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# Abstract regularized equilibria: application to Becker's household behavior theory

J. X. Cruz Neto[1] · J. O. Lopes[1] · A. Soubeyran[2] · J. C. O. Souza[1]

**Abstract**
In this paper, we consider an abstract regularized method with a skew-symmetric mapping as regularization for solving equilibrium problems. The regularized equilibrium problem can be viewed as a generalized mixed equilibrium problem and some existence and uniqueness results are analyzed in order to study the convergence properties of the algorithm. The proposed method retrieves some existing one in the literature on equilibrium problems. We provide some numerical tests to illustrate the performance of the method. We also propose an original application to Becker's household behavior theory using the variational rationality approach of human dynamics.

**Keywords** Equilibrium problem · Variational rationality · Desires · Traps · Household behavior · Resource allocation problems

## 1 Introduction

Let $K \subset \mathbb{R}^m$ be a nonempty, closed and convex set and consider a bifunction $F : K \times K \to \mathbb{R}$ such that

$(P1)$ $F(x, x) = 0$ for all $x \in K$;
$(P2)$ $F(\cdot, y) : K \longrightarrow \mathbb{R}$ is upper semicontinuous for all $y \in K$;
$(P3)$ $F(x, \cdot) : K \longrightarrow \mathbb{R}$ is convex and lower semicontinuous for all $x \in K$.

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The equilibrium problem (EP) consists of finding $x^* \in K$ such that

$$F(x^*, y) \geq 0, \quad \forall\, y \in K. \tag{1}$$

This problem will be denoted by $EP(F, K)$ and its solution set by $S(F, K)$. Equilibrium problems have been widely studied both on the issue dealing with conditions for existence of solutions (e.g., Iusem et al. 2009; Iusem and Sosa 2003; Konnov and Dyabilkin 2011; Chadli et al. 2015) and algorithmic purposes (see, for instance Bento et al. 2016; Burachik and Kassay 2012; CruzNeto et al. 2019; Flåm and Antipin 1996; Iusem and Nasri 2007; Iusem and Sosa 2010; Konnov 2003; Langenberg 2012; Mordukhovich et al. 2012; Moudafi 2007, 1999; Nguyen et al. 2009; Oliveira et al. 2013; Hieu 2018; Mashreghi and Nasri 2012). It represents a rather general and suitable format for investigation and solution of various problems arising in mathematical physics, economics, operations research, game theory, and other fields; see for instance (Blum and Oettli 1994; Muu and Oettli 1992; Panicucci et al. 2009). The term equilibrium problem was coined in Blum and Oettli (1994), but the problem itself was studied in much early work of Fan (1972). As pointed out in Blum and Oettli (1994), EP includes as particular case scalar and vector optimization problems, saddle-point (minimax) problems, variational inequalities, Nash equilibria problems, complementarity problems and fixed point problems.

One of the most popular strategy for solving (EP) is the so-called regularization method. The Tikhonov regularization method for ill-posed problems is well-known for minimization, monotone inclusion and fixed-point problems. This approach was introduced for solving an equilibrium problems by Moudafi (1999), see also Moudafi and Théra (1999) and Iusem and Sosa (2010). This iterative method solves at each iteration the regularized $EP(F_k, K)$, i.e., given $x^k$ it computes $x^{k+1}$ as follows:

$$x^{k+1} \in S(F_k, K), \tag{2}$$

where

$$F_k(x, y) = F(x, y) + \gamma_k \langle x - x^k, y - x \rangle \tag{3}$$

and $\{\gamma_k\}$ is a positive bounded auxiliar sequence. This kind of method was studied for instance in Bento et al. (2016), Burachik and Kassay (2012), Blum and Oettli (1994) and Konnov (2003) in different contexts and it consists of solving at each iteration a regularized (EP) instead of the original problem. Burachik and Kassay (2012) generalized the method (2) by using Bregman functions on the regularization term as follows:

$$F_k(x, y) = F(x, y) + \gamma_k \langle \nabla\phi(x) - \nabla\phi(x^k), y - x \rangle, \tag{4}$$

where $\phi$ is Bregman function; see Burachik and Kassay (2012) for more details. If $\phi(x) = \frac{1}{2}||x||^2$ then (4) reduces to (3). In order to generalize (4), Bento et al. (2016) proposed the following regularization in (2):

$$F_k(x, y) = F(x, y) + \lambda_k \langle \nabla_1 d(x, x^k), y - x \rangle, \tag{5}$$

where $\nabla_1 d$ stands to the gradient of the mapping of $d$ with respect to the first variable. They established convergence results for a more general setting of bilevel equilibrium problems which include equilibrium problems. Oliveira et al. (2013) considered existence of solutions of (EP) using a Tikhonov-type regularization given by $\tilde{F}(x, y) = F(x, y) + \lambda g(x, y)$, where $g : K \times K \to \mathbb{R}$ is a strongly monotone bifunction and $\lambda > 0$.

A more general class of problem than (EP) is the so-called Mixed Equilibrium Problem (MEP). This problem was studied by Ceng and Yao (2008), see also Yao et al. (2009) and

related literatures. The mixed equilibrium problem consists of finding $x^* \in K$ such that

$$F(x^*, y) + \psi(y) - \psi(x^*) \geq 0, \quad \forall\, y \in K, \tag{6}$$

where $\psi : K \longrightarrow \mathbb{R}$ is a given function. This problem is denoted by $MEP(F, K)$. We refer to Muu and Quy (2019), Oyewole et al. (2020) and Shamshad and Nisha (2019) for a variants of this problem. Note that, if $\psi \equiv 0$, then the mixed equilibrium problem (6) reduces to the equilibrium problem (1). A more general version of (6) can be considered as finding $x^* \in K$ such that

$$F(x^*, y) + \varphi(x^*, y) - \varphi(x^*, x^*) \geq 0, \quad \forall\, y \in K, \tag{7}$$

where $\varphi : K \times K \longrightarrow \mathbb{R}$ is a given function. We call this problem of generalized mixed equilibrium problem (GMEP). An inexact version of problem (7), in the sense that each subproblem is solved inexactly, can be considered as follows: given $\delta \geq 0$, a point $x^* \in K$ is a $\delta$-solution of the inexact mixed equilibrium problem if it satisfies

$$F(x^*, y) + \varphi(x^*, y) - \varphi(x^*, x^*) \geq -\delta, \quad \forall\, y \in K. \tag{8}$$

In this case, if $\varphi \equiv 0$, then (8) becomes the inexact equilibrium problem studied by Hung and Muu (2012). We denote the inexact equilibrium problem (8) by $IEP(F, K, \delta)$.

The aim of this work is twofold. Firstly, we analyze the existence and uniqueness of solutions for a mixed equilibrium problem given by (7) in order to propose an abstract regularized algorithm for solving equilibrium problems such that it generalizes some existing regularized methods in the literature. The motivation to consider this kind of regularization comes from application in behavioral sciences using recent Variational Rationality (VR) approach of human dynamics by Soubeyran (2009, 2010, 2016, 2019a, b). In this context, the regularization term can be seen as a crude formulation of the complex concept of resistance to change; see for instance Bento and Soubeyran (2015), CruzNeto et al. (2020), Bento et al. (2019, 2018) and Soubeyran and Souza (2020). As a second contribution, we give an application in behavioral sciences of our abstract regularization approach of equilibrium problems. We propose a dynamic version of the famous static Becker's "allocation of time problem" using the VR approach; see Becker (1965).

In the seminal work Becker (1965), "A theory of the allocation of time", the Nobel prize Becker (Becker 1965) revolutionized the modeling of household behavior. Following the splendid comments of Chappori and Lewbel (1965), " Becker (1965) is not the first study to consider time use in the home, e.g., he cites Jacob Mincer (1962), who considers a married woman's time trade-off between housework and paid work. Nor is it quite the first to propose a household production function, in which purchased goods are converted into commodities like meals that generate utility. An earlier household production article he does not cite is Terence Gorman (1980), which was written in 1956 and widely circulated as a working paper for decades before it was finally published. However, what Becker does uniquely is to merge goods consumption with time use in the production of household utility. Previous models of labour supply considered consumption and leisure as distinct goods that separately provide utility. In contrast, Becker emphasizes that there are many different types of time use, just as there are many types of consumption goods, and that different types of time use and consumption goods combine in different ways to yield commodities, e.g., prepared meals, from which we get utility. He then draws a variety of important implications from the observation that various types of time and consumption combine into a single household objective function with a single overall budget constraint. In doing so, Becker (along with Mincer) created the foundational modeling framework for virtually all modern household level analyses of consumption and time use, in what was sometimes called the "New Home

Economics". Becker's model is in some ways more general than those in common use today, in particular, he emphasizes that different types of time (e.g., weekends versus weekdays) should have different shadow prices within the household. This is in contrast to the vast majority of models today that associate a single observable wage rate to each individual's time.

The modern literature has generalized Becker in two main directions. A first strand of the literature considers unitary models which, like Becker's original time allocation theory, treat the household as maximizing a single utility function under a resource constraint. They are often dynamic and forward looking. A second strand of the literature follows a second Becker's paper (Becker 1974), where the allocation of resources within a household is determined by bargaining among household members, with outcomes that may therefore depend on the determinants of the bargaining power of each household member. We will not examine this second extension.

The paper is organized as follows. In Sect. 2, we recall some basic concepts and results in equilibrium problems. In Sect. 3, we show an existence result for our regularized bifunction. In Sect. 4, we present a generalized regularization algorithm for solving an equilibrium problem and study its convergence analysis. The method is performed in some academic examples in Sect. 5. An original application to Becker's household behavior theory is given in Sect. 6. Finally, some conclusions are presented.

## 2 Preliminaries

In this section, we recall some definitions and known results which will be used in the following sections. Usually, existence results for $EP(f, K)$ require either boundedness of the feasible set $K$ or certain coercivity conditions. We recall the classical Ky Fan inequality result from Fan (1972) in case of Minimax Problems.

**Theorem 1** *Assume that $F$ satisfies the Conditions $(P1)$–$(P3)$ and $K$ is compact. Then $S(F, K)$ is nonempty.*

Recently, many efforts were concentrated for obtaining coercivity conditions, which guarantee existence of solutions on unbounded feasible sets; see e.g. Iusem et al. (2009) and Burachik and Kassay (2012) and references therein. For such coercivity conditions it has been assumed beyond the first three basic assumptions on equilibrium problems, i.e., $(P1)$–$(P3)$, one has been considered the following properties:

$(P4)$ $F(x, y) + F(y, x) \leq 0$ for all $x, y \in K$ (monotone bifunction);
$(P5)$ For any sequence $\{z^n\} \subset K$ with $\lim_{n\to\infty} \|z^n\| = \infty$, there exists $u \in K$ and $n_0 \in \mathbb{N}$ such that $F(z^n, u) \leq 0, \quad \forall\, n \geq n_0$.

Thus, under all these assumptions they obtained a well known result of existence of solution for the problem $EP(F, K)$; see (Iusem et al. 2009, Theorem 4.2). For sake of completeness, we present a version of the proof (adapted to the Euclidean space) which requires the following result proved by Ky Fan in 1961 in the context of convex feasibility problems; see (Fan 1961, Lemma 1).

**Lemma 1** *Let $Y$ be a nonempty subset of $\mathbb{R}^m$ and, for each $y \in Y$, consider a closed subset $C(y)$ of $\mathbb{R}^m$. If the following two conditions hold:*

*(C1) the convex hull of any finite subset $\{x_1, \ldots, x_k\}$ of $Y$ is contained in $\cup_{i=1}^{k} C(x_i)$;*

*(C2) $C(y)$ is compact for at least some $y \in Y$;*

*then $\cap_{y\in Y} C(y) \neq \emptyset$.*

**Theorem 2** *Assume that $F$ satisfies the Conditions $(P1)$–$(P5)$. Then $S(F, K)$ is nonempty.*

***Proof*** Let $n \in \mathbb{N}$ be arbitrary. Define $K_n := \{x \in K : \|x\| \leq n\}$ (since $K_n$ is nonempty for sufficiently large $n$, in what follows, for the sake of simplicity, we suppose without loss of generality that $K_n$ is nonempty for all $n \in \mathbb{N}$, furthermore it is compact) and $L_F(n, y) := \{x \in K_n : F(y, x) \leq 0\}$ which is compact (closed from (P2) and bounded due to $L_F(n, y) \subset K_n$).

We intend to invoke Lemma 1 with $Y = K_n$, $C(y) = L_F(n, y)$, and thus, we must check the validity of its hypotheses. Firstly, we verify $(C1)$. For this, take $x_1, \ldots, x_k \in K_n$ and $\lambda_1, \ldots, \lambda_k \in [0, 1]$ such that $\sum_{i=1}^k \lambda_i = 1$, and denote by $\bar{x} = \sum_{i=1}^k \lambda_i x_i$. We must verify that $\bar{x}$ belongs to $\cup_{i=1}^k L_F(n, x_i)$, i.e., that $\bar{x} \in K_n$ and that $F(x_i, \bar{x}) \leq 0$ for some $i$. Indeed, $\bar{x} \in K_n$ from convexity of $K_n$ and $\|\bar{x}\| \leq \sum_{i=1}^k \lambda_i \|x_i\| \leq n \sum_{i=1}^k \lambda_i = n$. Now, from $(P1)$ and $(P3)$, we have $0 = F(\bar{x}, \bar{x}) \leq \sum_{i=1}^k \lambda_i F(\bar{x}, x_i)$, so $F(\bar{x}, x_i) \geq 0$ for some $i$, thus by $(P4)$ $F(x_i, \bar{x}) \leq 0$. In the sequel, we verify $(C2)$. Since $C(y) = L_F(n, y)$ is compact (as seen previously) and nonempty (since $F(y, y) = 0$ by $(P1)$, implies $y \in C(y)$) for all $y \in K_n$, and therefore $(C2)$ holds. Thus, we are within the hypotheses of Lemma 1, and we conclude that $\cap_{y\in K_n} L_F(n, y)$ is nonempty for each $n \in \mathbb{N}$, so that for each $n$ we may choose $x^n \in \cap_{y\in K_n} L_F(n, y)$. We distinguish two cases:

(i) There exists $p \in \mathbb{N}$ such that $\|x^p\| < p$, or
(ii) For all $n \in \mathbb{N}$, $\|x^n\| = n$ . In this case, assumption $(P5)$ ensures the existence of $u \in K$ and $n_0$ such that $F(x^n, u) \leq 0$ for all $n \geq n_0$. Take $p \geq n_0$ such that $\|u\| < p$ where $F(x^p, u) \leq 0$.

In both cases $(i)$ or $(ii)$, we will show that $x^p$ is a solution of $EP(F, K)$. Since $x^p \in \cap_{y\in K_p} L_F(p, y)$, we have $F(y, x^p) \leq 0$, for all $y \in K_p$, by $(P4)$ we get $F(x^p, y) \geq 0$, for all $y \in K_p$. Thus, we only need to show that $F(x^p, w) \geq 0$, for all $w \in K/K_p$. Let $\bar{y}$ given by $\bar{y} = x^p$ (case $(i)$) or $\bar{y} = u$ (case $(ii)$), so $F(x^p, \bar{y}) \leq 0$. Now, take any $w \in K/K_p$, thus there exists $t \in (0, 1)$ such that $z_t := (1 - t)\bar{y} + tw$ belongs to $K_p$ and $F(x^p, z_t) \geq 0$. By $(P3)$, we have

$$0 \leq F(x^p, z_t) \leq (1 - t)F(x^p, \bar{y}) + tF(x^p, w) \leq tF(x^p, w)$$

which implies that $F(x^p, w) \geq 0$, and hence, we conclude that $F(x^p, y) \geq 0$, for all $y \in K$, i.e., $x^p$ is a solution of $EP(F, K)$. □

The following definition was introduced by Antipin (1996) which encompasses a large class of functions that has interesting properties and can be used to make well-defined an equilibrium problem.

**Definition 1** A mapping $\varphi : K \times K \to \mathbb{R}$ is said to be skew-symmetric if

$$(A1)\ \ \varphi(x, x) - \varphi(x, y) - \varphi(y, x) + \varphi(y, y) \geq 0,\ \forall x, y \in K.$$

If the inequality holds strictly, for all $x, y \in K$ such that $x \neq y$, we say that $\varphi$ is strictly skew-symmetric.

Skew-symmetric functions have some properties similar to monotonicity of the gradient and non-negativity of the second derivative of a convex function. For more properties and applications of skew symmetric functions we refer to Antipin (1996).

We will consider the following assumptions on the mapping $\varphi$:

$(A2)$ $\varphi(x,\cdot): K \longrightarrow \mathbb{R}$ is convex in the second argument, for all $x \in K$;
$(A3)$ $\varphi: K \times K \longrightarrow \mathbb{R}$ is continuous.

***Example 1*** Let $f: \mathbb{R}^m \to \mathbb{R}$ and $g: \mathbb{R}^m \to \mathbb{R}_+$ be a convex functions. Define the bifunctions $\varphi_i: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, $i = 1, 2$, given by $\varphi_1(x, y) = f(y) - f(x)$ and $\varphi_2(x, y) = g(x)g(y)$. One can check that they satisfy assumptions (A1)–(A3). Indeed,

$$\varphi_1(x, x) - \varphi_1(x, y) - \varphi_1(y, x) + \varphi_1(y, y) = 0$$

and

$$\varphi_2(x, x) - \varphi_2(x, y) - \varphi_2(y, x) + \varphi_2(y, y) = (g(x) - g(y))^2 \geq 0.$$

Conditions (A2) and (A3) directly follow from the convexity of $f$ and $g$.

We recall that a set-valued operator $T: \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is called monotone (resp. strongly monotone with constant $\gamma > 0$) if

$$\langle x - y, u - v\rangle \geq 0, \quad \left(\langle x - y, u - v\rangle \geq \gamma ||x - y||^2\right)$$

for $u \in T(x)$ and $v \in T(y)$. A monotone operator is maximal if (considered as a graph) is it not strictly contained in any other monotone operator on $\mathbb{R}^m$. We denote the domain of $T$ by dom $(T)$ and it is given by dom $(T) = \{x \in \mathbb{R}^m : T(x) \neq \emptyset\}$.

***Example 2*** Let $T: \mathbb{R}^m \to \mathbb{R}^m$ be a single-valued maximal monotone operator and define $\varphi(x, y) = \langle T(x), y\rangle$. Then the condition (A1) is valid, i.e.,

$$\varphi(x, x) - \varphi(x, y) - \varphi(y, x) + \varphi(y, y) = \langle T(x) - T(y), x - y\rangle \geq 0, \quad \forall x, y \in \text{dom}\,(T).$$

Moreover, one can easily check that $\varphi$ is convex in the second argument (condition (A2)) and continuous (condition (A3)).

Given $x \in \mathbb{R}^n$, we will denote by

$$\Upsilon(x) = ||x - \arg\min_{y \in K}\{F(x, y) + \frac{1}{2}||y - x||^2\}||.$$

The next proposition will be used in Sect. 5. It measures the quality of a candidate to solution of an (EP). It is a well known characterization of a solution of an equilibrium problem; see Iusem (2011).

**Proposition 1** *A point $x^* \in S(F, K)$ if and only if $\Upsilon(x^*) = 0$.*

## 3 Existence and uniqueness of solution for GMEP

Let $F$ be a bifunction, $\varphi$ a regularization mapping and a constant $\lambda > 0$. We will consider the auxiliar bifunction $\bar{F}: K \times K \to \mathbb{R}$ defined by

$$\bar{F}(x, y) = F(x, y) + \lambda\left(\varphi(x, y) - \varphi(x, x)\right). \tag{9}$$

Before present our results it is worth to mention some important examples of regularization existing in the literature which satisfy the assumptions made in the regularization $\varphi$ for solving the equilibrium problem (1).

- Moudafi (1999), Moudafi and Théra (1999) and Iusem and Sosa (2010), the authors considered the regularized auxiliar bifunction given by

$$\tilde{F}(x, y) = F(x, y) + \lambda\langle x - \bar{x}, y - x\rangle,$$

where $\bar{x} \in K$ is fixed. Defining $\varphi(x, y) = \langle x - \bar{x}, y - x\rangle$, one can check that $\varphi$ is a skew-symmetric. Moreover, it satisfies the conditions (A2) and (A3).
- Burachik and Kassay (2012), the following regularized bifunction $\tilde{F}$ is considered

$$\tilde{F}(x, y) = F(x, y) + \gamma\langle\nabla\phi(x) - \nabla\phi(\bar{x}), y - x\rangle,$$

where $\phi$ is a Bregman function. Note that defining $\varphi(x, y) = \langle\nabla\phi(x) - \nabla\phi(\bar{x}), y - x\rangle$, we have that $\varphi$ is skew-symmetric, and furthermore it satisfies the conditions (A2) and (A3).
- CruzNeto et al. (2019), the authors studied the following regularized equilibrium problems

$$\tilde{F}(x, y) = F(x, y) + \gamma(D_\phi(y, \bar{x}) - D_\phi(x, \bar{x})),$$

where $D_\phi(x, y) = \phi(x) - \phi(y) - \langle\nabla\phi(y), x - y\rangle$ is the Bregman distance associated to the Bregman function $\phi$. Considering $\varphi(x, y) = D_\phi(y, \bar{x}) - D_\phi(x, \bar{x})$ we have that $\varphi$ is a skew-symmetric mapping which satisfies the conditions (A2) and (A3).
- Oliveira et al. (2013) considered the following Tikhonov-type regularization

$$\tilde{F}(x, y) = F(x, y) + \lambda g(x, y),$$

where $g : K \times K \to \mathbb{R}$ is a strongly monotone bifunction which satisfies (A2), $g(\cdot, y)$ is upper semicontinuous and $g(x, \cdot)$ is lower semicontinuous, for all $x, y \in K$. Note that a strongly monotone bifunction is a (strongly) skew-symmetric mapping. It is worth to mention that we will consider both existence of solution and algorithmic approach for solving (EP) in a more general context than in Oliveira et al. (2013).

From now on, we consider $K \subset \mathbb{R}^m$ a closed and convex set, $F : K \times K \to \mathbb{R}$ a bifunction such that conditions $(P1)$–$(P4)$ hold and $\varphi$ be a generalized regularization mapping such that assumptions $(A1)$–$(A3)$ hold.

As shown by Iusem et al. (2009) condition $(P5)$ is a sufficient condition for the existence of solutions of (EP). Next, we consider a less restrictive condition than (P5) (here called $(P6)$) which was introduced by Burachik and Kassay (2012).

**Theorem 3** *Let $\bar{x} \in K$ be fixed. We have that $\bar{F}$ satisfies conditions $(P1)$–$(P4)$. Moreover, if for every sequence $\{z^n\} \subset K$ such that $\lim_{n\to\infty} \|z^n\| = \infty$ implies*

$$(P6) \quad \liminf_{n\to\infty} \left[F(\bar{x}, z^n) + \lambda\left(\varphi(z^n, z^n) - \varphi(z^n, \bar{x})\right)\right] > 0,$$

*then $\bar{F}$ satisfies condition $(P5)$.*

***Proof*** One can easily check that $\bar{F}$ satisfies $(P1)$–$(P3)$ having in mind that $K \subset \operatorname{int}\mathcal{D}$, where $\mathcal{D} = \operatorname{dom}\varphi$. Now, we claim that $\bar{F}$ satisfies $(P4)$. Indeed, we have

$$\begin{aligned}\bar{F}(x, y) + \bar{F}(y, x) &= F(x, y) + F(y, x)\\ &\quad + \lambda\left(\varphi(x, y) - \varphi(x, x) + \varphi(y, x) - \varphi(y, y)\right) \le 0,\end{aligned}$$

where the inequality comes from the fact that $F$ is monotone due to $(P4)$, $\varphi$ is skew-symmetric from $(A1)$ and $\lambda > 0$. Thus, the first part of the assertion is proved. Next, we will show that

$\bar{F}$ satisfies $(P5)$ under the assumption $(P6)$. Let $\{z^n\} \subset K$ be an arbitrary sequence such that $\lim_{n\to\infty} \|z^n\| = \infty$. From (9), we have

$$\bar{F}(z^n, \bar{x}) = F(z^n, \bar{x}) + \lambda \left(\varphi(z^n, \bar{x}) - \varphi(z^n, z^n)\right).$$

Since $F$ is monotone from $(P4)$, the following inequality holds

$$\begin{aligned}\bar{F}(z^n, \bar{x}) = F(z^n, \bar{x}) + \lambda \left(\varphi(z^n, \bar{x}) - \varphi(z^n, z^n)\right) \leq \\ - \left[F(\bar{x}, z^n) + \lambda \left(\varphi(z^n, z^n) - \varphi(z^n, \bar{x})\right)\right].\end{aligned} \tag{10}$$

From $(P6)$, we have that there exists $n_0 \in \mathbb{N}$ such that

$$F(z^n, \bar{x}) + \lambda \left(\varphi(z^n, z^n) - \varphi(\bar{x}, z^n)\right) > 0,$$

for all $n \geq n_0$, which means from (10) that $\bar{F}(z^n, \bar{x}) < 0$, for all $n \geq n_0$. This completes the proof. □

***Remark 1*** One can easily check that if $\varphi$ satisfies (A1), i.e., $\varphi$ is skew-symmetric and $\bar{x} \in K$, then the following condition (P6*) implies (P6), and hence, from Theorem 3 it implies (P5):

$$(P6^*) \quad \liminf_{n\to\infty} \left[F(\bar{x}, z^n) + \lambda \left(\varphi(\bar{x}, z^n) - \varphi(\bar{x}, \bar{x})\right)\right] > 0.$$

Next, we provide a class of mapping $\varphi$ in which condition (P6) holds, and hence, condition (P5) is verified. We consider the very important class of strongly skew-symmetric functions $\varphi$, i.e., it satisfies the following assumption:

$(A4)$ $\varphi(x, x) - \varphi(x, y) - \varphi(y, x) + \varphi(y, y) \geq \gamma \|y - x\|^2$, for some $\gamma > 0$.

**Theorem 4** *Assuming either*

(i) *$K$ is bounded, or*
(ii) *$\varphi$ is strongly skew-symmetric.*

*Then, $\bar{F}$ satisfies condition $(P5)$.*

***Proof*** Using Theorem 3 it is enough to check that condition $(P6)$ holds under (i) or (ii). If (i) is satisfied, i.e., if $K$ is bounded, then one can easily check that condition $(P6)$ holds since there is no $\{z^n\} \subset K$ such that $||z^n|| \to \infty$ as $n \to \infty$. Hence, it is enough to prove that (ii) implies condition $(P6)$. Take a sequence $\{z^n\} \subset K$ such that $||z^n|| \to \infty$ and $\bar{x} \in K$ fixed. From (A4), we have

$$F(\bar{x}, z^n) + \lambda(\varphi(z^n, z^n) - \varphi(z^n, \bar{x})) \geq F(\bar{x}, z^n) + \lambda(\varphi(\bar{x}, z^n) - \varphi(\bar{x}, \bar{x})) + \gamma \|z^n - \bar{x}\|^2. \tag{11}$$

Since the mapping $F(x, \cdot)$ is convex for all $x \in K$, then for any $u \in \partial F(\bar{x}, \cdot)(\bar{x})$, we have

$$F(\bar{x}, z^n) \geq F(\bar{x}, \bar{x}) + \langle u, z^n - \bar{x}\rangle = \langle u, z^n - \bar{x}\rangle, \tag{12}$$

taking into account that $F(\bar{x}, \bar{x}) = 0$. Again, since $\varphi(x, \cdot)$ is convex for all $x \in K$, for any $v \in \partial\varphi(\bar{x}, \cdot)(\bar{x})$ and $\lambda > 0$, we obtain

$$\lambda(\varphi(\bar{x}, z^n) - \varphi(\bar{x}, \bar{x})) \geq \lambda\langle v, z^n - \bar{x}\rangle. \tag{13}$$

Adding inequalities (12) and (13), we have

$$F(\bar{x}, z^n) + \lambda(\varphi(\bar{x}, z^n) - \varphi(\bar{x}, \bar{x})) \geq \langle u, z^n - \bar{x}\rangle + \lambda\langle v, z^n - \bar{x}\rangle. \tag{14}$$

Using (14) in (11), we have

$$F(\bar{x}, z^n) + \lambda(\varphi(z^n, z^n) - \varphi(z^n, \bar{x})) \geq \langle u, z^n - \bar{x}\rangle + \lambda\langle v, z^n - \bar{x}\rangle + \gamma \|z^n - \bar{x}\|^2,$$

$$\geq -\|w\|\|z^n - \bar{x}\| + \gamma\|z^n - \bar{x}\|^2 \tag{15}$$

where $w = u + \lambda v$ and applying the Cauchy-Schwartz inequality. Using the fact that $||z^n|| \to \infty$ in (15), then there exists $n_0 \in \mathbb{N}$ such that the right-hand side of (15) is positive for all $n \geq n_0$, and hence, we conclude (P6). Therefore, the desired result follows from Theorem 3. □

***Example 3*** Let us consider $\varphi(x, y) = \langle T(x), y\rangle$, where $T : \mathbb{R}^m \to \mathbb{R}^m$ is a strongly maximal monotone (single-valued) mapping with modulus $\tau > 0$. One can prove that $\varphi$ satisfies $(A4)$ with $\gamma = \tau$.

***Example 4*** The Tikhonov-type regularization analyzed in Oliveira et al. (2013) is given by $\varphi(x, y) = g(x, y)$, where $g : K \times K \to \mathbb{R}$ is a $\beta$-strongly monotone bifunction, i.e., $g(x, y) + g(y, x) \leq -\beta||x-y||^2$, for all $x, y \in K$. It generalizes the regularization considered in Iusem and Sosa (2010), Moudafi (1999) and Moudafi and Théra (1999), where the regularization auxiliar bifunction is given by $g(x, y) = \langle x - \bar{x}, y - x\rangle$ is strongly monotone with $0 < \beta \leq 1$. In this case, one can easily check that $\varphi$ satisfies (A4) with $\gamma = \beta$.

The next result establishes the existence and uniqueness of the solution of the GMEP (7) which can also be viewed as the regularized equilibrium problem $EP(\bar{F}, K)$.

**Theorem 5** *The following assertions hold:*

(i) *If condition* $(P6)$ *holds, then* $EP(\bar{F}, K)$ *admits at least one solution;*
(ii) *If* $\varphi$ *is strictly skew-symmetric, then* $EP(\bar{F}, K)$ *admits at most one solution.*

*Altogether, if condition* $(P6)$ *holds and* $\varphi$ *is strictly skew-symmetric, then* $EP(\bar{F}, K)$ *has a unique solution.*

***Proof*** From Theorem 3, we have that $\bar{F}$ satisfies condition $(P5)$. Thus, from Theorem 2, we obtain that $EP(\bar{F}, K)$ has a solution. This proves the first assertion. To prove (ii), we assume that both $x_1$ and $x_2$ solve $EP(\bar{F}, K)$. Hence,

$$0 \leq \bar{F}(x_1, x_2) = F(x_1, x_2) + \lambda(\varphi(x_1, x_2) - \varphi(x_1, x_1)). \tag{16}$$
$$0 \leq \bar{F}(x_2, x_1) = F(x_2, x_1) + \lambda(\varphi(x_2, x_1) - \varphi(x_2, x_2)). \tag{17}$$

Adding (16) with (17), we get

$$0 \leq F(x_1, x_2) + F(x_2, x_1) - \lambda(\varphi(x_1, x_1) - \varphi(x_2, x_1) - \varphi(x_1, x_2) + \varphi(x_2, x_2)).$$

Since $\lambda > 0$, $F$ is monotone and $\varphi$ is strictly skew-symmetric, then it follows from last inequality that $x_1 = x_2$ and the second assertion is proved. The last statement is a direct combination of (i) and (ii). □

***Remark 2*** Note that item (ii) of the previous results holds if, in particular, $\varphi$ is strongly skew-symmetric (condition (A4)). This is the case when $\varphi$ is a Tikhonov-type regularization as given in Oliveira et al. (2013); see Example 4. Therefore, our last result generalizes (Oliveira et al. 2013, Theorem 3.2). This is an important generalization because there are strongly skew-symmetric functions which are not a bifunction (strongly monotone); see, for instance, Example 3.

## 4 Generalized regularization algorithm

Recall that we are assuming that $F$ satisfies conditions $(P1)$–$(P4)$ and $\varphi$ verifies assumptions $(A1)$–$(A3)$. In this section, we additionally assume that condition $(P5)$ holds. As we prove in last section, there are some sufficient condition to guarantee condition $(P5)$ is valid, for instance, if condition (P6) holds, or $K$ is bounded, or $\varphi$ is strongly skew-symmetric.

Next, we define and analyze the convergence of an inexact algorithm for solving $EP(F, K)$. Given two sequences of parameters $\{\lambda_k\}$ and $\{\delta_k\}$ such that $0 \leq \lambda_k < \bar{\lambda}$, for some $\bar{\lambda}$, and $\delta_k \geq 0$. Let $\{x^k\} \subset K$ be a sequence such that $x^k$ is a solution of the inexact equilibrium problem $IEP(F_k, K, \delta_k)$, where $F_k : K \times K \to \mathbb{R}$ is given by $F_k(x, y) = F(x, y) + \lambda_k (\varphi(x, y) - \varphi(x, x))$. This means that $x^k \in K$ satisfies

$$F(x^k, y) + \lambda_k \left(\varphi(x^k, y) - \varphi(x^k, x^k)\right) \geq -\delta_k, \forall\, y \in K. \tag{18}$$

In the sequel, we consider $\{x^k\}$ the sequence generated by (18). It is worth to mention that the well definition of $\{x^k\}$ follows from Theorem 5.

**Theorem 6** *Assume that* $\lim_{\to\infty} \lambda_k = 0$ *and* $\lim_{\to\infty} \delta_k = 0$.

(i) *If the sequence* $\{x^k\}$ *has a cluster point, then it belongs to* $S(F, K)$*;*
(ii) *Additionally, if* $\varphi$ *satisfies condition (A4) and* $S(F, K)$ *is non-empty, then* $\{x^k\}$ *is bounded.*

***Proof*** Let $x^*$ be an accumulation point of $\{x^k\}$ and $\{x^{k_j}\}$ be a subsequence converging to $x^*$. From (18), we have

$$F_k(x^k, y) = F(x^k, y) + \lambda_k \left(\varphi(x^k, y) - \varphi(x^k, x^k)\right) \geq -\delta_k, \quad \forall\, y \in K, \forall k \in \mathbb{N}.$$

In particular the inequality above is valid for all $k_j$. Thus, taking the limit with $k_j \to \infty$ and using that $\lambda_k \to 0$ and $\delta_k \to 0$, we get $F(x^*, y) \geq 0$, for all $y \in K$. Now, let us prove the second assertion. Let $x^k$ be a solution of $IEP(F_k; K, \delta_k)$ and $x^* \in S(F, K)$. Then,

$$F_k(x^k, x^*) = F(x^k, x^*) + \lambda_k(\varphi(x^k, x^*) - \varphi(x^k, x^k)) + \delta_k \geq 0$$

and, from $(P4)$, we have that $F(x^k, x^*) \leq 0$, respectively. Combining last two inequalities, we have

$$\lambda_k(\varphi(x^k, x^*) - \varphi(x^k, x^k)) + \delta_k \geq 0. \tag{19}$$

One has that $\lambda_k \leq \bar{\lambda}$ and $\delta_k \leq \bar{\delta}$, for some positive real number $\bar{\lambda}, \bar{\delta} > 0$, for all $k \in \mathbb{N}$ and hence, by (19), we have

$$\varphi(x^k, x^*) - \varphi(x^k, x^k) + \frac{\bar{\delta}}{\bar{\lambda}} \geq 0. \tag{20}$$

Since $\varphi$ is strongly skew-symmetric with modulus $\gamma > 0$, we obtain

$$\varphi(x^*, x^*) - \varphi(x^k, x^*) - \varphi(x^*, x^k) + \varphi(x^k, x^k) \geq \gamma ||x^k - x^*||^2. \tag{21}$$

Adding inequalities (20) and (21), we have

$$\varphi(x^*, x^*) - \varphi(x^*, x^k) \geq \gamma ||x^k - x^*||^2 - \frac{\bar{\delta}}{\bar{\lambda}}. \tag{22}$$

By (A2), we have that $\varphi(x^*, \cdot)$ is convex and this implies that

$$\varphi(x^*, x^k) - \varphi(x^*, x^*) \geq \langle v, x^k - x^* \rangle, \tag{23}$$

where $v \in \partial\varphi(x^*, x^*)$. Thus, adding inequalities (22) and (23), we obtain

$$\gamma||x^k - x^*||^2 + \langle v, x^k - x^*\rangle \leq \frac{\bar{\delta}}{\bar{\lambda}}.$$

The assertion directly follows from last inequality. □

***Remark 3*** In Oliveira et al. (2013, Theorem 3.3) is proved that if $\{x^k\}$ satisfying (18) with $\lim_{\to\infty} \lambda_k = 0$, $\delta_k = 0$, for all $k \in \mathbb{N}$, and $\varphi$ is a strongly monotone bifunction such that $\limsup_{||y||\to\infty} \frac{\varphi(x,y)}{||x-y||} < +\infty$, then $\{x^k\}$ is bounded if, and only if, $EP(F, K)$ is non-empty. Our previous result extends this result and enables a broad class of regularization including the one proposed in Oliveira et al. (2013) to solve equilibrium problems. Example 3 shows a class of regularization which fit with our set of assumption and does not verify the hypothesis in Oliveira et al. (2013, Theorem 3.3).

## 5 Numerical experiments

In this section, we illustrate our method for solving some academic problems considered in Santos and Scheimberg (2011) and Santos and Souza (2020) using different kind of regularizations. The algorithms have coded in MATLAB using the subroutine "*quadprog*" on a notebook 8 GB Intel Core i7 Dual-Core. For all tests, both inner and outer stop criteria is $\|x^{k,j} - x^{k,j+1}\| < \epsilon$, for some $\epsilon > 0$. The subproblems (18) are solved applying Projected Subgradient Method (PSM) given by

$$x^{k,j+1} = P_C\left[x^{k,j} - \frac{\beta_j}{\gamma_j} g^j\right], \quad \text{with} \quad \beta_j \geq 0, \sum \beta_j = +\infty, \ \sum \beta_j^2 < +\infty$$

with $g^j \in \partial_y F_k(x^{k,j}, x^{k,j})$, $\gamma_j = \max\{1, \|g^j\|\}$, where $\partial_y F_k$ stands to the subdifferential of $F_k$ with respect the variable $y$; see Santos and Scheimberg (2011).

It is worth to mention that regularization such as (3), (4) and (5) are well known and very popular for regularization methods in optimization. An important computational issue about these regularization is that when, for instance in equilibrium problems, the subproblem are solved with PSM which takes $g^j \in \partial_y F_k(x^{k,j}, x^{k,j})$ one has that the regularization plays no role in the algorithm. This happens because using these regularizations we have $\partial_y F_k(x^{k,j}, x^{k,j}) = \partial_y F(x^{k,j}, x^{k,j})$.

Next, we perform the proposed method comparing the performance of the regularization (3) with other regularizations which the previous mentioned issue does not happens, i.e., $\partial_y F_k(x^{k,j}, x^{k,j}) \neq \partial_y F(x^{k,j}, x^{k,j})$. In the sequel, we consider the following regularization, for all $k \in \mathbb{N}$:

I. $\varphi(x, y) = \lambda_k\langle x - x^k, y - x\rangle$;
II. $\varphi(x, y) = \lambda_k\langle T(x), y\rangle$, where $T(x) = \nabla h(x)$ with $h(x) = x^2$ as in Example 2;
III. $\varphi(x, y) = \lambda_k(\langle T(x), y\rangle + \psi(x) - \psi(y))$, where $T(x) = \nabla h(x)$ with $h(x) = x^2$ and $\psi(t) = \log t$ as in Oliveira et al. (2013, Remark 3.2).

In Tables 1, 2 and 3, we report the number of iterates (k) until the method satisfies the stopping rule $\|x^{k,j} - x^{k,j+1}\| < \epsilon$. It is worth to mention that for a general regularization it is not proved that if $x^{k+1} = x^k$, then the method returns a solution. However, we use this classical stopping rule followed by the accuracy $\Upsilon^* = \Upsilon(x^k)$ in the last iterate $x^k$ showing how far from the solution the method stopped. The minimum, maximum and the median of the cpu time until the method stop is also reported. In Figs. 1 and 2, we show the behaviour

**Table 1** Example 5: 10 times with random initial points in $[-10, 10] \times [-10 \times 10]$, $\lambda_k = 1$ and $\epsilon = 10^{-4}$

| Regularization | Iterations (k) | Min CPU time (s) | Max CPU time (s) | Median CPU time (s) | Solution found |
|---|---|---|---|---|---|
| I | 4 | 0.2432 | 0.5086 | 0.3411 | (0.5, 0.5) |
| II | 3 | 0.0806 | 0.0910 | 0.0836 | (0.5, 0.5) |
| III | 2 | 0.0889 | 0.1052 | 0.0977 | (0.5, 0.5) |

**Table 2** Example 6: 10 times with random initial points in $[-10, 10] \times [-10 \times 10]$, $\lambda_k = 0.001$ and $\epsilon = 10^{-3}$

| Regularization | Iterations (k) | Min CPU time (s) | Max CPU time (s) | Median cpu time (s) | Solution found |
|---|---|---|---|---|---|
| I | 14.2 | 0.4337 | 4.3946 | 2.7958 | (5.0042, 8.9971) |
| II | 14 | 3.5532 | 3.9570 | 3.6818 | (5.0063, 8.9954) |
| III | 15 | 2.7884 | 2.9132 | 2.8244 | (5.0077, 8.9941) |

**Table 3** Example 7: the initial point $x^0 = (1, 3, 1, 1, 2)$ and $\epsilon = 10^{-3}$ as in Santos and Scheimberg (2011) and $\lambda_k = 0.001$

| Regularization | Iterations (k) | CPU time (s) | $\Upsilon(x^k)$ |
|---|---|---|---|
| I | 2 | 0.4015 | 4.6e–04 |
| II | 2 | 0.4103 | 9.1e–04 |
| III | 2 | 0.4026 | 1.1e–03 |

of method using three different regularizations for 10 randomly initial points for Example 5 and 6, respectively.

We consider different choice for the sequence $\lambda_k$ setting constant for all $k \in \mathbb{N}$. We take $\beta_k = 100/3k$ and $\gamma_k = \max\{3, \|g^k\|\}$ in the PSM.

***Example 5*** Consider the 2-dimensional nonsmooth equilibrium problem defined by the bifunction $F(x, y) = |y_1| - |x_1| + y_2^2 - x_2^2$ and the constraint set given by $K = \left\{x \in \mathbb{R}^2_+ ;\ x_1 + x_2 = 1\right\}$. One can check that $F$ is monotone and the solution set is the single point $x^* = (\frac{1}{2}, \frac{1}{2})$.

***Example 6*** Consider the 2-dimensional variational inequality problem given by $F(x) = (2x_1 + \frac{8}{3}x_2 - 34, 2x_2 + \frac{5}{4}x_1 - 24.25)$ and the constraint set defined by $K = \{x \in \mathbb{R}^2 : 0 \le x_i \le 10,\ i = 1, 2\}$. The solution set of this problem is a single point $x^* = (5, 9)$.

***Example 7*** Consider the equilibrium problem where the constraint set $K$ is given by

$$K = \left\{ x \in \mathbb{R}^5 : \sum_{i=1}^{5} x_i \ge -1, \quad -5 \le x_i \le 5, \quad i = 1, 2, \ldots, 5 \right\}$$

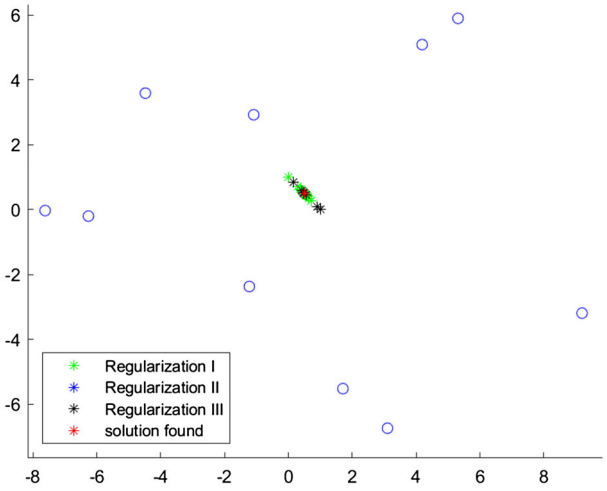


**Fig. 1** The initial points, iterates of the sequence using three regularizations and the solution found

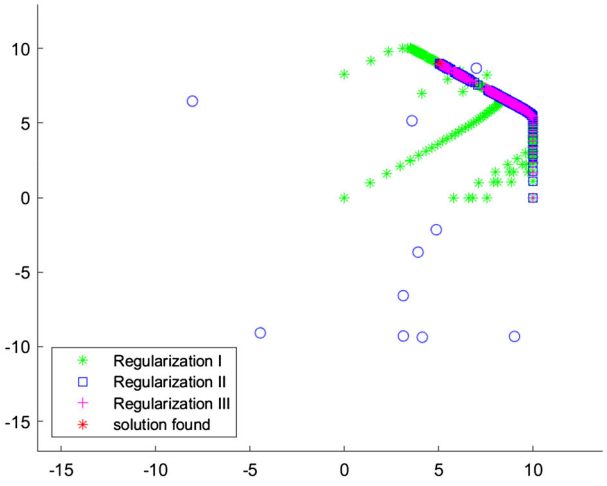


**Fig. 2** The initial points, iterates of the sequence using three regularizations and the solution found

and the bifunction $F$ is of the form $F(x, y) = \langle Px + Qy + q,\ y - x\rangle$, in which the matrices $P$, $Q$ and the vector $q$ are given by

$$P = \begin{bmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix};\quad Q = \begin{bmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \text{ and } q = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{bmatrix}.$$

## 6 Application to Becker's household behaviour theory

In this last section we give an application in behavioral sciences of our abstract regularization approach to solve equilibrium problems. We propose a dynamic version of the famous static Becker's "allocation of time problem" (Becker 1965), using a recent variational rationality approach of human dynamics (Soubeyran 2009, 2010, 2016, 2019a, b).

### 6.1 Household behavior

#### 6.1.1 Becker's allocation of time model. The static case.

Consider the Becker's household model and, for easy reading, let us adapt the notations to our framework. Let $U(Z_1, \ldots, Z_m)$ be the utility function of a household over the quantities of final commodities $(Z_1, \ldots, Z_m) \in \mathbb{R}^m_+$. This household produces at home these final goods. He uses the vector of inputs $y_j = (r_j, t_j)$ to produce the quantity $Z_j = h_j(y_j)$ of each final good $j$. Each $r_j$ is a bundle (quantity vector) of intermediate goods purchased at the vector of prices $p_j$, and $t_j$ is a bundle of time use quantities. Becker defined $t_j$ as a vector also, to distinguish between, e.g., daytime from nighttime hours, or weekdays from weekends. Moreover he assigns a vector of wage rates $w_j$ to $t_j$, thereby assuming that, e.g., the cost of a unit of time on weekends and weekdays would generally be different (Chappori and Lewbel 1965). Rather than have one budget constraint for goods and another one for time, Becker supposes that the household can trade-off time for money, and so only faces the single budget constraint $\sum_{i=1}^m (p_j r_j + w_j t_j) \leq S$, where $S > 0$ represents a full income. In this setting the household chooses a vector of resources $y$ to maximize an 'earnings' function $g(y) = U[h_1(y_1), \ldots, h_m(y_m)]$, where $y = (y_1, \ldots, y_m)$ is a vector of resources (inputs and times) subject to his household's single budget constraint and to the production functions for each commodity. Then he solves the static resource allocation problem $\sup\{g(y), \quad y \in K\}$, where $K = \left\{y \in X = \mathbb{R}^m_+, \langle q, y\rangle \leq S\right\}$, $q = (q_1, \ldots, q_m)$ and $q_j = (p_j, w_j) > 0$, $j \in J = \{1, 2, .., m\}$.

In this application, we are interested in a dynamic formulation of this famous model, using the recent variational rationality approach of human dynamics (Soubeyran 2009, 2010, 2016, 2019a, b).

### 6.2 The variational rationality approach: an experience dependent formulation

#### 6.2.1 Stay and change human dynamics

The (VR) variational rationality approach (Soubeyran 2009, 2010, 2016, 2019a, b) modelizes and unifies a lot of different models of stay and change human dynamics which appeared in behavioral sciences (economics, management sciences, psychology, sociology, political sciences, decision theory, game theory, artificial intelligence, etc.). Stays refer to static phases where, each period, an agent performs again and again the same bundle of activities, exploiting existing capabilities. This leads to the formation of temporary habits, routines, rules and norms, etc.... Changes represent transformation phases (including need's recognition, exploration, evaluation, learning and training change activities) where an agent stops doing some activities, continues doing others and starts doing new activities. These changes help to form and break habits and routines, etc.... This dynamical approach considers entities (an agent,

an organization or several interacting agents), which are, at the beginning of the story, in an undesirable initial position. Being unable to reach immediately (one period) their final desired positions (desires), they try, each period, to improve their current position, to progressively approach and reach their desired ends (desires). The goal of this approach is to examine the transition problem: how such entities will accept to follow an acceptable and feasible transition before being able to reach their final desires; how they accept to overcome a lot of intermediate obstacles, difficulties and resistances to change; how a high initial motivation to change, a succession of not too high intermediate sacrifices and high enough intermediate satisfactions can sustain their motivation to change during the transition in order to persevere until reaching the final desired position.

This (VR) approach admits a lot of variants, based on the same short list of general principles and concepts. The five main concepts refer to moves (changes and stays), worthwhile moves, worthwhile transitions made of a succession of worthwhile moves, variational traps, worthwhile to approach and reach but not worthwhile to leave, and desires not advantageous to leave.

### 6.2.2 Feasible activities

**Bundles of activities.** The variational rationality approach considers bundles of activities $x \in X$. They refer to different lists of elementary activities (tasks) which can be done within a period. In the present application relative to an allocation of activities problem (Becker 1965), $x = (r, t) = (x_1, \ldots, x_m) \in X$ is a bundle of elementary activities $x_j = (r_j, t_j)$, $j \in J = \{1, \ldots, m\}$, where $t = (t_1, \ldots, t_m)$ and $r = (r_1, \ldots, r_m)$. In this context $(r_j, t_j)$ is the task $x_j =$ "spending the specific time $t_j \in \mathbb{R}_+$ to use the bundle of inputs $r_j \in \Gamma_j$ in order to produce the quantity $Z_j = h_j(x_j) \in \mathbb{R}_+$ of the home final good $j \in J = \{1, \ldots, m\}$". That is, the home production function is $Z = h(x)$.

**Set of feasible bundles of activities** The Becker's resource constraint. $\sum_{j=1}^{m}(p_j r_j + w_j t_j) = \langle \alpha, x \rangle \leq S$ defines a bounded convex polytope

$$K = \{x \in X, \langle \alpha, x \rangle \leq S\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_j., \ldots, \alpha_m) \in \mathbb{R}^m_{++}$ with $\alpha_j = (p_j, w_j) > 0$.

### 6.2.3 Moves

**Definition** Consider two successive periods; the current period $k+1$ and the previous period $k$. In this simplified version of the variational rationality approach a current move $m = (\underline{x}, y)$ goes from $\underline{x} = x^k =$ having done the bundle of activities $x$ in the previous period $k$ to $y = x^{k+1} =$ doing the bundle of activities $y$ in the current period $k+1$. This move is a change if $y \neq x$ and a stay if $y = x$. In general, a move is $m = (\underline{x}, \omega, y)$ where $\omega \in \Omega$ modelizes a transition stage, that is a preparation stage where an agent chooses and becomes able to do the bundle of activities $y$ within the current period.

**Extensive and intensive moves** Soubeyran (2019a, b) make the important distinction between extensive and intensive moves. In this paper, this distinction will play a major role for the definition of resistance to move (see later). When doing a move, an agent, each period, either stops, continues and starts doing some activities (extensive move), or he performs, each period, the same list of activities, but more or less of each (intensive move). In the first case we move, i) for some $j \in J$ from $x_j > 0$ to $y_j = 0$, for others $j$ from $x_j = 0$

to $y_j > 0$ and for the last $j$ we have $x_j = y_j > 0$. In the second case all $x_j > 0$ and $y_j > 0$ remain strictly positive.

When we consider moves, we start to consider a dynamic version of Becker's allocation of time problem. Then, we depart from Becker's model which is static. To match with the mathematical formulation of equilibrium problems our dynamic version of Becker's model will only treat the case of intensive moves.

### 6.2.4 Worthwhile balances and worthwhile moves

**An experience dependent variational rationality structure**. The (VR) approach starts with a general definition of a worthwhile change: a change is worthwhile if motivation to change rather than to stay is "high enough" with respect to resistance to change rather than to stay. This definition allows a lot of variants, given the different aspects of motivation to move (more than one hundred theories/aspects of motivations exist in Psychology), the different aspects of resistance to move and different formulations of the sentence "being high enough". To cover all these different aspects (Soubeyran 2009, 2010, 2016, 2019a, b) defined a general variational rationality (VR) structure which modelizes successively activities, moves, utility and costs of moving, advantages and inconveniences to move, motivation and resistance to move, worthwhile balances and finally worthwhile moves. Two special cases emerge where utility and costs of moving do not depend of the whole move $m = (\underline{x}, \omega, y)$, but depend only of the end $y$ of this move, or of the beginning $\underline{x}$ and the end $y$ of this move, but not of the transition $\omega$. First, the simplest case where utility $g(y) \in \mathbb{R}$ and costs $C(y) \in \mathbb{R}_+$ depend only of doing a bundle of activities $y$. Second, the intermediate case where utility $g_s(y)$ and costs $C_s(y)$ depend of the internal state and the external environment $s$ of the agent, including, among other things, different reference points (for example the status quo $\underline{x}$ and aspiration levels $x^*$), experience, etc. To match with the mathematical part of this paper, this application considers a simplified intermediate case (which is still quite complicated), where

(i) a move $m = (\underline{x}, y)$ does not specify the preparation phase, that is, the transition $\omega$ which goes from $\underline{x}$ = having done the bundle of activities $x$ in the previous period to being able to do the bundle of activities $y$ in the current period ;
(ii) the internal state of the agent and the state of his external environment is limited to his experience $s = e$ ;
(iii) experience $e$ depends only of what the agent have done in the previous period, that is, $e = \underline{x}$. This is the Markov case;
(iv) motivation and resistance to move are identified with advantages and inconveniences to move. This defines a linear VR structure;
(v) costs of moving which are the sum of two costs, costs of being able to do and costs to do a bundle of activities $y$, refer only to costs to do. This is the case when changes are intensive (see above).

For the justifications of all these simplifications see Soubeyran (2019a, b).

**Formulation** Then, within this simplified VR structure:

1. $g_{\underline{x}}(y) \in \mathbb{R}$ represents the utility of doing a bundle of activities $y$ in the current period, after having done a bundle of activities $x$ in the previous period;
2. $\mathfrak{C}_{\underline{x}}(y) \in \mathbb{R}$ is the cost of being able to do a bundle of activities $y$ in the current period, after having done a bundle of activities $x$ in the previous period;
3. $C_{\underline{x}}(y) \in \mathbb{R}$ is the cost of doing a bundle of activities $y$ in the current period, after having done a bundle of activities $x$ in the previous period;

4. $C(\underline{x}, y) = \mathfrak{C}_{\underline{x}}(y) + C_{\underline{x}}(y)$ is the cost of moving from $x$ to $y$;
5. $A_{\underline{x}}(x, y) = g_{\underline{x}}(y) - g_{\underline{x}}(x)$ defines advantages to move (change rather than stay);
6. $L_{\underline{x}}(x, y) = -A_{\underline{x}}(x, y)$ represents losses to move;
7. $I_{\underline{x}}(x, y) = C(\underline{x}, y) - C(\underline{x}, x)$ defines inconveniences to move (change rather than stay);
8. $M_{\underline{x}}(x, y) = U_{\underline{x}}\left[A_{\underline{x}}(x, y)\right] \in \mathbb{R}_+$ is the utility $U_{\underline{x}}\left[A_{\underline{x}}\right]$ of advantages to move $A_{\underline{x}} = A_{\underline{x}}(x, y) \geq 0$. It defines motivation to change rather than to stay;
9. $R_{\underline{x}}(x, y) = D_{\underline{x}}\left[I_{\underline{x}}(x, y)\right] \in \mathbb{R}_+$ is the dis-utility $D_{\underline{x}}\left[I_{\underline{x}}\right]$ of inconveniences to move $I_{\underline{x}} = I_{\underline{x}}(x, y) \geq 0$. It defines resistance to change rather than to stay;
10. $B_{\underline{x},\xi}(x, y) = M_{\underline{x}}(x, y) - \xi R_{\underline{x}}(x, y), \xi > 0$ represents a worthwhile balance between motivation and resistance to move;
11. Then, a move $m = (\underline{x}, y)$ is worthwhile if $B_{\underline{x},\xi}(x, y) \geq 0$.

We suppose that:

(a) costs $\mathfrak{C}_{\underline{x}}(y)$ of being able to move from $\underline{x}$ to $y$ are zero for all intensive moves $m = (\underline{x}, y)$. That is, $\mathfrak{C}_{\underline{x}}(y) = 0$ for all $y \in K$ such that the move $m = (\underline{x}, y)$ is intensive;
(b) the values of advantages and inconveniences to move are linear. That is, $U_{\underline{x}}\left[A_{\underline{x}}(x, y)\right] = A_{\underline{x}}(x, y)$ and $R_{\underline{x}}(x, y) = D_{\underline{x}}\left[I_{\underline{x}}(x, y)\right] = I_{\underline{x}}(x, y)$ for all $x, y \in X$.

Then, worthwhile balances become

$$B_{\underline{x},\xi}(x, y) = A_{\underline{x}}(x, y) - \xi I_{\underline{x}}(x, y) = -\left[L_{\underline{x}}(x, y) + \xi I_{\underline{x}}(x, y)\right].$$

### 6.2.5 Ends: desires and traps

The variational rationality approach (Soubeyran 2009, 2010, 2016, 2019a, b defines successively the following aspects of an end position. The bundle of activities $x^*$ is:

- a desire if it is not advantageous to move away from it. That is, if

$$A_{\underline{x}^*}(x^*, y) \leq 0, \quad \forall y \in K.$$

The bundle of activities $x_*$ is,

- a weak stationary trap if it is not worthwhile to move away from it. That is, if $B_{\underline{x}^*,\xi}(x^*, y) \leq 0$, for all $y \in K$.
- a strong stationary trap if it is strictly not worthwhile to move away from it. That is, if $B_{\underline{x}^*,\xi}(x^*, y) < 0$, for all $y \in K, y \neq x_*$.
- an almost (inexact) stationary trap if it is not enough worthwhile to move away from it. That is, if $B_{\underline{x}^*,\xi}(x^*, y) \leq \delta$, for all $y \in K$ and some $\delta > 0$.

The VR approach defines also variational traps which are worthwhile to reach from an initial position, but not worthwhile to leave. In the present paper, we will not consider these variational traps because the generalized regularization algorithm of this paper does not tell something about them.

### 6.2.6 Transitions as successions

After having defined worthwhile moves, desires and traps, the VR approach considers different human dynamics. Here are some.

**Definition** A stay and change human dynamic is a succession of moves $m_k = (x^k, x^{k+1})$, $k = 0, 1, \ldots$, where $k + 1$ is the current period and $k$ is the previous period. This transition

can be a succession of worthwhile moves, or a succession of stationary or almost stationary traps, or a succession of variational traps, etc. To fit with the mathematical part of the paper we will consider a succession of almost stationary traps $x^0, \ldots, x^k, x^{k+1}, \ldots$ such that, for each $k$, $B_{\underline{x}^k,\xi_{k+1}}(x^k, y) \leq \delta_k$ with $\delta_k > 0$ and for all $y \in K$. This means that

$$\begin{aligned} B_{\underline{x}^k,\xi_{k+1}}(x^k, y) &= A_{\underline{x}^k}(x^k, y) - \xi_{k+1} I_{\underline{x}^k}(x^k, y) \\ &= -\left[L_{\underline{x}^k}(x^k, y) + \xi_{k+1} I_{\underline{x}^k}(x^k, y)\right] \\ &\leq \delta_k, \end{aligned}$$

that is, $L_{\underline{x}^k}(x^k, y) + \xi_{k+1}\left[C(x^k, y) - C(x^k, x^k)\right] \geq -\delta_k, \quad \forall y \in K$.

In this context, each period $k+1$, an agent moves from a previous trap to a new one. To be able to do that, he must break the old trap. A way to do that is to choose a new worthwhile ratio $\xi_{k+1} \neq \xi_k$.

**Definition** An aspiration point of a human dynamic $\left\{x^k\right\}$ is a bundle of activities $x_* \in X$ such that it exists $k_0 \in \mathbb{N}$ such that $A_{\underline{x}^k}(x^k, x_*) \geq 0$, or $B_{\underline{x}^k,\xi_{k+1}}(x^k, x_*) \geq 0$, for all $k \geq k_0$. This means that, after a succession of moves, an agent wants or finds worthwhile to approach $x_*$.

### 6.3 Dynamic Markov allocation problems with resistance to move

We are now in a good position to show how, very surprisingly, our abstract regularization method can modelize a dynamic formulation of the famous Becker's household's resource allocation problem.

#### 6.3.1 A static experience dependent resource allocation model

We start by giving a generalized (but still static) version of the static Becker's model (1965) given at the beginning of this section. We suppose that,

(a) the household utility function $g_{\underline{x}}(y)$ is experience dependent. That is, the utility of consuming the bundle of final goods $y$ in the current period depends of how much of all these final goods the consumer consumed in the previous period, i.e., $\underline{x}$. For example a linear utility function with experience dependent weights is $g_{\underline{x}}(y) = \langle \alpha(\underline{x}), y \rangle$, where $\alpha(\underline{x}) = (\alpha_1(\underline{x}), \ldots, \alpha_m(\underline{x})) \in \mathbb{R}^m_+$.
(b) the household chain of production does not produce directly final goods from market goods and labor. In a first upstream stage market goods and labor allow to produce at home intermediate goods. In a second downstream stage intermediate goods, ingredients and labor allow to produce at home final goods. The production of intermediate home goods adds home costs of production $C_{\underline{x}}(y) \in \mathbb{R}_+$ which are experience dependent.

The set of resource constraints $K$ includes now different kind of times spend to produce intermediate goods. Examples of experience dependent production costs are $C_{\underline{x}}(y) = \beta(\underline{x}) \|y\|^2$, $\beta(\underline{x}) \in \mathbb{R}_+$ or $C_{\underline{x}}(y) = \langle \beta(\underline{x}), y \rangle$, where $\beta(\underline{x}) = (\beta_1(\underline{x}), \ldots, \beta_l(\underline{x})) \in \mathbb{R}^l_+$. In this setting the static maximization program of the household is $\sup\left\{g_{\underline{x}}(y) - \xi C_{\underline{x}}(y),\ y \in K\right\}$, $\xi > 0$.

### 6.3.2 Equilibrium and regularized equilibrium problems as desires and traps problems

Let us show how, in the context of the Becker's time allocation problem Becker (1965), the variational rationality approach helps to identify equilibria to desires and generalized regularized equilibria to stationary traps.

**Losses of moving** They are the opposite of advantages to move, i.e.,

$$F(x, y) = L(x, y) = -A_{\underline{x}}(x, y) = g_{\underline{x}}(x) - g_{\underline{x}}(y),$$

while advantages to move are $A_{\underline{x}}(x, y) = g_{\underline{x}}(y) - g_{\underline{x}}(x)$. Then, when moving from the status quo position $x^k$ to $y$ in the current period $k+1$, losses of moving are $F_k(x^k, y) = -A_{\underline{x}^k}(x^k, y) = g_{\underline{x}^k}(x) - g_{\underline{x}^k}(y)$.

Hypothesis (P2) and (P3) suppose that the experience dependent utility function $\psi(\cdot, \cdot) : (x, y) \in X \longmapsto \psi(x, y) = g_{\underline{x}}(y) \in \mathbb{R}$ is, respectively

- lower semicontinuous with respect to the first variable $y$, for all $y \in K$;
- concave and upper semicontinuous in the second variable $y$, for all $x \in K$.

Hypothesis (P1) means that advantages to stay are zero, that is, $A_{\underline{x}}(x, x) = g_{\underline{x}}(x) - g_{\underline{x}}(x) = 0$.

Hypothesis (P4) supposes that if advantages to move from $x$ to $y$ are positive, then, there is no advantages to move from $y$ to $x$.

Hypothesis (P5) supposes that any dynamic which goes to infinity have an aspiration point. This is a position that, after some time, an agent wants or finds worthwhile to approach. Hence, the merit of such unbounded dynamics is to do not wander.

**Generalized regularization terms** They represent home production costs $C_{\underline{x}}(y) = C(\underline{x}, y) = \varphi(x, y)$. Then, the difference $\varphi(x, y) - \varphi(x, x)$ refers to inconveniences to move. That is, $\varphi(x, y) - \varphi(x, x) = I_{\underline{x}}(x, y) = C(\underline{x}, y) - C(\underline{x}, x)$.

Hypothesis (A1) supposes that, if inconveniences to move from $x$ to $y$ are positive, then, inconveniences to move from $y$ to $x$ are negative, that is, $I_{\underline{x}}(x, y) + I_{\underline{y}}(y, x) \leq 0$, for all $x, y \in K$. This is true if $C_{\underline{x}}(y) - C_{\underline{x}}(x) + C_{\underline{y}}(x) - C_{\underline{y}}(y) \leq 0$, for all $x, y \in K$.

Hypothesis (A2) and (A3) suppose that experience dependent costs of production $\varphi(\cdot, \cdot) : (x, y) \in X \longmapsto \varphi(x, y) = C_{\underline{x}}(y) \in \mathbb{R}$ are convex in the second argument $y$ (the usual case of decreasing returns) and are continuous (a natural hypothesis).

Hypothesis (A4) means that $I_{\underline{x}}(x, y) + I_{\underline{y}}(y, x) \leq -\gamma \|y - x\|^2, \gamma > 0$, for all $x, y \in K$. That is, if inconveniences to move from $x$ to $y$ are positive, then, inconveniences to move from $y$ to $x$ are strongly negative.

Example 1 considers strictly convex costs $\varphi(x, y) = C_{\underline{x}}(y) = \|y\|^2$ while example 2 considers the linear case $\varphi(x, y) = C_{\underline{x}}(y) = \langle T(x), y \rangle$ with experience dependent weights $T(x) = \beta(\underline{x})$. These costs satisfy hypothesis (A1), (A2) and (A3).

**Regularized losses** They are $F_\lambda(x, y) = F(x, y) + \lambda [\varphi(x, y) - \varphi(x, x)]$. They represent the opposite of worthwhile balances

$$B_{\underline{x},\lambda}(x, y) = [-F(x, y)] - \lambda I_{\underline{x}}(x, y) = A_{\underline{x}}(x, y) - \lambda I_{\underline{x}}(x, y).$$

A regularized loss $F_\lambda(x^k, y) = f(x^k, y) + \lambda_k \left[\varphi(x^k, y) - \varphi(x^k, x^k)\right]$ refers to the worthwhile balance $B_{\underline{x}^k,\lambda_{k+1}}(x^k, y)$. **Equilibria and regularized equilibria as desires and traps** At this stage of the presentation, it is evident that equilibria represent desires, while regularized equilibria refer to stationary traps.

### 6.3.3 The dynamic household problem: results

The mathematical part of the paper gives three different kinds of results. In term of the variational rationality approach, Theorems 3 and 4 give sufficient conditions for the existence of aspiration points, Theorem 5 gives sufficient conditions for the existence of desires, and the proof of Theorem 6 helps to show when a succession of stationary traps is an improving worthwhile to move dynamic, made of a succession of worthwhile moves going from a stationary trap to the next one. A striking result being that such worthwhile successions of stationary traps modelize the famous “Unfreeze, change, refreeze" Lewin’s behavioral change model Lewin (1947a, b, 1951) which is at the origin of a huge literature on “change management" and behavioral interventions in management sciences and health theory (more than 80 different models and variants). More precisely, here are our results in the context of behavioral sciences.

1. Theorems 3 and 4 give sufficient conditions which guarantee, for each unbounded succession of moves, the existence of aspirations points “worthwhile to approach" after doing a sufficient amount of moves. These conditions require that worthwhile balances satisfy conditions (P1)–(P4) and (P5). Condition (P5) is satisfied if, either condition (P6) works, or $K$ is bounded or costs to do are strongly skew-symmetric. $K$ is bounded is a weak hypothesis. It means that, each period, the household have limited resources.
2. A behavioral interpretation of $(P6^*)$ is the following. This hypothesis means that, given $\bar{x}$, for any unbounded human dynamic $\{z^n\}$, there exists a period $n_0$ such that, after this period, it is not worthwhile to move away from $\bar{x}$ to each successive $\{z^n\}$ . This means that $\bar{x}$ is a kind of local stationary trap for this dynamic.
3. Theorem 5 shows the existence and unicity of traps if advantages to move satisfy assumptions (P1)–(P4), costs to do satisfy assumptions (A1)–(A4) and condition (P6) holds.
4. Theorem 6 shows the existence of desires as accumulation points of a succession of almost worthwhile moves if condition (P5) is satisfied.

## 7 Conclusion

We have proposed a generalized regularization method for solving equilibrium problems. At each subproblem, the proposed method solves a generalized mixed equilibrium problem. Under some mild assumptions we have proved the well-definition and the convergence of the method in the sense that every accumulation point of the sequence generated by the method is a solution of the equilibrium problem even if the subproblem are solved inexactly. The proposed method retrieves some existing ones in the literature on equilibrium problems enlarging the application of this kind of regularization method. An original application to Becker’s household behavior theory has given using the variational rationality approach of human dynamics as well as some numerical experiments.

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