situations,…) on a Hadamard manifold:

(i) the standard static equilibrium problem given in (3.1);
(ii) a new and dynamic regularized equilibrium problem given in (4.4) which, as observed in the very beginning of the convergence analysis, consists to find in the current period $k+1$, a choice $x^{k+1} \in \Omega$ such that

$$F_{\lambda_k, x^k}(x^{k+1}, y) = F(x^{k+1}, y) + \lambda_k I_{x^k}(x^{k+1}, y) \geq 0, \quad y \in \Omega,$$

where the specific regularized term is given by

$$I_{x^k}(x^{k+1}, y) := \left[d^2(x^k, y) - d^2(x^k, x^{k+1})\right], \quad \lambda_k > 0$$

is a weight, x^{k+1} is the current choice, y is a current alternative choice, and $x^k \in \Omega$ is the last choice. Under specific assumptions, this paper shows, first, the existence of a static equilibrium problem (3.1). Then, it shows that the solution $x^k \in \Omega$ of the dynamic equilibrium problem converges to a static equilibrium $x^* \in \Omega$.

5.1 Desires: the duality between equilibrium points and strong aspiration points

Let us see, using a recent variational rationality formalization of acceptable stays and changes approach and avoidance dynamics which unifies a lot of them in different disciplines, see Soubeyran (2009, 2010, 2016, 2019a, b), why a dual view of an equilibrium problem is so important for applications in Behavioral Sciences. Consider an active agent who derives the utility $V(y) \in \mathbb{R}$ to perform an action $y \in \Omega$, where his highest utility level $\overline{V} = \sup\{V(y) : y \in \Omega\} < +\infty$ is finite, because of bounded needs. In Applied Mathematics and very often in Behavioral Sciences, agents are supposed to minimize a cost function, or to try to decrease their level of unsatisfaction $U(y) \in \mathbb{R}$. The (VR) approach defines unsatisfaction as unsatisfied needs, i.e, the difference $U(y) = \overline{V} - V(y) \geq 0$. We will consider such unsatisfaction levels, given the Mathematical audience.

In a static context, the agent will not regret to have done action x^* if there is a loss

$$F(x^*, y) = U(y) - U(x^*) \geq 0,$$

to change, moving from doing x^* to do any other action $y \in \Omega$ with a highest unsatisfaction level. This view supposes that the agent is, at the very beginning of the story, right where he wants to be, at the maximum of utility $V(x^*)$, i.e, at the lowest level of unsatisfaction $U(x^*)$. But this is not very realistic. In real life, agent are not, initially, exactly where they want to be. Usually, they start to be in an unsatisfactory position, at $y = x^0 \in \Omega$, where $U(y) > U(x^*)$. This means that, initially, an agent will find an advantage to change $A(y, x^*) = U(y) - U(x^*) > 0$ from $y = x^0$ to x^*. In this case, $A(y, x^*) = F(x^*, y) > 0$.

Then, the (VR) approach completely reverses the logic of the equilibrium problem. It focuses first the attention on agents who aspire for better and considers advantages to change to a better situation, and only, at the end, on agents who fear for less, and consider losses to change to a worst situation! Moreover, unfortunately, most of the time, an agent does not know, at least at the very beginning, what can be the best action to do, $x^* \in \Omega$. To hope to know it, he cannot escape to enter, if possible, in an improving dynamic, where each period, his unsatisfaction moves from $U(x^k)$ in the last period to a lower level $U(x^{k+1}) \leq U(x^k)$ in the current period, enjoying the advantage to change $A(x^k, x^{k+1}) = U(x^k) - U(x^{k+1}) \geq 0$. Then, the agent will follow an improving transition of stays and changes, where, each period, the agent will stay, $x^{k+1} = x^k$, if he fails to find an improving action, or will change, $x^{k+1} \neq x^k$, in the opposite case of success.

This dual dynamic context considers both:

(i) the equilibrium concept, relative to losses to change (static negative view), where an agent, being there, at x^*, will prefer to stay there, than to move away, because moving from x^* to any $y \in \Omega$ will generate the loss

$$F(x^*, y) = U(y) - U(x^*) \geq 0;$$

(ii) the dual concept of strong aspiration, relative to advantages to change (dynamic positive view), where the agent being at $y = x^0 \neq x^*$, will prefer to move to the aspiration point x^* rather than to stay at y, because moving from any y to x^* will generate the advantage to change

$$A(y, x^*) = U(y) - U(x^*) = F(x^*, y) \geq 0.$$

Then, the dual equilibrium problem is: find $x^* \in X$ such that:

$$A(y, x^*) \geq 0, \quad y \in \Omega,$$

i.e., $A(x^*, y) = -A(y, x^*) \leq 0$ for all $y \in \Omega$.

This dual view helps to define desires as an equilibrium (to want what you have) and a strong aspiration point (to hope to have what you want, starting from anywhere). Of course, in this very simple formulation of a non "reference dependent" utility function, we have $A(x, y) = -F(x, y)$, i.e., if there is an advantage to change from x to y, there is an opposite loss to move the other way. This is not the case if utility functions are reference dependent; see Soubeyran (2009, 2010, 2016, 2019a, b).

5.2 The regularized equilibrium function as an instance of a worthwhile to change payoff

We are now in a good position to see what represents, in the context of the variational rationality approach, the regularized equilibrium function given in (4.3). The previous discussion on losses to change $F(x, y)$ and their opposites, advantages to change $A(x, y) = -F(x, y)$ directs our attention to the opposite of the regularized equilibrium function,

$$\Delta_{\lambda,z}(x, y) = -F_{\lambda,z}(x, y) = A(x, y) - \lambda I_z(x, y),$$

where $I_z(x, y) = \left[d^2(z, y) - d^2(z, x)\right]$ denotes the regularization term in (4.3). More explicitly, in a dynamic context, we move the attention from

$$F_{\lambda_k, x^k}(x^{k+1}, y) = F_{x^k}(x^{k+1}, y) + \lambda_k I_{x^k}(x^{k+1}, y)$$

to its opposite,

$$\Delta_{\lambda_k,x^k}(x^{k+1},y) = -F_{\lambda_k,x^k}(x^{k+1},y) = A_{x^k}(x^{k+1},y) - \lambda_k I_{x^k}(x^{k+1},y).$$

It turns out that $\Delta_{\lambda,z}(x,y)$ is a specific, but interesting instance of a worthwhile to change payoff, a central concept of the variational rationality approach, where an agent balances, each current period $k+1$, advantages $A(x^{k+1},y)$ and inconveniences to change $I_z(x^{k+1},y)$, using the balancing weight λ_k. This means that an hypothetic change from x^{k+1} to y in the current period is not (strictly) worthwhile, i.e, $\Delta_{\lambda_k,x^k}(x^{k+1},y) \leq 0$ if advantages to change are high enough (the weight λ being hight enough) with respect to inconveniences to change, i.e, $A_{x^k}(x^{k+1},y) - \lambda_k I_{x^k}(x^{k+1},y) \leq 0$. In the specific context of this paper, the (VR) approach (see Soubeyran 2009, 2010) defines inconveniences to change as follows. They represent, in the current period, the difference $I_{x^k}(x^{k+1},y) := C_{x^k}(x^k,y) - C_{x^k}(x^k,x^{k+1})$ between costs to be able to change $C_{x^k}(x^k,y) \in \mathbb{R}_+$ from the last position $z = x^k$ to the alternative position y, and costs to be able to change $C_{x^k}(x^k,x^{k+1})$ from the last position $z = x^k$ to a current position x^{k+1}. Because they refer to a lot of different situations, in different disciplines, where resistance to change and inertia matter, these costs are not easy to define in a precise way. For a carefull formulation of these costs to be able to change and several examples, see Soubeyran (2015). In the present paper, $C_{x^k}(x^k,y) = d^2(x^k,y)$, where $d(z,y)$ is the geodesic distance on a Hadamard manifold.

Given that $F_{\lambda_k,x^k}(x^{k+1},x^{k+1}) = 0$, the current condition $F_{\lambda_k,x^k}(x^{k+1},y) \geq 0$, for all $y \in \Omega$ means that, each current period $k+1$, the agent minimizes his current regularized equilibrium function, i.e,

$$x^{k+1} = \text{argmin}_{y\in\Omega} F_{\lambda_k,x^k}(x^{k+1},y).$$

Then, the current opposite condition $\Delta_{\lambda_k,x^k}(x^{k+1},y) \leq 0$, for all $y \in \Omega$, means that, each current period, the agent maximizes his current worthwhile to change balance, i.e., $x^{k+1} = \text{argmax}_{y\in\Omega} \Delta_{\lambda_k,x^k}(x^{k+1},y)$.

5.3 The algorithm as an instance of a worthwhile transition

In the current period $k+1$, the current equilibrium condition (4.4) can be written, taking the opposite,

$$\Delta_{\lambda_k,x^k}(x^{k+1},y) \leq 0, \quad y \in \Omega.$$

Take $y = x^k$, $k \in \mathbb{N}$. Then,

$$\Delta_{\lambda_k,x^k}(x^k,x^{k+1}) = -\Delta_{\lambda_k,x^k}(x^{k+1},x^k) = F_{\lambda_k,x^k}(x^{k+1},y) \geq 0$$

shows that it is worthwhile to change, $x^{k+1} \in W_{\lambda_k,x^k}(x^k)$, $k \in \mathbb{N}$, from x^k in the last period to x^{k+1} in the current period. This shows that Algorithm 1 defines a worthwhile stay and change transition, where, each period, advantages to change are higher enough with respect to inconveniences to change, i.e,

$$A_{x^k}(x^k,x^{k+1}) \geq \lambda_k I_{x^k}(x^k,x^{k+1}).$$

Then, inconveniences to change $I_{x^k}(x^k,x^{k+1}) = d^2(x^k,x^{k+1})$ and advantages to change $A_{x^k}(x^k,x^{k+1})$ are non negative all along the path of worthwhile stays and changes.

5.4 The two main results show how to reach desires, escaping to a succession of temporary variational traps

The concept of trap appears in an informal way in a lot of different disciplines in Behavioral Sciences; for a short survey see Soubeyran (2015). The variational rationality approach gives a general and formal definition. A (permanent) variational trap is the end of a worthwhile stay and change transition $x^{k+1} \in W_{\lambda_k, x^k}(x^k)$, $k \in \mathbb{N}$. More precisely, a (permanent) variational trap is, both,

(i) an aspiration point, more or less easy to approach, i.e, x^k converges to x^* as k goes to $+\infty$, and reaches it, $x^* \in W_{\lambda_k, x^k}(x^k)$, $k \in \mathbb{N}$, following a worthwhile stay and change transition, $x^{k+1} \in W_{\lambda_k, x^k}(x^k)$, $k \in \mathbb{N}$;
(ii) a stationary trap (a static equilibrium with inconveniences to change), difficult to leave, this is, $W_{\lambda_*, x^*}(x^*) = \{x^*\}$, where $\lambda_* > 0$ is the end worthwhile to change ratio.

Variational traps can be permanent or temporary. In the present paper, they are temporary, for $k \in \mathbb{N}$, $W_{\lambda_k, x^k}(x^k) = \left\{x^k\right\}$, because changing the worthwhile to change ratio λ_k breaks the current variational trap.

The present paper shows:

(A) the existence and reachability of desires (both, as equilibria and strong aspiration points) as the ends of a succession of temporary variational traps;
(B) Fejér convergence of a succession of temporary variational traps to some desire (equilibrium). This means that costs to be able to reach an equilibrium decrease;
(C) costs to be able to change which go to zero (see the proof of Theorem 3). Then, resistance to change disappears in the long run.

5.5 Behavioral hypothesis

Let us give a behavioral interpretation of the main hypothesis of this paper. The Assumption 1 (monotonicity) implies that if there is a loss $F(x, y) \geq 0$ to move from x to y, then, there is a gain to go in the reverse way, because bifunction $F(y, x) \leq -F(x, y) \leq 0$, for all $x, y \in \Omega$. This is a very natural hypothesis and the "optimal size of the firm problem" presented in Cruz Neto et al. (2018a) can be used to illustrate this situation. With regarding Assumption 2, it is a kind of coercivity assumption which says that if costs to be able to change $C(x^0, x^k) = d^2(x^0, x^k)$ go to infinity as k goes to infinity, then, there exists $x^* \in \Omega$ and $k_0 \in \mathbb{N}$ such that $F(x^k, x^*) \leq 0$, i.e., $A(x^k, x^*) \geq 0$ for all $k \geq k_0$. In behavioral term, this means that it exists a kind of weak aspiration point along a sequence when costs to be able to change go to infinity.

In this paper, in view of applications, there are two specific behavioral hypo-thesis, namely:

(i) costs to be able to change, given by $C(x, y) = d^2(x, y)$, are symmetric, because $d(y, x) = d(x, y)$. This is a strong assumption that must be removed in future researches. An immediate way to remove it is to consider costs to be able to change $C(x, y) = q^2(x, y)$ where $q(x, y) \geq 0$ is a quasi distance such that $q(x, y) = 0$ iff $y = x$ and $q(x, z) \leq q(x, y) + q(y, z)$, for all $x, y, z \in \Omega$. Then, we can suppose that when there is an advantage to change from x to y, i.e, $A(x, y) \geq 0$, costs to be able to change from x to y are higher than the reverse, i.e, $C(x, y) \geq C(y, x)$. This is a natural hypothesis which supposes that it is more costly to improve than the reverse. In this case $d(x, y) = \max\{q(x, y), q(y, x)\}$ is a distance, which satisfies the symmetric axiom

all along a worthwhile transition, which is improving, because advantages to change $A(x^k, x^{k+1}) \geq 0$ are non negative on such a transition (see above the paragraph "The algorithm as an instance of a worthwhile transition");

(ii) losses and advantages to change functions $F(x, y) = -A(x, y)$ are not reference dependent. In the general case they depend on experience (in the Markov case, on the last action $x^k = z$).

Finally, as seen in the introduction, the choice of a Hadamard manifold is useful in Behavioral Sciences, because it helps to deal with inevitable constraints (to save space, see, for example, Cruz Neto et al. 2014).

6 Conclusions

In this paper, we provided a sufficient condition to obtain the existence of solutions of EP and we proposed a new proximal algorithm for EP in Hadamard manifolds. We gave an application in the context of a recent unifying approach of a lot of stability and change theories in Behavioral Sciences, the "Variational rationality approach of human behavior", to the desire problem in Psychology, which refers to how to escape to a succession of temporary traps to finally succeed to reach his desires.

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References

Batista, E. E. A., Bento, G. C., & Ferreira, O. P. (2016). Enlargement of monotone vector fields and an inexact proximal point method for variational inequalities in Hadamard manifolds. *Journal of Optimization Theory and Applications*, *170*(3), 916–931.

Bento, G. C., & Melo, J. G. (2012). Subgradient method for convex feasibility on Riemannian manifolds. *Journal of Optimization Theory and Applications*, *152*(3), 773–785.

Bianchi, M., & Schaible, S. (1996). Generalized monotone bifunctions and equilibrium problems. *Journal of Optimization Theory and Applications*, *90*, 31–43.

Blum, E., & Oettli, W. (1994). From optimization and variational inequalities to equilibrium problems. *Mathematical Studies*, *63*, 123–145.

Bulavsky, V. A., & Kalashnikov, V. V. (1998). A Newton-like approach to solving an equilibrium problem. *Annals of Operations Research*, *81*, 115–128.

Colao, V., López, G., Marino, G., & Martín-Márquez, V. (2012). Equilibrium problems in Hadamard manifolds. *Journal of Mathematical Analysis and Applications*, *388*, 61–77.

Cruz Neto, J. X., Ferreira, O. P., & Lucambio Pérez, L. R. (2002). Contributions to the study of monotone vector fields. *Acta Mathematica Hungarica*, *94*(4), 307–320.

Cruz Neto, J. X., Ferreira, O. P., Lucambio Pérez, L. R., & Németh, S. Z. (2006). Convex and monotone-transformable mathematical programming problems and a proximal-like point method. *Journal of Global Optimization*, *35*(1), 53–69.

Cruz Neto, J. X., Jacinto, F. M. O., Soares, P. A, Jr., & Souza, J. C. O. (2018b). On maximal monotonicity of bifunctions on Hadamard manifolds. *Journal of Global Optimization*, *72*(3), 591–601.

Cruz Neto, J. X., Oliveira, P. R., Soares, P. A, Jr., & Soubeyran, A. (2014). Proximal point method on Finslerian manifolds and the effort accuracy trade-off. *Journal of Optimization Theory and Applications*, *162*(3), 873–891.

Cruz Neto, J. X., Oliveira, P. R., Soubeyran, A., & Souza, J. C. O. (2018a). A generalized proximal linearized algorithm for DC functions with application to the optimal size of the firm problem. *Annals of Operations Research,* 1–27.

Cruz Neto, J. X., Santos, P. S. M., & Soares, P. A, Jr. (2016). An extragradient method for equilibrium problems on Hadamard manifolds. *Optimization Letters*, *10*, 1327–1336.

Cruz Neto, J. X., Santos, P. S. M., & Souza, S. S. (2013). A sufficient descent direction method for quasiconvex optimization over Riemannian manifolds. *Pacific Journal of Optimization*, *8*(4), 803–815.

Facchinei, F., & Kanzow, C. (2010). Generalized Nash equilibrium problems. *Annals of Operations Research*, *175*, 177–211.

Ferreira, O. P., Lucambio Pérez, L. R., & Németh, S. Z. (2005). Singularities of monotone vector fields and an extragradient-type algorithm. *Journal of Global Optimization*, *31*, 133–151.

Ferreira, O. P., & Oliveira, P. R. (2002). Proximal point algorithm on Riemannian manifold. *Optimization*, *51*, 257–270.

Haesen, S., Sebekovic, A., & Verstraelen, L. (2003). Relations between intrinsic and extrinsic curvatures. *Kragujevac Journal of Mathematics*, *25*, 139–145.

Iusem, A. N., Kassay, G., & Sosa, W. (2009). On certain conditions for the existence of solutions of equilibrium problems. *Mathematical Programming Series B*, *116*, 259–273.

Iusem, A. N., & Sosa, W. (2003). New existence results for equilibrium problems. *Nonlinear Analysis*, *52*, 621–635.

Iusem, A. N., & Sosa, W. (2010). On the proximal point method for equilibrium problems in Hilbert spaces. *Optimization*, *59*, 1259–1274.

Konnov, I. V. (2003). Application of the proximal method to non-monotone equilibrium problems. *Journal of Optimization Theory and Applications*, *119*, 317–333.

Kristály, A. (2014). Nash-type equilibria on Riemannian manifolds: A variational approach. *Journal de Mathématiques Pures et Appliquées*, *101*(5), 660–688.

Kristály, A., Li, C., López-Acedo, G., et al. (2016). What do 'Convexities' imply on Hadamard manifolds? *Journal of Optimization Theory and Applications*, *170*(3), 1068–1074.

Ledyaev, Yu S, & Zhu, Q. J. (2007). Nonsmooth analysis on smooth manifolds. *Transactions of the American Mathematical Society*, *359*, 3687–3732.

Li, C., Lopéz, G., & Martín-Márquez, V. (2009). Monotone vector fields and the proximal point algorithm on Hadamard manifolds. *Journal of the London Mathematical Society*, *79*(2), 663–683.

Li, C., Lopéz, G., Martín-Márquez, V., & Wang, J. H. (2011a). Resolvents of set valued monotone vector fields in Hadamard manifolds. *Set-Valued and Variational Analysis*, *19*(3), 361–383.

Li, C., Lopéz, G., Wang, X., & Yao, J. C. (2019). Equilibrium problems on Riemannian manifolds with applications. *Journal of Mathematical Analysis and Applications*, *473*, 866–891.

Li, C., Mordukhovich, B. S., Wang, J., & Yao, J. C. (2011b). Weak sharp minima on Riemannian manifolds. *SIAM Journal on Optimization*, *21*(4), 1523–1560.

Li, C., & Yao, J. C. (2012). Variational inequalities for set-valued vector fields on Riemannian manifolds: Convexity of the solution set and the proximal point algorithm. *SIAM Journal on Control and Optimization*, *50*(4), 2486–2514.

Li, S. L., Li, C., Liou, Y. C., & Yao, J. C. (2009). Existence of solutions for variational inequalities on Riemannian manifolds. *Nonlinear Analysis*, *71*, 5695–5706.

Lewin, K. (1935). *A dynamic theory of personality*. New York: McGraw-Hill.

Lewin, K. (1936). *Principles of topological psychology*. New York: McGraw-Hill.

Lewin, K. (1938). *The conceptual representation and measurement of psychological forces*. Durham: Duke University Press.

Lewin, K. (1951). *Field theory in social science*. New York: Harper.

Moudafi, A. (1999). Proximal point algorithm extended for equilibrium problems. *Journal of Natural Geometry*, *15*, 91–100.

Németh, S. Z. (2003). Variational inequalities on Hadamard manifolds. *Nonlinear Analysis*, *52*, 1491–1498.

Ng, L. (1995). Classical and modern formulations of curvature. *Mathematics, 230a*.

Pany, G., Mohapatra, R. N., & Pani, S. (2018). Solution of a class of equilibrium problems and variational inequalities in FC spaces. *Annals of Operations Research*, *269*, 565–582.

Quiroz, E. A. P. (2013). An extension of the proximal point algorithm with Bregman distances on Hadamard manifolds. *Journal of Global Optimization*, *56*, 43–59.

Rapcsák, T. (1997). Smooth nonlinear optimization in $\mathbb{R}^n$. In *Nonconvex optimization and its applications* (Vol. 19). Dordrecht: Kluwer Academic Publishers.

Sakai, T. (1996). *Riemannian geometry*, Transl. Math. Monogr. (Vol. 149). Providence: Amer. Math. Soc.

Soubeyran, A. (2009). Variational rationality, a theory of individual stability and change: worthwhile and ambidextry behaviors. Preprint. GREQAM, Aix Marseille University.

Soubeyran, A. (2010). Variational rationality and the “unsatisfied man”: Routines and the course pursuit between aspirations, capabilities and beliefs. Preprint. GREQAM, Aix Marseille University.

Soubeyran, A. (2015). Variational rationality. Traps, or desires, as ends of stay and change worthwhile transitions, GREQAM, AMSE, Aix-Marseille University.

Soubeyran, A. (2016). Variational rationality. A theory of worthwhile stay and change approach-avoidance transitions ending in traps, Preprint, GREQAM-AMSE, Aix Marseille University.

Soubeyran, A. (2019). Variational rationality. 1. An adaptive theory of the unsatisfied man, Preprint. AMSE. Aix-Marseille University.

Soubeyran, A. (2019). Variational rationality. 2. A general theory of goals and intentions as satisficing worthwhile moves, Preprint. AMSE, Aix-Marseille University.

Tang, G., Zhou, L., & Huang, N. (2013). The proximal point algorithm for pseudomonotone variational inequalities on Hadamard manifolds. *Optimization Letters*, *7*, 779–790.

Udriste, C. (1994). *Convex functions and optimization methods on Riemannian manifolds, mathematics and its applications* (Vol. 297). Dordrecht: Kluwer Academic Publishers.

Yang, W. H., Zhang, L.-H., & Song, R. (2013). Optimality conditions for the nonlinear programming problems on Riemannian manifolds. *Pacific Journal of Optimization*, *10*(2), 415–434.