

*Princeton Summer School on Combustion*

*June 20 - June 26, 2021*

**Structure and Dynamics  
of  
Combustion Waves in Premixed Gases**

Paul Clavin

Aix-Marseille Université  
ECM & CNRS (IRPHE)

**Lecture VIII**  
**Thermo-acoustic instabilities. Vibratory flames**

*Copyright 2021 by Paul Clavin*

*This material is not to be sold, reproduced or distributed  
without permission of the owner, Paul Clavin*

# Lecture 8: Thermo-acoustic instabilities

## Lecture 8-1. Rayleigh criterion

*Acoustic waves in a reactive medium*

*Sound emission by a localized heat source*

*Linear growth rate*

## Lecture 8-2. Admittance & transfer function

*Flame propagating in a tube*

*Pressure coupling*

*Velocity and acceleration coupling*

## Lecture 8-3. Vibratory instability of flames

*Acoustic re-stabilisation and parametric instability (Mathieu's equation)*

*Flame propagating downward (sensitivity to the Markestein number)*

*Bunsen flame in an acoustic field*

# VIII-I) Rayleigh criterion



## Acoustic waves in a reactive medium

Ideal gas

$$p = (c_p - c_v)\rho T$$

$$\frac{c_p}{c_p - c_v} \frac{Dp}{Dt} = c_p T \frac{D\rho}{Dt} + c_p \rho \frac{DT}{Dt}$$

$$D/Dt \equiv \partial/\partial t + \mathbf{u} \cdot \nabla$$

$$a^2 = (c_p/c_v)(c_p - c_v)T$$

$$\rho c_p \frac{D}{Dt} T = \frac{c_p}{c_p - c_v} \frac{D}{Dt} p - \frac{c_v}{c_p - c_v} a^2 \frac{D}{Dt} \rho$$

Energy conservation

$$\rho c_p \frac{DT}{Dt} = \frac{Dp}{Dt} + \nabla \cdot (\lambda \nabla T) + \sum_j Q^{(j)} \dot{W}^{(j)}$$

$$\frac{c_v}{c_p - c_v} \frac{Dp}{Dt} - \frac{c_v}{c_p - c_v} a^2 \frac{D\rho}{Dt} = \nabla \cdot (\lambda \nabla T) + \sum_j Q^{(j)} \dot{W}^{(j)}$$

elimination of T

$$Dp/Dt - a^2 D\rho/Dt = \dot{q}_\gamma$$

isentropic acoustic  
 $\delta p = a^2 \delta \rho$

$$\dot{q}_\gamma \equiv (\gamma - 1) \left[ \nabla \cdot (\lambda \nabla T) + \sum_j Q^{(j)} \dot{W}^{(j)} \right] \quad \gamma \equiv c_p/c_v$$

$\dot{q}_\gamma(\mathbf{r}, t)$  = heat transfer + heat release

(rate of energy transfert per unit volume)

$$\mathrm{D}p/\mathrm{D}t - a^2 \mathrm{D}\rho/\mathrm{D}t = \dot{q}_\gamma$$

$$\dot{q}_\gamma \equiv (\gamma - 1) \left[ \nabla \cdot (\lambda \nabla T) + \sum_j Q^{(j)} \dot{W}^{(j)} \right] \quad \gamma \equiv c_p/c_v$$

Linearization around a uniform state  $\nabla \bar{a} \approx 0$ ,

$$\rho = \bar{\rho} + \rho', \quad p = \bar{p} + p', \quad \mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}', \quad \dot{q}_\gamma = \bar{\dot{q}_\gamma} + \dot{q}'_\gamma$$

Mean flow velocity neglected in front of the sound speed  $\bar{\mathbf{u}} \cdot \nabla \approx 0$

$$\frac{1}{\rho} \frac{\mathrm{D}\rho}{\mathrm{D}t} = -\nabla \cdot \mathbf{u}, \quad \rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p \quad \Rightarrow \quad \partial \rho'/\partial t = -\bar{\rho} \nabla \cdot \mathbf{u}', \quad \bar{\rho} \partial \mathbf{u}'/\partial t = -\nabla p',$$

Approximations  
 $a, c_p, c_v$ ; constant

$$\partial/\partial t \quad \partial p'/\partial t - \bar{a}^2 \partial \rho'/\partial t = \dot{q}'_\gamma$$

elimination of  $\rho'$

$$\partial^2 p'/\partial t^2 - \bar{a}^2 \Delta p' = \partial \dot{q}'_\gamma / \partial t$$

## Sound emission by a localized heat source in free space

classical problem  
of acoustics

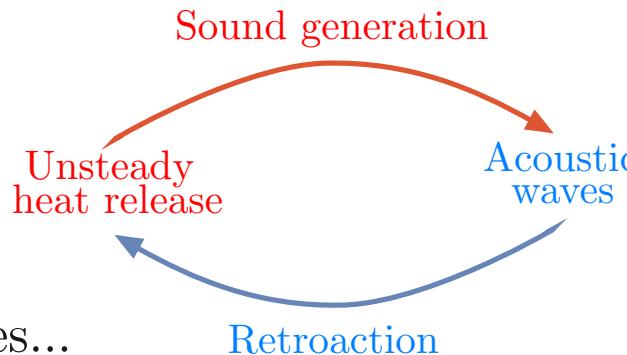
$$\text{acoustic wavelength} \gg \text{size of the combustion zone} \quad \partial \dot{q}'_\gamma(\mathbf{r}, t)/\partial t = \delta(\mathbf{r}) \ddot{\Omega}(t) \quad \ddot{\Omega}(t) \equiv \partial \dot{\Omega}(t)/\partial t, \quad \dot{\Omega}(t) = \iiint \dot{q}'_\gamma(\mathbf{r}', t) d^3 \mathbf{r}'$$

$$\text{Green's retarded propagator} \quad (1/\bar{a}^2) \partial^2 G/\partial t^2 - \Delta G = \delta(\mathbf{r}) \delta(t), \quad G(\mathbf{r}, t) = \bar{a} \delta(r - \bar{a}t)/4\pi r$$

$$\text{spherical geometry} \quad p'(\mathbf{r}, t) = \frac{1}{4\pi \bar{a}^2} \iiint \frac{1}{r'} \frac{\partial}{\partial t} \dot{q}'_\gamma(\mathbf{r}', t - r/\bar{a}) d^3 \mathbf{r}' = \frac{\ddot{\Omega}(t - r/\bar{a})}{4\pi \bar{a}^2 r}, \quad r = |\mathbf{r}|$$

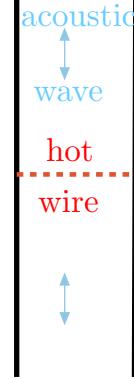
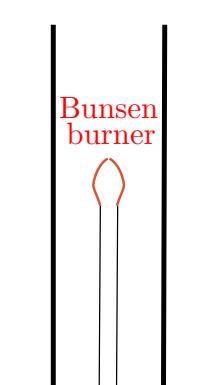
# Liner growth rate

retro-action loop:



Rocket engines, gas turbines...

Singing flame



Simplest retro-action mechanism: pressure coupling + 1-D geometry

$$\partial^2 p' / \partial t^2 - \bar{a}^2 \Delta p' = \partial \dot{q}'_\gamma / \partial t$$

$$\Rightarrow \frac{d^2 \tilde{p}_k}{dt^2} - \frac{b}{\tau_{ins}} \frac{d \tilde{p}_k}{dt} + \bar{a}^2 k^2 \tilde{p}_k = 0$$

$$\tilde{p}_k = e^{\sigma t} \quad 2\sigma\tau_{ins} = b \pm \sqrt{b^2 - 4\bar{a}^2 k^2 \tau_{ins}^2}$$

$$\frac{1}{\tau_{ins}} \ll \omega_k = \bar{a}k$$

$\text{Im}(\sigma) = \omega_k + \dots$ ,  $\text{Re}(\sigma) = b/(2\tau_{inst}) + \dots \Rightarrow \begin{cases} b > 0 \text{ fluctuations of heat release and pressure in phase: instability} \\ b < 0 \text{ fluctuations of heat release and pressure out of phase: stability} \end{cases}$

More general retro-action mechanism

$$\delta \dot{q}'_\gamma(x, t) = \frac{1}{\tau_{ins}} \int_{-\infty}^t b(t-t') \delta p'(x, t') t'$$

$$b(\tau) = \int_{-\infty}^{+\infty} r(\omega) e^{i\omega\tau_d(\omega)} e^{i\omega\tau} d\omega + \text{c.c.} \quad r(\omega) > 0 \quad \omega\tau_d(\omega) \text{ is the phase lag}$$

$$-\pi/2 < \omega_k \tau_d(\omega_k) < +\pi/2 : \text{Instability}$$

Nonlinear study: limit cycles in the unstable case

# Lecture 8: Thermo-acoustic instabilities

## Lecture 8-1. Rayleigh criterion

*Acoustic waves in a reactive medium*

*Sound emission by a localized heat source*

*Linear growth rate*

## Lecture 8-2. Admittance & transfer function

*Flame propagating in a tube*

*Pressure coupling*

*Velocity and acceleration coupling*

## Lecture 8-3. Vibratory instability of flames

*Acoustic re-stabilisation and parametric instability (Mathieu's equation)*

*Flame propagating downward (sensitivity to the Markestein number)*

*Bunsen flame in an acoustic field*

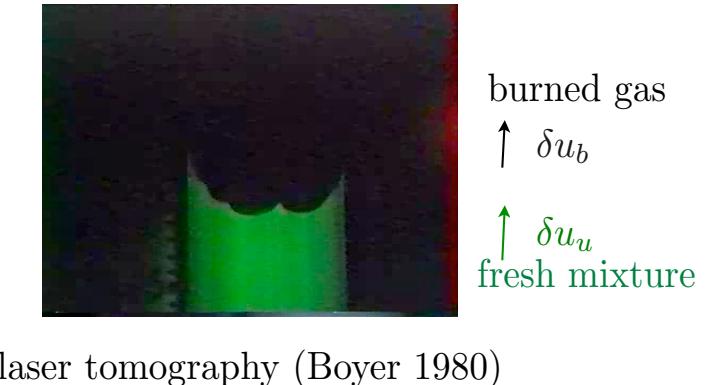
# Admittance & transfer function

## Flame propagating in a tube

thickness of the flame brush  $\ll$  acoustic wavelength

gas expansion  $\Rightarrow$  jump of the fluctuations of the flow velocity (acoustics)

$$(\delta u_b - \delta u_u)/U_L = O(1)$$



jump of pressure across the flame brush is negligible

$$\text{acoustic pressure} \\ \delta p = \rho a \delta u$$

$$(\delta p_b - \delta p_u)/p = O(U_L/a)$$

$$p/\rho a^2 = O(1)$$

$\delta p_f$  : fluctuation of the pressure at the flame

averaged (per period) flux of combustion energy transferred to the acoustic waver

$$\dot{\mathcal{E}}_t = \overline{(\delta u_b - \delta u_u)\delta p_f}$$

mass conservation (quasi-isobaric combustion)

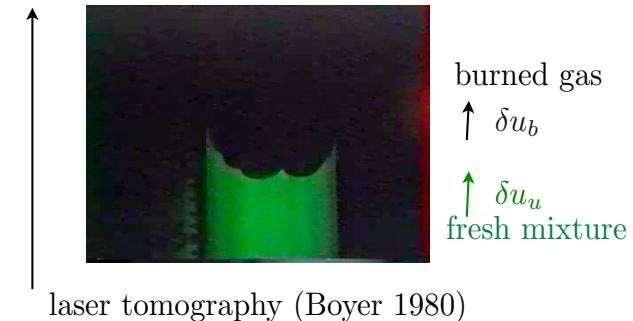
$$\nabla \cdot \mathbf{u} = \frac{1}{T} \frac{DT}{Dt} = \frac{\dot{q}_\gamma / (\gamma - 1)}{\rho c_p T} = \frac{\dot{q}_\gamma}{\rho a^2}$$

$$(\delta u_b - \delta u_u) = \int_{\text{flame brush}} \frac{\delta \dot{q}_\gamma}{\rho a^2} dx$$

$$\dot{\mathcal{E}}_t = \overline{(\delta u_b - \delta u_u) \delta p_f}$$

$$(\delta u_b - \delta u_u) = \int_{\text{flame brush}} \frac{\delta \dot{q}_\gamma}{\rho a^2} dx$$

## Pressure coupling



Definition of the admittance function  $\mathcal{Z}(\omega)$

$$\delta u(t) = \operatorname{Re} [\hat{u}(\omega) e^{i\omega t}] \quad \delta p(t) = \operatorname{Re} [\hat{p}(\omega) e^{i\omega t}]$$

$$\dot{\mathcal{E}}_t = \frac{1}{4\rho_b a_b} (\mathcal{Z} \hat{p}_f \hat{p}_f^* + \mathcal{Z}^* \hat{p}_f \hat{p}_f^*) = \frac{1}{2} [\operatorname{Re} \mathcal{Z}(\omega)] \frac{|\hat{p}_f|^2}{\rho_b a_b}$$

$$(\hat{u}_b - \hat{u}_u) = \mathcal{Z}(\omega) \hat{p}_f / \rho_b a_b \quad \Leftarrow \text{(Rayleigh: } \delta \dot{q}_\gamma \text{ v.s. } \delta p)$$

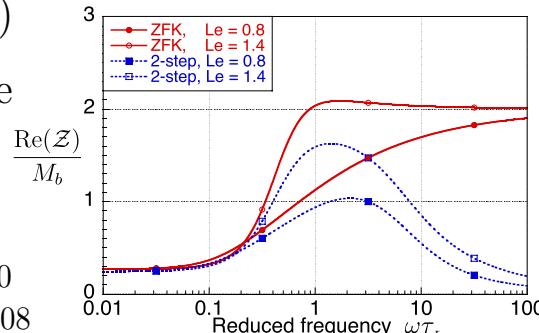
$$\text{instability : } \operatorname{Re}(\mathcal{Z}) > 0$$

Analytical study of a planar flame submitted to a fluctuation of pressure ( $\beta \rightarrow \infty$ )  $\delta T_f/T_f \propto \delta p_f/p_f$

$$|\mathcal{Z}| = O(M_b)$$

gaseous flame

Clavin et al. 1990  
Clavin Searby 2008

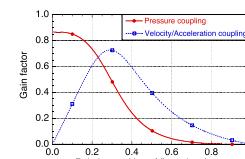


$$\frac{\tau_a}{\tau_{ins}} \propto (\gamma - 1) M_b \frac{E}{k_B T_b}$$

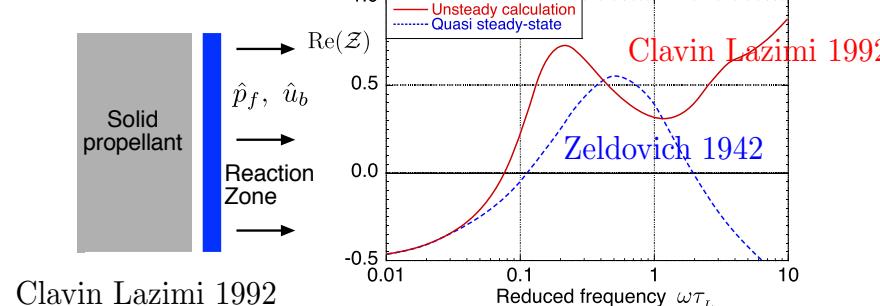
coeff depends on the position in the tube as  $\delta p_f$  does

weak coupling

$$\tau_a / \tau_{ins} \ll 1$$



Solid propellant



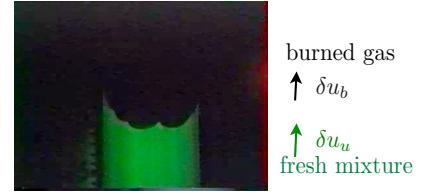
Clavin Lazimi 1992

$$\delta u(t) = \operatorname{Re} [\hat{u}(\omega) e^{i\omega t}]$$

$$\delta p(t) = \operatorname{Re} [\hat{p}(\omega) e^{i\omega t}]$$

## Velocity and acceleration coupling

fluctuating velocity  $\Rightarrow$  modification to flame geometry  
 $\Rightarrow$  fluctuation of heat release through the flame surface



**Transfer function for a flame in a tube**  $T_r(\omega)$

$$(\hat{u}_b - \hat{u}_u) = T_r(\omega) \hat{u}_u$$

$$\dot{\mathcal{E}}_t = (1/4)(T_r \hat{u}_u \hat{p}_f^* + T_r^* \hat{u}_u^* \hat{p}_f)$$

$$\hat{u} \hat{p}_f^* = -\hat{u}^* \hat{p}_f$$

$$\dot{\mathcal{E}}_t = \operatorname{Im} T_r(\omega) (i \hat{u}_u \hat{p}_f^*) / 2$$

phase quadrature (acoustic mode of a tube)

Real number (sign depends on position)

## Weakly cellular flame propagating downward in an acoustic wave

acceleration of a curved flame  $\Rightarrow$  modulation of the flame surface  $S = \int dy \sqrt{1 + \alpha_y'^2}$

$$\int \delta \dot{q} dx = \rho_u U_L c_p (T_b - T_u) \delta S / S_o \quad \delta u_b - \delta u_u = \int \frac{\delta \dot{q}}{\rho a^2} dx \quad \Rightarrow \quad \delta u_b - \delta u_u = (T_b/T_u - 1) U_L \delta S / S_o$$

fluctuation of heat release rate/ cross-section area

Consider a curved front slightly perturbed

$$x = \alpha(y, t)$$

$$\alpha(y, t) = \tilde{\alpha}(t) \cos(ky)$$

$$\tilde{\alpha}(t) = \tilde{\alpha}_0 + \hat{\alpha}_1 e^{i\omega t} + \text{c.c}$$

unperturbed  $\tilde{\alpha}_0$       perturbation  $\hat{\alpha}_1 e^{i\omega t}$

$$k \tilde{\alpha}_0 \ll 1 \quad |\tilde{\alpha}_1| \ll \tilde{\alpha}_0 \quad (\text{linear response ok}) \Rightarrow \quad \delta S / S_o = (k^2/2) \tilde{\alpha}_0 \hat{\alpha}_1 e^{i\omega t} + \text{c.c.}$$

$\tilde{\alpha}_1$  vs  $\hat{u}_u$  ?

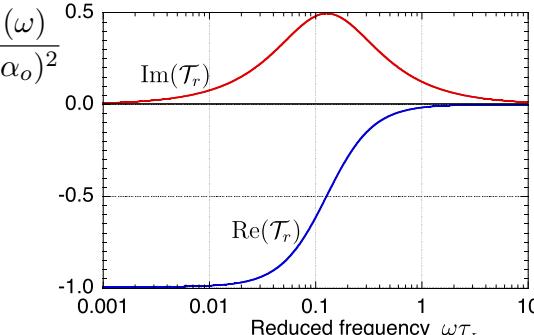
$$g'(t) = \operatorname{Re} [i\omega \hat{u}_u e^{i\omega t}] \quad \bar{g} > 0$$

$$\text{lecture IV: } \left(1 + \frac{\rho_b}{\rho_u}\right) \frac{d^2 \tilde{\alpha}}{dt^2} + 2(U_L k) \frac{d\tilde{\alpha}}{dt} - \left(\frac{\rho_u}{\rho_b} - 1\right) k \left[ -\frac{\rho_b}{\rho_u} [\bar{g} + g'(t)] + U_L^2 k \left(1 - \frac{k}{k_m}\right) \right] \tilde{\alpha} = 0$$

Analytical expression ( $k = k_c$ )

(Pelcé Rochewerger 1992)

ok for the primary instability



## Lecture 8: Thermo-acoustic instabilities

### Lecture 8-1. Rayleigh criterion

*Acoustic waves in a reactive medium*

*Sound emission by a localized heat source*

*Linear growth rate*

### Lecture 8-2. Admittance & transfer function

*Flame propagating in a tube*

*Pressure coupling*

*Velocity and acceleration coupling*

### Lecture 8-3. Vibratory instability of flames

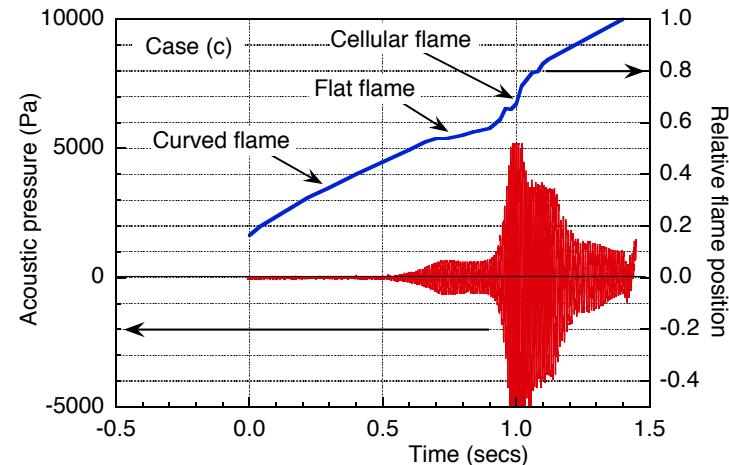
*Acoustic re-stabilisation and parametric instability (Mathieu's equation)*

*Flame propagating downward (sensitivity to the Markestein number)*

*Bunsen flame in an acoustic field*

# VIII-3) Vibratory instability of flames

primary instability + re-stabilisation + parametric instability



## Acoustic re-stabilisation and parametric instability

Markstein 1964

$$\tau' \equiv t/\tau_h, \quad \tau_h \equiv 1/(U_L k), \quad \varpi \equiv \omega \tau_h, \quad = (\omega \tau_L)/(k d_L) \quad \kappa = k d_L$$

$$v_b \equiv \rho_u / \rho_b > 1$$

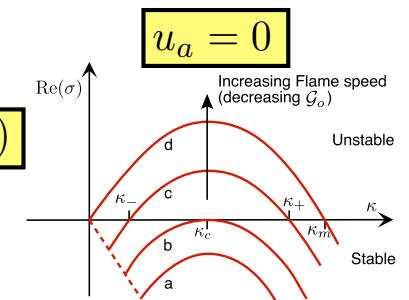
$$B \equiv \frac{d^2 \tilde{\alpha}}{d\tau'^2} + 2B \frac{d\tilde{\alpha}}{d\tau'} + [-D + \varpi^2 C \cos(\varpi \tau')] \tilde{\alpha} = 0$$

$$C \equiv \left( \frac{v_b - 1}{v_b + 1} \right) \frac{u_a}{\varpi},$$

$$D \equiv v_b \left( \frac{v_b - 1}{v_b + 1} \right) \frac{N}{\kappa}$$

$$g'(t) = \omega u_a U_L \cos(\omega t)$$

$$N(\kappa) \equiv -G_o + \kappa - \kappa^2 / \kappa_m \quad G_o \equiv v_b^{-1} |g| d_L / U_L^2$$

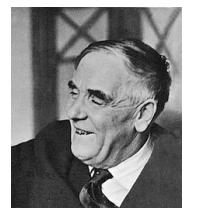


Mathieu's equation. Kapitza pendulum

$$t \equiv \varpi \tau' \quad Y(t) \equiv e^{B\tau'} \tilde{\alpha}$$

$$\frac{d^2 Y}{dt^2} + \{\Omega + h \cos(t)\} Y = 0$$

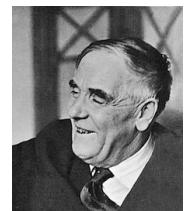
$$\Omega = -\frac{(D + B^2)}{\varpi^2} \quad h = C$$



Kapitza 1951

## Mathieu's equation. Kapitza pendulum

$$\frac{d^2Y}{dt^2} + \{\Omega + h \cos(t)\} Y = 0$$

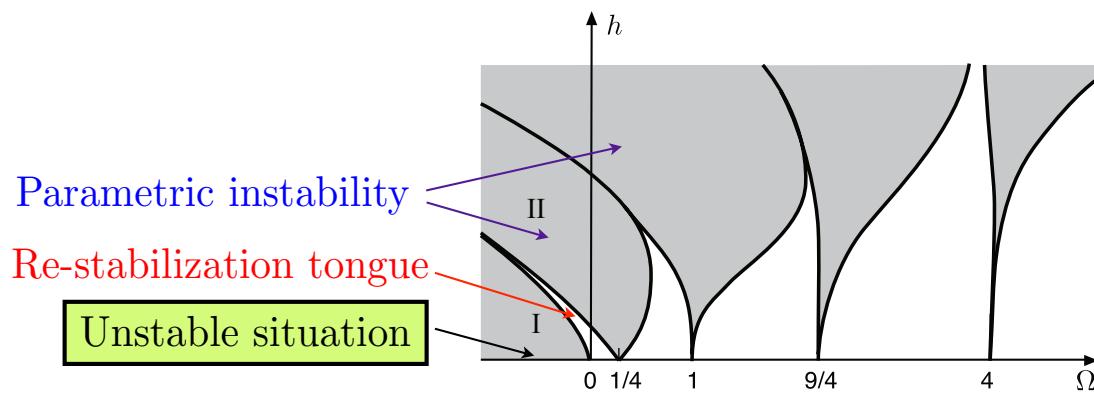
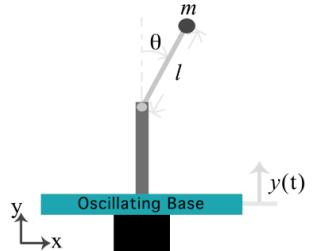


Faraday 1831

Kapitza 1951

$\Omega > 0$  : Oscillator whose frequency  $\sqrt{\Omega}$  is modulated      Parametric instability (Faraday 1831)

$\Omega < 0$  : Re-stabilization of an unstable position of a pendulum by oscillations (Kapitza 1951)



Parametric instability

Re-stabilization tongue

Unstable situation

Stability limits of the solutions to Mathieu's equation  
White regions: stable. Grey regions unstable

### Flame propagating downward

$$\frac{d^2\tilde{\alpha}}{d\tau'^2} + 2B \frac{d\tilde{\alpha}}{d\tau'} + [-D + \varpi^2 C \cos(\varpi\tau')] \tilde{\alpha} = 0$$

$\downarrow g'(t) = \omega u_a U_L \cos(\omega t)$

Markstein 1960

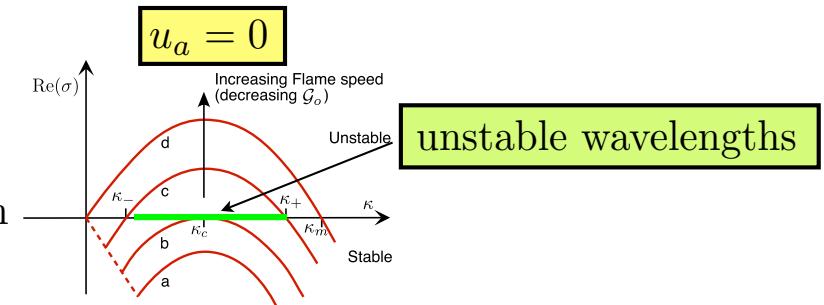
$$u_{aI}^{*2} \approx 2v_b \frac{(v_b+1)}{(v_b-1)} \left(1 - \frac{U_{Lc}}{U_L}\right), \quad k_I^* \approx \frac{1}{2} \frac{U_{Lc}}{U_L}$$

$$u_{aII}^* = 2v_b / (v_b - 1)$$

Bychkov 1999

Clavin 2015

ok with experiments Searby Rochewerger 1991

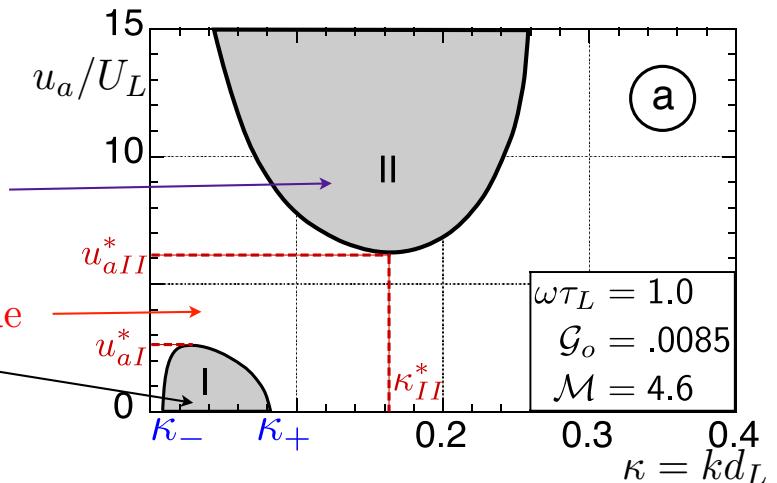


unstable wavelengths

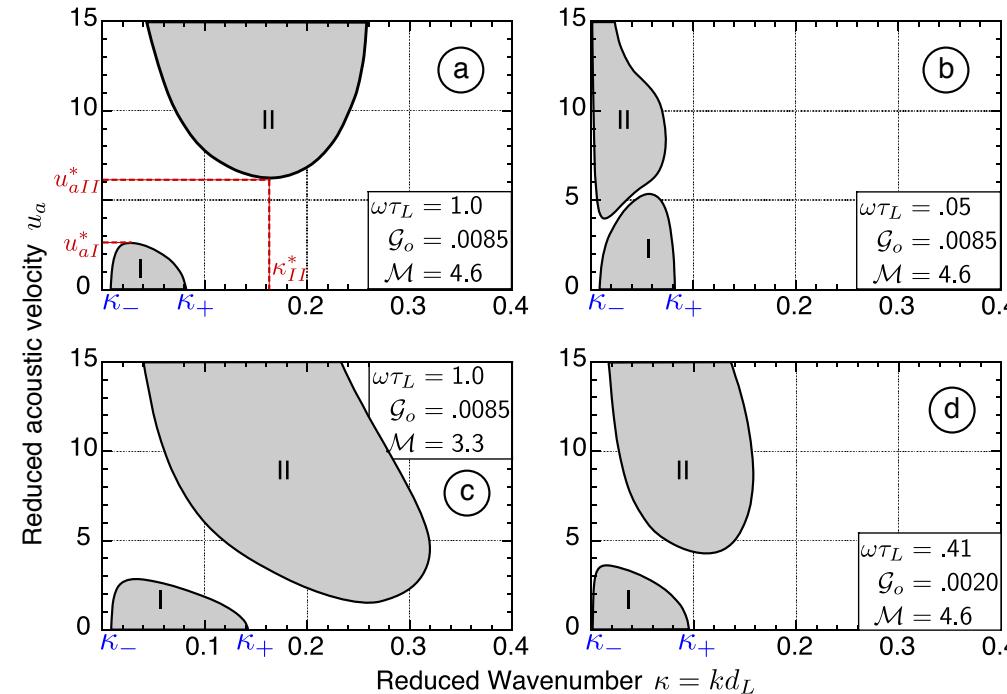
Parametric instability

$u_{aI}^* < u_a < u_{aII}^*$   
Re-stabilization tongue

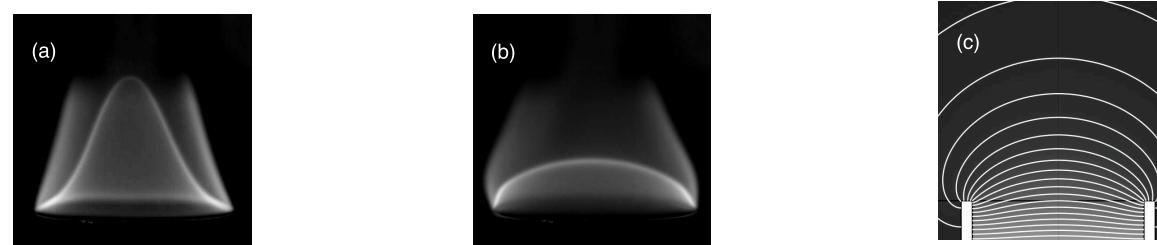
unstable wavelengths



# Sensitivity of the acoustic instability to the Markstein number an the acoustic frequency



Flattening of Bunsen flames in an acoustic field (Hahnemann Ehret 1943, Durox et al. 1997, Baillot et al. 1999)



Rich Bunsen methane flame

+ intense axial acoustic field

140 Hz

acoustic equipotential surface