

*Princeton Summer School on Combustion*

*June 20 - June 26, 2021*

# **Lectures on Dynamics of Combustion Waves in Premixed Gases**

Professor Paul Clavin

Aix-Marseille Université  
ECM & CNRS (IRPHE)

## **Lecture XIII Stability analysis of shock waves**

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# Lecture 13 : Stability analysis of shock waves

## 13-1. Acoustic waves and entropy-vorticity wave

*Linearized Euler equations*

*Linearized flow field*

## 13-2. Analyses

*Dispersion relation for general materials*

*Classification of normal modes*

*Spontaneous emission of sound and instability*

*Stability of shocks in ideal gases*

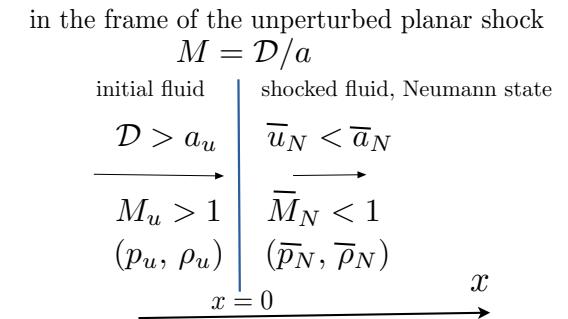
*Stability of reacting shocks*

# Acoustic waves and entropy-vorticity wave

Shock wave  $\approx$  hydrodynamic discontinuity + Rankine-Hugoniot conditions

thickness  $\approx$  mean free path

The flow of shocked gas in a planar wave is uniform



$\mathcal{D} > a_u \Rightarrow$  the upstream flow in a wrinkled shock is not perturbed  
 supersonic wave

Flow velocity  $\bar{u}_N$  is sufficiently large  $\Rightarrow$  the diffusive fluxes are negligible:

compressed gas:

$$Ds/Dt = 0 \Leftrightarrow \partial s/\partial t + \underline{u} \cdot \nabla s = 0$$

no entropy production in the compressed gas

The entropy of shocked gas is modified at the Neumann state of a wrinkled shock

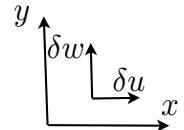
$$\Rightarrow \nabla s(\mathbf{r}, t) \neq 0$$

entropy production inside the shock thickness: entropy jump !

# Linearized Euler equations

(written in 2-D for simplicity. Extension to 2-D is straightforward)

$$u = \bar{u}_N + \delta u, \quad w = \delta w, \quad \rho = \bar{\rho}_N + \delta \rho, \quad p = \bar{p}_N + \delta p$$



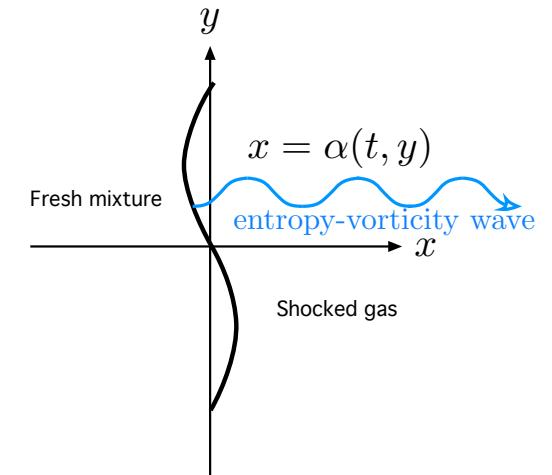
*Compressed gas*

$$\frac{1}{\bar{\rho}_N} \frac{D}{Dt} \delta \rho + \frac{\partial}{\partial x} \delta u + \frac{\partial}{\partial y} \delta w = 0,$$

$$D/Dt = \partial/\partial t + \bar{u}_N \partial/\partial x \quad \bar{\rho}_N \frac{D}{Dt} \delta u = -\frac{\partial}{\partial x} \delta p, \quad \bar{\rho}_N \frac{D}{Dt} \delta w = -\frac{\partial}{\partial y} \delta p,$$

$$\partial p / \partial \rho|_{s=\text{cst}} \equiv a \quad \frac{D}{Dt} \delta s = 0 \Rightarrow \frac{D}{Dt} \delta p = \bar{a}_N^2 \frac{D}{Dt} \delta \rho,$$

isentropic: no entropy production but propagation of entropy from the shock



*Wave equation for the pressure (d'Alembert equation)*

$$\text{eliminating } \delta \rho \quad \Rightarrow \quad \frac{1}{\bar{\rho}_N \bar{a}_N^2} \frac{D}{Dt} \delta p + \frac{\partial}{\partial x} \delta u + \frac{\partial}{\partial y} \delta w = 0$$

$$\text{eliminating } \delta u \text{ and } \delta w \quad \Rightarrow \quad \boxed{\frac{D^2}{Dt^2} \delta p - \bar{a}_N^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \delta p = 0}$$

the pressure fluctuations are fully propagated by acoustic waves in the shocked gas moving at constant velocity  $\bar{u}_N$

# Linearized flow field

*Flow splitting*

acoustic wave + vorticity wave

$$\delta p = \delta p^{(a)}, \quad \boxed{\begin{aligned} \delta u &= \delta u^{(a)} + \delta u^{(i)} \\ \delta w &= \delta w^{(a)} + \delta w^{(i)} \end{aligned}}$$

$$\bar{\rho}_N \left[ \frac{\partial}{\partial t} + \underline{u}_N \cdot \nabla \right] \underline{u}^{(a)} = -\underline{\nabla} p$$

the pressure fluctuations are fully propagated by acoustic waves in the shocked gas moving at constant velocity  $\bar{u}_N$

*Wave equation for the pressure (d'Alembert equation)*

eliminating  $\delta\rho$   $\Rightarrow \frac{1}{\bar{\rho}_N \bar{a}_N^2} \frac{D}{Dt} \delta p + \frac{\partial}{\partial x} \delta u^{(a)} + \frac{\partial}{\partial y} \delta w^{(a)} = 0$

eliminating  $\delta u$  and  $\delta w$   $\Rightarrow \boxed{\frac{D^2}{Dt^2} \delta p - \bar{a}_N^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \delta p = 0}$

*Entropy-vorticity wave (isobaric)*

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{u}_N \frac{\partial}{\partial x}$$

$$D\delta u^{(i)}/Dt = 0$$

$$D\delta w^{(i)}/Dt = 0$$

$\Leftrightarrow$

$$\frac{\partial \delta u^{(i)}}{\partial t} + \bar{u}_N \frac{\partial \delta u^{(i)}}{\partial x} = 0$$

$$\frac{\partial \delta w^{(i)}}{\partial t} + \bar{u}_N \frac{\partial \delta w^{(i)}}{\partial x} = 0$$

$$\delta u^{(i)}(y, t - x/\bar{u}_N)$$

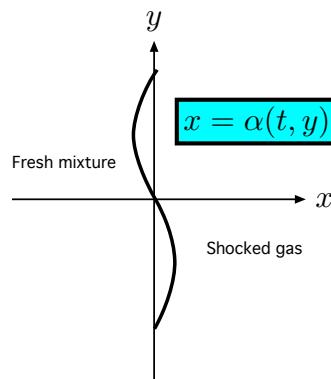
$$\delta w^{(i)}(y, t - x/\bar{u}_N)$$

$$\frac{1}{\bar{\rho}_N \bar{a}_N^2} \cancel{\frac{D}{Dt}} \delta p + \frac{\partial}{\partial x} \delta u^{(i)} + \frac{\partial}{\partial y} \delta w^{(i)} = 0$$

$$\Rightarrow \frac{\partial}{\partial x} \delta u^{(i)} + \frac{\partial}{\partial y} \delta w^{(i)} = 0 \Rightarrow \boxed{\frac{\partial}{\partial t} \delta u^{(i)} = \bar{u}_N \frac{\partial}{\partial y} \delta w^{(i)}}$$

# Linearized flow field

Normal-mode analysis



$$\alpha(y, t) = \hat{\alpha} e^{iky + \sigma t}$$

$$\delta f(x, y, t) = \tilde{f}(x) e^{iky + \sigma t}$$

$k \in \text{Re}$   
 $\sigma \in \text{Z}$  σ(k) ?

$$\delta p = \tilde{p}_N \exp(i l_{\pm} x + i k y + \sigma t)$$

2nd-order algebraic eq.

$$l_{\pm}(\sigma, k) \quad (\sigma + i l_{\pm} \bar{u}_N)^2 + \bar{a}_N^2 (l_{\pm}^2 + k^2) = 0 \iff \frac{D}{Dt^2} \delta p - \bar{a}_N^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \delta p = 0 \quad \text{wave eq.}$$

No length-scale other than  $|k|^{-1}$  in the problem

$$i \frac{l_{\pm}}{|k|} = \frac{\bar{M}_N S \pm \sqrt{1 + S^2}}{\sqrt{1 - \bar{M}_N^2}} \quad \text{with} \quad S \equiv \frac{\sigma}{\bar{a}_N |k|} \frac{1}{\sqrt{1 - \bar{M}_N^2}},$$

$$\begin{aligned} \bar{\rho}_N \frac{D}{Dt} \delta u &= -\frac{\partial}{\partial x} \delta p, & \tilde{u}^{(a)} &= -\frac{i l_{\pm} \bar{u}_N}{\sigma + i l_{\pm} \bar{u}_N} \frac{\tilde{p}_N}{\bar{\rho}_N \bar{u}_N} e^{i l_{\pm} x}, & x = 0 : \delta p &= \delta p_N(y, t) = \tilde{p}_N e^{i k y + \sigma t} \\ \bar{\rho}_N \frac{D}{Dt} \delta w &= -\frac{\partial}{\partial y} \delta p, & \tilde{w}^{(a)} &= -\frac{i k \bar{u}_N}{\sigma + i l_{\pm} \bar{u}_N} \frac{\tilde{p}_N}{\bar{\rho}_N \bar{u}_N} e^{i l_{\pm} x}, & & \text{RH conditions} \end{aligned}$$

$$\begin{aligned} \tilde{u}^{(i)} &= \left[ \tilde{u}_N + \frac{i l_{\pm} \bar{u}_N}{\sigma + i l_{\pm} \bar{u}_N} \frac{\tilde{p}_N}{\bar{\rho}_N \bar{u}_N} \right] e^{-\sigma x / \bar{u}_N}, \\ \tilde{w}^{(i)} &= \left[ \tilde{w}_N + \frac{i k \bar{u}_N}{\sigma + i l_{\pm} \bar{u}_N} \frac{\tilde{p}_N}{\bar{\rho}_N \bar{u}_N} \right] e^{-\sigma x / \bar{u}_N}, \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \delta u^{(i)}(y, t - x / \bar{u}_N) \\ \delta w^{(i)}(y, t - x / \bar{u}_N) \end{aligned}$$

Rankine-Hugoniot conditions

Incompressibility condition

$$\frac{\partial}{\partial t} \delta u^{(i)} = \bar{u}_N \frac{\partial}{\partial y} \delta w^{(i)}$$

$\Rightarrow$

$$-\frac{(i l_{\pm} \sigma + \bar{u}_N k^2)}{(\sigma + i l_{\pm} \bar{u}_N)} \frac{\tilde{p}_N}{\bar{\rho}_N \bar{u}_N} - \sigma \frac{\tilde{u}_N}{\bar{u}_N} + i k \tilde{w}_N = 0$$

# Lecture 13 : Stability analysis of shock waves

## 13-1. Acoustic waves and entropy-vorticity wave

*Linearized Euler equations*

*Linearized flow field*

## 13-2. Analyses

*Dispersion relation for general materials*

*Classification of normal modes*

*Spontaneous emission of sound and instability*

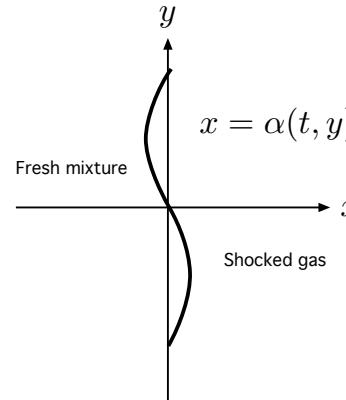
*Stability of shocks in ideal gases*

*Stability of reacting shocks*

# Dispersion relation for general materials

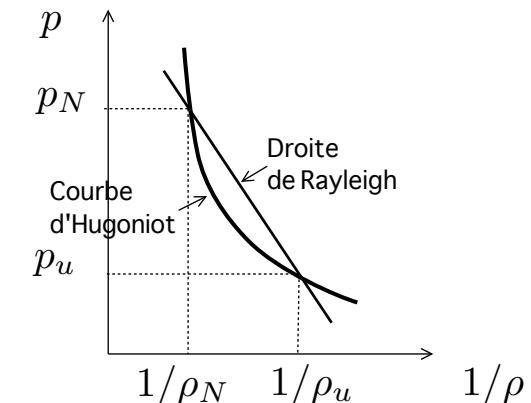
*Rankine Hugoniot relations (general material)*

$$\begin{array}{l|l} M = \mathcal{D}/a & \\ \text{initial fluid} & \text{shocked fluid, Neumann state} \\ \hline \mathcal{D} > a_u & u_N < a_N \\ \hline M_u > 1 & M_N < 1 \\ (p_u, \rho_u) & (p_N, \rho_N) \end{array}$$



$$\alpha(t, y) = \hat{\alpha} e^{\sigma t + iky}$$

2 parameters for the material:  $r$  and  $n$



non-dimensional parameters

$$r \equiv -\frac{(\rho_u \mathcal{D})^2}{d\bar{p}_N/d\bar{\rho}_N^{-1}} > 0,$$

$$\bar{M}_N \leftrightarrow n \equiv \frac{\bar{\rho}_N}{\rho_u} \frac{\bar{M}_N^2}{(1 - \bar{M}_N^2)}$$

$$\delta p_N = \frac{1}{r} \left( \frac{\rho_u}{\bar{\rho}_N} \right)^2 \bar{\mathcal{D}}^2 \delta \rho_N$$

Linear rate      Quadratic equation for  $\sigma^2/\bar{a}_N^2 k^2$        $\Leftarrow$        $\pm 2\bar{M}_N S \sqrt{1 + S^2} = (1 + r)S^2 + (1 - r)n$

$$aS^4 + 2bS^2 + c = 0,$$

$$S^2 \equiv \frac{\sigma^2}{\bar{a}_N^2 k^2} \frac{1}{(1 - M_N^2)}$$

$$a \equiv (1 + r)^2 - 4\bar{M}_N^2, \quad b \equiv (1 - r^2)n - 2\bar{M}_N^2, \quad c \equiv (1 - r)^2 n^2 > 0.$$

# Classification of normal modes

$$\alpha(t, y) = \hat{\alpha} e^{\sigma t + iky}$$

$$aS^4 + 2bS^2 + c = 0,$$

$$S^2 \equiv \frac{\sigma^2}{\bar{a}_N^2 k^2} \frac{1}{(1 - M_N^2)}$$

$$a \equiv (1+r)^2 - 4\bar{M}_N^2, \quad b \equiv (1-r^2)n - 2\bar{M}_N^2, \quad c \equiv (1-r)^2 n^2 > 0.$$

$\text{Re}(\sigma) < 0$  : stable mode exponentially damped

$$e^{-|\text{Re}(\sigma)|t}$$

$\text{Re}(\sigma) > 0$  : unstable mode exponentially amplified

$$e^{|\text{Re}(\sigma)|t}$$

$S^2 < 0$  :  $\text{Re}(\sigma) = 0, \quad \omega \equiv \text{Im}(\sigma) \neq 0$  neutral **oscillatory** modes

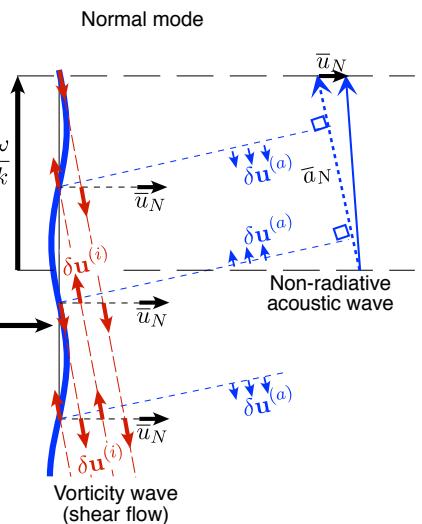
longitudinal component of the velocity (unperturbed shock) of the **sound wave**:

$$\mathbf{e}_x \cdot (\bar{u}_N \mathbf{e}_x - \bar{a}_N \mathbf{e}_K) = \bar{u}_N - \bar{a}_N \frac{l}{\sqrt{l^2 + k^2}}$$

Neutral **oscillatory** modes

Spontaneous generation of sound. **Radiating** condition:  $\bar{u}_N \sqrt{l^2 + k^2} - \bar{a}_N l > 0$

**Non-radiating** condition:  $\bar{u}_N \sqrt{l^2 + k^2} - \bar{a}_N l < 0$



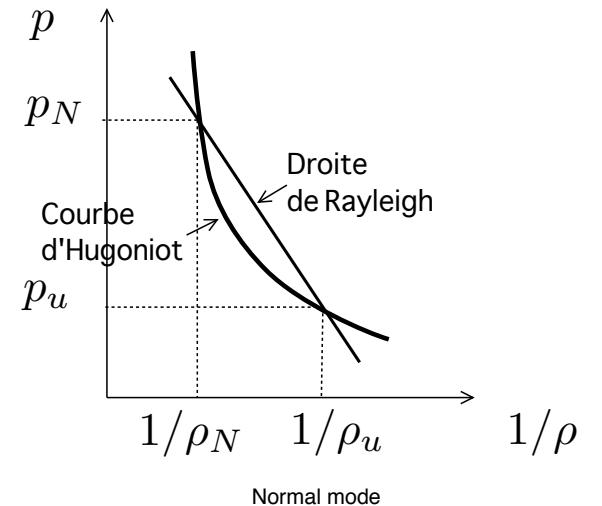
unstable  $t^n$ ,  $n > 0$   
stable  $1/t^n$

# Classification of normal modes

2 non-dimensional parameters

$$r \equiv -\frac{(\rho_u \bar{D})^2}{d\bar{p}_N/d\bar{\rho}_N^{-1}} > 0,$$

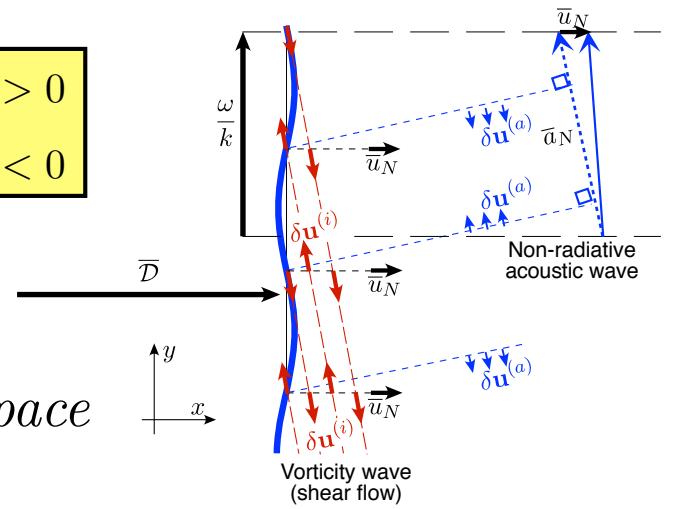
$$n \equiv \frac{\bar{\rho}_N}{\rho_u} \frac{\bar{M}_N^2}{(1 - \bar{M}_N^2)}$$



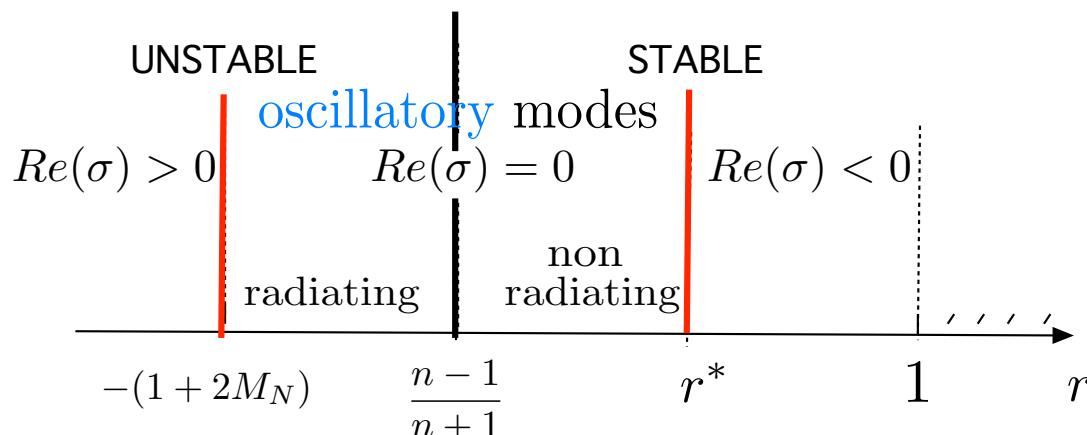
Neutral oscillatory modes

Spontaneous generation of sound. Radiating condition:  $\bar{u}_N \sqrt{l^2 + k^2} - \bar{a}_N l > 0$

Non-radiating condition:  $\bar{u}_N \sqrt{l^2 + k^2} - \bar{a}_N l < 0$



*Classification of the normal modes in the parameters space*



$$r^* = \frac{n - \sqrt{(1 - M_N^2) \left( 1 - \frac{\rho_u}{\rho_N} \right)}}{n + 1}$$

# Stability of shocks in ideal gases

Polytropic gas,  $\gamma = \text{cst.}$

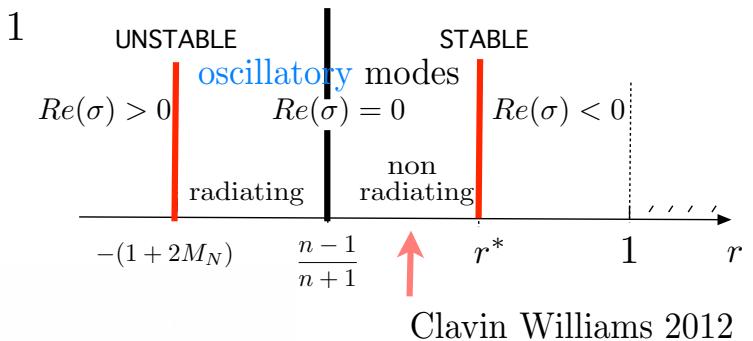
$$\frac{u_N}{\mathcal{D}} = \frac{\rho_u}{\rho_N} = \frac{(\gamma - 1)M_u^2 + 2}{(\gamma + 1)M_u^2}, \quad \frac{p_N}{p_u} = \frac{2\gamma M_u^2 - (\gamma - 1)}{(\gamma + 1)}, \quad M_N^2 = \frac{(\gamma - 1)M_u^2 + 2}{2\gamma M_u^2 - (\gamma - 1)}$$

$$r \equiv -\frac{(\rho_u \overline{\mathcal{D}})^2}{d\bar{p}_N/d\bar{\rho}_N^{-1}} = \frac{1}{\overline{M}_u^2}, \quad n \equiv \frac{\bar{\rho}_N}{\rho_u} \frac{\overline{M}_N^2}{(1 - \overline{M}_N^2)} = \frac{\overline{M}_u^2}{(\overline{M}_u^2 - 1)}$$

$$\pm 2\overline{M}_N S \sqrt{1 + S^2} = (1 + r)S^2 + (1 - r)n, \Rightarrow \pm 2S\overline{M}_N \sqrt{1 + S^2} = S^2 \left( 1 + \overline{M}_u^{-2} \right) + 1$$

$$\overline{M}_u > 1, \quad \gamma > 1 \quad \Rightarrow \quad (n - 1)/(n + 1) < r < r^*$$

$$r^* = \frac{n - \sqrt{(1 - M_N^2)(1 - \frac{\rho_u}{\rho_N})}}{n + 1} \quad \frac{1}{2\overline{M}_u^2 - 1} < \frac{1}{\overline{M}_u^2} < \frac{\overline{M}_u^2 - (\overline{M}_u^2 - 1)^2 \sqrt{2\overline{M}_u^{-2}[2\gamma\overline{M}_u^2 - (\gamma - 1)]^{-1}}}{2\overline{M}_u^2 - 1}$$



Shock waves in polytropic gases have neutral modes with non-radiating acoustic waves

They are stable with a relaxation of initial disturbances in power laws  $1/t^{3/2}$

OK with experiments

Freeman 1957 Lapworth 1959

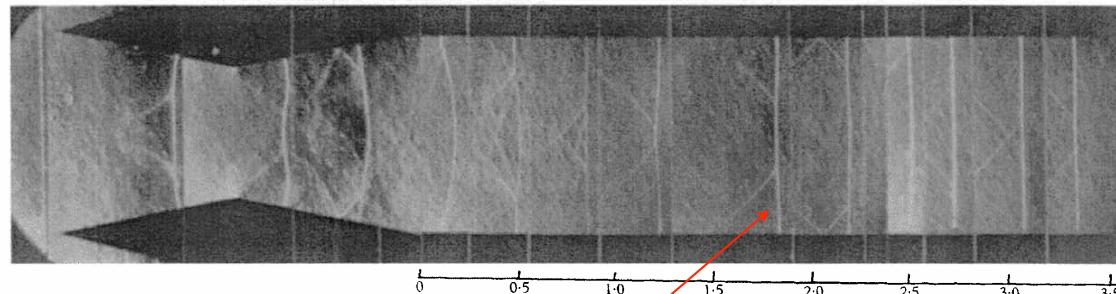


FIGURE 6 (plate 1). Photographs of shock wave at  $M_s = 1.41$ .



Formation of Mach stems  
(see next lecture)

# Stability of reacting shocks

Clavin Williams 2009 2013

Reacting shocks = detonations considered as an hydrodynamic discontinuity

thickness = 0: no modification of the inner structure

$$r \equiv -\frac{\rho_u \mathcal{D}^2}{dp_b/d\rho_b^{-1}} = \bar{M}_u^2 \frac{1 + \bar{\mathcal{V}}_b}{1 - \bar{M}_u^2 \bar{\mathcal{V}}_b} \quad \bar{\mathcal{V}}_b = [\bar{M}_u^{-2} - (1 + \chi)]/2 \quad \chi \equiv \sqrt{(1 - \bar{M}_u^{-2})^2 - 4\mathcal{Q}\bar{M}_u^{-2}} \quad r = \frac{(1 - \chi) + \frac{1}{\bar{M}_u^2}}{(1 + \chi) + \frac{1}{\bar{M}_u^2}}$$

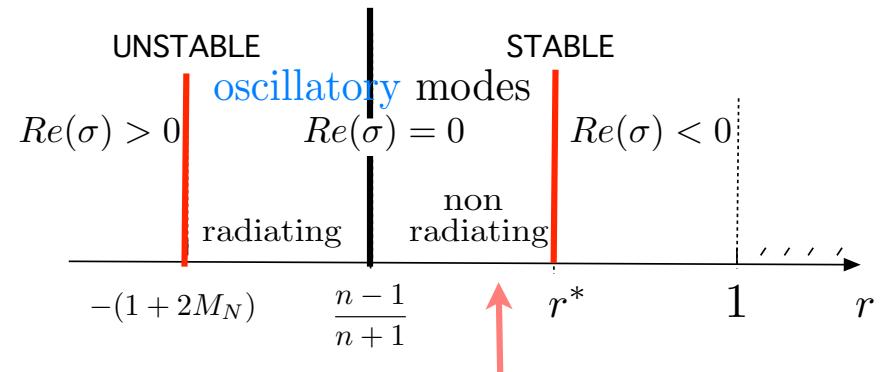
$\xrightarrow[0 \leqslant \chi < 1]{\text{CJ wave}}$   
overdriven regime

$$\chi = \sqrt{(1 - \bar{M}_u^{-2})^2 - (1 - \bar{M}_{u_{CJ}}^{-2})^2 \bar{M}_{u_{CJ}}^2 \bar{M}_u^{-2}}$$

$$n \equiv \left(\frac{\bar{\rho}_b}{\rho_u}\right) \frac{\bar{M}_b^2}{1 - \bar{M}_b^2} = \frac{1}{\bar{M}_u^{-2} - 1 - 2\bar{\mathcal{V}}_b} = \frac{1}{\chi}$$

$$r = \frac{(1 - \chi) + \frac{1}{\bar{M}_u^2}}{(1 + \chi) + \frac{1}{\bar{M}_u^2}} = \left(\frac{n-1}{n+1}\right) \left[ \frac{1 + \frac{1}{\bar{M}_u^2(1-\chi)}}{1 + \frac{1}{\bar{M}_u^2(1+\chi)}} \right]$$

$$\boxed{\left(\frac{n-1}{n+1}\right) \leqslant r}$$



Clavin Williams 2009  $\neq$  Majda Rosales 1983

Overdriven reacting shocks in polytropic gases have neutral modes with non-radiating acoustic waves

They are **stable** with a relaxation of initial disturbances in **power laws**

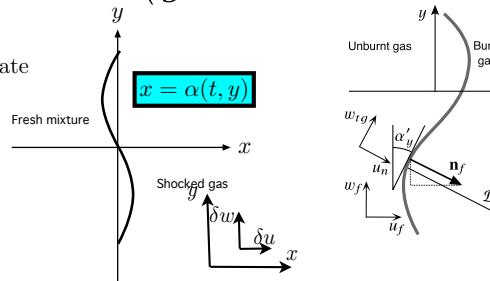
For the CJ marginal regime the acoustic waves in the burned gas propagate in the direction parallel to the unperturbed planar solution

Details of the calculation

## (details of the calculation)

## Rankine Hugoniot relations (general material)

$$\begin{array}{l|l} M = \mathcal{D}/a & \text{shocked fluid, Neumann state} \\ \text{initial fluid} & \\ \mathcal{D} > a_u & u_N < a_N \\ \hline M_u > 1 & M_N < 1 \\ (p_u, \rho_u) & (p_N, \rho_N) \end{array}$$



non-dimensional parameters

$$r \equiv -\frac{(\rho_u \mathcal{D})^2}{d\bar{p}_N/d\bar{\rho}_N^{-1}} > 0,$$

$$\bar{M}_N \leftrightarrow n \equiv \frac{\bar{\rho}_N}{\rho_u} \frac{\bar{M}_N^2}{(1 - \bar{M}_N^2)}$$

$$\delta p_N = \frac{1}{r} \left( \frac{\rho_u}{\bar{\rho}_N} \right)^2 \bar{\mathcal{D}}^2 \delta \rho_N$$

Jump conditions p.5 lecture IV

mass  $\rho_N(u_N - \partial\alpha/\partial t - w_N \partial\alpha/\partial y) = \rho_u(\bar{\mathcal{D}} - \partial\alpha/\partial t)$   $\Rightarrow \delta\rho_N \bar{u}_N + \bar{\rho}_N(\delta u_N - \partial\alpha/\partial t) = -\rho_u \partial\alpha/\partial t$

tangential momentum  $w_N = (\bar{\mathcal{D}} - u_N)\alpha'_y$   $\Rightarrow \delta w_N = (\bar{\mathcal{D}} - \bar{u}_N)\partial\alpha/\partial y$

tangent to the Hugoniot curve (geometrical construction)

longitudinal momentum  $p_N - p_u = -m^2 \left( \frac{1}{\rho_N} - \frac{1}{\rho_u} \right)$   $\Rightarrow \frac{\delta p_N}{\bar{p}_N} = \frac{2}{\bar{m}} \left( 1 - \frac{p_u}{\bar{p}_N} \right) \rho_u \frac{\partial\alpha}{\partial t} + \frac{\bar{m}^2}{\bar{p}_N \bar{\rho}_N} \frac{\delta \rho_N}{\bar{\rho}_N}$

$$\left. \begin{aligned} \Rightarrow \frac{\delta p_N}{\bar{p}_N} &= -2 \frac{\left( 1 - \frac{p_u}{\bar{p}_N} \right)}{(1-r)} \frac{\partial\alpha/\partial t}{\bar{\mathcal{D}}}, & \frac{\delta \rho_N}{\bar{\rho}_N} &= -2 \left( \frac{\bar{\rho}_N}{\rho_u} - 1 \right) \frac{r}{(1-r)} \frac{\partial\alpha/\partial t}{\bar{\mathcal{D}}} \\ \frac{\delta u_N}{\bar{u}_N} &= \left( \frac{\bar{\rho}_N}{\rho_u} - 1 \right) \frac{1+r}{(1-r)} \frac{\partial\alpha/\partial t}{\bar{\mathcal{D}}}, & \frac{\delta w_N}{\bar{u}_N} &= \left( \frac{\bar{\rho}_N}{\rho_u} - 1 \right) \frac{\partial\alpha}{\partial y} \end{aligned} \right\}$$

Linear rate

$$\pm \sqrt{S^2 + 1} \frac{\tilde{p}_N}{\bar{\rho}_N \bar{a}_N \bar{u}_N} = -S \frac{\tilde{u}_N}{\bar{u}_N} + \frac{ik\tilde{w}_N}{|k|\bar{u}_N} \frac{\bar{M}_N}{\sqrt{1 - \bar{M}_N^2}}$$

$$\alpha(y, t) = \hat{\alpha} e^{iky + \sigma t} \quad \frac{\partial\alpha}{\partial t} = \sigma\alpha \quad \frac{\partial\alpha}{\partial y} = ik\alpha$$

$$\Rightarrow \pm 2\bar{M}_N S \sqrt{1 + S^2} = (1+r)S^2 + (1-r)n$$

$$S \equiv \frac{\sigma}{\bar{a}_N |k|} \frac{1}{\sqrt{1 - \bar{M}_N^2}}$$

taking the square

$$aS^4 + 2bS^2 + c = 0,$$

$$S^2 \equiv \frac{\sigma^2}{\bar{a}_N^2 k^2} \frac{1}{(1 - \bar{M}_N^2)}$$

$$a \equiv (1+r)^2 - 4\bar{M}_N^2, \quad b \equiv (1-r^2)n - 2\bar{M}_N^2, \quad c \equiv (1-r)^2 n^2 > 0.$$

Quadratic equation for  $\sigma^2/\bar{a}_N^2 k^2$

(details of the calculation)

# Dispersion relation for general materials

*Compatibility condition*

$$\left( \frac{\partial}{\partial t} + \bar{u}_N \frac{\partial}{\partial x} \right)^2 \delta p - \bar{a}_N^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \delta p = 0$$

$$-\frac{(il_{\pm}\sigma + \bar{u}_N k^2)}{(\sigma + il_{\pm}\bar{u}_N)} \frac{\tilde{p}_N}{\bar{\rho}_N \bar{u}_N} - \sigma \frac{\tilde{u}_N}{\bar{u}_N} + ik\tilde{w}_N = 0$$

$$(\sigma + il_{\pm}\bar{u}_N)^2 + \bar{a}_N^2(l_{\pm}^2 + k^2) = 0 \Rightarrow \sigma^2 + 2i\sigma l_{\pm}\bar{u}_N - l_{\pm}^2 \bar{u}^2 + \bar{a}_N^2(l_{\pm}^2 + k^2) = 0$$

$$i \frac{l_{\pm}}{|k|} = \frac{\bar{M}_N S \pm \sqrt{1+S^2}}{\sqrt{1-\bar{M}_N^2}}$$

$$\text{with } S \equiv \frac{\sigma}{\bar{a}_N |k|} \frac{1}{\sqrt{1-\bar{M}_N^2}}$$

$$-(il_{\pm}\bar{u}_N \sigma + \bar{u}_N^2 k^2) = \sigma^2 + il_{\pm}\sigma \bar{u}_N + (\bar{a}_N^2 - \bar{u}_N^2)(l_{\pm}^2 + k^2)$$

$$\begin{aligned} &= (\sigma + il_{\pm}\bar{u}_N) \left[ \sigma - \left( 1 - \bar{M}_N^2 \right) (\sigma + il_{\pm}\bar{u}_N) \right] \\ &= (\sigma + il_{\pm}\bar{u}_N) \left[ \sigma \bar{M}_N^2 - \left( 1 - \bar{M}_N^2 \right) il_{\pm}\bar{u}_N \right] \end{aligned}$$

$$-(il_{\pm}\bar{u}_N \sigma + \bar{u}_N^2 k^2) = -(\sigma + il_{\pm}\bar{u}_N) \sqrt{1 - \bar{M}_N^2} [\pm \sqrt{1+S^2}] |k| \bar{u}_N$$

$$-\frac{(il_{\pm}\sigma + \bar{u}_N k^2)}{(\sigma + il_{\pm}\bar{u}_N)} \frac{\tilde{p}_N}{\bar{\rho}_N \bar{u}_N} - \sigma \frac{\tilde{u}_N}{\bar{u}_N} + ik\tilde{w}_N = 0 \Rightarrow -\sqrt{1 - \bar{M}_N^2} [\pm \sqrt{1+S^2}] |k| \bar{u}_N \frac{\tilde{p}_N}{\bar{\rho}_N \bar{u}_N} - \sigma \frac{\tilde{u}_N}{\bar{u}_N} + ik\tilde{w}_N = 0$$

$$\pm \sqrt{S^2 + 1} \frac{\tilde{p}_N}{\bar{\rho}_N \bar{a}_N \bar{u}_N} + S \frac{\tilde{u}_N}{\bar{u}_N} - \frac{ik\tilde{w}_N}{|k| \bar{u}_N} \frac{\bar{M}_N}{\sqrt{1 - \bar{M}_N^2}} = 0$$

(Buckmaster Ludford 1988. Clavin et al. 1997)

The Rankine Hugoniot relations yields an equation for  $S \propto \frac{\sigma}{\bar{a}_N |k|}$

$$\frac{\delta p_N}{\bar{p}_N} \propto \frac{\dot{\alpha}_t}{\bar{u}_N}, \quad \frac{\tilde{p}_N}{\bar{p}_N} \propto i\sigma \frac{\hat{\alpha}}{\bar{u}_N}$$

$$\frac{\delta u_N}{\bar{u}_N} \propto \frac{\dot{\alpha}_t}{\bar{u}_N}, \quad \frac{\tilde{u}_N}{\bar{u}_N} \propto i\sigma \frac{\hat{\alpha}}{\bar{u}_N}$$

$$\frac{\delta w_N}{\bar{u}_N} \propto \alpha'_y, \quad \frac{\tilde{w}_N}{\bar{u}_N} \propto ik\hat{\alpha}$$

*Downstream boundary condition*

$x \rightarrow \infty :$  bounded condition (in the unstable case,  $\operatorname{Re}\sigma > 0$ )

$$\delta p = \tilde{p}_N \exp(il_{\pm}x + iky + \sigma t)$$

selection of the the sign in  $l_{\pm}$  such that  $e^{il_{\pm}x}$  does not diverge

# Spontaneous emission of sound and instability

Yakov Kontorovich 1954-57

## Oscillatory neutral modes

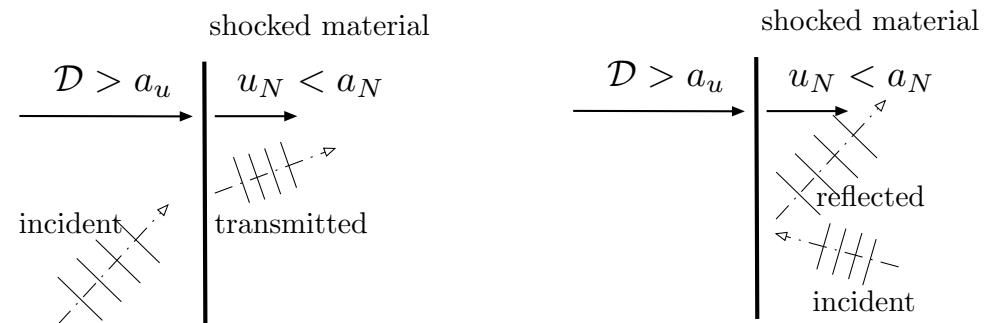
$$\pm 2\bar{M}_N S \sqrt{1+S^2} = (1+r)S^2 + (1-r)n$$

neutral oscillatory mode

$$S = i\Omega, \quad \Omega > 1$$

$$\Rightarrow \begin{cases} \text{radiating waves: } l/|k| = [\bar{M}_N \Omega - \sqrt{\Omega^2 - 1}] / \sqrt{1 - \bar{M}_N^2}, / \sqrt{1 - \bar{M}_N^2}, \\ \quad 2\bar{M}_N \Omega \sqrt{\Omega^2 - 1} = -\Omega^2 (1+r) + (1-r)n > 0, \\ \text{non-radiating waves: } l/|k| = [\bar{M}_N \Omega + \sqrt{\Omega^2 - 1}] / \sqrt{1 - \bar{M}_N^2}, \\ \quad -2\bar{M}_N \Omega \sqrt{\Omega^2 - 1} = -\Omega^2 (1+r) + (1-r)n < 0 \end{cases}$$

*Transmitted or reflected sound wave*



If a normal mode is radiating the response of the shock diverges when the reflected (or transmitted) waves matched the radiating normal mode

$$\frac{\text{reflected}}{\text{incident}} \rightarrow \frac{\tilde{p}_r}{\tilde{p}_i} = -\frac{[2\bar{M}_N \Omega \sqrt{\Omega^2 - 1} - \Omega^2 (1+r) + (1-r)n]}{[-2\bar{M}_N \Omega \sqrt{\Omega^2 - 1} - \Omega^2 (1+r) + (1-r)n]} \leftarrow \text{denominator goes through 0}$$

A neutral oscillatory mode that is radiating is considered as unstable

D'Yakov Kontorovich 1954-57

*Power laws of neutral modes*

Square root in the dispersion relation  $\pm 2\bar{M}_N \Omega \sqrt{\Omega^2 - 1} - \Omega^2 (1+r) + (1-r)n = 0 \Rightarrow$  cut in the complex plane  $\Rightarrow$  power laws  
Bates 2007

Damping ( $n < 0$ ) or amplification ( $n > 0$ ) involving power laws  $t^n$   
can be described by Laplace transform not by Fourier transform

Neutral modes with non-radiating acoustic waves relax following a power law in time  $t^n \quad n < 0$

Neutral modes with a radiating acoustic wave is unstable according a power law in time  $t^n \quad n > 0$