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Structure and Dynamics of Combustion Waves in Premixed Gases

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Lecture X Supersonic waves

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Lecture 10 : Supersonic waves

10-1. Background

Model of hyperbolic equations for the formation of discontinuity

Riemann invariants

Rankine-Hugoniot conditions for shock waves

Mikhelson (Chapman-Jouguet) conditions for detonations

10-2. Inner structure of a weak shock wave

Formulation

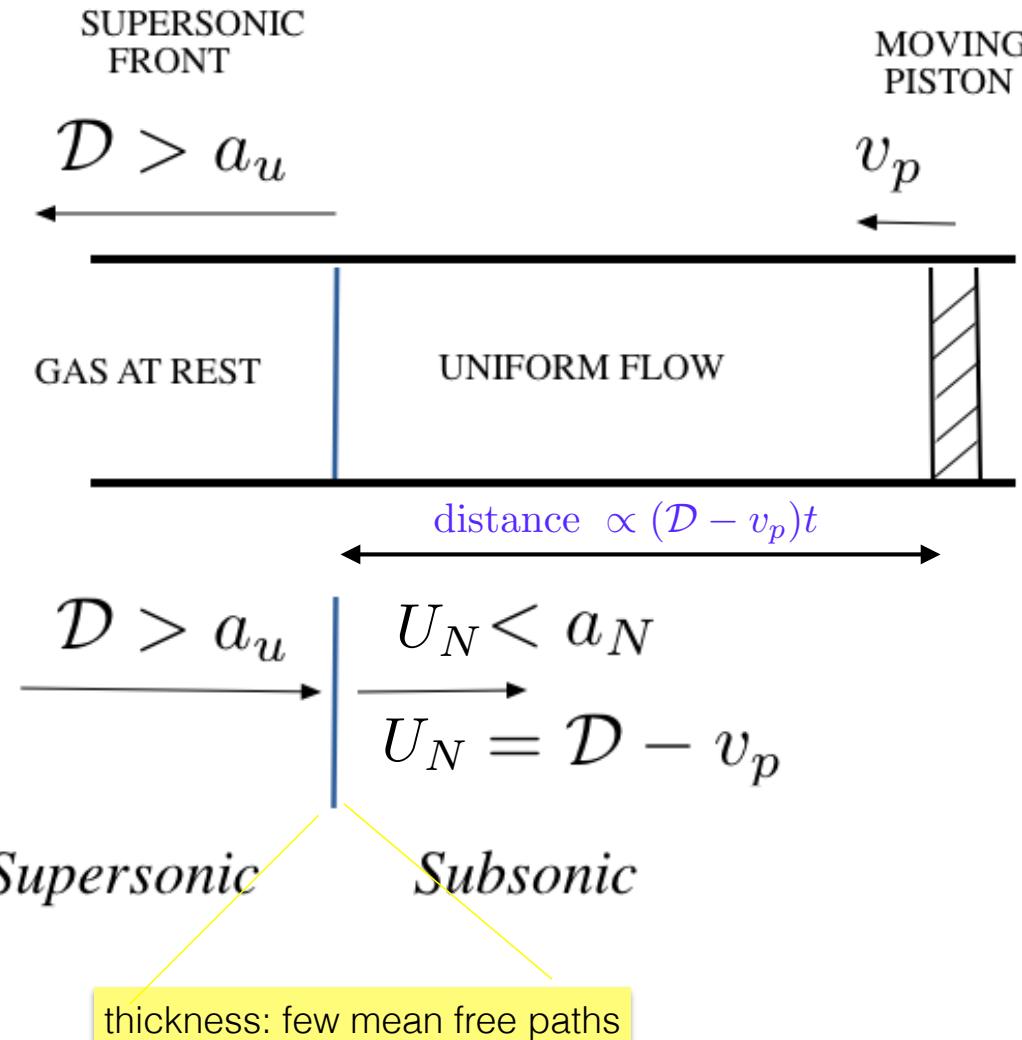
Dimensional analysis

Analysis

10-3. ZND structure of detonations

10-4. Selection mechanism of the CJ wave

PLANAR SHOCK WAVE INERT GAS



Poisson 1808, Stokes 1848, **Riemann** 1860, Rankine 1869, **Hugoniot** 1889, Rayleigh 1910

SHOCK WAVE AS A SINGULARITY OF THE EULER EQUATIONS

shock wave \approx discontinuity in the solution of the Euler equations

Model of hyperbolic equations for the formation of discontinuities

$$a(u)$$

$$\boxed{\partial u / \partial t + a(u) \partial u / \partial x = 0}$$

$$t = 0 : \quad u = u_o(x)$$

$$u = u(x, t) ?$$

Simple case: linear equation

$$a = a_o = \text{cst.} :$$

$$\boxed{u = u_o(x - a_o t)}$$

propagation at constant velocity **without deformation**

Nonlinear equation $a(u)$ $da/du \neq 0$

Method of characteristics



Riemann 1860

The solution is conserved along any trajectory $dx/dt = a(u)$ in the phase plan (x, t) .

$$u = u(\color{red}{x(t)}, t) \quad \color{red}{du/dt = 0}$$

$u(x, t)$ is **constant** along the **straight lines** $\boxed{x = a_o t + x_o}$: $u = u_o$ $u_o \equiv u(a_o)$

Finite-time singularity

Nonlinear equation

$$\frac{\partial u}{\partial t} + a(u) \frac{\partial u}{\partial x} = 0$$

$$t = 0 : \quad u = u_o(x)$$

$$u = u(x, t) ?$$

Method of characteristics

$$\text{trajectories } x(t) : \quad dx/dt = a(u) \quad u = u(\mathbf{x}(t), t) \quad du/dt = 0$$



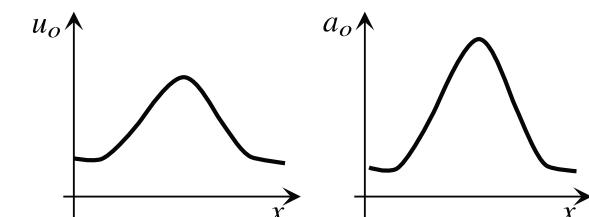
$$u(x, t) = \text{cst.} \Rightarrow a(x, t) = \text{cst.}$$

$u(x, t)$ is **constant** along the **straight lines**; $t = 0 : x = a_o t + x_o, u = u_o$

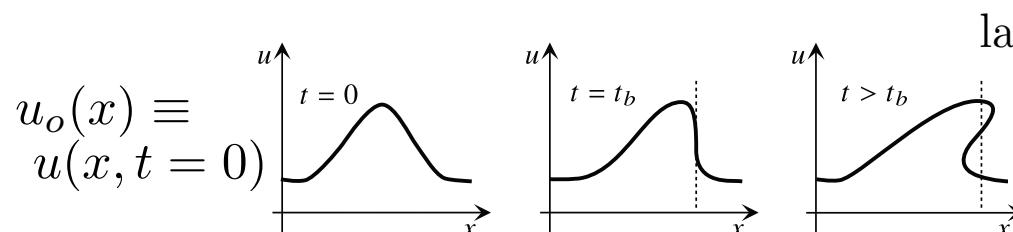
$$x = a_o t + x_o$$

$$u_o \equiv u(a_o)$$

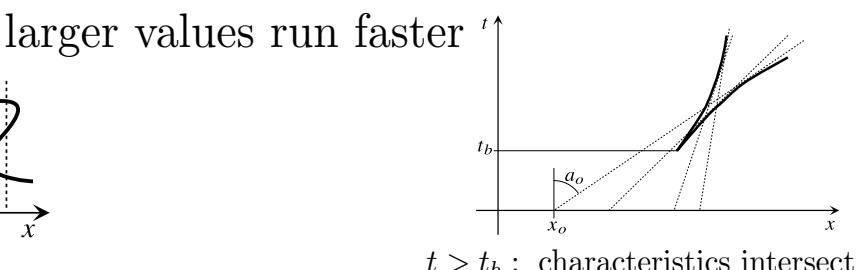
Speed increases with increasing u , $da/du > 0$



\Rightarrow formation of singularities after a finite time



$t > t_b$: multivalued solution. Wave breaking

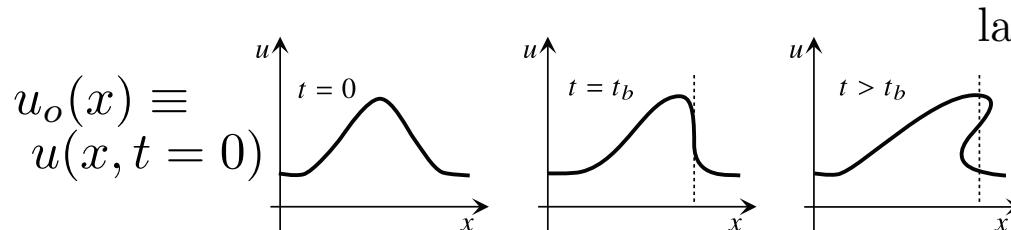
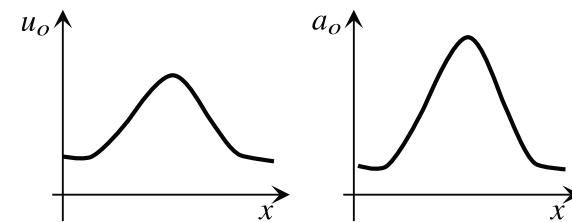


$t > t_b$: characteristics intersect

$$\partial u / \partial t + a(u) \partial u / \partial x = 0$$

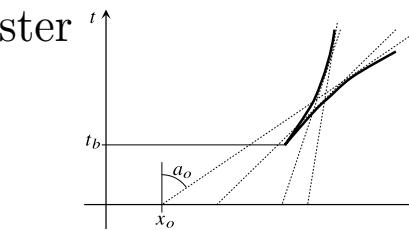
Speed increases with increasing u , $\frac{da}{du} > 0$

\Rightarrow formation of singularities after a finite time



$t > t_b$: multivalued solution. Wave breaking

larger values run faster



$t > t_b$: characteristics intersect

$$u(x, t) = u_o(x_o(x, t)), \quad x(x_o, t) = a_o(x_o)t + x_o,$$

trajectory

$$\frac{\partial x}{\partial x_o} = 1 + t(\frac{da_o}{dx_o}) \quad \frac{\partial x_o}{\partial x} = [1 + t(\frac{da_o}{dx_o})]^{-1}$$

$$\frac{\partial u}{\partial x} = \frac{\partial x_o}{\partial x} \frac{du_o}{dx_o} \quad \frac{\partial u}{\partial x} = \frac{\frac{du_o}{dx_o}}{[1 + t(\frac{da_o}{dx_o})]} \quad \text{diverges at time } t = \frac{1}{-\frac{da_o}{dx_o}} \quad \text{where } \frac{da_o}{dx_o} < 0$$

$t_b \equiv$ time of wave breaking (shortest time for the divergence of $\partial u / \partial x$)

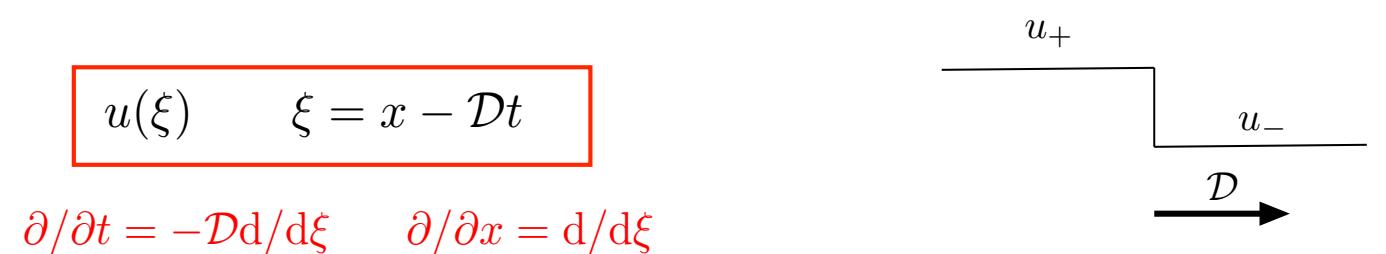
$$t_b = \frac{1}{\max |da_0/dx_0|}$$

Discontinuous solutions

$$\partial u / \partial t + a(u) \partial u / \partial x = 0 \quad \text{conservative form} \quad \partial u / \partial t + \partial j / \partial x = 0$$

$$j(u) \quad dj/du = a(u) \quad j(u) \equiv \int_{u_-}^u a(u') du'$$

Are step functions $u_+ \neq u_-$ propagating at constant velocity \mathcal{D} solutions ?



$$-\mathcal{D} \frac{du}{d\xi} + \frac{dj}{d\xi} = 0 \quad j - \mathcal{D}u \quad \text{is a conserved scalar}$$

$$j(u_+) - \mathcal{D}u_+ = j(u_-) - \mathcal{D}u_- \quad \mathcal{D} = \frac{j(u_+) - j(u_-)}{u_+ - u_-}$$

Infinite numbers of solutions !! Ill posed problem

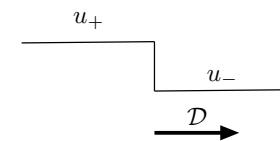
$$f(u) \times \left(-\mathcal{D} \frac{du}{d\xi} + \frac{dj}{d\xi} \right) = 0$$

$$G - \mathcal{D}F = \text{ cst.} \quad \mathcal{D} = \frac{G(u_+) - G(u_-)}{F(u_+) - F(u_-)} \quad \text{where} \quad \frac{dF}{du} \equiv f(u) \quad \frac{dG}{du} \equiv f(u)a(u)$$

Discontinuous solutions

$$\partial u / \partial t + a(u) \partial u / \partial x = 0$$

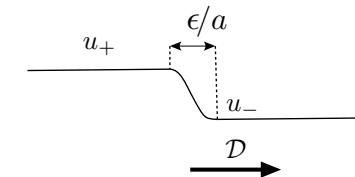
Are step functions $u_+ \neq u_-$ propagating at constant velocity \mathcal{D} solutions ?



$$u(\xi) \quad \xi = x - \mathcal{D}t$$

Infinite numbers of solutions !! Ill posed problem

adding a dissipative term makes the problem well posed



$$u(\xi) \quad \xi = x - \mathcal{D}t \quad \partial u / \partial t + a(u) \partial u / \partial x = \epsilon \partial^2 u / \partial x^2, \quad \epsilon > 0$$

$\partial / \partial t = -\mathcal{D}d/d\xi \quad \partial / \partial x = d/d\xi$

dimension $\epsilon = \text{length}^2/\text{time}$

$$\frac{d}{d\xi} \left[-u\mathcal{D} + j(u) - \epsilon \frac{du}{d\xi} \right] = 0 \quad \epsilon \frac{du}{d\xi} = j(u) - u\mathcal{D} + \text{cst} \quad \frac{\xi}{\epsilon} = \int_0^u \frac{du}{j(u) - u\mathcal{D} + \text{cst}} \quad j(u) \equiv \int_{u_-}^u a(u') du'$$

$\xi = \pm\infty : du/d\xi = 0 \Rightarrow$ 2 expressions of the cst that should be equal \Rightarrow a single value of \mathcal{D}

$$\mathcal{D} = \frac{j(u_+) - j(u_-)}{u_+ - u_-} \quad \text{independent of } \epsilon !$$

$$j(u_+) - u_+\mathcal{D} + \text{cst} = 0 \\ j(u_-) - u_-\mathcal{D} + \text{cst} = 0$$

$u(\xi)$ continuous function

$\lim_{\epsilon \rightarrow 0} u =$ step function

$a(u) = u$: Burgers equation. Analytical solution to the initial value problem

Riemann invariants (1860)

Euler equation + constant entropy + ideal gas: $a^2 = \frac{dp}{d\rho} = \gamma \frac{p}{\rho}$ $\frac{p}{\rho^\gamma} = \text{cst}$ $2 \frac{da}{a} = \frac{dp}{p} - \frac{d\rho}{\rho} \Rightarrow 2 \frac{da}{a} = (\gamma - 1) \frac{d\rho}{\rho}$

$$\lambda \times \frac{\partial}{\partial t} \rho + u \frac{\partial}{\partial x} \rho + \rho \frac{\partial}{\partial x} u = 0$$

$$\frac{a^2}{\rho} \frac{\partial}{\partial x} \rho + \frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u = 0$$

$$\lambda \frac{\partial}{\partial t} \rho + \left(\lambda u + \frac{a^2}{\rho} \right) \frac{\partial}{\partial x} \rho + \frac{\partial}{\partial t} u + (\lambda \rho + u) \frac{\partial}{\partial x} u = 0$$

choose λ such that $\lambda u + a^2/\rho = \lambda(\lambda\rho + u)$, $\Rightarrow \lambda = \pm a/\rho$



Riemann 1860

$$\lambda\rho + u = u \pm a$$

2 characteristics :

$$\lambda d\rho + du = 0$$

invariants :

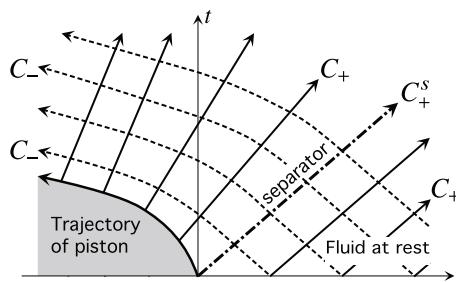
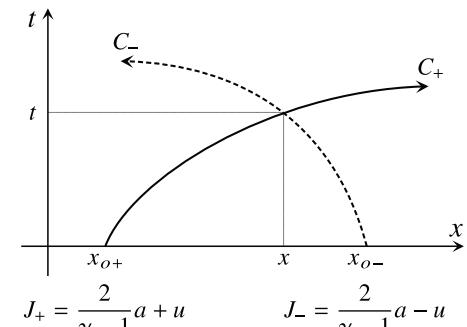
$$\pm \frac{a}{\rho} d\rho + du = 0$$

$$\lambda \left[\frac{\partial}{\partial t} + (\lambda \rho + u) \frac{\partial}{\partial x} \right] \rho + \left[\frac{\partial}{\partial t} + (\lambda \rho + u) \frac{\partial}{\partial x} \right] u = 0$$

$$C_+ : \frac{dx}{dt} = u + a, \quad C_- : \frac{dx}{dt} = u - a$$

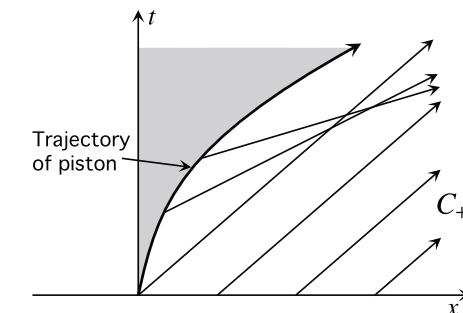
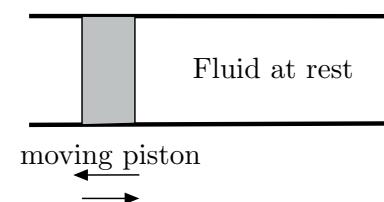
$$C_+ : \frac{a}{\rho} d\rho + du = 0 \quad C_- : \frac{a}{\rho} d\rho - du = 0$$

$$J_+ \equiv \frac{2}{\gamma - 1} a + u = \text{cst sur } C_+ \quad J_- \equiv \frac{2}{\gamma - 1} a - u = \text{cst sur } C_-$$



Rarefaction wave

Simple waves
 C_+ are straight lines



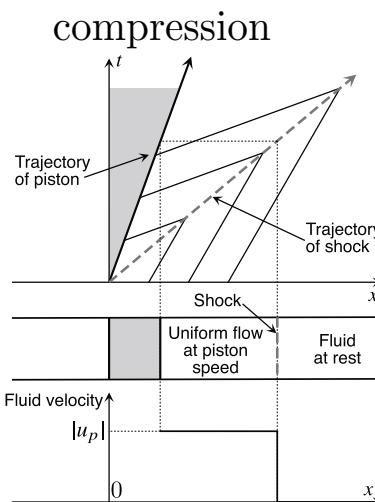
Compression wave
formation of a singularity: shock wave

Centred waves

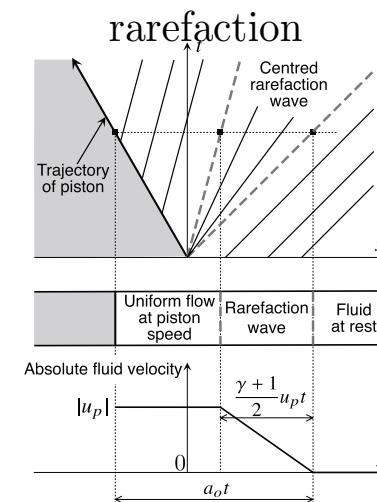
$t < 0$: piston velocity = 0

$t > 0$: piston velocity = cst $\neq 0$

Sel-similar solutions x/t



shock wave at constant velocity $>$ piston velocity



no discontinuity of u
discontinuity of du/dx propagating at the local sound speed

fully unsteady process: thickness $\nearrow u_p t$

JUMP CONDITIONS ACROSS A SHOCK WAVE

Shock waves



Rankine 1870



Hugoniot 1880

Rankine-Hugoniot conditions for shock waves (1870 – 1880)

Eqs for the conservation of mass, momentum and energy

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \quad \frac{\partial(\rho u)}{\partial t} = -\frac{\partial}{\partial x} \left(p + \rho u^2 - \mu \frac{\partial u}{\partial x} \right) \quad \frac{\partial(\rho e_{tot})}{\partial t} = -\frac{\partial}{\partial x} \left[\rho u(h + u^2/2) - \lambda \frac{\partial T}{\partial x} - \mu u \frac{\partial u}{\partial x} \right]$$

Initial state		Shocked gas
$\rho_u \quad p_u$		$\rho_N \quad p_N$
$\xrightarrow{\mathcal{D}}$	$\xrightarrow{u_N}$	$m \equiv \rho_u \mathcal{D} = \rho_N u_N \quad p_u + \frac{m^2}{\rho_u} = p_N + \frac{m^2}{\rho_N} \quad h_N - h_u + (u_N^2 - \mathcal{D}^2)/2 = 0$

written in the **moving frame** of the shock at velocity \mathcal{D}
steady problem

$$p_u - p_N = m^2 \left[\frac{1}{\rho_N} - \frac{1}{\rho_u} \right] \quad h_u - h_N = \frac{m^2}{2} \left[\frac{1}{\rho_N^2} - \frac{1}{\rho_u^2} \right]$$

$$h(\rho_u, p_u) - h(\rho_N, p_N) + \frac{1}{2} \left(\frac{1}{\rho_u} + \frac{1}{\rho_N} \right) (p_N - p_u) = 0$$

Hugoniot curve $(p - 1/\rho)$

$$h(p, \rho) - h(p_u, \rho_u) - \frac{1}{2} \left(\frac{1}{\rho_u} + \frac{1}{\rho} \right) (p - p_u) = 0$$

Michelson-Rayleigh line

$$p - p_u = -m^2 \left(\frac{1}{\rho} - \frac{1}{\rho_u} \right)$$

Ideal (polytropic) gas $\gamma \equiv c_p/c_v$

$$p = (c_p - c_v)\rho T, \quad h = c_p T = \frac{\gamma}{\gamma - 1} \frac{p}{\rho}$$

$$h(p, \rho) - h(p_u, \rho_u) - \frac{1}{2} \left(\frac{1}{\rho_u} + \frac{1}{\rho} \right) (p - p_u) = 0$$

$$\mathcal{P} \equiv \frac{(\gamma + 1)}{2\gamma} \left(\frac{p}{p_u} - 1 \right), \quad \mathcal{V} \equiv \frac{(\gamma + 1)}{2} \left(\frac{\rho_u}{\rho} - 1 \right)$$

$(\mathcal{P} + 1)(\mathcal{V} + 1) = 1$
Hugoniot curve

$\mathcal{P} = -M_u^2 \mathcal{V}$
Michelson-Rayleigh line

$p - p_u = -m^2 \left(\frac{1}{\rho} - \frac{1}{\rho_u} \right)$

quadratic equation for \mathcal{V} , 2 solutions: $\mathcal{V} = 0$, $\mathcal{V} = \mathcal{V}_N$

Shocked gas (Neumann state) vs M_u

$\frac{u_N}{\mathcal{D}} = \frac{\rho_u}{\rho_N} = \frac{(\gamma - 1)M_u^2 + 2}{(\gamma + 1)M_u^2}$	$\frac{p_N}{p_u} = \frac{2\gamma M_u^2 - (\gamma - 1)}{(\gamma + 1)}$	$\frac{T_N}{T_u} = \frac{[2\gamma M_u^2 - (\gamma - 1)][(\gamma - 1)M_u^2 + 2]}{(\gamma + 1)^2 M_u^2}$
$M_N^2 = \frac{(\gamma - 1)M_u^2 + 2}{2\gamma M_u^2 - (\gamma - 1)}$	$\Leftrightarrow 2\gamma M_u^2 M_N^2 - (\gamma - 1)(M_u^2 + M_N^2) - 2 = 0$	

Initial state
 $\rho_u \quad p_u$

$\xrightarrow{\mathcal{D}}$

Shocked gas
 $\rho_N \quad p_N$

$\xrightarrow{u_N}$

General comments

The Hugoniot curve is tangent to the isentropic

the entropy change along the Hugoniot curve is of third order $\Rightarrow M_u \equiv \mathcal{D}/a_u > 1$

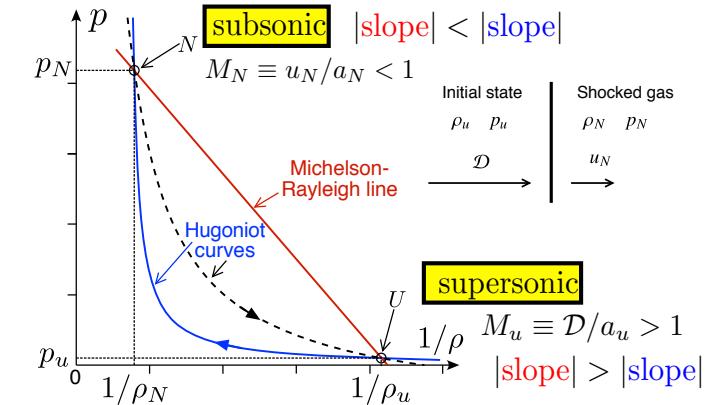
$$\delta s = s - s_u \quad \delta p = p - p_u \quad \delta s \nearrow \delta p \nearrow$$

$$\delta s = \frac{1}{12} \frac{1}{T_u} \left(\frac{\partial^2 (1/\rho)}{\partial p^2} \right)_s (\delta p)^3$$

The Hugoniot relation is not an iso-function of state

$$h(p, \rho) - h(p_u, \rho_u) - \frac{1}{2} \left(\frac{1}{\rho_u} + \frac{1}{\rho} \right) (p - p_u) = 0 \quad \text{cannot be written in the form} \quad \mathcal{H}(1/\rho, p) = \mathcal{H}(1/\rho_u, p_u)$$

$$h(p, \rho) - h(p_N, \rho_N) - \frac{1}{2} \left(\frac{1}{\rho_N} + \frac{1}{\rho} \right) (p - p_N) = 0 \neq h(p, \rho) - h(p_u, \rho_u) - \frac{1}{2} \left(\frac{1}{\rho_u} + \frac{1}{\rho} \right) (p - p_u) = 0$$



Rarefaction shock does not exist. The entropy of the fluid increases through the shock (Irreversibility)
 can be proved for weak shock by the entropy balance $\rho u \frac{ds}{dx} = \frac{d}{dx} \left(\frac{\lambda}{T} \frac{dT}{dx} \right) + \dot{\omega}_s \quad \dot{\omega}_s > 0$
 or by the H-theorem using the Boltzmann equation

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Formulation

Dimensional analysis

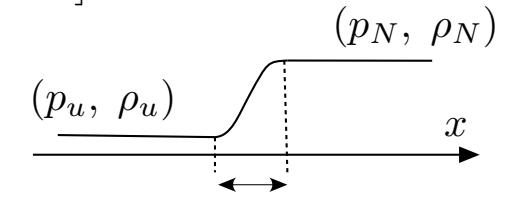
Analysis

10-3. ZND structure of detonations

10-4. Selection mechanism of the CJ wave

Inner structure of a weak shock

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \quad \frac{\partial(\rho u)}{\partial t} = -\frac{\partial}{\partial x} \left(p + \rho u^2 - \mu \frac{\partial u}{\partial x} \right) \quad \frac{\partial(\rho e_{tot})}{\partial t} = -\frac{\partial}{\partial x} \left[\rho u(h + u^2/2) - \lambda \frac{\partial T}{\partial x} - \mu u \frac{\partial u}{\partial x} \right]$$



Formulation

(reference frame attached to the shock wave)

$$\boxed{\rho u = m,} \quad p + \rho u^2 - \mu \frac{du}{dx} = \text{cst.} \quad m \left(h + \frac{u^2}{2} \right) - \lambda \frac{dT}{dx} - \mu u \frac{du}{dx} = \text{cst.}$$

$$x \rightarrow -\infty : \quad p = p_u, \quad \rho = \rho_u, \quad u = \mathcal{D} \quad \quad x \rightarrow \infty : \quad dp/dx = 0, \quad dp/dx = 0, \quad du/dx = 0$$

$$m = \rho_u \mathcal{D} \quad \frac{p}{\rho} = (\gamma - 1) c_v T \quad h = \frac{\gamma}{\gamma - 1} \frac{p}{\rho}$$

Two coupled equations for p and $v \equiv 1/\rho, m$ given ($u = mv, \quad c_v T = pv/(\gamma - 1)$)

$$(p - p_u) + m^2(v - v_u) = \mu m \frac{dv}{dx}, \quad \frac{\gamma}{\gamma - 1}(pv - p_u v_u) - \frac{1}{2}(p - p_u)(v + v_u) = \frac{\gamma}{\gamma - 1} \frac{\lambda}{mc_p} \frac{d(pv)}{dx} + \frac{\mu m}{2}(v - v_u) \frac{dv}{dx}$$

$\frac{1}{2}(u^2 - u_u^2) = \frac{m^2}{2}(v + v_u)(v - v_u)$

$x \rightarrow \infty : \quad dp/dx = 0, \quad dv/dx = 0$

$= \frac{1}{2}(v + v_u) \left[-(p - p_u) + \mu m \frac{dv}{dx} \right]$

Dimensional analysis

$$u/a \xrightarrow{\text{speed of sound}} O(1),$$

$$\mu/\rho \xrightarrow{\text{mean free path}} \ell/a$$

$$\mu/\rho = \text{viscous diffusion coefficient} \approx \ell/a \Rightarrow \text{thickness of shock waves} \approx \text{mean free path}$$

kinetic theory of gases

$$\cancel{\rho u^2} \quad \cancel{\mu \frac{du}{dx}}$$

macroscopic equations not valid ?

ok for weak shock !

$$M_u \equiv \mathcal{D}/a_u > 1$$

Analysis for

$$\epsilon \equiv M_u - 1 \ll 1$$

(weak shock)

$$\mathcal{D}/a_u = 1 + \epsilon$$

$$(p - p_u) + m^2(v - v_u) = \mu m \frac{dv}{dx}, \quad m = \rho_u \mathcal{D}$$

$$\frac{\gamma}{\gamma-1}(pv - p_u v_u) - \frac{1}{2}(p - p_u)(v + v_u) = \frac{\gamma}{\gamma-1} \frac{\lambda}{mc_p} \frac{d(pv)}{dx} + \frac{\mu m}{2} (v - v_u) \frac{dv}{dx}$$

$x \rightarrow -\infty : p = p_u, v = v_u \quad x \rightarrow \infty : dp/dx = 0, dv/dx = 0$

Non-dimensional equations

$$v \equiv 1/\rho \quad \nu \equiv (v - v_u)/v_u = O(\epsilon) \quad \pi \equiv (p - p_u)/p_u = O(\epsilon)$$

anticipating

$$\begin{cases} d\nu/d\xi = O(\epsilon^2) \\ d\pi/d\xi = O(\epsilon^2) \end{cases}$$

$$\lambda/mc_p = D_{Tu}/(\rho_u \mathcal{D}) \approx D_{Tu}/(\rho_u a_u) \quad \Pr \equiv \mu/(\rho_u D_{Tu})$$

mean free path

$$\xi \equiv x/\ell \quad \ell \equiv D_{Tu}/a_u$$

$$a_u = \sqrt{\gamma p_u / \rho_u}$$

$$M_u^2 = 1 + 2\epsilon + ..$$

$$\begin{cases} \frac{1}{\gamma}\pi + (1+2\epsilon)\nu = \Pr \frac{d\nu}{d\xi} + O(\epsilon^3), \\ \left(\frac{\gamma+1}{2\gamma}\right)\pi\nu + \frac{1}{\gamma}\pi + \nu = \frac{d\pi}{d\xi} + \frac{d\nu}{d\xi} + O(\epsilon^3), \end{cases} \quad \text{valid up to order } \epsilon^2$$

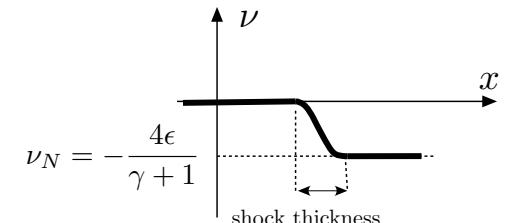
$$\frac{1}{\gamma}\pi + \nu = \Pr \frac{d\nu}{d\xi} - 2\epsilon\nu + O(\epsilon^3), \quad \Rightarrow \quad \left(\frac{\gamma+1}{2\gamma}\right)\pi\nu + \Pr \frac{d\nu}{d\xi} - 2\epsilon\nu = \frac{d\pi}{d\xi} + \frac{d\nu}{d\xi} + O(\epsilon^3),$$

$$\pi = -\gamma\nu + O(\epsilon^2), \quad \Rightarrow$$

$$[(\gamma-1) + \Pr] \frac{d\nu}{d\xi} = \left(\frac{\gamma+1}{2}\nu + 2\epsilon\right)\nu$$

Rankine-Hugoniot

$$\nu_N = -4\epsilon \frac{1}{\gamma+1} \quad \pi_N = -4\epsilon \frac{\gamma}{\gamma+1}$$

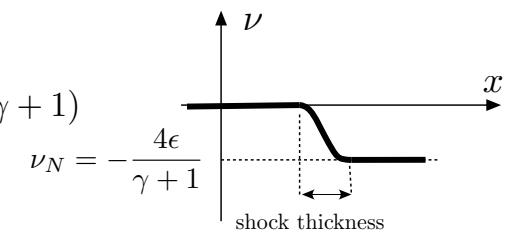


$M_u \equiv \mathcal{D}/a_u > 1$ **Analysis for** $\epsilon \equiv M_u - 1 \ll 1$ (weak shock)

$$[(\gamma - 1) + \text{Pr}] \frac{d\nu}{d\xi} = \left(\frac{\gamma + 1}{2} \nu + 2\epsilon \right) \nu$$

$$\frac{2}{\gamma + 1} [(\gamma - 1) + \text{Pr}] \frac{d\nu}{d\xi} = \nu(\nu - \nu_N) \leq 0$$

$\xi = -\infty$: initial state, $\nu = 0$, $\xi = +\infty$: shocked gas, $\nu = \nu_N = -4\epsilon/(\gamma + 1)$



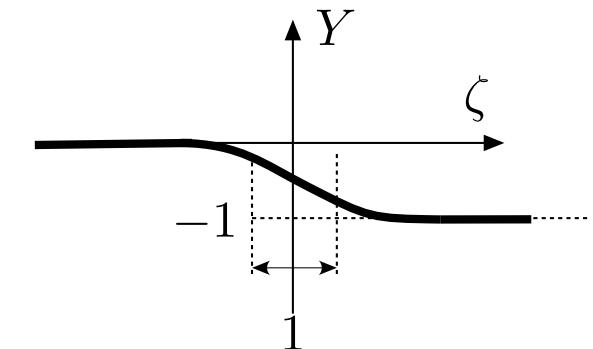
$$Y \equiv \frac{(\gamma + 1)}{4\epsilon} \nu \in [0, -1]$$

$$\zeta \equiv \frac{2}{[(\gamma + 1) + \text{Pr}]} \epsilon \xi = \frac{2}{[(\gamma + 1) + \text{Pr}]} \frac{x}{(\ell/\epsilon)}$$

$$\zeta = O\left(\frac{x}{\ell/\epsilon}\right)$$

$$\frac{dY}{d\zeta} = Y(Y + 1) < 0$$

$\zeta = -\infty$: $Y = 0$, $\zeta = +\infty$: $Y = -1$



$$\zeta = \int \frac{dY}{Y(Y + 1)}$$

$$Y(\zeta) = -\frac{e^\zeta}{e^\zeta + 1}$$

shock thickness = mean free path/ $(M_u - 1)$

microscopic length if $M_u - 1 = O(1)$

macroscopic length if $(M_u - 1) \ll 1$

$$x = O(l/\epsilon)$$

Lecture 10 : Supersonic waves

10-1. Background

Model of hyperbolic equations for the formation of discontinuity

Riemann invariants

Rankine-Hugoniot conditions for shock waves

Mikhelson (Chapman-Jouguet) conditions for detonations

10-2. Inner structure of a weak shock wave

Formulation

Dimensional analysis

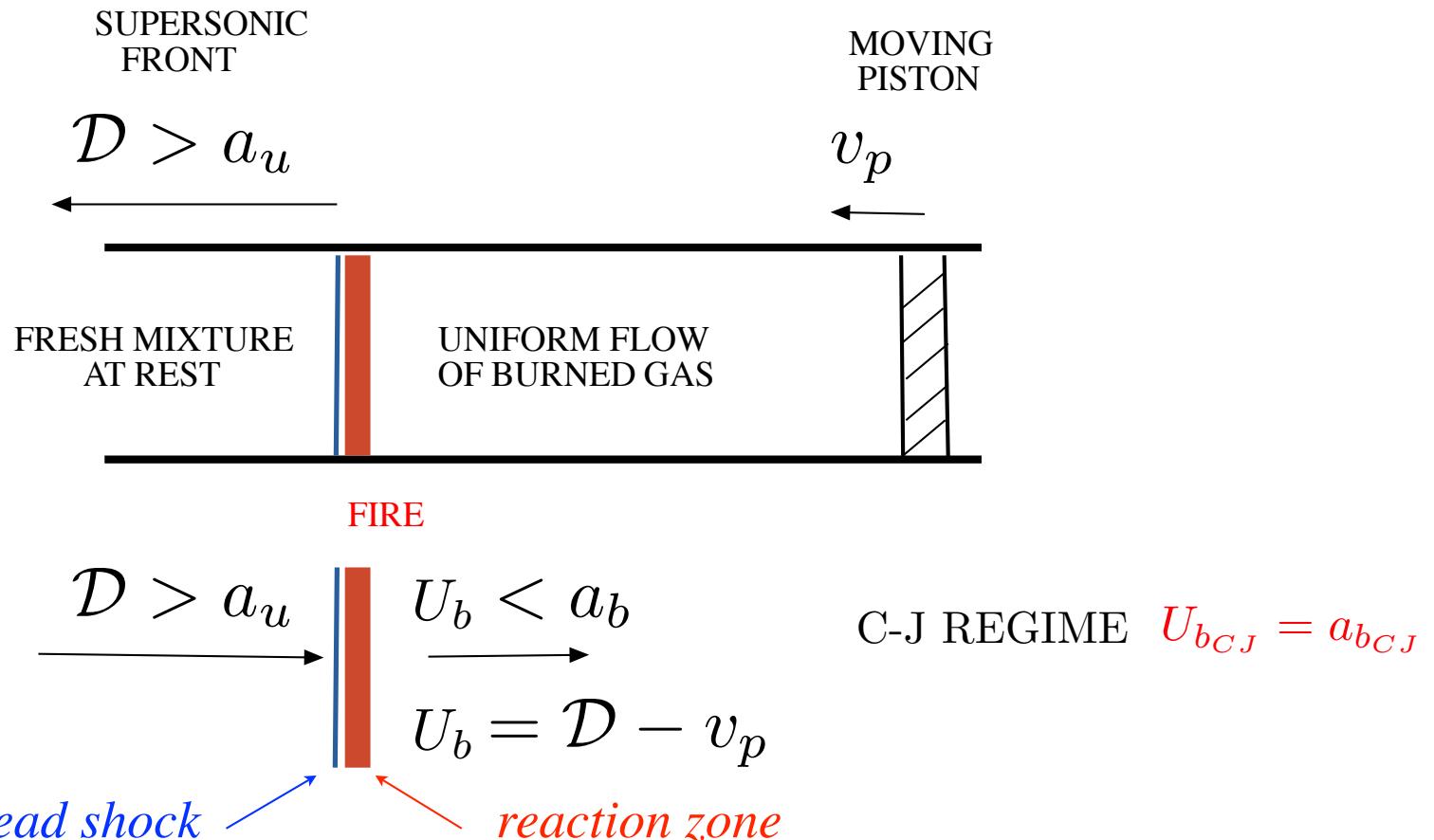
Analysis

10-3. ZND structure of detonations

10-4. Selection mechanism of the CJ wave

OVERDRIVEN DETONATION REACTING GAS

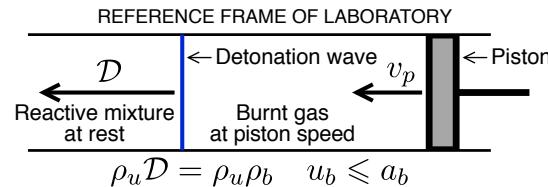
PISTON SUPPORTED SUPERSONIC WAVE



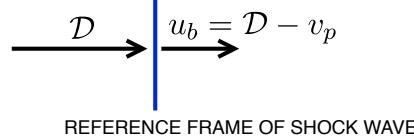
Abel 1870, Berthelot et Vielle 1881, Mallard et Le Chatelier 1881, **Mikhelson** 1893, Chapman 1899, Jouguet 1904,
Vielle 1900, **Zel'dovich** 1940, von Neumann 1942, Döring 1943,

JUMP ACROSS A DETONATION

Mikhelson condition for the CJ detonation (1893)



$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x}, \quad \frac{\partial(\rho u)}{\partial t} = -\frac{\partial}{\partial x} \left(p + \rho u^2 - \mu \frac{\partial u}{\partial x} \right), \quad \frac{\partial(\rho e_{tot})}{\partial t} = -\frac{\partial}{\partial x} \left[\rho u \left(c_p T + \frac{u^2}{2} - q_m \psi \right) - \lambda \frac{\partial T}{\partial x} - \mu u \frac{\partial u}{\partial x} \right]$$



$$\mathcal{P} \equiv \frac{(\gamma+1)}{2\gamma} \left(\frac{p}{p_u} - 1 \right), \quad \mathcal{V} \equiv \frac{(\gamma+1)}{2} \left(\frac{\rho_u}{\rho} - 1 \right) \quad \mathcal{Q} \equiv \frac{\gamma+1}{2} \frac{q_m}{c_p T_u}$$

$$\begin{aligned} (\mathcal{P} + 1)(\mathcal{V} + 1) &= 1 + \mathcal{Q} \\ \mathcal{P} &= -M_u^2 \mathcal{V} \end{aligned}$$

quadratic equation for \mathcal{V}

$$M_u^2 \mathcal{V}^2 + (M_u^2 - 1)\mathcal{V} + \mathcal{Q} = 0$$

supersonic combustion wave

$$M_u \equiv \mathcal{D}/a_u > 1$$

Lower bound of propagation velocity $\mathcal{D} = \mathcal{D}_{CJ}$
(called Chapman-Jouguet 1899 – 1904)

$$(M_u^2 - 1)^2 \geqslant 4\mathcal{Q}M_u^2$$

$$M_u \geqslant M_{u_{CJ}} \equiv \sqrt{\mathcal{Q}} + \sqrt{\mathcal{Q} + 1},$$

Mikhelson (1893)

In the CJ wave the velocity of the burned gas is sonic in the frame of the wave $u_{bCJ} = a_{bCJ}$ (self-sustained wave)
(Rayleigh line is tangent)

In the overdriven detonations $\mathcal{D} > \mathcal{D}_{CJ}$ the velocity of the burned gas is subsonic in the frame of the wave $u_b < a_b$
(piston-supported detonation)

Vieille conjecture (1900)

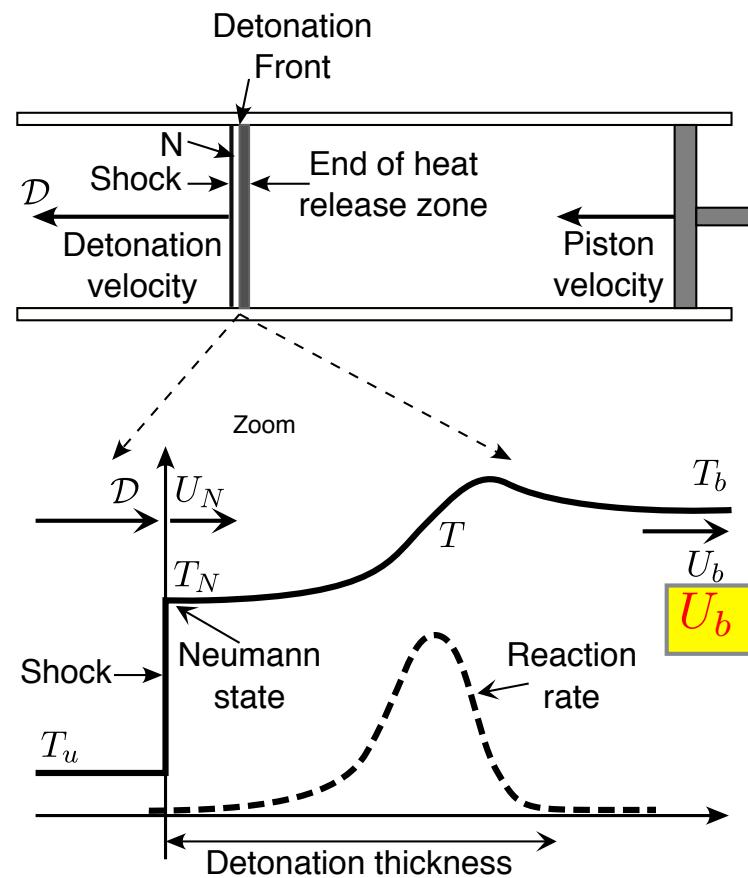
detonation = inert shock wave followed by a exothermal reaction zone

$U \rightarrow N \rightarrow B$
(large Arrhenius factor)



Paul Vieille (1900)

Planar detonation



thickness \approx few mms

Arrhenius law $e^{-E/k_B T}$

$$\text{reaction rate} \approx \frac{e^{-E/k_B T}}{\tau_{\text{coll}}} \quad \frac{E}{k_B T} \approx 10$$

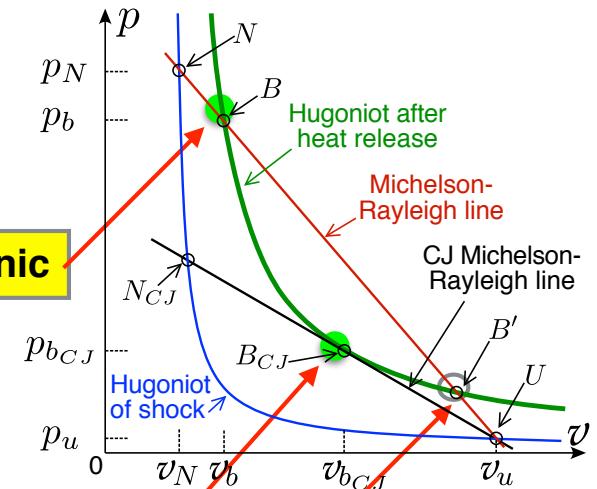
elastic collisions

Overdriven regime / Self sustained wave

Marginal solution

the so called Chapman-Jouguet wave

Mikhel'son
(1893)



1910)
non physical
under ordinary conditions

Sonic condition

C-J detonation

ZND structure of detonations

Zeldovich (1940) Neumann (1942) Döring (1944)



Zeldovich

Combustion = large activation energy

Orders of magnitude

$$\frac{E}{k_B T_N} \gg 1 \Rightarrow \frac{1}{\tau_r(T_N)} \approx \frac{e^{-E/k_B T_N}}{\tau_{coll}} \ll \frac{1}{\tau_{coll}} \quad \frac{u_N}{a_N} = O(1), \quad d_N \equiv u_N \tau_r(N) \gg a_N \tau_{coll} \approx \ell$$

thickness of the reaction zone \gg thickness of the lead (inert) shock

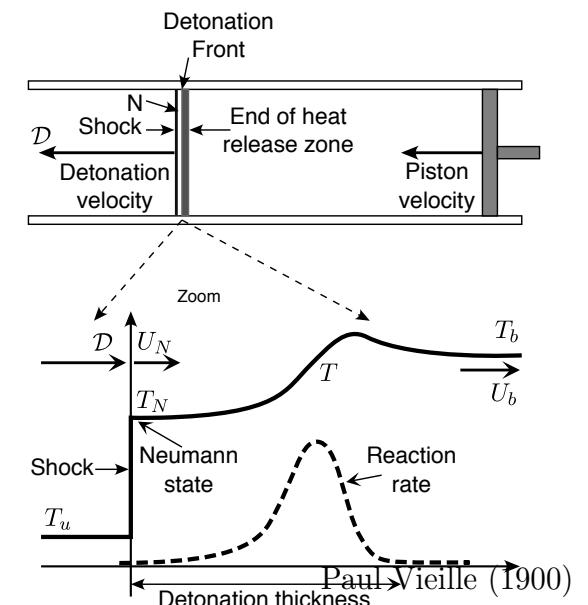
structure of the detonation: inert shock followed by a much larger reaction zone



conjectured by Vieille (1900)

$$\frac{D_T}{d_N^2} \approx \frac{a_N^2 \tau_{coll}}{d_N^2} \approx \frac{\tau_{coll}}{(\tau_r(T_N))^2} \approx \frac{e^{-E/k_B T_N}}{\tau_r(T_N)} \ll \frac{1}{\tau_r(T_N)}$$

diffusion rate \ll reaction rate \Rightarrow diffusion terms are negligible



ZND structure of detonations

Zeldovich (1940) Neumann (1942) Döring (1944)



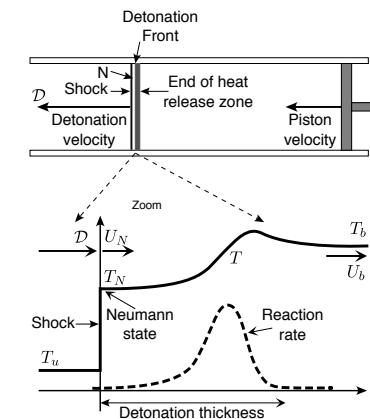
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$$\frac{D_T}{d_N^2} \approx \frac{a_N^2 \tau_{coll}}{d_N^2} \approx \frac{\tau_{coll}}{(\tau_r(T_N))^2} \approx \frac{e^{-E/k_B T_N}}{\tau_r(T_N)} \ll \frac{1}{\tau_r(T_N)}$$

diffusion rate \ll reaction rate \Rightarrow diffusion terms are negligible

Formulation

Reference frame of the lead shock ($x = 0$)



$$\begin{aligned} \frac{d(\rho u)}{dx} &= 0 & \frac{dp}{dx} + \rho u \frac{du}{dx} &= 0 \\ \frac{\gamma}{\gamma - 1} \frac{d}{dx} \left(\frac{p}{\rho} \right) + u \frac{du}{dx} - q_m \frac{d\psi}{dx} &= 0 & \rho u \frac{d\psi}{dx} &= \rho \frac{\dot{w}(T, \psi)}{\tau_r(T_N)} \end{aligned}$$

$$\begin{aligned} \psi \in [0, 1] \quad \dot{w}(T, \psi = 1) &= 1 \\ \dot{w}(T, \psi = 0) &= 0. \end{aligned}$$

$$x = 0 : \quad u = u_N, \quad \rho = \rho_N, \quad p = p_N, \quad \psi = 1, \quad \dot{w} = 1$$

$$x \rightarrow \infty : \quad u = u_b, \quad \rho = \rho_b, \quad p = p_b, \quad \psi = 0, \quad \dot{w} = 0$$

detonation thickness $d_N = u_N \tau_r(T_N)$

$$a^2 = \gamma \frac{p}{\rho} \quad \text{Elimination of } p \text{ and } \rho$$

$$\frac{d}{dx} \left(\frac{p}{\rho} \right) = p \frac{d}{dx} \left(\frac{1}{\rho} \right) + \frac{1}{\rho} \frac{dp}{dx} = \frac{p}{\rho u} \frac{du}{dx} + \frac{1}{\rho} \frac{dp}{dx} = \frac{a^2}{\gamma u} \frac{du}{dx} - u \frac{du}{dx} \quad \Rightarrow \quad \frac{\gamma}{\gamma - 1} \frac{d}{dx} \left(\frac{p}{\rho} \right) + u \frac{du}{dx} = \frac{1}{(\gamma - 1)u} (a^2 - u^2) \frac{du}{dx}$$

$$(a^2 - u^2) \frac{du}{dx} = (\gamma - 1) q_m u \frac{d\psi}{dx}, \quad \Rightarrow \quad \boxed{\frac{du^2}{d\psi} = 2(\gamma - 1) q_m \frac{u^2}{(a^2 - u^2)}}$$

Phase portrait in the plan $\psi - u^2$

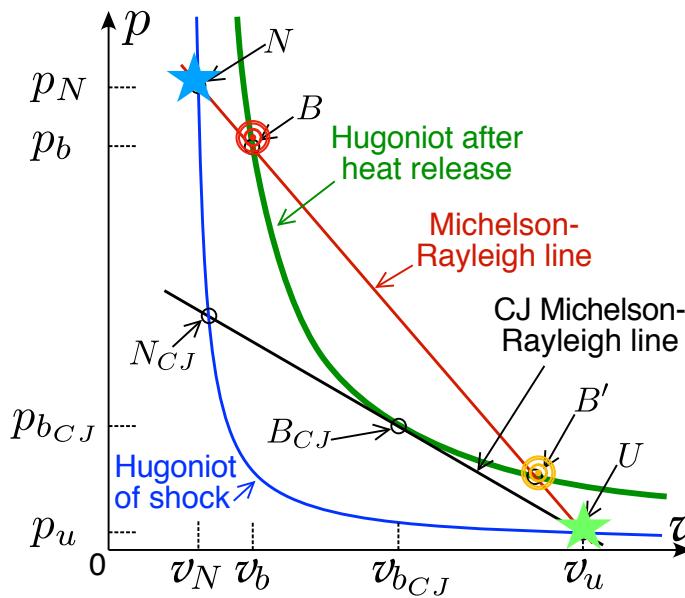
$$\frac{du^2}{d\psi} = 2(\gamma - 1)q_m \frac{u^2}{(a^2 - u^2)}$$

Initial state $\psi = 0 : u^2 = \mathcal{D}^2, a^2 = a_u^2$ Neumann state $\psi = 0 : u^2 = u_N^2, a^2 = a_N^2$

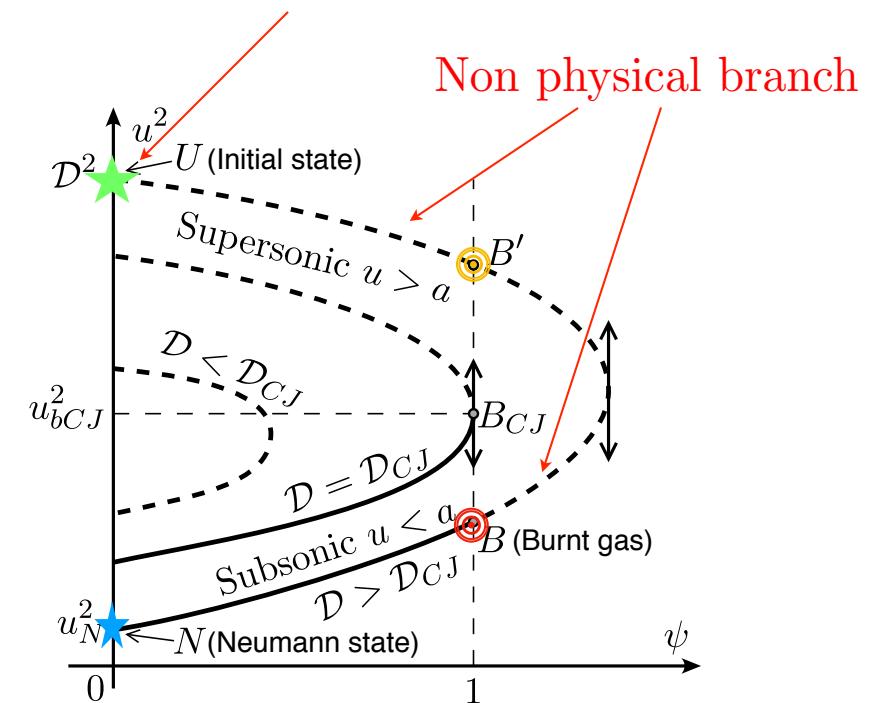
energy eq.

$$\frac{1}{\gamma - 1}a^2 + \frac{1}{2}u^2 - q_m\psi = \frac{1}{\gamma - 1}a_u^2 + \frac{1}{2}\mathcal{D}^2$$

$C_p T$



zero reaction rate
no supersonic wave without a leading shock



OK with the Vieille's conjecture

Lecture 10 : Supersonic waves

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Formulation

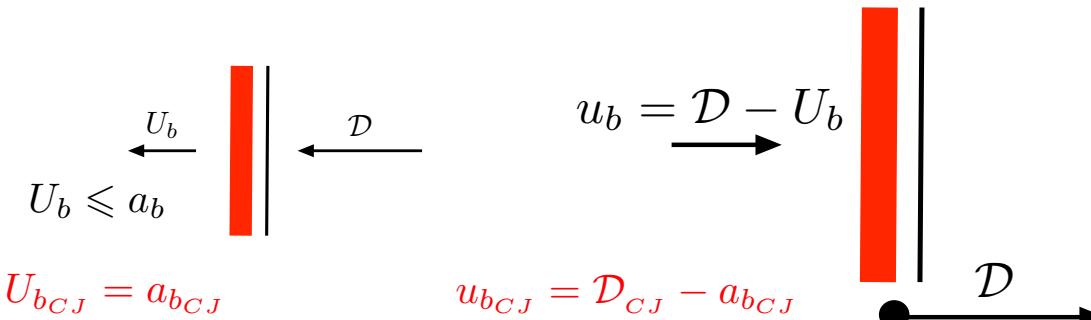
Dimensional analysis

Analysis

10-3. ZND structure of detonations

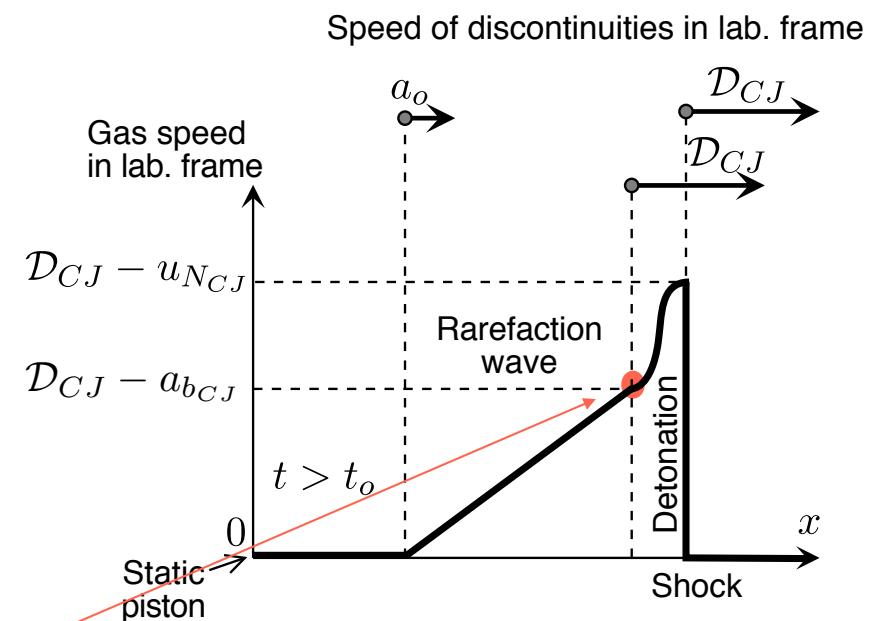
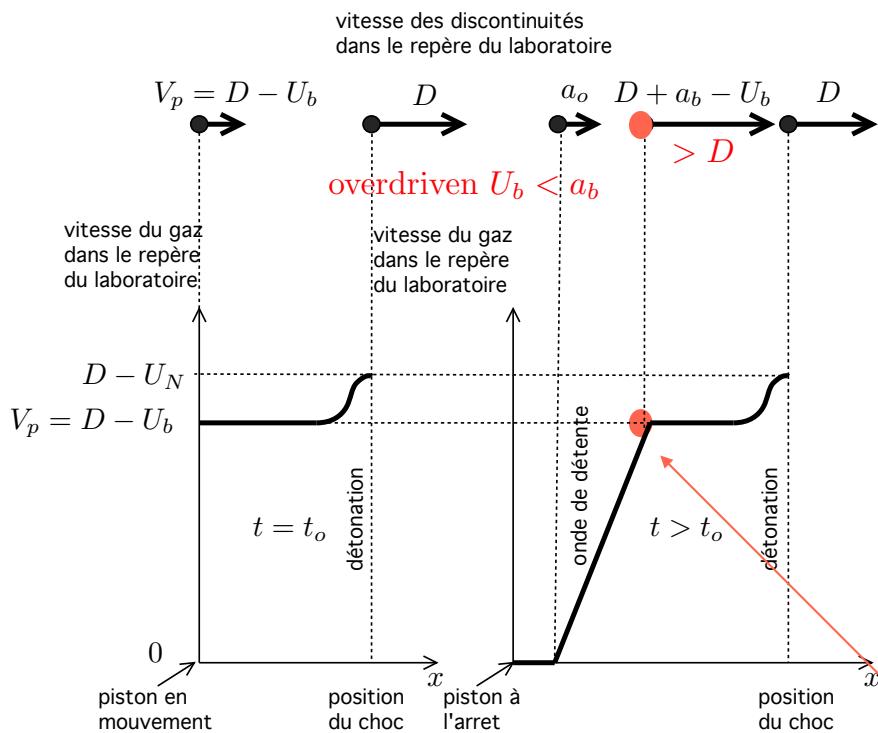
10-4. Selection mechanism of the CJ wave

C-J Detonation = self-propagating wave

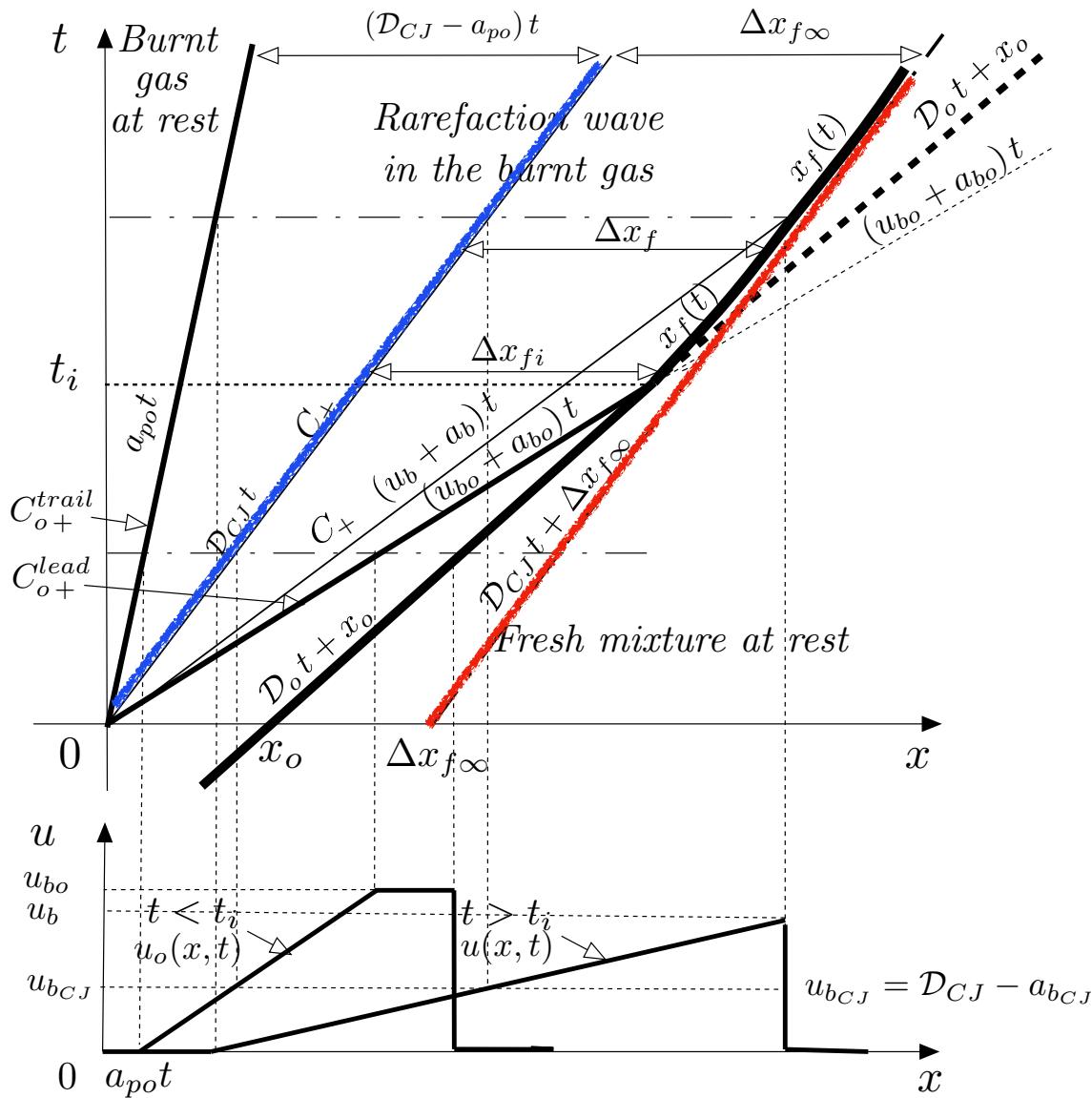


Selection mechanism

Rarefaction wave in the burnt gas when the piston is suddenly stopped



leading edge of the rarefaction wave = speed of sound /gas flow



$$\sqrt{\frac{dx_f}{dt} - \mathcal{D}_{CJ}} \propto \frac{x_f}{t}$$

$$\frac{\mathcal{D}_{CJ}}{x_f(t) - \mathcal{D}_{CJ}t} \propto \frac{1}{t} + \text{constant}$$

$$\lim_{t \rightarrow \infty} x_f(t) = \mathcal{D}_{CJ}t + \text{constant}$$