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On the long-run fluctuations of inheritance in two-sector OLG models[☆]

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Abstract

This paper provides a long-run cycle perspective to explain the behavior of the annual flow of inheritance. Based on the low- and medium frequency properties of long time bequests series in Sweden, France, UK, and Germany, we explore the extent to which a two-sector Barro-type OLG model is consistent with such empirical regularities. As long as agents are sufficiently impatient and preferences are non-separable, we show that endogenous fluctuations are likely to occur through two mechanisms, which can generate independently or together either period-2 cycles or Hopf bifurcations. The first mechanism relies on the elasticity of intertemporal substitution or equivalently the sign of the cross-derivative of the utility function whereas the second rests on sectoral technologies through the sign of the capital intensity difference across two sectors. Furthermore, building on the quasi-palindromic nature of the degree-4 characteristic equation, we derive some meaningful sufficient conditions associated to the occurrence of complex roots and a Hopf bifurcation in a two-sector OLG model.

JEL classification: C62, E32, O41

Keywords: Two-sector overlapping generations model Altruism; Bequest; Endogenous fluctuations; Quasi-palindromic polynomial; Periodic and quasi-periodic cycles

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¹ Pierre Cartigny was my Ph.D. advisor. I met him first when I was a student and we became friends during my PhD. We have had deep and intense discussions about multisector optimal growth models. He has been a wonderful teacher and co-author. I will greatly miss him. Carine Nourry was a very close friend. I met her in 1994 as a student during her Master and we started to work jointly during her PhD. I remember as if it was yesterday how we discussed

1. Introduction

In a very influential contribution, [Piketty \(2011\)](#) shows that inherited wealth has again a prominent role for life-cycle income, especially with respect to human capital and labor income. One explanation is based on the so-called $r - g$ theory of [Piketty \(2011\)](#). Especially, the gap between the (steady-state) growth rate g and the rate of return on private wealth r might explain the empirical dynamics of bequests.² Based on the extensive data work

complex dynamics in optimal growth models with altruistic and non-altruistic agents in our shared office in Marseille. She was a distinguished scholar and a wonderful co-author. I will also greatly miss my dear friend.

² Using a simple generic overlapping generations model of wealth accumulation, growth, and inheritance, [Piketty \(2011\)](#) argues that the gap between the (steady-state) growth rate g and the rate of return on private wealth r is the core argument of this resurgence. On the one hand, when g is large and $g > r$, the new wealth accumulated out of current income, and thus human capital, contributes more to life-cycle income than past inherited wealth, especially when it is grounded on low (past) income relative to today’s income. Accordingly, inheritance flows remain a small fraction of national income. On the other hand, when growth is low such that $r > g$, inherited wealth is capitalized at a faster (growth) rate than national income and becomes dominant with respect to current income. Consequently, inheritance flows become a larger

of Piketty (2011), particularly with long-dated administrative tax records, two key empirical features are the very pronounced U-shaped pattern of the inheritance (relative to income or wealth) variable in France (Piketty, 2011), Germany (Shinke, 2012), UK (Atkinson, 2018), Sweden (Ohlsson et al., 2020), and Switzerland (Brühlhart et al., 2018) and the fact that the annual bequest flow is slowly recovering its level after the First World War.³

In this paper, we show that the dynamics of the inheritance flow can be consistent with the predictions of long-run (stochastic) limit cycles and that a Barro-type bequests model can generate such endogenous fluctuations.⁴ Notably the economy can converge to a stable (long-run) cyclical path where some macroeconomic variables oscillate indefinitely around the steady-state, and thus bequest flows can go back and forth to a low, steady or high level. Moreover, when examined over long periods, historical series of inheritance flows (Piketty, 2011) look part of a long cycle through its U-shaped pattern, especially when extracting its medium to long-run component.⁵ In a broader perspective, this descriptive fact is to be reconciled with recent papers (e.g., Beaudry et al. (2015, 2017, 2020) and Growiec et al. (2015, 2018)) that challenge the seminal contributions of Granger (1966) and Sargent (1987): macroeconomic variables do not display (very) pronounced peaks at business cycles frequencies and thus data are not supportive of strong internal boom-bust cycles.⁶ For instance, Beaudry et al. (2015, 2020) show the existence of a recurrent peak in several spectral densities of US trendless macroeconomic data suggesting the presence of periodicities at medium term irrespective of the exogenous cyclical forces. At least their results run counter the empirical irrelevance of endogenous fluctuations.⁷ We build on these results to reconcile the predictions of our model with empirical evidence.

Looking at long-date inheritance flows data for Sweden (Ohlsson et al., 2020) as well as France, UK and Germany (Piketty, 2011), we first provide a quantitative assessment of the long-run cyclical behavior of bequests as a share of national income. In so doing, we proceed in two steps. First, using the low-frequency methodology of Müller and Watson (2016, 2017) and band pass filters at low frequencies, we extract the long-run component of the inheritance variable of interest and then characterize the business, medium- and long-run fluctuations. Notably, empirical evidence suggest that movements of the inheritance variable are dominated by medium and low frequencies and are characterized by a slow cyclical mean reversion. Second, following Beaudry

proportion of national income. These arguments have been challenged by Jones (2015) and Acemoglu and Robinson (2015), among others. See also Piketty (2015).

³ Long series are not generally available for all OECD countries.

⁴ Deterministic limit cycles have a long tradition in economics. Especially the seminal contributions of Benhabib and Nishimura (1979, 1985) have shown that even in standard models featuring forward-looking agents and a competitive equilibrium structure, the steady state or balanced growth path was inherently unstable and thus deterministic (endogenous) fluctuations were easily obtained as soon as the fundamental nonlinear structure of the model was taken into account. More recently, Beaudry et al. (2015, 2017, 2020) have put forward the existence of endogenous stochastic limit cycles, i.e. a deterministic cycle where the stochastic component is essentially an i.i.d. process, that can generate alternate periods of booms and busts (see also Benhabib and Nishimura (1989)). Recent strands of the literature that discuss the emergence of limit cycles include contributions on innovation-cycles and growth (Matsuyama, 1999; Growiec et al., 2018), on endogenous credit cycles in OLG models (Azariadis and Smith, 1998; Myerson, 2012; Gu et al., 2013), on endogenous learning- and bounded rationality-based business fluctuations (Hommes, 2013).

⁵ For further evidence, see Section 2.

⁶ Comin and Gertler (2006) first provide evidence of medium-term cycles (with a periodicity between 8 and 50 years). See also Correa-López and De Blas (2012) for an application to medium-term technology cycles.

⁷ In the same vein, Growiec et al. (2018) conclude that the labor's share of GDP exhibits medium-run swings. See also Charpe et al. (2020).

et al. (2015, 2020), we make use of the spectral density to identify peak ranges over some frequency intervals, and thus to provide some support of recurrent (medium and long-run) cyclical fluctuations at that frequency (interval). Moreover, we test the presence of a shape restriction on the spectral density, i.e. the statistical significance of the “peak range” of the spectral density at a given frequency interval with respect to a flat prior. Our results strongly support the view of medium-term business fluctuations. While the presence of a peak range does not necessarily imply strong endogenous cyclical forces, it suggests that data cannot, at least, contradict the existence of endogenous (stochastic) limit cycles.⁸ We also show that the inheritance flow and the national income experience the same periodicity as that of the inheritance income ratio and this suggests that the cyclical component of both the inheritance ratio and the two corresponding level variables are dominated by medium-term fluctuations of a comparable periodicity.

Based on this empirical assessment, we view the existence of such limit cycles for the inheritance flow (in level or as a share of national income) as a complementary interpretation of Piketty (2011) and thus model and rationalize it by formally characterizing the corresponding complex dynamics.

Capitalizing on Michel and Venditti (1997), we do so through the lens of a two-sector overlapping generation (henceforth, OLG) model with a pure consumption good and one capital good, and a constant population of finitely-lived agents. We consider the decentralized problem where altruistic agents determine their life-cycle consumptions, savings and inheritance to their children. As long as bequests are strictly positive across generations, the optimal solution is described by a dimension-four dynamical system. The degree-4 characteristic polynomial associated to the linearized dynamical system around the steady state appears to have a quasi-palindromic structure. It is then possible to explicitly solve it and provide a complete assessment of its characteristic roots.⁹ To the best of our knowledge, it is the first available proposition that exploits the quasi-palindromic property of the characteristic equation in the literature of macroeconomic dynamic models.¹⁰ This result is the cornerstone of our paper. First, we show that the steady state with strictly positive bequests can be either saddle-point stable or (totally) unstable. In the latter case, endogenous cycles can occur. Second, it is well-known since Benhabib and Nishimura (1979) that the existence of a Hopf bifurcation in models featuring forward-looking agents and a competitive equilibrium structure requires the consideration of at least three sectors and thus of dimension-four dynamical systems.¹¹ However, as shown in several contributions, e.g. Magill (1977, 1979a,b) and Magill and Scheinkman (1979), the curse of dimensionality prevents the derivation of meaningful sufficient conditions for the existence of complex characteristic roots.

Our paper shows that only two sectors are sufficient in an OLG economy and it makes one step further to a better understanding of the occurrence of complex roots.¹² We provide indeed

⁸ See Dufourt et al. (2015) where the Hopf bifurcation is also shown to be relevant from an empirical perspective in two-sector infinite-horizon models with productive externalities and sunspot fluctuations.

⁹ If $P(x) = \sum_{i=0}^4 a_i x^i$ is a polynomial of degree 4 and $a_i = a_{n-i}$ for $i = 0, \dots, 2$, then P is palindromic (or reciprocal). If $P(x) = a_0 x^4 + a_1 x^3 + a_2 x^2 + a_3 mx + a_4 m^2$ for some constant $m \neq 0$, then $P(mx) = \frac{x^4}{m^2} P(\frac{m}{x})$ and P is quasi-palindromic. Importantly, using the change of variables $z = x + \frac{m}{x}$ in $\frac{P(x)}{x^2}$ produces a quadratic equation.

¹⁰ A similar characteristic polynomial has been obtained by de la Croix and Michel (1999) in an optimal growth model with inherited tastes but they do not refer to the quasi-palindromic property and they do not provide a complete solving of the characteristic roots.

¹¹ See also Cartigny and Venditti (1995) and Venditti (1996).

¹² It is important to mention that both the non separability of the utility function and the OLG structure of the model are necessary to get complex roots.

clear-cut sufficient conditions for the existence of complex roots leading to a Hopf bifurcation, and as far as we know, this is the first time such conditions are exhibited in the literature. Third, we can identify two key mechanisms that lead to quasi-periodic cycles through a Hopf bifurcation. The first one is based on the properties of preferences, and especially the sign of the cross-derivative of the utility function or equivalently the elasticity of intertemporal substitution. The second rests on sectoral technologies through the sign of the capital intensity difference across sectors.¹³

Furthermore, these preference and technology-based mechanisms can either generate endogenous fluctuations independently or self-sustain themselves and thus amplify or mitigate (long-run) limit cycles. In the case of non-strictly concave preferences, mild perturbations of either the capital intensity difference across sectors or of the elasticity of intertemporal substitution lead to a flip bifurcation and thus to persistent period-2 cycles.¹⁴ The global dynamics can then be described as the product of two cycles implying complex properties of the optimal path.¹⁵ On the one hand, the elasticity of intertemporal substitution must be large enough to allow a substitution effect between the first and second period consumptions while satisfying the standard transversality conditions and the convergence towards the period-two cycle. On the other hand, the consumption good sector needs to be capital intensive to generate fluctuations of the capital stock.

In contrast, these two mechanisms can no longer be separated in the case of strictly concave preferences, and a Hopf bifurcation can occur with quasi-periodic cycles under some intermediate and plausible values for the elasticity of current consumption, the elasticity of intertemporal substitution and an upper threshold condition for the sectoral elasticities of capital-labor substitution.¹⁶ After providing conditions for the existence of strictly stationary bequests, we argue that both bequests and bequests as a share of GDP can be characterized by optimal periodic and quasi-periodic cycles. This sharply contrasts with the predictions of the standard Barro model in which the optimal path monotonically converges toward the steady state if the life-cycle utility function of a representative generation living over two periods is additively separable.

These results provide a theoretical support for our empirical findings of Section 2 suggesting the existence of medium-term business cycles of inheritance. They also have strong policy implications. Indeed, in contrast to Piketty (2011) where the condition $r > g$ leading to an inherently unstable behavior of the distribution of wealth requires a tax on bequest to avoid such a pattern, the recent rise in property income can be interpreted as an optimal phenomenon that may reverse automatically later on. The rise of inequality can then be only temporary and may not call for any redistribution policy and bequest taxation.¹⁷

Indeed, in a standard two-sector optimal growth model with one pure consumption and one investment good that can also be consumed, complex roots cannot occur even if the utility function defined over the pair of consumptions is non-separable and strictly concave.

¹³ The capital intensity difference across sectors has long been identified as a key driver of the dynamic properties of two-sector optimal growth models.

¹⁴ The consumption good is assumed to be more capital intensive than the investment good (see Benhabib and Nishimura (1985)).

¹⁵ The quasi-palindromic polynomial can be factored as the product of two order-2 polynomials where one quadratic polynomial captures only the preference-based mechanism and the other only that based on technology.

¹⁶ Kalra (1996) and Reichlin (1992) provide conditions for the existence of Hopf cycles in two-sector OLG models but do not take into account bequests. See also Ghiglino and Tvede (1995) for the analysis of endogenous cycles in general OLG models, and Ghiglino (2005) for the analysis of the link between wealth inequality and endogenous fluctuations in a two-sector model.

¹⁷ Such a conclusion could be further explored in a model à la Blanchard (1985)-Weil (1989) with heterogeneous overlapping families of infinitely-lived agents.

The paper is organized as follows. Section 2 discusses the empirical relevance of interpreting the dynamics of inheritance flows (in level or as a share of national income) of long-dated Sweden and France annual data as a long endogenous cycle. Section 3 presents the two-sector model with non-additively separable preferences, defines the decentralized solution with altruistic agents, proves the existence of a steady state with positive bequests, and finally derives the characteristic polynomial from which the stability analysis is conducted. The existence of endogenous cycles is discussed in Section 4 considering successively period-two cycles under the assumption of a non-strictly concave utility function and quasi-periodic cycles under a Hopf bifurcation with a strictly concave utility function. In Section 5 we consider a general class of homogeneous preferences and sectoral Cobb–Douglas technologies to illustrate all our main results. Concluding comments are provided in Section 6 and all the proofs are contained into a final Appendix.

2. Long-run cycles and inheritance: an empirical assessment

In this section, we explore some empirical properties of the inheritance variable (relative to national income) using the recent data set in Sweden proposed by Ohlsson et al. (2020).¹⁸ Our treatment is notable in three dimensions. First, we make use of the recent low-frequency approach initiated by Müller and Watson (2016, 2017) to identify and estimate the long-run component of the bequests variable. Second, following the recent contributions of Beaudry et al. (2020), Charpe et al. (2020), and Growiec et al. (2018) for other macro variables, we highlight the frequency decomposition of the inheritance variable and the predominance of the medium- and long-term components and test the presence of a significant peak at these frequency bands. We also provide evidence for two level variables of the inheritance ratio, i.e. the inheritance flow and the national income. At the same time, it is worth emphasizing that we only discuss *indirect* empirical evidence on the variable of interest, which is consistent with the occurrence of endogenous fluctuations (e.g., the emergence of a period-2 cycle or of (quasi-) periodic (long-run) cycles), and do not attempt to explain the changes of the main determinants of the inheritance variable, nor the comovements with other macro variables.¹⁹

2.1. The historical series of bequests in Sweden

The annual inheritance flow as a fraction of national income in Sweden is presented in Fig. 1.²⁰ The sample covers the period

¹⁸ It is worth emphasizing that the bequest variable corresponds to post-mortem transfers, rather than inter-vivos transfers. From an empirical point of view, this choice is motivated by the absence of long-dated inter-vivos transfers series and the fact that it is consistent with our theoretical model (see Remark 2 in Section 3.3).

¹⁹ A more structural approach will require to simulate and estimate the OLG model in the presence of stochastic limit cycles (i.e., a deterministic limit cycle where the stochastic component is essentially an i.i.d. process). This could be done by determining the topological normal form for the flip (respectively, Hopf) bifurcation using Taylor expansions (see Kuznetsov (1998)) or perturbation methods (e.g., Galizia (2018)). We leave this issue for further research.

²⁰ It corresponds to their Fig. 5. The methodology and data construction are explained in Ohlsson et al. (2020) as well as Piketty (2011). Notably, using Eq. (1) of Piketty (2011), the inheritance flow as a ratio of national income is defined by:

$$\frac{B_t}{Y_t} = \mu_t \times m_t \times \frac{W_t}{Y_t}$$

where B_t , Y_t , W_t , μ_t and m_t denote respectively the aggregate inheritance flow, the aggregate national income, the aggregate private wealth, the mortality rate and the ratio between average wealth of the deceased and average wealth of the living. For further details, see Section 3 and the technical appendix of Piketty (2011), Piketty and Zucman (2015), and Ohlsson et al. (2020).

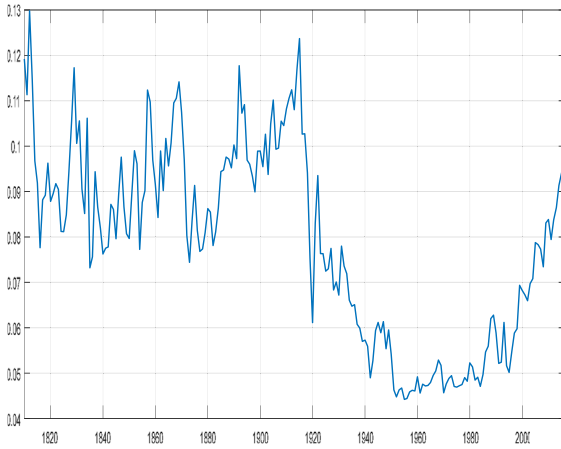


Fig. 1. Inheritance as a ratio of national income in Sweden 1810–2016.

1810–2016. A visual inspection suggests that the historical series arguably looks part of a long cycle. Indeed the inheritance variable displays medium- to long-run swings: it was about 8–13% of national income between 1810 and 1915 with some marked fluctuations, then down to less than 5% in 1960, and back up to about 8.5% by 2010 with a U-shaped curve starting from the end of the First World War. Therefore, the inheritance variable seems to be consistent with a persistence and long-lasting mean-reversal process.²¹

Before turning to the extraction of the long-run component and its characterization, one key issue regards the stationarity of the series, especially for the existence of the spectral density. Standard stationary tests report mixed evidence, due to the sample size, the presence of a structural break and the corresponding low power of unit root tests to distinguish between non-stationary and near-stationary stochastic processes. Notably, the (sequential) multi-break points test of Bai and Perron (2003a,b, 2012) reports evidence for the presence of one structural break in 1923.²² Accordingly, we implement the unit root test proposed by Perron (1990), which captures the changing mean of the bequests variable and thus the structural break. We reject the null of nonstationarity at conventional significance level.²³ Accordingly, the inheritance variable is specified in level, but is adjusted to account for the presence of a breakpoint around 1923, and thus to insure the weak stationarity of the series.²⁴

²¹ It is worth noting that similar patterns are observed in France and the UK.

²² We use a 15% trimming, an upper bound of five structural breaks, and a 5% significance level

²³ Using the additive-outlier model and the innovational-outlier model Perron, 1990, the student-based statistic of the null hypothesis of nonstationarity is greater (in absolute value) than the critical value at 5% (-3.34) for $\lambda = 0.5$ (i.e., when the structural break occurs at the middle of the sample). As a robustness check, we also study three parametric models of persistence. First, we estimate the local-to-unit (LTU) AR model (e.g., Phillips (1987)) in which $x_t = \mu + u_t$, and $u_t = \rho_T u_{t-1} + v_t$ for all t with an autoregressive coefficient $\rho_T = 1 - c/T$ and v_t is a weakly stationary process. Second, we also consider the “local-level” model in which the initial series is the sum of $I(0)$ and $I(1)$ processes. Finally, we also implement a fractional model in which $(1-L)^d x_t = v_t$ where L is the lag operator, $-0.5 < d < 1.5$ and v_t is a weakly stationary process.

²⁴ We first estimate the regression $\text{inh}_t = \alpha + \beta_1 \times \mathbb{I}_{t=1922} + \beta_2 \times \mathbb{I}_{t=1923} + \beta_3 \times \mathbb{I}_{t=1924} + u_t$ by the ordinary least squares method, where $\mathbb{I}_{t=\text{year}}$ is a dummy variable that equals one when $t = \text{year}$ and 0 otherwise. Then the adjusted inheritance variable is defined, for $t = 1, \dots, T$, as $\text{inh}_t^* = \hat{\alpha} + \hat{u}_t$, where $\hat{\alpha}$ is the OLS estimate of the intercept and \hat{u}_t denote the residuals. Results are robust when considering a larger window around the structural break.

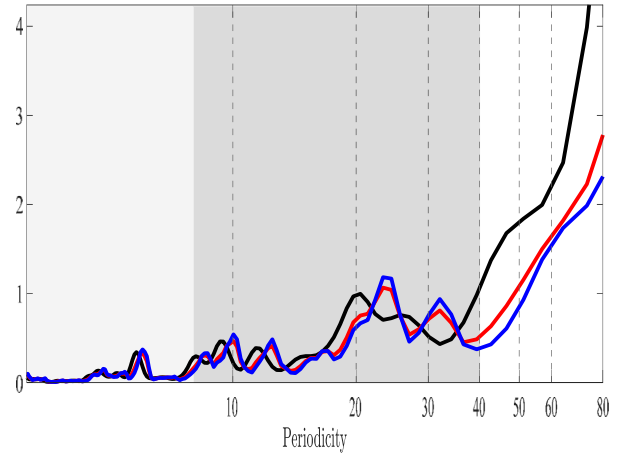


Fig. 2. Periodogram of the demeaned inheritance variable. **Note:** The black, red and blue solid lines represent the periodogram based on Hanning, Blackman–Tukey and Parzen window, respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

In this respect, Fig. 2 reports the periodogram of the demeaned (adjusted) inheritance variable.²⁵ We report three nonparametric power spectral density estimates by applying a Blackman–Tukey window (red line), a Parzen window (blue line), and a Hanning window (black line) to the covariogram of the filtered series before using a fast Fourier transform algorithm. As discussed by Comin and Gertler (2006), it is generally considered that the high-frequency component captures the periodicity below 8 years, i.e. business fluctuations and noise, the medium-frequency component the periodicity between 8 and 40 to 50 years, and the low-frequency (long-run) component the periodicity above 40 to 50 years. In this respect, as to be expected, the periodogram displays a typical hump shape and a peak at large periodicity, which are characteristic of persistent and near-stationary stochastic processes.²⁶ Moreover, it is clear that the inheritance variable is driven by frequency movements beyond the business cycle, and thus by medium and low frequencies.

2.2. The long-run component of bequests

To further discuss the relative contribution of the frequency movements, we now proceed with the estimation of the long-run component. Fig. 3 displays the long-run component (left panel) of the inheritance variable using three filtered series – the cyclical gap component (right panel), including short-run and medium-term fluctuations, being the difference between the raw series and the filtered series. The first long-run component is obtained from the approximate low bandpass filter of Baxter and King (1999).²⁷ We filter the high- and medium-frequency

²⁵ When the periodogram displays a substantial peak at a given frequency or a peak range, this provides some support of recurrent (e.g., medium-term) cyclical fluctuations at that frequency (range).

²⁶ Conversely, the spectral density displays a peak around the zero frequency, i.e. at low frequency.

²⁷ We also implement the asymmetric filter proposed by Christiano and Fitzgerald (2003, 1999). As a robustness analysis, we also transform our data using the first difference operator and apply the Baxter–King filter. This requires in turn to cumulate the filtered series. On the other hand, using the appropriate asymptotic results, the low-frequency approach of Müller and Watson (2016, 2017) remains valid for $I(1)$ models. Therefore we consider the level specification as a more plausible way to extract the long-run component, possibly at the expense of a loss of efficiency for the Baxter–King filter.

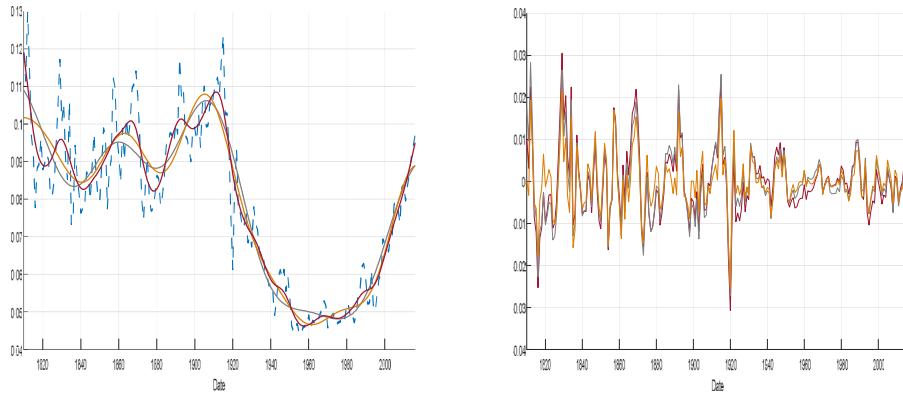


Fig. 3. Long-run and cyclical component of the inheritance variable. **Note:** Left panel (resp., right panel) reports the long-run component (resp., the cyclical component). The red, orange and grey solid lines represent the HP, Baxter-King, and cosine-based measures of the long-run (resp., cyclical) component. The blue dotted line is the inheritance series. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 1
Contribution of the long-run component.

	q	Long-run component	Short- to medium-term component
Cosine projection	4	79.02%	20.98%
	8	83.80%	16.20%
	12	84.99%	15.01%
HP filter		87.65%	12.35%
Baxter-King filter		84.13%	15.87%
Quadratic trend		40.75%	59.25%

Note: The long-run component is estimated using the low frequency approach of Müller and Watson (2016, 2017) using different cosine basis $q = 4, 8,$ and 12 , the HP filter, the Baxter-King filter, and a quadratic trend. The short- to medium-term component is defined as the difference between the initial series and the long-run component.

with periodicity below 40 years.²⁸ On the other hand, the second trend is constructed using the methodology proposed by Müller and Watson (2016, 2017), namely by extracting long-run sample information after isolating a small number of low-frequency trigonometric weighted averages.²⁹ In so doing, we project the series into a constant and twelve (q) cosine functions with periods $\frac{2T}{j}$ for $j = 1, \dots, 12$ in order to capture the variability for periods longer than 35 years ($2T/q$). Importantly, one advantage of the low-frequency approach of Müller and Watson (2016, 2017) relative to bandpass or other moving average filters, is that it is applicable beyond the $I(0)$ assumption.³⁰ Finally, for sake of comparison, the third long-run component is obtained after adjusting a standard one-sided HP filter.

Fig. 3 suggests that the long-run component, as measured by the three filtered series, is a main driver of the total variability of the bequests variable, and displays a nonlinear pattern. It is further highlighted in Table 1: the contribution of the long-run component and thus low frequencies account for a very substantial part of the variance of the inheritance variable, irrespective of the filtered series.

Interestingly, the long-run component is slowly mean-reverting. Notably, the half-time, i.e. the time for the expected

²⁸ Qualitative conclusions remain robust when considering other periodicities.

²⁹ Let $\{x_t, t = 1, \dots, T\}$ denote a (scalar) time series. Let $\Psi(s) = [\Psi_1(s), \dots, \Psi_q(s)]'$ denote a \mathbb{R}^q -valued function with $\Psi_j(s) = \sqrt{2} \cos(js\pi)$, and let $\Psi_T = [\Psi(\frac{1-0.5}{T}), \Psi(\frac{2-0.5}{T}), \dots, \Psi(\frac{T-0.5}{T})]'$ denote the $T \times q$ matrix after evaluating $\Psi(\cdot)$ at $s = \frac{t-0.5}{T}$, for $t = 1, \dots, T$. The low-frequency projection is the fitted series from the OLS regression of $[x_1, \dots, x_T]$ onto a constant and Ψ_T .

³⁰ For an extensive discussion about the relationship of this approach with spectral analysis, the scarcity of low-frequency information, and the relevance of the approximation using a small q , see Müller and Watson (2016)

value of the inheritance ratio to reach the middle value between the current value and the (long-run) mean, is nearby 20 years.³¹ It can be reconciliated with the presence of long-run cycles.

2.3. The medium-term cyclical component of bequests

Capitalizing on the long-run component, we now consider the gap component (Fig. 3, right panel) defined as the inheritance ratio variable in deviation of the long-run (stochastic) trend. We then focus on the periodogram and highlight in dark (resp., light) grey the band of frequencies corresponding to periodicities from 8 to 40 years (resp., 1 to 8 years) in Fig. 4. Irrespective of the smoothing method (Hanning, Parzen or Blackman-Tukey), one dominant feature is the distinct peak in the spectral density around 25 years. This suggests that the inheritance variable ratio exhibits important recurrent cyclical phenomena at approximately 25 to 30-year intervals. Following Beaudry et al. (2020), we formally test the presence of a shape restriction on the spectral density: we consider a “peak range” for 20–35 years and test the null hypothesis of a flat spectral density against a “peak range”. We strongly reject at 5 percent level that the spectrum is flat in the “peak range”. This result is robust when considering a narrow “peak range”. Accordingly, after filtering for the long-run component, the adjusted inheritance variable is predominantly dominated by medium-term fluctuations.

To further explore the properties of the inheritance ratio variable and some implications of our model, we now look at the medium-term cyclical component of both the inheritance flow and the national income. One key issue is that the raw data is non-stationary as there is clearly an upward trend for both variables (Fig. 5). Indeed, the spectral density requires that the (Fourier transform of the) autocorrelation function is defined, which is not the case for a nonstationary process.

Instead of using a first-difference transformation, we rather capitalize on the low-frequency approach of Müller and Watson (2016, 2017), which remains valid for non-stationary variables.³² Therefore, we proceed as before: we first implement the cosine-based projection to extract the long-run component of both variables and then construct the cyclical component as the difference

³¹ The half-time is defined by $HT = -\log(2)/\log(\hat{\phi})$ where $\hat{\phi}$ is the OLS estimate of the autoregressive parameter of the regression: $\text{inh}_t^{\text{LR}} = \mu + \phi \text{inh}_{t-1}^{\text{LR}} + \eta_t$, where inh_t^{LR} is the long-run component of the adjusted inheritance variable using the Baxter-King, HP- or cosine-based transformation.

³² One issue of the first-difference filter is that it heavily emphasizes movements at the highest frequencies and deemphasizes those at lower frequencies, and this can lead to mask some properties of the cyclical component that are present at lower frequencies.

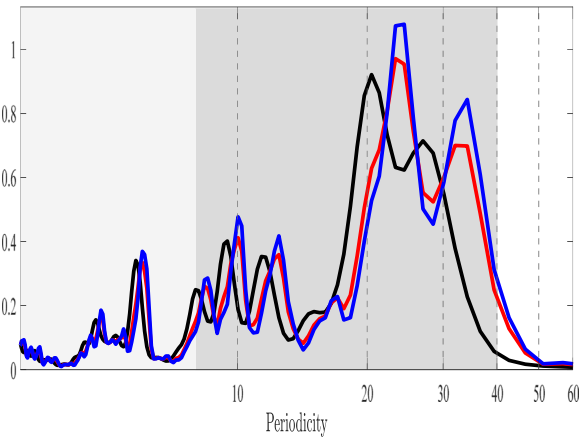


Fig. 4. Short- to medium-run components. **Note:** The black, red and blue solid lines represent the periodogram based on Hanning, Blackman–Tukey and Parzen window, respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

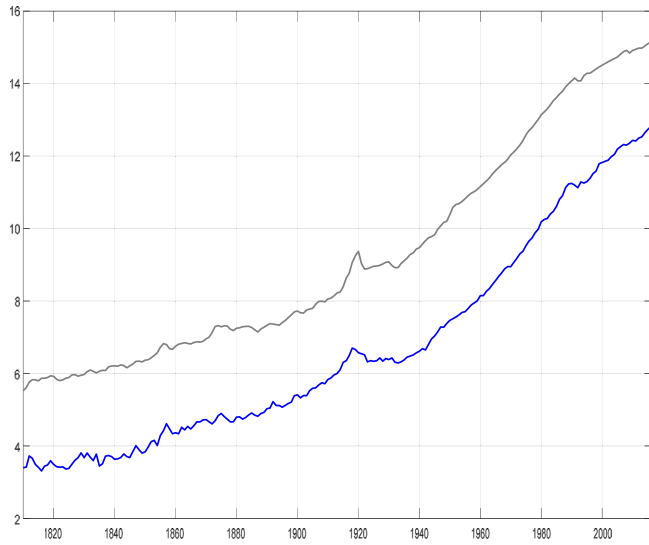


Fig. 5. Inheritance flows and national income in Sweden (1810–2016). **Note:** The grey and blue solid lines represent the log of national income and the log of inheritance flows, respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

between the raw series and the filtered series.³³ Fig. 6 then reports the periodogram of the cyclical component of the inheritance flow and the national income. Irrespective of the smoothing method, there is a distinct peak in the spectral density around 25 years for both variables. All in all, both variables experience the same periodicity as that of the inheritance income ratio and this suggests that the cyclical component of both the inheritance ratio and the two corresponding level variables are dominated by medium-term fluctuations of a comparable periodicity.

2.4. International evidence

Using the data set of Piketty (2011), Atkinson (2018) and Shinke (2012), we also conduct some empirical analysis for France, the UK and Germany. We only discuss the results for

³³ Due to the non-stationarity of the two series, results are more sensitive to the choice of the cosine basis than in the case of the inheritance ratio. Fig. 6 displays evidence when $q = 16$.

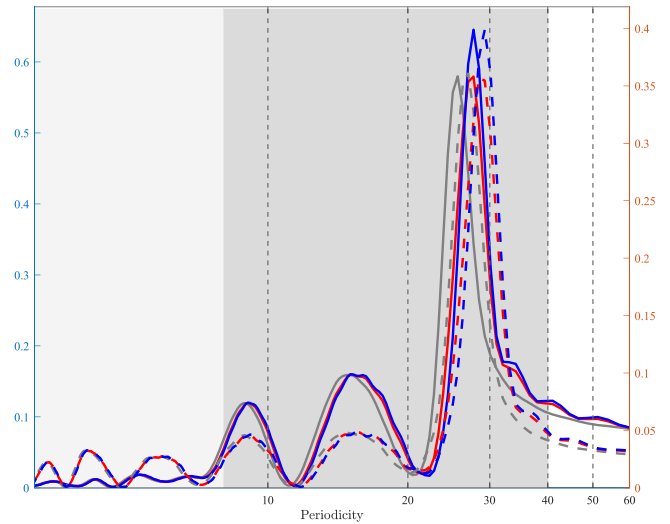


Fig. 6. Short- to medium-run components of the inheritance flows and national income variables in Sweden. **Note:** The grey, red and blue solid lines represent the periodogram based on Hanning, Blackman–Tukey and Parzen window for the (cosine-based) cyclical component of the inheritance flow (scaled by 10^{-4}) using the left y-axis. The dotted lines are those of the cyclical component of the national income (scaled by 10^{-5}) using the right y-axis. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

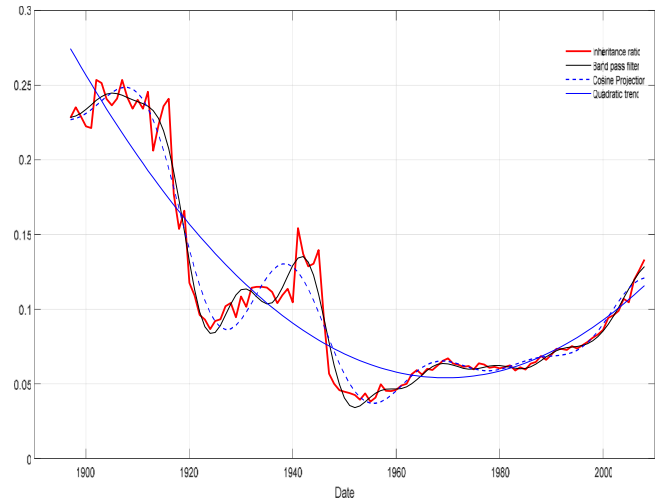


Fig. 7. Inheritance ratio and long-run components in France.

France : results are qualitatively similar for other countries.³⁴ Fig. 7 depicts the inheritance flow as a fraction of national income for France using annual data from 1897 through 2008, and three filter-based long-run components. The inheritance variables display medium- to long-run swings: it was about 20–25% of national income between 1820 and 1910, down to less than 5% in 1950, and back up to about 15% by 2010.

Using the same methodology, we extract the long-run component of the inheritance ratio variable.³⁵ Irrespective of the filtering method, this component strongly contributes to explain the fluctuations of the initial series and thus low frequencies are the main driver.

³⁴ Empirical evidences for UK and Germany are available upon request.

³⁵ The initial series of inheritance is also adjusted to account for the presence of a breakpoint after the first World War.

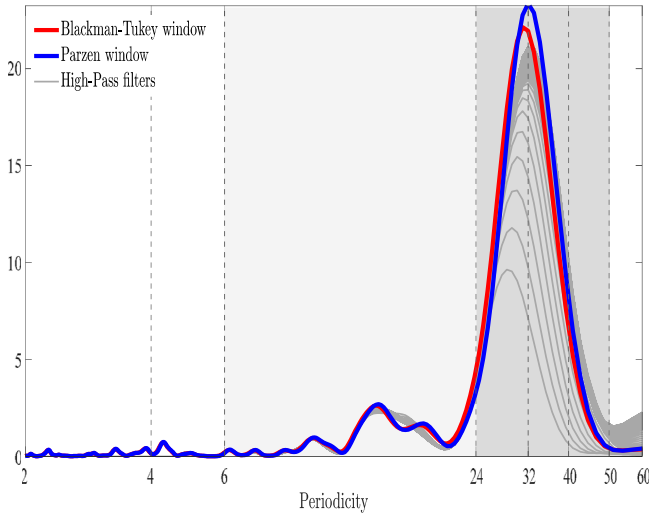


Fig. 8. Periodogram of the inheritance ratio variable.

Finally, Fig. 8 displays the periodogram of the projection-based cyclical component (using $q = 6$), as well as nonparametric power spectral density estimates. For comparison, we also report on the same axes the spectra when first passing the series through various high bandpass filters that remove some remaining low-frequency. We highlight in dark grey the band of frequencies corresponding to periodicities from 24 to 40 (or 50) years. Irrespective of the method used, one dominant feature is the distinct peak in the spectral density around 32 years and the local hump in its neighborhood. This suggests that the inheritance ratio variable exhibits important recurrent cyclical phenomena at approximately 30-year intervals, which is again consistent with the occurrence of (stochastic) limit cycles.³⁶ Finally, we closely examine the cyclical component of the inheritance flow and national income for both countries and find similar evidence as in Section 2.4: movements in both variables are mainly driven by medium frequencies, with a periodicity nearby 30 years.³⁷

2.5. Summary

Empirical evidence provides support that movements in the inheritance variable (as a share of national income) are dominated by medium and low frequencies, and these movements are characterized by slow cyclical mean-reversion. This is also the case when looking at the two level variables defining the inheritance ratio. In this respect, these results run counter the empirical irrelevance of endogenous fluctuations. Accordingly, it would be interesting to capture the existence of endogenous limit cycles in a bequest model.

3. The model with altruistic agents and a bequest motive

3.1. Production

We consider a two-sector economy with one pure consumption good y_0 and one capital good y . Each good is produced with

³⁶ We also test the presence of a shape restriction on the spectral density and consider a “peak range” for 24–40 years under the null hypothesis of a flat spectral density against a “peak range”. We strongly reject at 5 percent level that the spectrum is flat in the “peak range”. This result is robust when considering a narrow “peak range”.

³⁷ Results are not reported but are available upon request.

a standard constant returns to scale technology:

$$y_0 = f^0(k_0, l_0), \quad y = f^1(k_1, l_1)$$

with $k_0 + k_1 \leq k$, k being the total stock of capital, and $l_0 + l_1 \leq 1$, the total amount of labor being normalized to 1.

Assumption 1. Each production function $f^i : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, $i = 0, 1$, is C^2 , increasing in each argument, concave, homogeneous of degree one and such that for any $x > 0$, $f_{k_i}^i(0, x) = f_{l_i}^i(x, 0) = +\infty$, $f_{k_i}^i(+\infty, x) = f_{l_i}^i(x, +\infty) = 0$.

For any given (k, y) , we define a temporary equilibrium by solving the following problem of optimal allocation of factors between the two sectors:

$$\begin{aligned} T(k, y) = \max_{k_0, k_1, l_0, l_1} & f^0(k_0, l_0) \\ \text{s.t.} & y \leq f^1(k_1, l_1) \\ & k_0 + k_1 \leq k \\ & l_0 + l_1 \leq 1 \\ & k_0, k_1, l_0, l_1 \geq 0. \end{aligned} \quad (1)$$

The value function $T(k, y)$ is called the social production function and describes the frontier of the production possibility set. Constant returns to scale of technologies imply that $T(k, y)$ is concave non strictly. We will assume in the following that $T(k, y)$ is at least C^2 .³⁸

Let p denote the price of the investment good, r the rental rate of capital and w the wage rate, all in terms of the price of the consumption good, it is straightforward to show that:

$$\begin{aligned} T_k(k, y) &= r(k, y), \quad T_y(k, y) = -p(k, y) \text{ and} \\ w(k, y) &= T(k, y) - r(k, y)k + p(k, y)y. \end{aligned} \quad (2)$$

We can also characterize the second derivatives of $T(k, y)$. Using the concavity property we have:

$$T_{kk}(k, y) = \frac{\partial r}{\partial k} \leq 0, \quad T_{yy}(k, y) = -\frac{\partial p}{\partial y} \leq 0.$$

As shown by Benhabib and Nishimura (1985), the sign of the cross derivative $T_{ky}(k, y)$ is given by the sign of the relative capital intensity difference between the two sectors. Denoting $a_{00} = l_0/y_0$, $a_{10} = k_0/y_0$, $a_{01} = l_1/y$ and $a_{11} = k_1/y$ the capital and labor coefficients in each sector, it is easy to derive from the constant returns to scale property that:

$$\frac{dp}{dr} = a_{01} \left(\frac{a_{11}}{a_{01}} - \frac{a_{10}}{a_{00}} \right) \equiv b \quad (3)$$

with b the relative capital intensity difference, and thus

$$T_{ky} = T_{yk} = -\frac{\partial p}{\partial r} \frac{\partial r}{\partial k} = -T_{kk}b.$$

The sign of both b and T_{ky} is positive if and only if the investment good is capital intensive. Note also that $T_{yy}(k, y)$ can be written as:

$$T_{yy} = -\frac{\partial p}{\partial r} \frac{\partial r}{\partial y} = T_{kk}b^2.$$

Remark 1. The derivative $dr/dp = b^{-1}$ is well-known in trade theory as the Stolper–Samuelson effect. Similarly, at constant prices, we can derive the associated Rybczinsky effect $dy/dk = b^{-1}$. We therefore find the well-known duality between the Rybczinsky and Stolper–Samuelson effects.

³⁸ A proof of the differentiability of $T(k, y)$ under Eq. (1) and non-joint production is provided in Benhabib and Nishimura (1979).

3.2. Preferences

The economy is populated by a constant population of finitely-lived agents.³⁹ In each period t , $N_t = N$ persons are born, and they live for two periods: they work during the first (with one unit of labor supplied) and they have preferences for consumption (c_t , when they are young, and d_{t+1} , when they are old) which are summarized by the utility function $u(c_t, d_{t+1})$ such that [Assumption 2](#) is satisfied.

Assumption 2. $u(c, d)$ is increasing in both arguments ($u_c(c, d) > 0$ and $u_d(c, d) > 0$), concave and C^2 over the interior of \mathbb{R}_+^2 .

Since total labor is normalized to 1, we consider from now on that $N = 1$. We also introduce a standard normality assumption between the two consumption levels.

Assumption 3. Consumptions c and d are normal goods.

We finally consider the following useful elasticities of consumptions:

$$\begin{aligned} \epsilon_{cc} &= -u_c/u_{cc}c > 0, & \epsilon_{cd} &= -u_c/u_{cd}d, & \epsilon_{dc} &= -u_d/u_{dc}c, \\ \epsilon_{dd} &= -u_d/u_{dd}d > 0 \end{aligned} \quad (4)$$

It is worth noting that the normality [Assumption 3](#) implies $1/\epsilon_{cc} - 1/\epsilon_{dc} \geq 0$ and $1/\epsilon_{dd} - 1/\epsilon_{cd} \geq 0$ and concavity in [Assumption 2](#) implies $1/(\epsilon_{cc}\epsilon_{dd}) - 1/(\epsilon_{dc}\epsilon_{cd}) \geq 0$. Taking these elasticities, the elasticity of intertemporal substitution between c_t and d_{t+1} writes:

$$\varsigma(c_t, d_{t+1}) = \frac{\frac{u_d(c_t, d_{t+1})/u_c(c_t, d_{t+1})}{c_t/d_{t+1}}}{\frac{\partial(u_d(c_t, d_{t+1})/u_c(c_t, d_{t+1}))}{\partial(c_t/d_{t+1})}} = \frac{1}{\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{dc}}} \geq 0. \quad (5)$$

This elasticity will be used as a parameter driving the existence of endogenous fluctuations.

3.3. The Barro formulation of altruism

We consider a decentralized economy composed of overlapping generations of parents loving their children. As in the [Barro \(1974\)](#) formulation, each agent is altruistic towards his descendant through a bequest motive.⁴⁰ Parents indeed care about their child's welfare by taking into account their child's utility into their own utility function. They are price-takers, considering as given the prices p_t , w_t and r_{t+1} as defined by (2), and determine their optimal decisions with respect to their budget constraints

$$w_t + p_t x_t = c_t + \zeta_t \text{ and } R_{t+1} \zeta_t = d_{t+1} + p_{t+1} x_{t+1} \quad (6)$$

with $R_{t+1} = r_{t+1}/p_t$ the gross rate of return, ζ_t the savings of young agents born in t and x_t the amount of bequest transmitted at time t by agents born in $t-1$. Note that bequest x_t is expressed as an investment good and requires the relative price p_t to enter the budget constraints.⁴¹ In each period, bequests must be non-negative:

$$x_t \geq 0 \text{ for all } t \geq 0 \quad (7)$$

Remark 2. We need here to provide comments on the difference between inter-vivos bequests and post-mortem ones. Clearly, the

³⁹ An increasing population could be considered without altering all our results.

⁴⁰ The co-existence of altruistic and non-altruistic agents as in [Nourry and Venditti \(2001\)](#) could also be considered.

⁴¹ Formulating bequest as a consumption good with budget constraints $w_t + x_t = c_t + \zeta_t$ and $R_{t+1} \zeta_t = d_{t+1} + x_{t+1}$ would not affect the results of our analysis. Indeed the optimality conditions (10)–(11) given below would remain unchanged.

Barro formulation refers to inter-vivos bequests as transmissions to offsprings are operated while old agents are still alive. On the contrary, the data used in [Section 2](#) of this paper are related to post-mortem inheritance. However, such a difference is not significant for two reasons. First, if we decompose both periods of life into two sub-periods, we can interpret the budget constraints (6) as bequests transmitted by an old household during the second sub-period just before his death and which is received by the offsprings in the second sub-period while they are close to retirement. Such a timing is rather plausible with what is generally observed for bequest behaviors. Second, it is well-known that inter-vivos transfers are an increasing part of inheritance as they benefit from decreased taxes which are supported by the donator parents. Since there is no taxation in our model, there is therefore no real difference between inter-vivos and post-mortem bequests.⁴²

An altruistic agent has a utility function given by the following Bellman equation

$$\begin{aligned} V_t(x_t) &= \max_{\{c_t, d_{t+1}, \zeta_t, x_{t+1}\}} \{u(c_t, d_{t+1}) + \beta V_{t+1}(x_{t+1})\} \\ &= \max_{\{c_t, d_{t+1}, \zeta_t, x_{t+1}\}} \sum_{t=0}^{+\infty} \beta^t u(c_t, d_{t+1}) \end{aligned} \quad (8)$$

subject to (6) and (7), and where $\beta \in (0, 1)$ is interpreted as the intergenerational degree of altruism. Note that β can also be viewed as a discount factor.

Substituting the expressions of c_t and $d_t + 1$ from the budget constraints (6) into the optimization problem (8) we get

$$\max_{\{\zeta_t, x_{t+1}\}} \sum_{t=0}^{+\infty} \beta^t u(w_t + p_t x_t - \zeta_t, R_{t+1} \zeta_t - p_{t+1} x_{t+1}) \quad (9)$$

The first order conditions are given by

$$u_c(c_t, d_{t+1})p_t - u_d(c_t, d_{t+1})r_{t+1} = 0, \quad (10)$$

$$\begin{aligned} \beta u_c(c_{t+1}, d_{t+2}) - u_d(c_t, d_{t+1}) &\leq 0 \text{ with an equality if} \\ x_t &> 0, \end{aligned} \quad (11)$$

with $p_t = -T_y(k_t, k_{t+1})$ and $r_{t+1} = T_k(k_{t+1}, k_{t+2})$. Assuming a complete depreciation within one period,⁴³ savings ζ_t is obtained from the production of the investment good and we get $\zeta_t = p_t y_t = p_t k_{t+1}$. Moreover, for any $t \geq 0$, total consumption is given by the social production function, i.e. $c_t + d_t = T(k_t, y_t)$.⁴⁴

Assuming for now that bequests are strictly positive (we will show later on that this can be the case at the steady state), the first order conditions (10)–(11) can be equivalently re-written as the following two difference equations of order two:

$$\begin{aligned} u_d(T(k_t, k_{t+1}) - d_t, d_{t+1}) - \beta u_c(T(k_{t+1}, k_{t+2}) - d_{t+1}, d_{t+2}) &= 0 \\ u_c(T(k_t, k_{t+1}) - d_t, d_{t+1})T_y(k_t, k_{t+1}) + \\ \beta u_c(T(k_{t+1}, k_{t+2}) - d_{t+1}, d_{t+2})T_k(k_{t+1}, k_{t+2}) &= 0. \end{aligned} \quad (12)$$

⁴² A possibility to get post-mortem transfers in a two-period OLG model would be to consider life-time uncertainty and accidental bequests (see for instance [Abel \(1985\)](#) and [Fuster \(1999\)](#)).

⁴³ Considering that in an OLG model one period is approximately 30 years, complete depreciation is a realistic assumption.

⁴⁴ As shown by [Weil \(1987\)](#), when the non-negativity constraints of bequests (7) hold with a strict inequality in order to preserve the link across generations, the altruistic problem is equivalent to a central planner problem maximizing a welfare function.

Considering some (given) initial conditions (d_0, k_0) , any path that satisfies Eqs. (12) together with the following transversality conditions,

$$\begin{aligned} \lim_{t \rightarrow +\infty} \beta^t u_d(c_t, d_{t+1}) p_{t+1} k_{t+1} &= 0 \text{ and} \\ \lim_{t \rightarrow +\infty} \beta^t u_d(c_t, d_{t+1}) d_{t+1} &= 0, \end{aligned} \quad (13)$$

is an optimal path.⁴⁵

Solving Eqs. (12) and considering the transversality conditions (13) then yield the equilibrium paths for capital $\{k_t\}_{t \geq 0}$ and second period consumption $\{d_t\}_{t \geq 0}$. The dynamics of first period consumption is obviously $c_t = T(k_t, k_{t+1}) - d_t$ and the dynamics of bequests is derived from the budget constraints (6) as follows

$$p_t x_t = r_t k_t - d_t \quad (14)$$

whereas the dynamics of bequests as a proportion of GDP is

$$\begin{aligned} \frac{p_t x_t}{\text{GDP}_t} &= \frac{p_t x_t}{T(k_t, k_{t+1}) + p_t y_t} \\ &= s(k_t, k_{t+1}) - \frac{d_t}{T(k_t, k_{t+1}) - T_y(k_t, k_{t+1}) k_{t+1}}. \end{aligned} \quad (15)$$

with

$$s(k_t, k_{t+1}) = \frac{T_k(k_t, k_{t+1}) k_t}{T(k_t, k_{t+1}) - T_y(k_t, k_{t+1}) k_{t+1}}$$

the share of capital income in GDP. We will then study the local dynamic properties of capital k_t and consumption d_t from the dynamical system (12) and derive in a second step that bequests and bequests as a proportion of GDP have similar dynamic properties, providing then a support for our empirical findings of Section 2.

3.4. Steady state

A steady state is defined as the stationary solution, $k_t = k^*$, $d_t = d^*$, $x_t = x^*$, for all t , of the following nonlinear system of equations:

$$\frac{u_d(T(k, k) - d, d)}{u_c(T(k, k) - d, d)} = \beta \quad (16)$$

$$\frac{T_y(k, k)}{T_k(k, k)} = \beta \quad (17)$$

$$x = T_k(k, k)k - d > 0 \quad (18)$$

We get the following result:

Proposition 1. Under Assumptions 1–3:

- (i) there exists a unique stationary capital stock k^* solution of Eq. (17).
- (ii) there exists a unique stationary consumption level d^* , with $d^* < T(k^*, k^*)$, solution of Eqs. (16) if

$$\begin{aligned} \lim_{d \rightarrow T(k^*, k^*)} \frac{u_d(T(k^*, k^*) - d, d)}{u_c(T(k^*, k^*) - d, d)} &= 0 \text{ and} \\ \lim_{d \rightarrow 0} \frac{u_d(T(k^*, k^*) - d, d)}{u_c(T(k^*, k^*) - d, d)} &= +\infty \end{aligned} \quad (19)$$

- (iii) under conditions (19), for any given $\beta \in (0, 1)$, the stationary bequests x^* are positive in the economy with degree of altruism equal to β if

$$\lim_{d \rightarrow T_k(k^*, k^*) k^*} \frac{u_d(T(k^*, k^*) - d, d)}{u_c(T(k^*, k^*) - d, d)} \in (0, \beta) \quad (20)$$

Proof. See Appendix A.1. \square

⁴⁵ It is worth noting that our model merges the formulations of Benhabib and Nishimura (1979) and Galor (1992), and is close to the optimal growth model with fertility choice of Barro and Becker (1989) where intergenerational transfers occur.

A 3-uple (k^*, d^*, x^*) is then defined to be the Modified Golden Rule. Finally, the stationary consumption of young agents is obtained from $c^* = T(k^*, k^*) - d^* > 0$. As bequests are positive at the steady state, then by continuity they are positive in a neighborhood of the steady state and we may study the local stability properties of the equilibrium path considering the dynamical system (12).

3.5. Characteristic polynomial

We are now in a position to derive the characteristic polynomial from total differentiation of Eqs. (12). Denoting $T(k^*, k^*) = T^*$, $T_k(k^*, k^*) = T_k^*$ and $T_{kk}(k^*, k^*) = T_{kk}^*$, let us introduce the elasticities of the consumption good's output and the rental rate with respect to the capital stock, all evaluated at the steady state, as given by:

$$\varepsilon_{ck} = T_k^* k^* / T^* > 0, \quad \varepsilon_{rk} = -T_{kk}^* k^* / T_k^* > 0. \quad (21)$$

Then Proposition 2 yields the degree-4 characteristic polynomial and discusses the multiplicity order of the possible (characteristic) roots.

Proposition 2. Under Assumptions 1–3, the degree-4 characteristic polynomial is given by

$$\mathcal{P}(\lambda) = \lambda^4 - \lambda^3 B + \lambda^2 C - \lambda \frac{B}{\beta} + \frac{1}{\beta^2} \quad (22)$$

with

$$B = -\frac{\beta}{b \varepsilon_{cc}} \frac{\varepsilon_{ck}}{\varepsilon_{rk}} \left(\frac{\varepsilon_{cc}}{\varepsilon_{dc}} - \frac{\varepsilon_{cd}}{\varepsilon_{dd}} \right) + \frac{\beta + b^2}{\beta b} + \frac{\varepsilon_{dc}}{\beta \varepsilon_{cc}} + \frac{\varepsilon_{cd}}{\varepsilon_{dd}} \quad (23)$$

$$C = -\frac{(1+\beta)}{b \varepsilon_{cc}} \frac{\varepsilon_{ck}}{\varepsilon_{rk}} \left(\frac{\varepsilon_{cc}}{\varepsilon_{dc}} - \frac{\varepsilon_{cd}}{\varepsilon_{dd}} \right) + \frac{\beta + b^2}{\beta b} \left(\frac{\varepsilon_{dc}}{\beta \varepsilon_{cc}} + \frac{\varepsilon_{cd}}{\varepsilon_{dd}} \right) + \frac{2}{\beta}$$

or equivalently

$$\begin{aligned} \mathcal{P}(\lambda) &= \left[\lambda^2 - \lambda \left(\frac{\varepsilon_{dc}}{\beta \varepsilon_{cc}} + \frac{\varepsilon_{cd}}{\varepsilon_{dd}} \right) + \frac{1}{\beta} \right] \frac{(\lambda b - 1)(\lambda \beta - b)}{\beta b} \\ &\quad + \lambda \left(\lambda - \frac{1}{\beta} \right) \left(\lambda - \frac{1}{\beta} \right) \frac{\beta}{b \varepsilon_{cc}} \frac{\varepsilon_{ck}}{\varepsilon_{rk}} \left(\frac{\varepsilon_{cc}}{\varepsilon_{dc}} - \frac{\varepsilon_{cd}}{\varepsilon_{dd}} \right) \end{aligned} \quad (24)$$

If λ is a characteristic root of (24), then $\bar{\lambda}$, $(\beta \lambda)^{-1}$ and $(\beta \bar{\lambda})^{-1}$ are also characteristic roots of (24). Moreover, at least two roots or a pair of complex conjugate roots have a modulus larger than one, and one of the following cases necessarily holds:

- (i) the four roots are real and distinct,
- (ii) the four roots are given by two pairs of non-real complex conjugates,
- (iii) there are two complex conjugates double roots or two real double roots.

Proof. See Appendix A.2. \square

Proposition 2 is of critical importance and several points are worth commenting. First, as initially proved by Levhari and Liviatan (1972) in multisector optimal growth models, it shows that if there exists a pair of complex characteristic roots $(\lambda, \bar{\lambda})$ solutions of the quartic polynomial (24), then a second pair of complex characteristic roots, $(\beta \lambda)^{-1}$ and $(\beta \bar{\lambda})^{-1}$, are also solutions of (24). Therefore, Proposition 2 proves that the 4 characteristic roots are either all real or all complex. Second, Proposition 2 also implies that at most two characteristic roots can have a modulus lower than 1 and thus that the steady state can be either saddle-point stable or totally unstable. Of course in this last case, endogenous cycles can occur. Third, under Assumption 2, the sign of the expression $\frac{\varepsilon_{cc}}{\varepsilon_{dc}} - \frac{\varepsilon_{cd}}{\varepsilon_{dd}}$ is given by the sign of the cross derivative u_{cd} , i.e. by the opposite of the sign of ε_{cd} , ε_{dc} , which is a crucial ingredient to determine the local stability properties of the steady state. Fourth, using Eq. (24), when the utility function is non-strictly concave, i.e. if $\frac{\varepsilon_{cc}}{\varepsilon_{dc}} - \frac{\varepsilon_{cd}}{\varepsilon_{dd}} = 0$, then the degree-4 polynomial simplifies to a product of two degree-2 polynomials.

Remark 3. The degree-4 characteristic polynomial (22) is a quasi-palindromic equation that can be solved explicitly, and its roots can be determined using only quadratic equations (see Appendix A.8 for details.). The same type of characteristic polynomial has been studied by de la Croix and Michel (1999) in an optimal growth model with inherited tastes. However, since they focus on the saddle-point property of the steady state, they do not consider the possibility of bifurcating complex roots and they do not need to refer to the quasi-palindromic structure in order to explicitly compute the four characteristic roots (see a more detailed discussion in Appendix A.8). As far as we know, this is the first time this type of detailed methodology is applied to macroeconomic dynamic models. Using the quasi-palindromic structure of the characteristic polynomial will allow us to provide explicit sufficient conditions for the occurrence of complex characteristic roots and prove the existence of a Hopf bifurcation.

Remark 4. If $b = 0$, the two sectors are characterized by the same technology and one gets the one-sector formulation with a two-dimensional dynamical system as considered in Michel and Venditti (1997). Indeed, the characteristic polynomial can be simplified as follows

$$\mathcal{P}(\lambda) = \lambda^2 - \lambda \frac{\frac{\epsilon_{dc}}{\beta \epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}} - \frac{(1+\beta) \epsilon_{ck}}{\epsilon_{cc} \epsilon_{rk}} \left(\frac{\epsilon_{cc} - \epsilon_{cd}}{\epsilon_{dc} - \epsilon_{dd}} \right)}{1 - \frac{\beta}{\epsilon_{cc}} \frac{\epsilon_{ck}}{\epsilon_{rk}} \left(\frac{\epsilon_{cc} - \epsilon_{cd}}{\epsilon_{dc} - \epsilon_{dd}} \right)} + \frac{1}{\beta}$$

The same conclusions on the existence of period-2 cycles as in Michel and Venditti (1997) are obviously derived.

Similarly, if the utility function is additively separable, i.e. $u_{cd} = u_{dc} = 0$, we get the two-sector optimal growth formulation with a two-dimensional dynamical system as considered in Benhabib and Nishimura (1979). The characteristic polynomial can indeed be simplified as follows

$$\mathcal{P}(\lambda) = \lambda^2 - \lambda(1 + \beta) \frac{\frac{\beta}{\epsilon_{cc}} \frac{\epsilon_{ck}}{\epsilon_{rk}} + (\beta + b^2)}{\frac{\beta}{\epsilon_{cc}} \frac{\epsilon_{ck}}{\epsilon_{rk}} + (1 + \beta)b} + \frac{1}{\beta}$$

The same conclusions on the existence of period-2 cycles as in Benhabib and Nishimura (1979) are then derived. In the limit case with $b = 0$ and $u_{cd} = u_{dc} = 0$, we are obviously back to the standard Ramsey (1928) model where the unique optimal path converges monotonically to the steady state.

4. Optimal endogenous fluctuations

As shown by Proposition 2, the structure of the characteristic polynomial crucially depends on whether the utility function is non-strictly concave, i.e. $\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}} = 0$, or strictly concave, i.e. $\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}} \neq 0$. We then consider these two configurations successively in the following two Sections.

4.1. Period-two cycles under non-strictly concave preferences: A separated mechanism

In this section we assume that the utility function is non-strictly concave.

Assumption 4. The utility function $u(c, d)$ is concave non-strictly, i.e. $\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}} = 0$.

In so doing, one can show that the characteristic roots cannot be complex.

Lemma 1. Under Assumptions 1–4, the characteristic roots are real.

Proof. See Appendix A.3. \square

Following simultaneously the same methodologies as in the two-sector optimal growth model and the optimal growth solution of the aggregate OLG model, we discuss the local stability properties of equilibrium paths depending both on the sign of the capital intensity difference across sectors b and the sign of the cross derivative u_{cd} , i.e. of the two elasticities ϵ_{cd} and ϵ_{dc} .

As a first step, Proposition 3 provides some simple conditions ensuring the saddle-point property with monotone convergence.

Proposition 3. Under Assumptions 1–4, if $b \geq 0$ and $\epsilon_{cd}, \epsilon_{dc} \geq 0$, i.e. $u_{cd} \leq 0$, then the equilibrium path is monotone and the steady-state (k^*, d^*) is a saddle-point.

Proof. See Appendix A.4. \square

We now show that convergence with oscillations and persistent competitive equilibrium cycles may occur under a quite large set of circumstances.

Proposition 4. Under Assumptions 1–4, the following results hold:

(i) When the investment good is capital intensive, i.e. $b \geq 0$, let $\epsilon_{cd}, \epsilon_{dc} < 0$, i.e. $u_{cd} > 0$. Then the steady state (k^*, d^*) is saddle-point stable with damped oscillations if and only if the elasticity of intertemporal substitution satisfies $\zeta \in (0, \underline{\zeta}) \cup (\bar{\zeta}, +\infty)$ with

$$\underline{\zeta} \equiv \frac{\epsilon_{cc}\beta}{1+\beta} \quad \text{and} \quad \bar{\zeta} \equiv \frac{\epsilon_{cc}}{2}$$

Moreover, when ζ crosses the bifurcation values $\underline{\zeta}$ or $\bar{\zeta}$, (k^*, d^*) undergoes a flip bifurcation leading to persistent period-2 cycles.

(ii) When $\epsilon_{cd}, \epsilon_{dc} \geq 0$, i.e. $u_{cd} \leq 0$, let the consumption good be capital intensive, i.e. $b < 0$. Then the steady state (k^*, d^*) is saddle-point stable with damped oscillations if and only if $b \in (-\infty, -1) \cup (-\beta, 0)$. Moreover, if there is some $\beta^* \in (0, 1)$ such that $b \in (-1, -\beta^*)$, then there exists $\bar{\beta} \in (0, 1)$ such that, when β crosses $\bar{\beta}$ from above, (k^*, d^*) undergoes a flip bifurcation leading to persistent period-2 cycles.

(iii) When the consumption good is capital intensive, i.e. $b < 0$, and $\epsilon_{cd}, \epsilon_{dc} < 0$, i.e. $u_{cd} > 0$, the steady state (k^*, d^*) is saddle-point stable with damped oscillations if and only if $b \in (-\infty, -1) \cup (-\beta, 0)$ and $\zeta \in (0, \underline{\zeta}) \cup (\bar{\zeta}, +\infty)$. Moreover, if there is some $\beta^* \in (0, 1)$ such that $b \in (-1, -\beta^*)$, then there exists $\bar{\beta} \in (0, 1)$ such that, when β crosses $\bar{\beta}$ from above or ζ crosses the bifurcation values $\underline{\zeta}$ or $\bar{\zeta}$, (k^*, d^*) undergoes a flip bifurcation leading to persistent period-2 cycles.

Proof. See Appendix A.5. \square

Proposition 4 provides two independent mechanisms leading to the existence of endogenous fluctuations. The first one is based on the properties of preferences through the sign of the cross derivative u_{cd} and is the most interesting in our context since it allows to generate period-2 cycles in a two-sector model even under a capital intensive investment good sector—a condition which is known since Benhabib and Nishimura (1985) to ensure monotone convergence in a standard two-sector optimal growth model.

To provide further economic insights, let us consider an instantaneous increase in the capital stock k_t . Using $c_t + d_t = T(k_t, y_t)$ and $T_k > 0$, it follows that c_t increases, and thus, taking that the marginal utility of second period consumption u_d is larger as $u_{dc} > 0$, a constant utility level $u(c_t, d_{t+1})$ can be obtained from a decrease of d_{t+1} . Using then the first equation in (12) and taking d_{t+2} as given, we get

$$\frac{\Delta c_{t+1}}{\Delta c_t} = \frac{u_{dc}}{u_{cc}\beta} + \frac{u_{dd}}{u_{cc}\beta} \frac{\Delta d_{t+1}}{\Delta c_t} < 0.$$

Finally, since $c_{t+1} + d_{t+1} = T(k_{t+1}, y_{t+1})$, total consumption at time $t + 1$ is lower, which in turn implies a lower capital

stock k_{t+1} when y_{t+1} holds constant. Endogenous fluctuations are thus generated from intertemporal consumption allocations based on some substitution effects between the first and second period consumptions. The important result is that the elasticity of intertemporal substitution needs to be large enough to allow sufficient substitution between c_t and d_{t+1} to generate aggregate oscillations, but should not be too large to be compatible with the transversality conditions (13) and a convergence process towards the period-two cycle.

The second mechanism is, as in the two-sector optimal growth model, based on the properties of sectoral technologies through the sign of the capital intensity difference across sectors. Following Benhabib and Nishimura (1985), we can use the Rybczinski and Stolper–Samuelson effects to provide a simple economic intuition for this result. Assume indeed that the consumption good is capital intensive, i.e. $b < 0$, and consider an instantaneous increase in the capital stock k_t . This results in two opposing mechanisms:

- On the one hand, the trade-off in production becomes more favorable to the consumption good, and the Rybczinsky effect implies a decrease of the output of the capital good y_t . This tends to lower both the investment and the capital stock in the next period k_{t+1} .

- On the other hand, in the next period the decrease of k_{t+1} implies again through the Rybczinsky effect an increase of the output of the capital good y_{t+1} . Indeed the decrease of k_{t+1} improves the trade-off in production in favor of the investment good which is relatively less intensive in capital and this tends to increase the investment and the capital stock in period $t+2$, k_{t+2} .

Of course, under both mechanisms, the existence of persistent fluctuations requires that the oscillations in consumption and relative prices must not present intertemporal arbitrage opportunities. Consequently, a minimum level of myopia, i.e. a low enough value for the degree of altruism (or discount rate) β , is thus necessary.

Note finally that in case (iii) of Proposition 4, both mechanisms hold at the same time. Interestingly, using both β and ζ as two bifurcation parameters allows to consider a co-dimension 2 bifurcation which corresponds to the flip bifurcation with a 1:2 resonance where two characteristic roots are equal to -1 simultaneously. As shown in Kuznetsov (1998), in such a configuration, under non-degeneracy conditions, the steady state is either saddle-point stable or elliptic. This last case may give rise to the existence of quasi-periodic cycles which are usually associated to a Hopf bifurcation.

4.2. Quasi-periodic cycles under strictly concave preferences: A mixed mechanism

As explained above, using a non-strictly concave utility function is convenient in the sense that it reduces the degree-4 characteristic polynomial to the product of two degree-2 polynomials. In such a framework, we have shown that the preference and technology mechanisms are separated and lead independently to the occurrence of endogenous fluctuations through period-two cycles. Relaxing this simplifying assumption, our objective here is to prove that the preference and technology mechanisms can mix together, and then complexify and amplify the possible endogenous fluctuations in the context of strictly concave preferences.

We start by providing general sufficient conditions allowing to rule out the existence of complex roots.

Proposition 5. *Under Assumptions 1–3, let the utility function $u(c, d)$ be strictly concave. Then the roots of the characteristic polynomial (24) are necessarily real in the following cases:*

(i) for any sign of ϵ_{cd} , ϵ_{cd} if the investment good sector is capital intensive, i.e. $b > 0$,

(ii) if $\epsilon_{cd}, \epsilon_{cd} > 0$ and the consumption good sector is capital intensive, i.e. $b < 0$.

Proof. See Appendix A.6. \square

Necessary conditions for the existence of complex roots are therefore based on the two mechanisms that generate endogenous fluctuations in the non-strictly concave case, namely $b < 0$ **and** $\epsilon_{cd}, \epsilon_{cd} < 0$. It is also important to notice that both the non separability of the utility function **and** the OLG structure are necessary to get complex roots.

In order to provide clear-cut conditions for the existence of complex characteristic roots and a Hopf bifurcation with quasi-periodic cycles, we consider a general class of homogeneous of degree $\gamma \leq 1$ utility functions. We introduce the share of first period consumption within total utility $\phi(c, d) \in (0, \gamma)$ defined by:

$$\phi(c, d) = \frac{u_c(c, d)c}{u(c, d)}. \quad (25)$$

Accordingly, the share of second period consumption within total utility is defined as $\gamma - \phi(c, d) \in (0, 1)$, and the first order condition (17) gives

$$\frac{c}{d} = \frac{\beta\phi}{\gamma - \phi} \quad (26)$$

Considering that $c + d = T(k, y)$ we get

$$d = \frac{(\gamma - \phi)T(k, y)}{\gamma - \phi(1 - \beta)} \quad (27)$$

Moreover, using (4), the elasticities of interest are given by

$$\begin{aligned} \epsilon_{cd} &= -\frac{\epsilon_{cc}}{1 - \epsilon_{cc}(1 - \gamma)}, \quad \epsilon_{dc} = -\frac{(\gamma - \phi)\epsilon_{cc}}{\phi[1 - \epsilon_{cc}(1 - \gamma)]}, \\ \epsilon_{dd} &= \frac{(\gamma - \phi)\epsilon_{cc}}{\phi - \epsilon_{cc}(1 - \gamma)(2\phi - \gamma)} \end{aligned} \quad (28)$$

We then need to impose a restriction on ϵ_{cc} to ensure concavity and the normality assumptions, namely $\epsilon_{cc} \leq \gamma/[\phi(1 - \gamma)] \equiv \bar{\epsilon}_{cc}$. Strict concavity is obtained if and only if $\gamma < 1$ and it is straightforward to get $\epsilon_{dd} > 0$ while $\epsilon_{cd}, \epsilon_{cd} < 0$ if and only if $\epsilon_{cc} < 1/(1 - \gamma) \equiv \tilde{\epsilon}_{cc} (< \bar{\epsilon}_{cc})$. Moreover, the elasticity of substitution between the two life-cycle consumption levels is now defined by:

$$\zeta(\phi) = \frac{\epsilon_{cc}(\gamma - \phi)}{\gamma - \phi\epsilon_{cc}(1 - \gamma)} \in (0, +\infty) \quad (29)$$

Notably, if $\epsilon_{cc} < \tilde{\epsilon}_{cc}$, then $\zeta(\phi) \in (0, \epsilon_{cc})$.

We first derive sufficient conditions to ensure saddle-point property of the steady state with real characteristic roots.

Proposition 6. *Let the utility function be homogeneous of degree $\gamma < 1$, and assume that $\epsilon_{cc} < \tilde{\epsilon}_{cc}$, $b \in (-\infty, -1) \cup (-\beta, 0)$ and*

$$-\frac{\epsilon_{ck}}{b\epsilon_{rk}} > 1 \quad (30)$$

Then there exist $0 < \underline{\zeta} \leq \bar{\zeta} < \epsilon_{cc}$ and $\hat{\epsilon}_{cc} \in (0, \tilde{\epsilon}_{cc})$ such that when $\zeta \in (0, \underline{\zeta}) \cup (\bar{\zeta}, \epsilon_{cc})$ the characteristic roots are real and the steady-state is saddle-point stable. Moreover,

(i) *when $\zeta \in (0, \underline{\zeta})$, the optimal path converges towards the steady state with oscillations if $\epsilon_{cc} \in (0, \hat{\epsilon}_{cc})$ or monotonically if $\epsilon_{cc} \in (\hat{\epsilon}_{cc}, \tilde{\epsilon}_{cc})$,*

(ii) *when $\zeta \in (\bar{\zeta}, \epsilon_{cc})$, the optimal path converges towards the steady state with oscillations.*

Proof. See Appendix A.7. \square

Condition (30) allows to get the existence of the bound $\hat{\epsilon}_{cc}$ and thus the occurrence of oscillations when the elasticity of intertemporal substitution is low, i.e. $\zeta \in (0, \underline{\zeta})$. This restriction

can be easily interpreted. Denoting σ_i the elasticity of capital-labor substitution in sector $i = 0, 1$ and using [Drugeon \(2004\)](#), we can relate the ratio of elasticities $\varepsilon_{ck}/\varepsilon_{rk}$ to an aggregate elasticity of substitution between capital and labor, denoted Σ , which is obtained as a weighted sum of the sectoral elasticities σ_i :⁴⁶

$$\frac{\varepsilon_{ck}}{\varepsilon_{rk}} = \left(\frac{T}{l_0^2} \right) \frac{s}{1-s} \frac{\Sigma}{GDP} \text{ with } \Sigma = \frac{GDP}{pykT} (pyk_0 l_0 \sigma_0 + Tk_1 l_1 \sigma_1), \quad (31)$$

$GDP = T + py$ and $s = rk/GDP$ the share of capital income in GDP. Therefore, oscillations when $\phi \in (\underline{\phi}, \bar{\gamma})$ are associated with a large aggregate elasticity of substitution between capital and labor *i.e.*, large enough sectoral elasticities of capital-labor substitution.

[Proposition 6](#) implies that the existence of complex roots, if any, requires to consider intermediate values for the elasticity of intertemporal substitution ζ between the first and second period consumptions, *i.e.* $\zeta \in (\underline{\zeta}, \bar{\zeta})$. As mentioned previously, the degree-4 characteristic polynomial (22) is a quasi-palindromic equation that can be solved explicitly, and its roots can be determined using only quadratic equations. As shown in [Appendix A.8](#), we apply this methodology in order to provide sufficient conditions for the occurrence of complex roots and a Hopf bifurcation.

We can indeed derive the following result:

Proposition 7. *Let the utility function be homogeneous of degree $\gamma < 1$, and assume that $\varepsilon_{cc} < \bar{\varepsilon}_{cc}$ and $b \in (-\beta, 0)$. Then there exist $\bar{b} \in (-\beta, 1)$, $\underline{\gamma} \in (0, 1)$, $\underline{\varepsilon}_{cc}, \bar{\varepsilon}_{cc} \in (0, \bar{\varepsilon}_{cc})$, $\bar{\varepsilon} > 0$ and four critical values $(\underline{\zeta} \leq \underline{\zeta}^c < \underline{\zeta}^H < \bar{\zeta}^H < \bar{\zeta}^c \leq \bar{\zeta})$ such that when $b \in (-\beta, \bar{b})$, $\gamma \in (\underline{\gamma}, 1)$, $\varepsilon_{cc} \in (\underline{\varepsilon}_{cc}, \bar{\varepsilon}_{cc})$ and*

$$-\frac{\varepsilon_{ck}}{b\varepsilon_{rk}} < \bar{\varepsilon} \quad (32)$$

the following results hold:

(i) the steady state (k^*, d^*) is saddle-point stable with damped oscillations if $\zeta \in (\underline{\zeta}^c, \underline{\zeta}^H) \cup (\bar{\zeta}^H, \bar{\zeta}^c)$,

(ii) when ζ crosses the bifurcation values $\underline{\zeta}^H$ or $\bar{\zeta}^H$, (k^*, d^*) undergoes a Hopf bifurcation leading to persistent quasi-periodic cycles.

Proof. See [Appendix A.8](#). \square

From a theoretical point of view, [Proposition 7](#) provides a strong conclusion as it shows that a Hopf bifurcation and quasi-periodic cycles can occur in a two-sector optimal growth framework as long as it is based on an OLG structure with non-separable and strictly concave preferences. The preference mechanism based on a substitutability effect between first and second period consumptions, and the technology mechanism based on a capital intensive consumption good sector feed each other when the utility function is strictly concave and amplify the endogenous fluctuations of capital and consumption. More complex quasi-periodic fluctuations can thus occur for intermediate values for the elasticity ε_{cc} and the elasticity of intertemporal substitution ζ between the first and second period consumptions, together with, using (31), a not too large value for the sectoral elasticities of capital-labor substitution.

[Propositions 4](#) and [7](#) have provided conditions for the existence of persistent cycles in output, capital and consumptions. However, as discussed in the data section, there is some evidence that the inheritance ratio and the inheritance flow are also driven by endogenous fluctuations. In this respect, [Proposition 8](#) now shows that both variables are characterized by persistent cycles

Proposition 8. *Under [Assumptions 1–3](#), the local stability properties provided in [Propositions 4](#) and [7](#) hold for both bequests as defined by (14) and bequests as a proportion of GDP as defined by (15). In particular, bequests and bequests as a proportion of GDP can be characterized by optimal periodic and quasi-periodic cycles.*

Proof. See [Appendix A.9](#). \square

[Proposition 8](#) proves the existence of long-run fluctuations of both bequests and bequests as a proportion of GDP, and provides a theoretical basis to explain the empirical evidence derived in [Section 2](#) showing that the annual inheritance flow as a fraction of national income displays medium- and long-run fluctuations. In contrast to the recent conclusion of [Piketty \(2011\)](#), we prove here that the recent increase of the ratio of inheritance to output can be interpreted as a portion of a long run cycle and not necessarily as a return to a steady state position characterizing the 19th and first half of the 20th century. As a consequence taxation of bequests may not be necessary as a medium- or long-lasting reversion of the ratio may occur later on.

5. Simple illustrations

To illustrate all our results, let us consider the general class of homogeneous of degree $\gamma \leq 1$ utility functions as described by the share of first period consumption within total utility $\phi(c, d) \in (0, \gamma)$ as defined by (25) and the elasticities (28). For the production side, we assume as in [Baierl et al. \(1998\)](#) that the consumption and investment goods are produced with Cobb–Douglas technologies as follows:

$$y_0 = k_0^{\alpha_0} l_0^{1-\alpha_0}, \quad y = k_1^{\alpha_1} l_1^{1-\alpha_1} \quad (33)$$

It can be shown that

$$b = \frac{\beta(\alpha_1 - \alpha_0)}{1 - \alpha_0} \quad (34)$$

5.1. Period-two cycles

We assume here that γ is set to one, which means that [Assumption 4](#) and $\varepsilon_{cc} < \bar{\varepsilon}_{cc}$ hold, and that the elasticity of intertemporal substitution between c and d satisfies $\zeta = \varepsilon_{cc}(1 - \phi) < \varepsilon_{cc}$. The following Corollary provides illustrations of all the cases of [Proposition 4](#).⁴⁷

Corollary 1. *Let the utility function be homogeneous of degree 1 and the sectoral production functions be given by (33) with $\alpha_0 > (1 + \alpha_1)/2 > \alpha_1$. Let $\beta_1 = (1 - \alpha_0)/(\alpha_0 - \alpha_1)$, $\beta_2 = (1 - \phi)/\phi$ and $\phi^* = (\alpha_0 - \alpha_1)/(1 - \alpha_1)$. Then the following cases hold:*

(i) If $\phi \leq 1/2$, the steady state (k^*, d^*) is saddle-point stable with damped oscillations if and only if $\beta \in (\beta_1, 1)$ and β_1 is a flip bifurcation value.

(ii) If $\phi > 1/2$, the steady state (k^*, d^*) is saddle-point stable with damped oscillations if and only if $\beta > \underline{\beta} = \max\{\beta_1, \beta_2\}$. Moreover, $\beta_1 \geq \beta_2$ if and only if $\phi \geq \phi^*$. Both β_1 and β_2 are flip bifurcation values implying that two flip bifurcations can occur successively when β is decreased from 1. More importantly, if $\phi = \phi^*$, then $\beta_1 = \beta_2 = \underline{\beta}$ and a co-dimension 2 flip bifurcation with a 1:2 resonance generically occurs.

Proof. See [Appendix A.10](#). \square

⁴⁶ The expression of Σ is derived from [Proposition 2](#) in [Drugeon \(2004\)](#).

⁴⁷ All the results of this Corollary can also be obtained using instead CES technologies with non unitary sectoral elasticities of capital-labor substitution. It can be shown indeed that the existence of a flip bifurcation is robust to a wide range of values for these parameters. A proof of this claim is available upon request.

Corollary 1 then shows that all our results on the existence of period-two cycle easily occur with a standard homogeneous utility function and Cobb–Douglas technologies. The most important illustration is provided by case (iv) with $\phi = \phi^*$. While providing a precise dynamic analysis of this co-dimension 2 bifurcation goes far beyond the objectives of this paper, it is worthwhile to mention that this case provides an interesting possibility of smooth endogenous fluctuations for the main aggregate variables which does not arise under a standard flip bifurcation. Indeed, while there does not *a priori* exist complex characteristic roots under a linear homogeneous utility function, [Kuznetsov \(1998\)](#) shows that under a 1:2 resonance, the steady state can be elliptic and a stable limit cycle, similar to those that arise under a Hopf bifurcation, can occur.

5.2. Quasi-periodic cycles

We assume here a strictly concave homogeneous utility function with $\gamma < 1$ and we focus on the case $\gamma = 0.98$. We also assume a standard value $\epsilon_{cc} = 1$ that satisfies $\epsilon_{cc} \in (\underline{\epsilon}_{cc}, \bar{\epsilon}_{cc})$. It is quite difficult to properly evaluate the degree of altruism β . However, if we consider that the amount of bequest transmitted to offsprings can be determined on the basis of the discounted value of wealth, we can use the annual discount factor which is often estimated to be around 0.96. Since one period in an OLG model is about 30 years, we may approximate the degree of altruism as $\beta = 0.96^{30} \approx 0.294$. For the sectoral Cobb–Douglas technologies we consider $\alpha_0 = 0.6$ and $\alpha_1 = 0.21$ so that the consumption good is capital intensive with $b \approx -0.28665$ close to $-\beta$.⁴⁸ The bounds exhibited in [Proposition 7](#) are equal to $\underline{\zeta}^c \approx 0.1194$ and $\bar{\zeta}^c \approx 0.6083$. We then find that the characteristic polynomial (44) admits four characteristic roots $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ that are complex conjugate by pair with $\lambda_1\lambda_2 > 1$ and $\lambda_3\lambda_4 < 1$ if $\zeta \in (\underline{\zeta}^c, \underline{\zeta}^H) \cup (\bar{\zeta}^H, \bar{\zeta}^c)$ while $\lambda_3\lambda_4 > 1$ if $\zeta \in (\underline{\zeta}^H, \bar{\zeta}^H)$, with $\underline{\zeta}^H \equiv 0.3193$ and $\bar{\zeta}^H \equiv 0.426$. Moreover $\lambda_3\lambda_4 = 1$ when $\zeta = \underline{\zeta}^H$ or $\bar{\zeta}^H$. As a result $\underline{\zeta}^H$ and $\bar{\zeta}^H$ are Hopf bifurcation values giving rise to quasi-periodic cycles in their neighborhood.

5.3. Positive bequests and endogenous fluctuations

Consider first the case of a linear homogeneous utility function ($\gamma = 1$) and assume as in [Corollary 1](#) that $\alpha_0 > (1 + \alpha_1)/2 > \alpha_1$. Using Eq. (27) and the expressions for the capital stock k^* , the interest rate r^* and the output for total consumption $T(k^*, k^*)$ in [Appendix A.10](#), we derive that $r^*k^* > d^*$ and thus $x^* > 0$ if and only if

$$\alpha_0\beta\phi > (1 - \phi)(1 - \alpha_0 - \beta\alpha_1) \quad (35)$$

We get the following results:

(i) If $2\alpha_1 > \alpha_0 > 1 - \alpha_1$ and $\beta > (1 - \alpha_0)/\alpha_1 \equiv \beta_3$, then $\beta_1 > \beta_3$ and $x^* > 0$ for any $\phi \in (0, 1)$. Then case i) of [Corollary 1](#) holds with positive bequests. Moreover, if $\phi \in (1/2, \phi_1)$ with $\phi_1 \equiv \alpha_1/(1 - \alpha_0 + \alpha_1)$, then $\phi^* \in (1/2, \phi_1)$, $\beta_1, \beta_2 > \beta_3$ and the case ii) of [Corollary 1](#) holds with positive bequests.

(ii) If $\alpha_0 < 1 - \alpha_1$, $\phi > 1/2$ and $\beta > (1 - \phi)/[\alpha_0\phi + \alpha_1(\gamma - \phi)] \equiv \beta_4$, then $x^* > 0$. Moreover, if $\min\{2\alpha_1, 1 - \alpha_1\} > \alpha_0 > \alpha_1$ then $\beta_1, \beta_2 > \beta_4$ and the case ii) of [Corollary 1](#) holds with positive bequests.

⁴⁸ The existence of a Hopf bifurcation can also be obtained using instead CES technologies with non unitary sectoral elasticities of capital-labor substitution. As in the case with period-two cycles, our conclusions are robust to a wide range of values for these parameters. A proof of this claim is also available upon request.

We have then proved that with a linear homogeneous utility function, the existence of period-two cycles and possibly of a co-dimension 2 flip bifurcation with a 1:2 resonance is compatible with positive bequests.

Let us finally illustrate the existence of quasi-periodic cycles under positive bequests when the utility function is homogeneous of degree $\gamma < 1$. The necessary and sufficient condition (35) for positive bequests $x^* > 0$ becomes

$$\alpha_0\phi\beta > (\gamma - \phi)(1 - \alpha_0 - \beta\alpha_1) > 0 \quad (36)$$

Considering the particular illustration in [Section 5.2](#) with $1 - \alpha_0 - \beta\alpha_1 > 0$ and $\alpha_0 > \alpha_1$, it follows from (36) that bequests are positive if and only if

$$\phi > \frac{\gamma(1 - \alpha_0 - \beta\alpha_1)}{1 - \alpha_0 + \beta(\alpha_0 - \alpha_1)} \equiv \tilde{\phi}_\gamma$$

which is equivalent to

$$\zeta < \frac{\alpha_0\beta\epsilon_{cc}}{1 - \alpha_0 + \beta(\alpha_0 - \alpha_1) - \epsilon_{cc}(1 - \gamma)(1 - \alpha_0 - \beta\alpha_1)} \equiv \tilde{\zeta}_\gamma$$

This condition can be satisfied only if

$$\epsilon_{cc} < \frac{1 - \alpha_0 + \beta(\alpha_0 - \alpha_1)}{(1 - \gamma)(1 - \alpha_0 - \beta\alpha_1)}$$

With $\gamma = 0.98$, $\beta = 0.294$, $\epsilon_{cc} = 1$, $\alpha_0 = 0.6$ and $\alpha_1 = 0.21$, we get $\tilde{\zeta}_\gamma \approx 0.3566 \in (\underline{\zeta}^H, \bar{\zeta}^H)$. It follows that positive bequests are compatible with quasi-periodic cycles. Indeed, the steady state, which is characterized by strictly positive bequests if $\zeta < \tilde{\zeta}_\gamma$, is saddle-point stable with damped oscillations if and only if $\zeta \in (\underline{\zeta}^c, \underline{\zeta}^H)$. Moreover, when ζ crosses the bifurcation values $\underline{\zeta}^H$ from below, the steady state undergoes a Hopf bifurcation leading to persistent quasi-periodic cycles and thus long-run fluctuations of bequests.

Remark 5. The conditions for bifurcating eigenvalues associated to both period-two cycles and quasi-periodic cycles, and which are based on the first-order approximation of the dynamical system, do not ensure the existence of stable cycles. Although some generic arguments allow to ensure a non-degeneracy property, the stability properties depend on higher order terms of the approximated dynamical system. As shown for instance in [Kalra \(1996\)](#), it is necessary to compute a projection of the dynamical system on the center manifold associated to the bifurcating eigenvalues. Besides the technical complexity of such computations, the saddle-point structure of our dimension-4 model unfortunately prevents from getting explicit results. In the case of a specific example with a CES utility function and Cobb–Douglas technologies, a possibility could be to determine numerically the center manifold for the flip (respectively, Hopf) bifurcation using Taylor expansions or perturbation methods (e.g., [Galizia \(2018\)](#)). However this is beyond the scope of the current paper.

6. Concluding comments

This paper explores the existence of limit cycles to explain the behavior of the annual flow of inheritance (in level or as a share of national income).

We have first provided some empirical evidence for medium- and long-run swings in the inheritance flows (in level and as a fraction of national income) in Sweden, France, UK, and Germany.⁴⁹ Notably, the contribution of the medium term component does not run counter the existence of endogenous (stochastic) cycles.

⁴⁹ For UK and Germany, empirical evidences are available upon request.

Based on this, using a two-sector Barro-type (Barro, 1974) OLG model with non-separable preferences and bequests, we have shown that two endogenous mechanisms, which can operate independently or together, can be identified as long as agents are sufficiently altruistic. The first mechanism relies on the elasticity of intertemporal substitution or equivalently the sign of the cross-derivative of the utility function whereas the second rests on sectoral technologies through the sign of the capital intensity difference across the two sectors. Accordingly, mild and plausible perturbations of these parameters can lead to endogenous fluctuations through period-2 cycles or Hopf bifurcations.

From a methodological point of view, we have exploited the quasi-palindromic nature of the characteristic equation associated to the optimal solution of a two-sector OLG model with altruistic parents and positive bequests to derive some meaningful sufficient conditions for the occurrence of complex roots.

A first avenue of future research would be to consider a stochastic version of our model and thus to characterize the existence of stochastic limit cycles. Another research perspective would be to study more deeply the econometrics of long-run/endogenous cycles.

Appendix

A.1. Proof of Proposition 1

Consider in a first step Eq. (16). Notice that the steady state value for k only depends on the characteristics of the technologies and is independent from the utility function. Moreover, this equation is equivalent to the equation which defines the stationary capital stock of a standard two-sector optimal growth model. The proof of Theorem 3.1 in Becker and Tsyganov (2002) restricted to the case of one homogeneous agent applies so that there exists one unique k^* solution of this equation.

Consider now Eq. (17) evaluated at k^* . We get:

$$\frac{u_d(T(k^*, k^*) - d, d)}{u_c(T(k^*, k^*) - d, d)} \equiv h(d) = \beta \quad (37)$$

The function $h(d)$ is defined over $(0, T(k^*, k^*))$ and satisfies

$$h'(d) = \frac{\frac{u_{dd}}{u_d} - \frac{u_{cd}}{u_c} + \frac{u_{cc}}{u_c} - \frac{u_{cd}}{u_d}}{u_c u_d} = -\beta \left[\frac{1}{d} \left(\frac{1}{\epsilon_{dd}} - \frac{1}{\epsilon_{cd}} \right) + \frac{1}{c} \left(\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{dc}} \right) \right]$$

Assumption 3 implies that $h'(d) < 0$. This monotonicity property together with the boundary conditions (19) finally ensure the existence and uniqueness of a solution $d^* \in (0, T(k^*, k^*))$ of Eq. (37).

Consider finally Eq. (18). Note that k^* does not depend on the utility function $u(c, d)$. The condition of positive stationary bequest ($x^* > 0$) is equivalent to $r^* k^* = T_k(k^*, k^*) k^* > d^*$. Then, under the boundary conditions (19), for any given $\beta \in (0, 1)$, the stationary consumption d^* is such that $x^* > 0$ if the boundary condition (20) also holds. \square

A.2. Proof of Proposition 2

Using (4) and the fact that at the steady state $-T_y^* = \beta T_k^*$, total differentiation of the first order Eqs. (12) gives after tedious but

straightforward computations:

$$\begin{aligned} & -\Delta k_t \frac{\beta T_k^* \epsilon_{cc}}{\epsilon_{dc}} + \Delta k_{t+1} \beta T_k^* \left(1 + \frac{\beta \epsilon_{cc}}{\epsilon_{dc}} \right) + \Delta d_t \frac{\beta \epsilon_{cc}}{\epsilon_{dc}} \\ & - \Delta d_{t+1} \beta \left(1 + \frac{\beta \epsilon_{cc} \epsilon_{cd}}{\epsilon_{dc} \epsilon_{dd}} \right) \\ & = \Delta k_{t+2} \beta^2 T_k^* - \Delta d_{t+2} \frac{\beta^2 \epsilon_{cc}}{\epsilon_{dc}} \\ & \Delta k_t \left(\frac{\beta T_k^{*2}}{\epsilon_{cc} c^* T_{kk}^*} - b \right) - \Delta k_{t+1} \left(\frac{\beta(1+\beta) T_k^{*2}}{\epsilon_{cc} c^* T_{kk}^*} - \beta - b^2 \right) \\ & - \Delta d_t \frac{\beta T_k^*}{\epsilon_{cc} c^* T_{kk}^*} \\ & + \Delta d_{t+1} \frac{\beta T_k^*}{\epsilon_{cc} c^* T_{kk}^*} \left(1 + \frac{\beta \epsilon_{cc}}{\epsilon_{dc}} \right) = -\Delta k_{t+2} \beta \left(\frac{\beta T_k^{*2}}{\epsilon_{cc} c^* T_{kk}^*} - b \right) \\ & + \Delta d_{t+2} \frac{\beta^2 T_k^*}{\epsilon_{dc} c^* T_{kk}^*} \end{aligned}$$

Denoting $\Delta \kappa_t = \Delta k_{t+1}$ and $\Delta \delta_t = \Delta d_{t+1}$, we get the following matrix expression of the previous linear system:

$$\begin{aligned} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \beta^2 T_k^* & -\frac{\beta^2 \epsilon_{cc}}{\epsilon_{dc}} \\ 0 & 0 & -\beta \left(\frac{\beta T_k^{*2}}{\epsilon_{cc} c^* T_{kk}^*} - b \right) & \frac{\beta^2 T_k^*}{\epsilon_{dc} c^* T_{kk}^*} \end{pmatrix} \begin{pmatrix} \Delta k_{t+1} \\ \Delta d_{t+1} \\ \Delta \kappa_{t+1} \\ \Delta \delta_{t+1} \end{pmatrix} \\ & = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\beta T_k^* \epsilon_{cc}}{\epsilon_{dc}} & \frac{\beta \epsilon_{cc}}{\epsilon_{dc}} & \beta T_k^* \left(1 + \frac{\beta \epsilon_{cc}}{\epsilon_{dc}} \right) & -\beta \left(1 + \frac{\beta \epsilon_{cc} \epsilon_{cd}}{\epsilon_{dc} \epsilon_{dd}} \right) \\ \frac{\beta T_k^{*2}}{\epsilon_{cc} c^* T_{kk}^*} - b & \frac{\beta T_k^*}{\epsilon_{cc} c^* T_{kk}^*} & -\frac{\beta(1+\beta) T_k^{*2}}{\epsilon_{cc} c^* T_{kk}^*} + \beta + b^2 & \frac{\beta T_k^*}{\epsilon_{cc} c^* T_{kk}^*} \left(1 + \frac{\beta \epsilon_{cc}}{\epsilon_{dc}} \right) \end{pmatrix} \\ & \quad \times \begin{pmatrix} \Delta k_t \\ \Delta d_t \\ \Delta \kappa_t \\ \Delta \delta_t \end{pmatrix} \\ & \Leftrightarrow \mathcal{A} \begin{pmatrix} \Delta k_{t+1} \\ \Delta d_{t+1} \\ \Delta \kappa_{t+1} \\ \Delta \delta_{t+1} \end{pmatrix} = \mathcal{B} \begin{pmatrix} \Delta k_t \\ \Delta d_t \\ \Delta \kappa_t \\ \Delta \delta_t \end{pmatrix} \text{ with } \mathcal{A} = \begin{pmatrix} I & 0 \\ 0 & \mathcal{A}_{22} \end{pmatrix} \text{ and} \\ & \quad \mathcal{B} = \begin{pmatrix} 0 & I \\ \mathcal{B}_{21} & \mathcal{B}_{22} \end{pmatrix} \end{aligned}$$

Matrix \mathcal{A} is invertible as $\det \mathcal{A} = \det \mathcal{A}_{22} = \beta^3 b \epsilon_{cc} / \epsilon_{dc}$, and we get

$$\mathcal{A}^{-1} = \begin{pmatrix} I & 0 \\ 0 & \mathcal{A}_{22}^{-1} \end{pmatrix} \text{ with } \mathcal{A}_{22}^{-1} = \begin{pmatrix} \frac{T_k^*}{\beta b \epsilon_{cc} c^* T_{kk}^*} & \frac{1}{\beta b} \\ \frac{\epsilon_{dc}}{\beta^2 \epsilon_{cc}} \left(\frac{\beta T_k^{*2}}{b \epsilon_{cc} c^* T_{kk}^*} - 1 \right) & \frac{\epsilon_{dc} T_k^*}{\beta b \epsilon_{cc}} \end{pmatrix}$$

The linearized dynamical system can then be expressed as follows

$$\begin{aligned} & \begin{pmatrix} \Delta k_{t+1} \\ \Delta d_{t+1} \\ \Delta \xi_{t+1} \\ \Delta \zeta_{t+1} \end{pmatrix} = \mathcal{A}^{-1} \mathcal{B} \begin{pmatrix} \Delta k_t \\ \Delta d_t \\ \Delta \xi_t \\ \Delta \zeta_t \end{pmatrix} \\ & = \begin{pmatrix} 0 & I \\ \mathcal{A}_{22}^{-1} \mathcal{B}_{21} & \mathcal{A}_{22}^{-1} \mathcal{B}_{22} \end{pmatrix} \begin{pmatrix} \Delta d_t \\ \Delta \xi_t \\ \Delta \zeta_t \end{pmatrix} \equiv J \begin{pmatrix} \Delta k_t \\ \Delta d_t \\ \Delta \xi_t \\ \Delta \zeta_t \end{pmatrix} \end{aligned}$$

Using (21), tedious but straightforward computations give the characteristic polynomial

$$\mathcal{P}(\lambda) = \lambda^4 - \lambda^3 B + \lambda^2 C - \lambda \frac{B}{\beta} + \frac{1}{\beta^2} \quad (38)$$

with

$$\begin{aligned} B &= -\frac{\beta}{b\epsilon_{cc}} \frac{\epsilon_{ck}}{\epsilon_{rk}} \left(\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}} \right) + \frac{\beta+b^2}{\beta b} + \frac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}} \\ C &= -\frac{(1+\beta)}{b\epsilon_{cc}} \frac{\epsilon_{ck}}{\epsilon_{rk}} \left(\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}} \right) + \frac{\beta+b^2}{\beta b} \left(\frac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}} \right) + \frac{2}{\beta} \end{aligned} \quad (39)$$

After simplifications we get the expression (24). It is worth noticing that B is equal to the trace of J while $1/\beta^2$ is the determinant of J .

Consider now that λ is a root of the characteristic polynomial (24), i.e. $\mathcal{P}(\lambda) = 0$. It follows obviously that if λ is complex then its conjugate $\bar{\lambda}$ is also a characteristic root. Let us then consider $\mathcal{P}((\beta\lambda)^{-1})$, namely

$$\begin{aligned} \mathcal{P}\left(\frac{1}{\beta\lambda}\right) &= \left[\frac{1}{\beta^2\lambda^2} - \frac{1}{\beta\lambda} \left(\frac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}} \right) + \frac{1}{\beta} \right] \frac{\left(\frac{b}{\beta\lambda} - 1\right)\left(\frac{1}{\lambda} - b\right)}{\beta b} \\ &+ \frac{1}{\beta\lambda} \left(\frac{1}{\beta\lambda} - 1 \right) \left(\frac{1}{\beta\lambda} - \frac{1}{\beta} \right) \frac{\beta}{b\epsilon_{cc}} \frac{\epsilon_{ck}}{\epsilon_{rk}} \left(\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}} \right) \\ &= \frac{1}{\beta^4\lambda^4} \left\{ \left[\lambda^2 - \lambda \left(\frac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}} \right) + \frac{1}{\beta} \right] \frac{(\lambda b - 1)(\lambda\beta - b)}{\beta b} \right. \\ &\left. + \lambda(\lambda - 1) \left(\lambda - \frac{1}{\beta} \right) \frac{\beta}{b\epsilon_{cc}} \frac{\epsilon_{ck}}{\epsilon_{rk}} \left(\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}} \right) \right\} = 0 \end{aligned}$$

It follows that $(\beta\lambda)^{-1}$ is also a characteristic root. The same argument applies for $(\beta\bar{\lambda})^{-1}$. It follows that the four characteristic roots are either all real, or given by two pairs of complex conjugates. Moreover, at least two roots or a pair of complex conjugate roots have a modulus larger than one. Considering that the structure of our model is similar to a standard optimal growth model, this result is the same as the one obtained by [Levhari and Liviatan \(1972\)](#).

The nature of the characteristic roots can be derived considering the following expressions:

$$\begin{aligned} \Delta &= \frac{256}{\beta^6} - \frac{192B^2}{\beta^5} - \frac{128C^2}{\beta^4} + \frac{288B^2C}{\beta^4} - \frac{60B^4}{\beta^4} - \frac{80B^2C^2}{\beta^3} + \frac{36B^4C}{\beta^3} \\ &- \frac{4B^6}{\beta^3} + \frac{16C^4}{\beta^2} - \frac{8B^2C^3}{\beta^2} + \frac{B^4C^2}{\beta^2} \\ D &= \frac{64}{\beta^2} - 16C^2 + 16B^2C - \frac{16B^2}{\beta} - 3B^4 \\ P &= 8C - 3B^2 \\ R &= B \left[B^2 + \frac{8}{\beta} - 4C \right] \end{aligned} \quad (40)$$

Since we already know that the characteristic roots are either all real, or all complex, we immediately derive that $\Delta \geq 0$. Tedious but straightforward computations also show that

$$\begin{aligned} D &= \frac{R}{B} \left[\frac{8}{\beta} - 3B^2 + 4C \right] \\ \Delta &= \frac{(\beta^2C^2 - 4\beta B^2 + 4\beta C + 4)R^2}{\beta^4 B^2} \end{aligned} \quad (41)$$

It follows that if $R = 0$ then $D = 0$ and $\Delta = 0$. This implies the following characterization of the roots:

(i) when $\Delta > 0$ the characteristic roots are real and distinct if $P < 0$ and $D < 0$, and given by two pairs of non-real complex conjugates if $P > 0$ or $D > 0$;

(ii) when $\Delta = R = D = 0$, there are two complex conjugates double roots or two real double roots depending on whether $P > 0$ or $P < 0$. \square

A.3. Proof of [Lemma 1](#)

Under [Assumption 4](#), let us denote the two degree-2 polynomials as follows

$$\mathcal{P}_1(\lambda) = \lambda^2 - \lambda \left(\frac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}} \right) + \frac{1}{\beta}, \quad \mathcal{P}_2(\lambda) = \frac{(\lambda b - 1)(\lambda\beta - b)}{\beta b} \quad (42)$$

The discriminant of $\mathcal{P}_1(\lambda)$ is equal to:

$$\Delta_1 = \left(\frac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}} + \frac{2}{\sqrt{\beta}} \right) \left(\frac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}} - \frac{2}{\sqrt{\beta}} \right)$$

Using (4) we get

$$\begin{aligned} \Delta_1 &= \left(\frac{1}{u_{cd}} \right)^2 \left(u_{cc} + \frac{2u_{cd}}{\sqrt{\beta}} + \frac{u_{dd}}{\beta} \right) \left(u_{cc} - \frac{2u_{cd}}{\sqrt{\beta}} + \frac{u_{dd}}{\beta} \right) \\ &= \left(\frac{1}{u_{cd}} \right)^2 \left(1 \quad \frac{1}{\sqrt{\beta}} \right) \begin{pmatrix} u_{cc} & u_{cd} \\ u_{dc} & u_{dd} \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{\sqrt{\beta}} \end{pmatrix} \\ &\times \begin{pmatrix} 1 & -\frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} u_{cc} & u_{cd} \\ u_{dc} & u_{dd} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{\beta}} \end{pmatrix} \end{aligned}$$

Under the concavity property in [Assumption 2](#), the Hessian matrix of the utility function $u(c, d)$ is quasi-negative definite which implies $\Delta_1 \geq 0$ and the associated characteristic roots are necessarily real. From $\mathcal{P}_2(\lambda)$ we obviously conclude that for any sign of the capital intensity difference b the associated characteristic roots are also necessarily real. \square

A.4. Proof of [Proposition 3](#)

Under [Assumptions 1–4](#), let $b \geq 0$ and $\epsilon_{cd}, \epsilon_{dc} \geq 0$, i.e. $u_{cd} \leq 0$. Using the fact that $\frac{\epsilon_{cc}}{\epsilon_{dc}} = \frac{\epsilon_{cd}}{\epsilon_{dd}}$, we derive the following expression

$$\mathcal{P}_1(\lambda) = \left(\lambda - \frac{\epsilon_{cc}}{\epsilon_{dc}} \right) \left(\lambda - \frac{\epsilon_{dc}}{\beta\epsilon_{cc}} \right) \quad (43)$$

The associated characteristic roots λ_1 and λ_2 are therefore both positive. Moreover we get:

$$\mathcal{P}_1(0) = \frac{1}{\beta} \geq 1$$

$$\mathcal{P}_1(1) = -\epsilon_{cc}\epsilon_{dc} \left(\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{dc}} \right) \left(\frac{1}{\beta\epsilon_{cc}} - \frac{1}{\epsilon_{dc}} \right)$$

The normality [Assumption 3](#) implies $\mathcal{P}_1(1) < 0$ and we conclude that the associated characteristic roots λ_1 and λ_2 are such that $\lambda_1 < 1$ and $\lambda_2 > 1$.

From $\mathcal{P}_2(\lambda)$, the associated characteristic roots λ_1 and λ_2 are both positive. Moreover we derive:

$$\mathcal{P}_2(0) = \frac{1}{\beta} \geq 1, \quad \mathcal{P}_2(1) = -\frac{(\beta-b)(1-b)}{\beta b}$$

From constant returns to scale, we get $wa_{01} + ra_{11} = p$ with $a_{01} = l_1/y$ and $a_{11} = k_1/y$. The second equation in (17) rewrites as $p = \beta r$. We then obtain after substitution in the previous equation $r(\beta - a_{11}) = wa_{01} > 0$ and thus

$$\beta - b = \frac{a_{00}(\beta - a_{11}) + a_{10}a_{01}}{a_{00}} > 0$$

When $b \geq 0$ we then necessarily have $b < \beta \leq 1$. It follows that $\mathcal{P}_2(0) < 0$ and we conclude that the associated characteristic roots λ_1 and λ_2 are such that $\lambda_1 < 1$ and $\lambda_2 > 1$. The steady state is therefore a saddle-point. \square

A.5. Proof of [Proposition 4](#)

(i) Under [Assumptions 1–4](#), let $b \geq 0$ and $\epsilon_{cd}, \epsilon_{dc} < 0$, i.e. $u_{cd} > 0$. As shown previously, we derive from $\mathcal{P}_2(\lambda) = 0$ that there exist two positive characteristic roots, one being lower than 1 and the other larger. From $\mathcal{P}_1(\lambda)$ as given by (43), the associated characteristic roots λ_1 and λ_2 are both negative. Moreover, we get:

$$\mathcal{P}_1(-1) = \left(1 + \frac{\epsilon_{cc}}{\epsilon_{dc}} \right) \left(1 + \frac{\epsilon_{dc}}{\beta\epsilon_{cc}} \right) = \frac{(\epsilon_{cc} + \epsilon_{dc})(\beta\epsilon_{cc} + \epsilon_{dc})}{\beta\epsilon_{cc}\epsilon_{dc}}$$

We conclude easily that

$$\mathcal{P}_1(-1) < 0 \Leftrightarrow \epsilon_{cc} \in (0, -\epsilon_{dc}) \cup (-\epsilon_{dc}/\beta, +\infty)$$

$$\mathcal{P}_1(-1) > 0 \Leftrightarrow \epsilon_{cc} \in (-\epsilon_{dc}, -\epsilon_{dc}/\beta)$$

It follows that the steady state is a saddle-point with damped oscillations when $\epsilon_{cc} \in (0, -\epsilon_{dc}) \cup (-\epsilon_{dc}/\beta, +\infty)$ and there exists a flip bifurcation with persistent period-2 cycles when ϵ_{cc} crosses the bifurcation values $-\epsilon_{dc}$ or $-\epsilon_{dc}/\beta$. Considering the expression of the elasticity of intertemporal substitution in consumption (5), these conditions can be equivalently stated in terms of ϵ . Namely, the steady state is a saddle-point with damped oscillations when $\zeta \in (0, \underline{\zeta}) \cup (\bar{\zeta}, +\infty)$ with $\underline{\zeta} = (\epsilon_{cc}\beta)/(1 + \beta)$ and $\bar{\zeta} = \epsilon_{cc}/2$, and there exists a flip bifurcation with persistent period-2 cycles when ζ crosses the bifurcation values $\underline{\zeta}$ or $\bar{\zeta}$.

(ii) Under [Assumptions 1-4](#), let $\epsilon_{cd}, \epsilon_{dc} \geq 0$, i.e. $u_{cd} \leq 0$, and $b < 0$. As shown previously, we derive from $\mathcal{P}_1(\lambda) = 0$ that there exist two positive characteristic roots, one being lower than 1 and the other larger. From $\mathcal{P}_2(\lambda)$, the associated characteristic roots λ_1 and λ_2 are both negative. Moreover we get:

$$\mathcal{P}_2(-1) = \frac{(1+b)(b+\beta)}{\beta b}$$

We conclude easily that

$$\mathcal{P}_1(-1) < 0 \Leftrightarrow b \in (-\infty, -1) \cup (-\beta, 0)$$

$$\mathcal{P}_1(-1) > 0 \Leftrightarrow b \in (-1, -\beta)$$

It follows that the steady state is a saddle-point with damped oscillations when $b \in (-\infty, -1) \cup (-\beta, 0)$. Moreover, if there is some $\beta^* \in (0, 1)$ such that $b \in (-1, -\beta^*)$, then there exists $\bar{\beta} \in (0, 1)$ such that, when β crosses $\bar{\beta}$ from above, (k^*, d^*) undergoes a flip bifurcation leading to persistent period-2 cycles.

(iii) The case where the consumption good is capital intensive, i.e. $b < 0$, and $\epsilon_{cd}, \epsilon_{dc} < 0$, i.e. $u_{cd} > 0$, is obviously derived from the two previous cases. \square

A.6. Proof of [Proposition 5](#)

The characteristic polynomial (24) can be expressed as follows

$$\begin{aligned} & \left[\lambda^2 - \lambda \left(\frac{\epsilon_{dc}}{\beta \epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}} \right) + \frac{1}{\beta} \right] \frac{(\lambda b - 1)(\lambda \beta - b)}{\beta b} \\ & = -\lambda(\lambda - 1) \left(\lambda - \frac{1}{\beta} \right) \frac{\beta}{b \epsilon_{cc}} \frac{\epsilon_{ck}}{\epsilon_{rk}} \left(\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}} \right) \end{aligned}$$

or equivalently, using the notations of [Lemma 1](#),

$$P_1(\lambda)P_2(\lambda) = P_3(\lambda)$$

with $P_3(\lambda)$ a degree-3 polynomial while $P_1(\lambda)P_2(\lambda)$ is a degree-4 polynomial. If these two polynomials intersect four times, then the four characteristic roots are real. To determine the number of intersections of these polynomials, we can use informations derived from the location of their respective roots. The roots of $P_3(\lambda) = 0$ are quite obvious, namely $\lambda_{31} = 0$, $\lambda_{32} = 1$ and $\lambda_{33} = 1/\beta$. Moreover, depending on the sign of $\epsilon_{cd}, \epsilon_{dc}$ we get - if $\epsilon_{cd}, \epsilon_{dc} < 0$, then $\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}} > 0$ and $\lim_{\lambda \rightarrow +\infty} P_3(\lambda) = -\infty$ while $\lim_{\lambda \rightarrow -\infty} P_3(\lambda) = +\infty$;

- if $\epsilon_{cd}, \epsilon_{dc} > 0$, then $\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}} < 0$ and $\lim_{\lambda \rightarrow +\infty} P_3(\lambda) = +\infty$ while $\lim_{\lambda \rightarrow -\infty} P_3(\lambda) = -\infty$;

The roots of $P_1(\lambda)P_2(\lambda) = 0$ are obviously given by the respective roots of $P_1(\lambda) = 0$ and $P_2(\lambda) = 0$.

(i) Assume first that $b > 0$. We have shown in the proof of [Proposition 3](#) that $b < \beta \leq 1$. The roots of $P_2(\lambda) = 0$ are then quite obvious, namely $\lambda_{21} = 1/b > 1$ and $\lambda_{22} = b/\beta < 1$. Finally, the roots of $P_1(\lambda) = 0$ are necessarily real and negative if $\epsilon_{cd}, \epsilon_{dc} < 0$, or positive if $\epsilon_{cd}, \epsilon_{dc} > 0$. Moreover, we have $\lim_{\lambda \rightarrow \pm\infty} P_1(\lambda)P_2(\lambda) = +\infty$ and $P_1(0)P_2(0) > 0$.

If $\epsilon_{cd}, \epsilon_{dc} < 0$, we derive from the above informations that $P_1(b/\beta)P_2(b/\beta) = 0 > P_3(b/\beta)$ while $P_1(1)P_2(1) < P_3(1) = 0$ implying a first intersection between $P_1(\lambda)P_2(\lambda)$ and $P_3(\lambda)$ in the positive orthant. Moreover, since $P_1(1/\beta)P_2(1/\beta) < P_3(1/\beta) = 0$

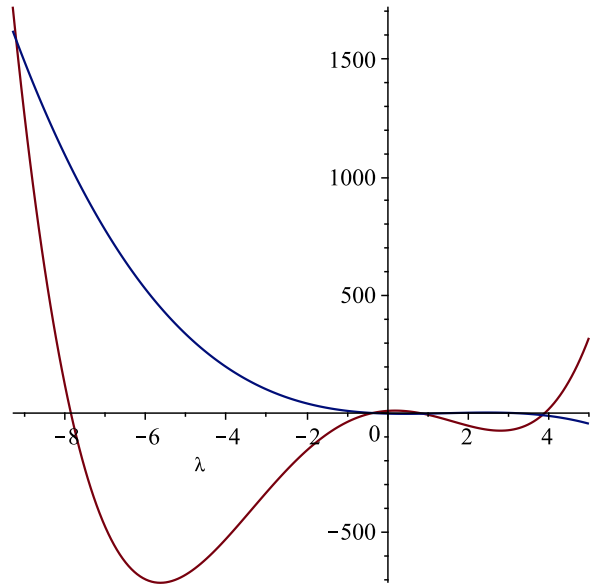


Fig. 9. Real roots when $b > 0$ and $\epsilon_{cd}, \epsilon_{dc} < 0$.

while $P_1(1/b)P_2(1/b) = 0 > P_3(b/\beta)$, we get a second intersection $P_1(\lambda)P_2(\lambda)$ and $P_3(\lambda)$ in the positive orthant. Since $P_1(0)P_2(0) > 0$, $P_1(\lambda)P_2(\lambda) = 0$ admits two roots in the negative orthant, $P_3(0) = 0$ and $P_3(\lambda)$ is an increasing function in the negative orthant, we conclude that there necessarily exists a third intersection between $P_1(\lambda)P_2(\lambda)$ and $P_3(\lambda)$ in the positive orthant. The last intersection, which also occurs in the negative orthant, is obtained because $\lim_{\lambda \rightarrow -\infty} P_1(\lambda)P_2(\lambda) > \lim_{\lambda \rightarrow -\infty} P_3(\lambda)$. Indeed $P_3(\lambda)$ a degree-3 polynomial while $P_1(\lambda)P_2(\lambda)$ is a degree-4 polynomial. We then get the following graphical illustration (see [Fig. 9](#)). It follows that the four roots of the characteristic polynomial (24) are real.

If $\epsilon_{cd}, \epsilon_{dc} > 0$, the roots of $P_3(\lambda) = 0$ and $P_2(\lambda) = 0$ are the same as before while the roots of $P_1(\lambda) = 0$ are now real and positive. Since $P_1(0)P_2(0) > 0$, $P_1(1/b)P_2(1/b) = 0$ and $P_1(1)P_2(1) > 0$, there necessarily exists a second root of $P_1(\lambda)P_2(\lambda) = 0$ between 0 and $1/b$ implying two intersections between $P_1(\lambda)P_2(\lambda)$ and $P_3(\lambda)$. The two others are obtained since $P_1(1/\beta)P_2(1/\beta) > P_3(1/\beta) = 0$, $P_1(b/\beta)P_2(b/\beta) = 0 < P_3(b/\beta)$ and $\lim_{\lambda \rightarrow +\infty} P_1(\lambda)P_2(\lambda) > \lim_{\lambda \rightarrow +\infty} P_3(\lambda)$. We then get the following graphical illustration (see [Fig. 10](#)).

Here again, it follows that the four roots of the characteristic polynomial (24) are real.

(ii) Assume now that $b < 0$ and $\epsilon_{cd}, \epsilon_{dc} > 0$. The roots of $P_2(\lambda) = 0$ become negative, namely $\lambda_{21} = 1/b < \lambda_{22} = b/\beta < 0$. We easily get $P_1(0)P_2(0) > 0$, $P_1(1)P_2(1) < P_3(1) = 0$, $P_1(1/\beta)P_2(1/\beta) < P_3(1/\beta) = 0$, $\lim_{\lambda \rightarrow +\infty} P_1(\lambda)P_2(\lambda) = +\infty$ and $\lim_{\lambda \rightarrow +\infty} P_3(\lambda) = -\infty$. It follows that there are three intersections between $P_1(\lambda)P_2(\lambda)$ and $P_3(\lambda)$ in the positive orthant. Moreover, we have $\lim_{\lambda \rightarrow -\infty} P_1(\lambda)P_2(\lambda) > \lim_{\lambda \rightarrow -\infty} P_3(\lambda)$ implying the existence of two additional intersections between $P_1(\lambda)P_2(\lambda)$ and $P_3(\lambda)$ in the negative orthant. We then get the following graphical illustration (see [Fig. 11](#)) and it follows that the four roots of the characteristic polynomial (24) are real. \square

A.7. Proof of [Proposition 6](#)

Using a homogeneous of degree $\gamma < 1$ utility function, the degree-4 characteristic polynomial as given by [Proposition 2](#)

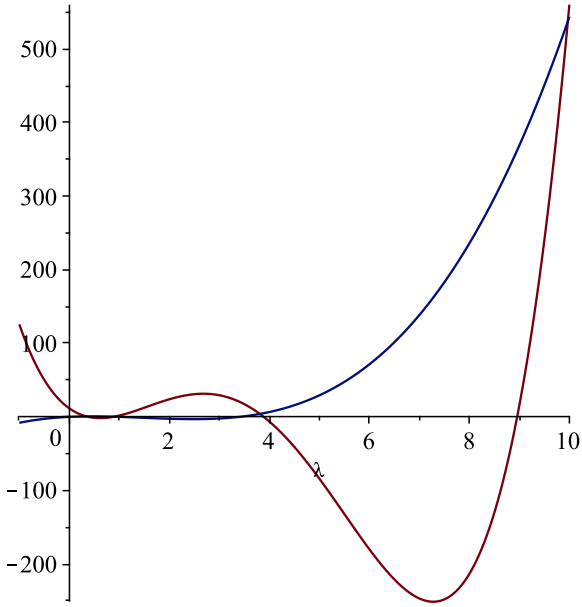


Fig. 10. Real roots when $b > 0$ and $\epsilon_{cd}, \epsilon_{dc} > 0$.

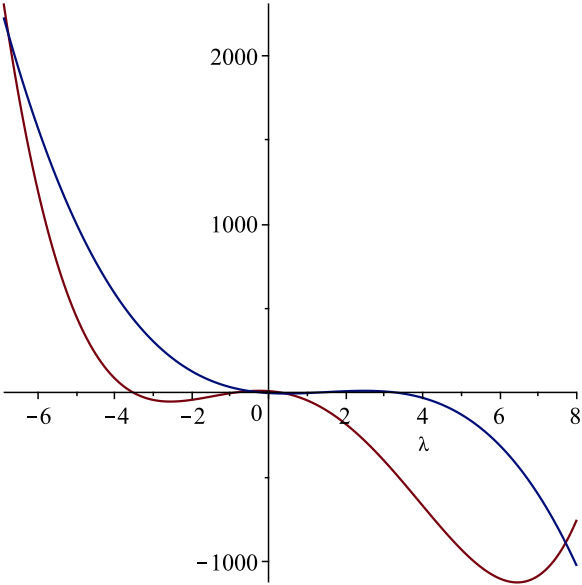


Fig. 11. Real roots when $b < 0$ and $\epsilon_{cd}, \epsilon_{dc} > 0$.

becomes

$$\begin{aligned} \mathcal{P}(\lambda) = & \left[\lambda^2 + \lambda \left(\frac{(\gamma-\phi)^2 + \beta\phi^2 - \beta\phi\epsilon_{cc}(1-\gamma)(2\phi-\gamma)}{\beta\phi(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma)]} \right) \right. \\ & + \frac{1}{\beta} \frac{(\lambda b-1)(\lambda\beta-b)}{\beta b} \\ & \left. + \lambda(\lambda-1) \left(\lambda - \frac{1}{\beta} \right) \frac{\beta}{b} \frac{\epsilon_{ck}}{\epsilon_{rk}} \frac{(1-\gamma)[\gamma-\epsilon_{cc}\phi(1-\gamma)]}{(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma)]} \right] \end{aligned} \quad (44)$$

and can be expressed as $\mathcal{Q}_1(\lambda) = \mathcal{Q}_2(\lambda)$ with

$$\begin{aligned} \mathcal{Q}_1(\lambda) & \equiv \frac{1}{\gamma-\phi} \left[\lambda^2(\gamma-\phi) + \lambda \left(\frac{(\gamma-\phi)^2 + \beta\phi^2 - \beta\phi\epsilon_{cc}(1-\gamma)(2\phi-\gamma)}{\beta\phi[1-\epsilon_{cc}(1-\gamma)]} \right) \right. \\ & \quad \left. + \frac{(\gamma-\phi)}{\beta} \right] \frac{(\lambda b-1)(\lambda\beta-b)}{\beta b} \\ \mathcal{Q}_2(\lambda) & \equiv -\frac{1}{\gamma-\phi} \lambda(\lambda-1) \left(\lambda - \frac{1}{\beta} \right) \frac{\beta}{b} \frac{\epsilon_{ck}}{\epsilon_{rk}} \frac{(1-\gamma)[\gamma-\epsilon_{cc}\phi(1-\gamma)]}{[1-\epsilon_{cc}(1-\gamma)]} \end{aligned}$$

Considering the limit $\phi \rightarrow \gamma$ we immediately conclude that one root λ_1 is necessarily real and equal to $\pm\infty$ and we get

$$\begin{aligned} \mathcal{Q}_1(\lambda) & = \lambda\gamma \frac{(\lambda b-1)(\lambda\beta-b)}{\beta b} \\ \mathcal{Q}_2(\lambda) & = -\lambda\gamma(\lambda-1) \left(\lambda - \frac{1}{\beta} \right) \frac{\beta}{b} \frac{\epsilon_{ck}}{\epsilon_{rk}} (1-\gamma) \end{aligned}$$

It follows that a second root λ_2 is real and equal to 0. Computing now the derivatives $\mathcal{Q}'_1(\lambda)$ and $\mathcal{Q}'_2(\lambda)$, and evaluating them at 0 gives

$$\begin{aligned} \mathcal{Q}'_1(0) & = \frac{\gamma}{\beta} \\ \mathcal{Q}'_2(0) & = -\frac{\gamma}{b} \frac{\epsilon_{ck}}{\epsilon_{rk}} (1-\gamma) \end{aligned}$$

It follows that $\mathcal{Q}'_1(0) \geq \mathcal{Q}'_2(0)$ if and only if $\epsilon_{cc} \leq \hat{\epsilon}_{cc}$ with

$$\hat{\epsilon}_{cc} \equiv -\frac{b}{(1-\gamma)} \frac{\epsilon_{rk}}{\epsilon_{ck}} \in (0, \tilde{\epsilon}_{cc})$$

Note that $\hat{\epsilon}_{cc} \in (0, \tilde{\epsilon}_{cc})$ if and only if

$$-\frac{\epsilon_{ck}}{b\epsilon_{rk}} > 1 \quad (45)$$

We conclude therefore that under condition (45) there exist two additional intersections between $\mathcal{Q}_1(\lambda)$ and $\mathcal{Q}_2(\lambda)$ implying that the two last characteristic roots λ_3, λ_4 are also real. Let us then assume that $b \in (-\infty, -1) \cup (-\beta, 0)$. We derive that

(i) if $\epsilon_{cc} < \hat{\epsilon}_{cc}$ then $\mathcal{Q}'_1(0) > \mathcal{Q}'_2(0)$ with $\mathcal{Q}_1(1/b) = \mathcal{Q}_2(b/\beta) = 0$ which implies that one intersection must occur between -1 and 0 , say $\lambda_3 \in (-1, 0)$. Moreover we derive also that $\lambda_1 = -\infty$ and $\lambda_4 < -1$;

(ii) if $\epsilon_{cc} \in (\hat{\epsilon}_{cc}, \tilde{\epsilon}_{cc})$ then $\mathcal{Q}'_1(0) < \mathcal{Q}'_2(0)$ with $\mathcal{Q}_2(1) = 0$ which implies that one intersection must occur between 0 and 1 , say $\lambda_3 \in (0, 1)$. Moreover we derive $\lambda_1 = +\infty$ and $\lambda_4 > 1$.

We then conclude by continuity that there exists $0 < \bar{\phi} < \gamma$ such that when $\phi \in (\bar{\phi}, \gamma)$, the above results hold with $\lambda_1 \in (-\infty, -1)$ and $\lambda_2 \in (-1, 0)$ when $\epsilon_{cc} < \hat{\epsilon}_{cc}$ or $\lambda_1 \in (1, \infty)$ and $\lambda_2 \in (0, 1)$ when $\epsilon_{cc} \in (\hat{\epsilon}_{cc}, \tilde{\epsilon}_{cc})$. Considering the expression of ζ as given by (29) which is a decreasing function of ϕ , we derive that there exists a corresponding value $\underline{\zeta} = \zeta(\bar{\phi})$, and it follows that the above results hold for $\zeta \in (0, \underline{\zeta})$.

Note now that the characteristic polynomial (44) can be also expressed as $\mathcal{Q}_1(\lambda) = \mathcal{Q}_2(\lambda)$ with

$$\begin{aligned} \mathcal{Q}_1(\lambda) & \equiv \frac{1}{\phi} \left[\lambda^2\phi + \lambda \left(\frac{(\gamma-\phi)^2 + \beta\phi^2 - \beta(\gamma-\phi)\epsilon_{cc}(1-\gamma)(2\phi-\gamma)}{\beta(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma)]} \right) + \frac{\phi}{\beta} \right] \\ & \quad \times \frac{(\lambda b-1)(\lambda\beta-b)}{\beta b} \\ \mathcal{Q}_2(\lambda) & \equiv -\frac{1}{\phi} \lambda(\lambda-1) \left(\lambda - \frac{1}{\beta} \right) \frac{\beta}{b} \frac{\epsilon_{ck}}{\epsilon_{rk}} \frac{\phi(1-\gamma)[\gamma-\epsilon_{cc}\phi(1-\gamma)]}{(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma)]} \end{aligned}$$

Considering the limit $\phi \rightarrow 0$ we immediately conclude that one root λ_1 is necessarily real and equal to $-\infty$ as $b < 0$, and we get

$$\begin{aligned} \mathcal{Q}_1(\lambda) & = \frac{\lambda\gamma^2}{\beta[1-\epsilon_{cc}(1-\gamma)]} \frac{(\lambda b-1)(\lambda\beta-b)}{\beta b} \\ \mathcal{Q}_2(\lambda) & = 0 \end{aligned}$$

It follows that $\lambda_2 = 0$, $\lambda_3 = 1/b$ and $\lambda_4 = b/\beta$ with one larger than -1 and the other lower than -1 as $b \in (-\infty, -1) \cup (-\beta, 0)$. We then conclude by continuity that there exists $0 < \underline{\phi} \leq \bar{\phi}$ such that when $\phi \in (0, \underline{\phi})$, the above results hold with $\lambda_1 \in (-\infty, -1)$ and $\lambda_2 \in (-1, 0)$. Considering again the expression of ζ as given by (29) which is a decreasing function of ϕ , we derive that there exists a corresponding value $\bar{\zeta} = \zeta(\underline{\phi}) \geq \underline{\zeta}$, and it follows that the above results hold for $\zeta \in (\bar{\zeta}, \epsilon_{cc})$. \square

A.8. Proof of Proposition 7

The expressions in (39) become here

$$\begin{aligned} B &= -\frac{\beta}{b} \frac{\varepsilon_{ck}}{\varepsilon_{rk}} \frac{(1-\gamma)[\gamma-\varepsilon_{cc}\phi(1-\gamma)]}{(\gamma-\phi)[1-\varepsilon_{cc}(1-\gamma)]} + \frac{\beta+b^2}{\beta b} \\ &\quad - \left(\frac{(\gamma-\phi)^2 + \beta\phi^2 - \beta\phi\varepsilon_{cc}(1-\gamma)(2\phi-\gamma)}{\beta\phi(\gamma-\phi)[1-\varepsilon_{cc}(1-\gamma)]} \right) \\ C &= -\frac{(1+\beta)}{b} \frac{\varepsilon_{ck}}{\varepsilon_{rk}} \frac{(1-\gamma)[\gamma-\varepsilon_{cc}\phi(1-\gamma)]}{(\gamma-\phi)[1-\varepsilon_{cc}(1-\gamma)]} \\ &\quad - \frac{\beta+b^2}{\beta b} \left(\frac{(\gamma-\phi)^2 + \beta\phi^2 - \beta\phi\varepsilon_{cc}(1-\gamma)(2\phi-\gamma)}{\beta\phi(\gamma-\phi)[1-\varepsilon_{cc}(1-\gamma)]} \right) + \frac{2}{\beta} \end{aligned} \quad (46)$$

As $\varepsilon_{cc} < \bar{\varepsilon}_{cc}$ and $b \in (-\infty, -1) \cup (-\beta, 0)$, we immediately get $C > 0$ for any $\phi \in (0, \gamma)$. Moreover, when $\varepsilon_{cc} = 0$, we get

$$B = \frac{\beta+b^2}{\beta b} - \frac{\beta}{b} \frac{\varepsilon_{ck}}{\varepsilon_{rk}} \frac{\gamma(1-\gamma)}{\gamma-\phi} - \left(\frac{(\gamma-\phi)^2 + \beta\phi^2}{\beta\phi(\gamma-\phi)} \right) < 0$$

for any $\phi \in (0, \gamma)$ if and only if

$$-\frac{\varepsilon_{ck}}{b\varepsilon_{rk}} < \frac{\gamma-\phi}{\beta\gamma(1-\gamma)} \left[\frac{(\gamma-\phi)^2 + \beta\phi^2}{\beta\phi(\gamma-\phi)} - \frac{\beta+b^2}{\beta b} \right] \quad (47)$$

As the right-hand-side of (47) is a decreasing function of ϕ , we conclude that it is always satisfied if

$$-\frac{\varepsilon_{ck}}{b\varepsilon_{rk}} < \frac{1}{\beta(1-\gamma)} \equiv \varepsilon^1 \quad (48)$$

with $\varepsilon^1 > 1$. Therefore, under condition (48) there exists $\bar{\varepsilon}_{cc}^1 \in (0, \bar{\varepsilon}_{cc})$ such that $B < 0$ for any $\phi \in (0, \gamma)$ if $\varepsilon_{cc} \in (0, \bar{\varepsilon}_{cc}^1)$.

Let us consider now the expression $P = 8C - 3B^2$. We derive from (46) that P is a hump-shaped function of ϕ over $(0, \gamma)$. When $\varepsilon_{cc} = 0$, we get

$$\begin{aligned} C &= -\frac{1+\beta}{b} \frac{\varepsilon_{ck}}{\varepsilon_{rk}} \frac{\gamma(1-\gamma)}{\gamma-\phi} - \frac{\beta+b^2}{\beta b} \left(\frac{(\gamma-\phi)^2 + \beta\phi^2}{\beta\phi(\gamma-\phi)} \right) + \frac{2}{\beta} \\ &\equiv -\frac{1+\beta}{b} x - \frac{\beta+b^2}{\beta b} z + \frac{2}{\beta} \\ B &= \frac{\beta+b^2}{\beta b} - z - \frac{\beta}{b} x \end{aligned} \quad (49)$$

and

$$P < -\frac{8(1+\beta)}{b} x - \left(\frac{\beta+b^2}{\beta b} + z \right)^2 - 2 \left[\left(\frac{\beta+b^2}{\beta b} \right)^2 - \frac{8}{\beta} + z^2 \right]$$

Straightforward computations yield $z \geq 2/\sqrt{\beta}$ and thus

$$\begin{aligned} \left(\frac{\beta+b^2}{\beta b} \right)^2 - \frac{8}{\beta} + z^2 &> \left(\frac{\beta+b^2}{\beta b} \right)^2 - \frac{4}{\beta} = \left(\frac{\beta+b^2}{\beta b} - \frac{2}{\sqrt{\beta}} \right) \\ &\quad \times \left(\frac{\beta+b^2}{\beta b} + \frac{2}{\sqrt{\beta}} \right) \\ &= \frac{(b-\sqrt{\beta})^2}{\beta b} \frac{(b+\sqrt{\beta})^2}{\beta b} > 0 \end{aligned}$$

for any $\phi \in (0, \gamma)$. Therefore, $P < 0$ for any $\phi \in (0, \gamma)$ when $\varepsilon_{cc} = 0$ if and only if

$$-\frac{\varepsilon_{ck}}{b\varepsilon_{rk}} < \frac{\gamma-\phi}{8(1+\beta)\gamma(1-\gamma)} \left\{ \left(\frac{\beta+b^2}{\beta b} + z \right)^2 + 2 \left[\left(\frac{\beta+b^2}{\beta b} \right)^2 - \frac{8}{\beta} + z^2 \right] \right\} \quad (50)$$

We can show that the right-hand-side of (50) is a U-shaped function of ϕ over $(0, \gamma)$ and there exists a unique minimum value $\varepsilon^2 > 1$ such that condition (50) holds if

$$-\frac{\varepsilon_{ck}}{b\varepsilon_{rk}} < \varepsilon^2 \quad (51)$$

It follows that under condition (51) there exists $\bar{\varepsilon}_{cc}^2 \in (1, \bar{\varepsilon}_{cc}^1)$ such that $P < 0$ for any $\phi \in (0, \gamma)$ if $\varepsilon_{cc} \in (0, \bar{\varepsilon}_{cc}^2)$.

Let us consider finally R and D as given by (40) and (41). Straightforward computations yield:

$$\begin{aligned} \lim_{\phi \rightarrow 0} B &= -\infty \quad \text{and} \quad \lim_{\phi \rightarrow 0} C = -\infty \quad \text{so that} \quad \lim_{\phi \rightarrow 0} R = -\infty \\ \text{and} \quad \lim_{\phi \rightarrow 0} D &= -\infty \end{aligned}$$

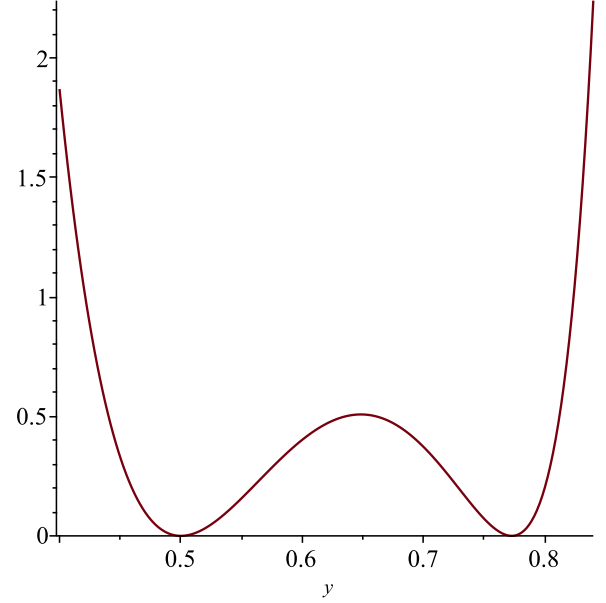


Fig. 12. The function $F(\phi)$ with its two roots ϕ_1 and ϕ_2 .

and there exists $\underline{\gamma}^1 \in (0, 1)$ such that when $\gamma \in (\underline{\gamma}^1, 1)$

$$\begin{aligned} \lim_{\phi \rightarrow \gamma} B &= -\infty \quad \text{and} \quad \lim_{\phi \rightarrow \gamma} C = -\infty \quad \text{so that} \quad \lim_{\phi \rightarrow \gamma} R = -\infty \\ \text{and} \quad \lim_{\phi \rightarrow \gamma} D &= -\infty \end{aligned}$$

We need now to show that there exists a subset of values of ϕ for which R and D can be positive. Let us consider the particular values $\varepsilon_{cc} = 0$, and $b = -\beta$. It follows from (49) that

$$B^2 + \frac{8}{\beta} - 4C = \left(z(\phi) - x - \frac{1+\beta}{\beta} \right)^2 - \frac{8(1+\beta)x}{\beta} \equiv F(\phi)$$

with

$$z(\phi) = \frac{(\gamma-\phi)^2 + \beta\phi^2}{\beta\phi(\gamma-\phi)} \quad \text{and} \quad x = \frac{\varepsilon_{ck}}{\varepsilon_{rk}} \frac{\gamma(1-\gamma)}{\gamma-\phi}$$

Obviously, $F(\phi) = 0$ can be solved through the degree two polynomial

$$z(\phi) - x - \frac{1+\beta}{\beta} = 2\sqrt{\frac{2(1+\beta)x}{\beta}}$$

It follows therefore that there exists $\underline{\gamma}^2 \in (0, 1)$ such that when $\gamma \in (\underline{\gamma}^2, 1)$ the two roots for which $F(\phi) = 0$ satisfy $\phi_1, \phi_2 \in (0, 1)$. In the particular case $\gamma = 1$, these roots are indeed such that

$$\phi_1 = \frac{1}{2} \quad \text{and} \quad \phi_2 = \frac{1}{1+\beta}$$

Moreover, there exists $\underline{\gamma}^3 \in (0, 1)$ such that when $\gamma \in (\underline{\gamma}^3, 1)$ there is a value $\phi_3 \in (\phi_1, \phi_2)$ such that $F'(\phi) = 0$ when $\phi = \phi_1, \phi_2, \phi_3$. Notice indeed that in the particular case $\gamma = 1$, we have

$$\phi_3 = \frac{1}{1+\sqrt{\beta}}$$

Obviously, $F(\phi) > 0$ when $\phi = \phi_3$. Note also that $\lim_{\phi \rightarrow 0} F(\phi) = \lim_{\phi \rightarrow 1} F(\phi) = +\infty$. As a result, we conclude that $F(\phi) \geq 0$ for any $\phi \in (0, 1)$ with the shape of Fig. 12.

Consider now $\gamma < 1$, $\varepsilon_{cc} > 0$ and the expressions of B and C as given by (46), and let us define

$$B^2 + \frac{8}{\beta} - 4C \equiv G(\varepsilon_{cc}, \gamma, \phi) \quad (52)$$

By continuity, there exists $\underline{\gamma}^4 \in (0, 1)$ close to 1 and $\bar{\phi}_3$ close to ϕ_3 such that for any given $\gamma \in (\underline{\gamma}^4, 1)$, $\partial G(\varepsilon_{cc}, \gamma, \bar{\phi}_3)/\partial \phi = 0$.

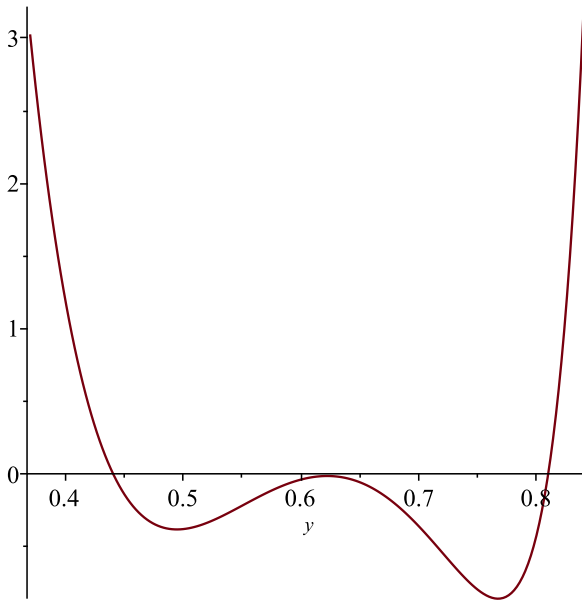


Fig. 13. The function $G(\epsilon_{cc}, \gamma, \phi)$.

Moreover, since $G(\epsilon_{cc}, \gamma, \phi)$ is a decreasing function of ϵ_{cc} with $\lim_{\epsilon_{cc} \rightarrow \bar{\epsilon}_{cc}^2} G(\epsilon_{cc}, \gamma, \phi) < 0$, we conclude that there exists $\underline{\epsilon}_{cc} \in (0, \bar{\epsilon}_{cc}^2)$ such that for any given $\gamma \in (\underline{\gamma}^4, 1)$, when $\epsilon_{cc} = \underline{\epsilon}_{cc}$ and $\phi = \bar{\phi}_3$ we have $G(\underline{\epsilon}_{cc}, \gamma, \bar{\phi}_3) = \partial G(\underline{\epsilon}_{cc}, \gamma, \bar{\phi}_3) / \partial \phi = 0$ such that the curve on Fig. 13.

We conclude therefore that there exist $\bar{b} \in (-\beta, 0)$, $\phi^c \in (0, \phi_1)$ and $\bar{\phi}^c \in (\phi_2, \gamma)$ such that if $\gamma \in (\max\{\underline{\gamma}^1, \underline{\gamma}^2, \underline{\gamma}^3, \underline{\gamma}^4\}, 1)$, $b \in (-\beta, \bar{b})$ and $\epsilon_{cc} \in (\underline{\epsilon}_{cc}, \bar{\epsilon}_{cc}^2)$, then $R > 0$ when $\phi \in (\phi^c, \bar{\phi}^c)$ and $R < 0$ when $\phi \in (0, \phi^c) \cup (\bar{\phi}^c, \gamma)$. Considering the expression of ζ as given by (29) which is a decreasing function of ϕ , we derive that there exists a corresponding value $\bar{\zeta}^c = \zeta(\phi^c)$ and $\underline{\zeta}^c = \zeta(\bar{\phi}^c)$, and it follows that $R > 0$ when $\zeta \in (\underline{\zeta}^c, \bar{\zeta}^c)$ and $R < 0$ when $\zeta \in (0, \underline{\zeta}^c) \cup (\bar{\zeta}^c, +\infty)$.

Let us consider now D . We have proved that for any given $\gamma \in (\underline{\gamma}^4, 1)$, if $\epsilon_{cc} \in (0, \bar{\epsilon}_{cc}^2)$ then $P < 0$ for any $\phi \in (0, \gamma)$. This implies that $-3B^2 < -8C$ and thus

$$\frac{8}{\beta} - 3B^2 + 4C < \frac{8}{\beta} - 4C = -4 \left\{ -\frac{(1+\beta)}{b} \frac{\epsilon_{ck}}{\epsilon_{rk}} \frac{(1-\gamma)[\gamma - \epsilon_{cc}\phi(1-\gamma)]}{(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma)]} - \frac{\beta + b^2}{\beta b} \left(\frac{(\gamma-\phi)^2 + \beta\phi^2 - \beta\phi\epsilon_{cc}(1-\gamma)(2\phi-\gamma)}{\beta\phi(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma)]} \right) \right\} < 0$$

It follows that if $\gamma \in (\max\{\underline{\gamma}^1, \underline{\gamma}^2, \underline{\gamma}^3, \underline{\gamma}^4\}, 1)$, $b \in (-\beta, \bar{b})$ and $\epsilon_{cc} \in (\underline{\epsilon}_{cc}, \bar{\epsilon}_{cc}^2)$, then D has the same sign as R for any $\phi \in (0, \gamma)$, and the characteristic roots are complex when $\phi \in (\phi^c, \bar{\phi}^c)$ and real when $\phi \in (0, \phi^c) \cup (\bar{\phi}^c, \gamma)$. Moreover, when $\phi = \phi^c$ or $\bar{\phi}^c$, $R = D = 0$. It follows therefore that the characteristic roots are complex when $\zeta \in (\underline{\zeta}^c, \bar{\zeta}^c)$ and real when $\zeta \in (0, \underline{\zeta}^c) \cup (\bar{\zeta}^c, \epsilon_{cc})$. Moreover, when $\zeta = \underline{\zeta}^c$ or $\bar{\zeta}^c$, $R = D = 0$.

As explained in Remark 1, the polynomial (38) belongs to the class of quasi-palindromic equation and the exact solutions can be computed. Dividing $\mathcal{P}(\lambda)$ by λ^2 gives

$$\frac{\mathcal{P}(\lambda)}{\lambda^2} = \lambda^2 + \left(\frac{1}{\lambda\beta}\right)^2 - B\left(\lambda + \frac{1}{\lambda\beta}\right) + C = 0$$

and denoting $z = \lambda + 1/(\lambda\beta)$ yields to the following degree-2 polynomial in z

$$\mathcal{P}(z) = z^2 - zB + C - \frac{2}{\beta}$$

The corresponding discriminant is then

$$\Delta_z = B^2 + \frac{8}{\beta} - 4C = \frac{R}{\beta}$$

and under the previous conditions we have $\Delta_z < 0$. The roots are then

$$z_1 = \frac{B + i\sqrt{-\frac{R}{\beta}}}{2} \text{ and } z_2 = \frac{B - i\sqrt{-\frac{R}{\beta}}}{2}$$

Plugging this into the definition of z gives the following two degree-2 polynomials in λ :

$$\lambda^2\beta - \lambda z_1\beta + 1 = 0 \text{ and } \lambda^2\beta - \lambda z_2\beta + 1 = 0 \quad (53)$$

It is worth noticing that a similar characteristic polynomial $\mathcal{P}(\lambda)$ as given by (38) has been considered by de la Croix and Michel (1999) in an optimal growth model with inherited tastes. While they do not refer to its quasi-palindromic structure, they find the same degree-2 polynomials (53) (see their equation (B.1) page 535). However, as they easily prove that the steady state is necessarily a saddle-point, they do not need to explicitly compute the four characteristic roots. In this paper, we need on the contrary to apply in detail the technique related to quasi-palindromic polynomials in order to provide precise conditions for bifurcating complex roots.

Denoting $\Delta_1 = (z_1\beta)^2 - 4\beta$ and $\Delta_2 = (z_2\beta)^2 - 4\beta$, straightforward computations give $\sqrt{\Delta_1}$ and $\sqrt{\Delta_2}$ as given in Box I, and we finally derive the characteristic roots $\lambda_1, \lambda_2, \lambda_3$ and λ_4 as given in Box II, with $\lambda_3 = 1/(\beta\lambda_1)$ and $\lambda_4 = 1/(\beta\lambda_2)$. The existence of a Hopf bifurcation amounts to show that the product $\lambda_1\lambda_2$ can cross the value 1 when the parameter ϕ is varied over the interval $(\phi^c, \bar{\phi}^c)$. Obviously we get

$$\lambda_1\lambda_2 = \left(\frac{B + \sqrt{\frac{B^2 + \frac{R}{\beta} - \frac{16}{\beta} + \sqrt{(B^2 + \frac{R}{\beta} - \frac{16}{\beta})^2 - 4BR}}{2}}}{4} \right)^2 \times \frac{B^2 - \frac{R}{\beta} - \frac{16}{\beta} + \sqrt{(B^2 + \frac{R}{\beta} - \frac{16}{\beta})^2 - 4BR}}{B^2 + \frac{R}{\beta} - \frac{16}{\beta} + \sqrt{(B^2 + \frac{R}{\beta} - \frac{16}{\beta})^2 - 4BR}}$$

By definition we know that if $\phi = \phi^c$ or $\bar{\phi}^c$, we get $R = 0$ and thus

$$\lambda_1\lambda_2 = \left(\frac{B + \sqrt{B^2 - \frac{16}{\beta}}}{4} \right)^2$$

Considering that $B < 0$, we then derive that $\lambda_1\lambda_2 < 1$ if and only if

$$B < -\frac{2(1+\beta)}{\beta} \quad (54)$$

But since $R = 0$, $B^2 = 4C - 8/\beta$ and, using (46) and assuming $b = -\beta$, inequality (57) becomes

$$\frac{\epsilon_{ck}}{\epsilon_{rk}} \frac{(1-\gamma)[\gamma - \epsilon_{cc}\phi(1-\gamma)]}{(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma)]} + \frac{(\gamma-\phi)^2 + \beta\phi^2 - \beta\phi\epsilon_{cc}(1-\gamma)(2\phi-\gamma)}{\beta\phi(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma)]} > \frac{1+\beta}{\beta} \quad (55)$$

When $\epsilon_{cc} = 0$, this inequality becomes

$$\frac{\epsilon_{ck}}{\epsilon_{rk}} \frac{\gamma(1-\gamma)}{(\gamma-\phi)} + \frac{\gamma-2\phi}{\phi(1-\phi)} \frac{\gamma-\phi(1+\beta)}{\beta} > 0 \quad (56)$$

There exists $\underline{\gamma}^5 \in (0, 1)$ such that when $\gamma \in (\underline{\gamma}^5, 1)$, (56) is obviously satisfied when $\phi = \phi^c$ or $\bar{\phi}^c$. Since the left-hand-side of inequality (55) is an increasing function of ϵ_{cc} , we conclude that $\lambda_1\lambda_2 < 1$ when $\gamma \in (\max\{\underline{\gamma}^1, \underline{\gamma}^2, \underline{\gamma}^3, \underline{\gamma}^4, \underline{\gamma}^5\}, 1)$, $b \in (-\beta, \bar{b})$, $\epsilon_{cc} \in (\underline{\epsilon}_{cc}, \bar{\epsilon}_{cc}^2)$ and $\phi = \phi^c$ or $\bar{\phi}^c$.

Tedious but straightforward computations also show that $\lambda_1\lambda_2$ is a hump-shaped function of ϕ over $(\phi^c, \bar{\phi}^c)$. Consider the critical values $\underline{\epsilon}_c$ and $\bar{\phi}_3$ previously defined such that when $\epsilon_{cc} = \underline{\epsilon}_{cc}$ and $\phi = \bar{\phi}_3$ we have $G(\underline{\epsilon}_{cc}, \gamma, \bar{\phi}_3) = \partial G(\underline{\epsilon}_{cc}, \gamma, \bar{\phi}_3) / \partial \phi = 0$ with

$$\sqrt{\Delta_1} = \frac{\beta \left(\sqrt{\frac{B^2 + \frac{R}{B} - \frac{16}{\beta} + \sqrt{\left(B^2 + \frac{R}{B} - \frac{16}{\beta}\right)^2 - 4BR}}{2}} + i \sqrt{\frac{B\sqrt{-\frac{R}{B}}}{B^2 + \frac{R}{B} - \frac{16}{\beta} + \sqrt{\left(B^2 + \frac{R}{B} - \frac{16}{\beta}\right)^2 - 4BR}}} \right)}{2}$$

$$\sqrt{\Delta_2} = \frac{\beta \left(\sqrt{\frac{B^2 + \frac{R}{B} - \frac{16}{\beta} + \sqrt{\left(B^2 + \frac{R}{B} - \frac{16}{\beta}\right)^2 - 4BR}}{2}} - i \sqrt{\frac{B\sqrt{-\frac{R}{B}}}{B^2 + \frac{R}{B} - \frac{16}{\beta} + \sqrt{\left(B^2 + \frac{R}{B} - \frac{16}{\beta}\right)^2 - 4BR}}} \right)}{2}$$

Box I.

$$\lambda_1 = \frac{B + \sqrt{\frac{B^2 + \frac{R}{B} - \frac{16}{\beta} + \sqrt{\left(B^2 + \frac{R}{B} - \frac{16}{\beta}\right)^2 - 4BR}}{2}} + i\sqrt{\frac{-R}{B}} \left[1 + \sqrt{\frac{B}{B^2 + \frac{R}{B} - \frac{16}{\beta} + \sqrt{\left(B^2 + \frac{R}{B} - \frac{16}{\beta}\right)^2 - 4BR}}} \right]}{4}$$

$$\lambda_2 = \frac{B + \sqrt{\frac{B^2 + \frac{R}{B} - \frac{16}{\beta} + \sqrt{\left(B^2 + \frac{R}{B} - \frac{16}{\beta}\right)^2 - 4BR}}{2}} - i\sqrt{\frac{-R}{B}} \left[1 + \sqrt{\frac{B}{B^2 + \frac{R}{B} - \frac{16}{\beta} + \sqrt{\left(B^2 + \frac{R}{B} - \frac{16}{\beta}\right)^2 - 4BR}}} \right]}{4}$$

$$\lambda_3 = \frac{B - \sqrt{\frac{B^2 + \frac{R}{B} - \frac{16}{\beta} + \sqrt{\left(B^2 + \frac{R}{B} - \frac{16}{\beta}\right)^2 - 4BR}}{2}} + i\sqrt{\frac{-R}{B}} \left[1 - \sqrt{\frac{B}{B^2 + \frac{R}{B} - \frac{16}{\beta} + \sqrt{\left(B^2 + \frac{R}{B} - \frac{16}{\beta}\right)^2 - 4BR}}} \right]}{4}$$

$$\lambda_4 = \frac{B - \sqrt{\frac{B^2 + \frac{R}{B} - \frac{16}{\beta} + \sqrt{\left(B^2 + \frac{R}{B} - \frac{16}{\beta}\right)^2 - 4BR}}{2}} - i\sqrt{\frac{-R}{B}} \left[1 - \sqrt{\frac{B}{B^2 + \frac{R}{B} - \frac{16}{\beta} + \sqrt{\left(B^2 + \frac{R}{B} - \frac{16}{\beta}\right)^2 - 4BR}}} \right]}{4}$$

Box II.

$G(\cdot)$ as defined by (52). We know that $\tilde{\phi}_3$ is in a neighborhood of $\phi_3 = 1/(1 + \sqrt{\beta})$. It follows that when $\epsilon_{cc} = \underline{\epsilon}_{cc}$ and $\phi = \tilde{\phi}_3$ we get again $R = 0$ and following the same argument as above we conclude that $\lambda_1 \lambda_2 > 1$ if and only if

$$B > -\frac{2(1+\beta)}{\beta} \quad (57)$$

Assuming $b = -\beta$ and $\phi = \phi_3$, this inequality is approximated by

$$\frac{\epsilon_{ck}}{\epsilon_{rk}} \frac{(1-\gamma)[1+\sqrt{\beta}-\epsilon_{cc}\phi(1-\gamma)]}{1-\epsilon_{cc}(1-\gamma)} + \frac{2-\epsilon_{cc}(1-\gamma)(1-\sqrt{\beta})}{1-\epsilon_{cc}(1-\gamma)} < \frac{1+\beta}{\sqrt{\beta}} \quad (58)$$

When $\gamma = 1$, this inequality is obviously satisfied. Therefore, there exists $\underline{\gamma}^6 < 1$ such that $\lambda_1 \lambda_2 > 1$ when $\gamma \in (\underline{\gamma}^6, 1)$, $\epsilon_{cc} = \underline{\epsilon}_{cc}$ and $\phi = \tilde{\phi}_3$. We conclude that there exists $\bar{\epsilon}_{cc}^3 \in (\underline{\epsilon}_{cc}, \bar{\epsilon}_{cc}^2]$, $\bar{\phi}^H \in (\underline{\phi}^c, \tilde{\phi}_3)$ and $\bar{\phi}^H \in (\tilde{\phi}_3, \bar{\phi}^c)$ such that when $\gamma \in (\max\{\underline{\gamma}^1, \underline{\gamma}^2, \underline{\gamma}^3, \underline{\gamma}^4, \underline{\gamma}^5, \underline{\gamma}^6\}, 1)$, $b \in (-\beta, \bar{b})$, $\epsilon_{cc} \in (\underline{\epsilon}_{cc}, \bar{\epsilon}_{cc}^3)$ then $\lambda_1 \lambda_2 < 1$ when $\phi \in (\underline{\phi}^c, \bar{\phi}^H) \cup (\bar{\phi}^H, \bar{\phi}^c)$ and $\lambda_1 \lambda_2 > 1$ when $\phi \in (\bar{\phi}^H, \bar{\phi}^H)$. The result follows denoting $\underline{\gamma} = \max\{\underline{\gamma}^1, \underline{\gamma}^2, \underline{\gamma}^3, \underline{\gamma}^4, \underline{\gamma}^5, \underline{\gamma}^6\}$, $\bar{\epsilon} = \min\{\bar{\epsilon}^1, \bar{\epsilon}^2\}$ and $\bar{\epsilon}_{cc} = \min\{\bar{\epsilon}_{cc}^1, \bar{\epsilon}_{cc}^2, \bar{\epsilon}_{cc}^3\}$. Considering one more time the expression of ζ as given by (29) which is a decreasing function of ϕ , we derive that there exists a corresponding value $\bar{\zeta}^H = \zeta(\bar{\phi}^H)$ and $\underline{\zeta}^H = \zeta(\bar{\phi}^H)$, and it follows that $\lambda_1 \lambda_2 < 1$ when $\zeta \in (\underline{\zeta}^c, \underline{\zeta}^H) \cup (\bar{\zeta}^H, \bar{\zeta}^c)$ and $\lambda_1 \lambda_2 > 1$ when $\zeta \in (\underline{\zeta}^H, \bar{\zeta}^H)$. \square

A.9. Proof of Proposition 8

Consider Eqs. (14) and (15) that give the dynamics of bequests and bequests as a proportion of GDP from the equilibrium paths of capital $\{k_t\}_{t \geq 0}$ and second period consumption $\{d_t\}_{t \geq 0}$, namely

$$p_t x_t = r_t k_t - d_t$$

$$\frac{p_t x_t}{GDP_t} = \frac{p_t x_t}{T(k_t, k_{t+1}) + p_t y_t} = s(k_t, k_{t+1}) - \frac{d_t}{T(k_t, k_{t+1}) - T_y(k_t, k_{t+1})k_{t+1}}. \quad (59)$$

with

$$s(k_t, k_{t+1}) = \frac{T_k(k_t, k_{t+1})k_t}{T(k_t, k_{t+1}) - T_y(k_t, k_{t+1})k_{t+1}}$$

the share of capital income in GDP. If $\{k_t\}_{t \geq 0}$ and $\{d_t\}_{t \geq 0}$ are characterized by periodic or quasi-periodic dynamics, this is also true for bequests. Indeed, consider first the case of period-2 cycles which are characterized for $\{k_t, d_t\}_{t \geq 0}$ by the existence of two pairs (k_1, d_1) and (k_2, d_2) such that $(k_t, d_t) = (k_1, d_1)$ and $(k_{t+1}, d_{t+1}) = (k_2, d_2)$. It follows that a period-2 cycle also exists for bequests as $p_t x_t = T_k(k_1, k_2)k_1 - d_1$ and $p_{t+1} x_{t+1} = T_k(k_2, k_1)k_2 - d_2$ since, as long as $(k_1, d_1) \neq (k_{t+1}, d_{t+1}) = (k_2, d_2)$, we get indeed $p_t x_t \neq p_{t+1} x_{t+1}$. A similar argument can be applied for quasi-periodic cycles which are characterized by a collection of distinct pairs $\{k_\tau, d_\tau\}_{\tau \geq 0}$ located on a circle (or more generally an ellipse). Depending on some coefficients associated to the projection of the dynamical system on the center manifold, this collection of pairs may be periodic, of period T , or quasi-periodic.

In both cases, for each pair (k_τ, d_τ) , we can find associated values of $p_\tau x_\tau$ that are also located on the ellipse and that possess the same periodic or quasi-periodic property.

Similarly, since the share of capital income in GDP, $s(k_t, k_{t+1})$, and the share of consumption of old agents in GDP, d_t/GDP_t , are generically non-constant and non-equal, if $\{k_t\}_{t \geq 0}$ and $\{d_t\}_{t \geq 0}$ are characterized by periodic or quasi-periodic dynamics, this is also true for bequests as a proportion of GDP. Indeed, consider again the case of period-2 cycles for $\{k_t, d_t\}_{t \geq 0}$ with two pairs (k_1, d_1) and (k_2, d_2) such that $(k_t, d_t) = (k_1, d_1)$ and $(k_{t+1}, d_{t+1}) = (k_2, d_2)$. It follows that a period-2 cycle also exists for bequests as a proportion of GDP as

$$\frac{p_t x_t}{GDP_t} = s(k_1, k_2) - \frac{d_1}{T(k_1, k_2) - T_y(k_1, k_2)k_2} \text{ and}$$

$$\frac{p_{t+1} x_{t+1}}{GDP_{t+1}} = s(k_2, k_1) - \frac{d_2}{T(k_2, k_1) - T_y(k_2, k_1)k_1}$$

As long as $(k_1, d_1) \neq (k_{t+1}, d_{t+1}) = (k_2, d_2)$, we get indeed

$$\frac{p_t x_t}{GDP_t} \neq \frac{p_{t+1} x_{t+1}}{GDP_{t+1}}$$

A similar argument can be applied for quasi-periodic cycles. As explained above, for each pair (k_τ, d_τ) , we can find associated values of $p_\tau x_\tau / GDP_\tau$ that are also located on the ellipse and that possess the same periodic or quasi-periodic property. \square

A.10. Proof of Corollary 1

Consider the Cobb–Douglas technologies as given by (33). We follow the same methodology as in Baierl et al. (1998). The Lagrangian associated with the optimization program (1) is:

$$\mathcal{L} = k_0^\alpha l_0^{1-\alpha_0} + w(1 - l_0 - l_1) + r(k - k_0 - k_1) + p \left[k_1^{\alpha_1} l_1^{1-\alpha_1} - y \right] \quad (60)$$

The first order conditions are:

$$r = \alpha_0 k_0^{\alpha_0-1} l_0^{1-\alpha_0} = p \alpha_1 k_1^{\alpha_1-1} l_1^{1-\alpha_1} \quad (61)$$

$$w = (1 - \alpha_0) k_0^{\alpha_0} l_0^{-\alpha_0} = p(1 - \alpha_1) k_1^{\alpha_1} l_1^{-\alpha_1} \quad (62)$$

Using $k_0 = k - k_1$, $l_0 = 1 - l_1$, and merging the above equations gives:

$$l_0^* = \frac{(1 - \alpha_0)\alpha_1(k - k_1^*)}{(\alpha_0 - \alpha_1)k_1^* + (1 - \alpha_0)\alpha_1 k} \quad (63)$$

$$l_1^* = \frac{\alpha_0(1 - \alpha_1)k_1^*}{(\alpha_0 - \alpha_1)k_1^* + (1 - \alpha_0)\alpha_1 k} \quad (64)$$

$$K_c^* = k - k_1^* \quad (65)$$

$$k_1^* = g(k, y) \equiv g \quad (66)$$

where

$$g(k, y) = \left\{ k_1 \in [0, k^{\alpha_1}] / y = \frac{[\alpha_0(1-\alpha_1)]^{1-\alpha_1} k_1}{[(1-\alpha_0)\alpha_1 k + (\alpha_0-\alpha_1)k_1]^{1-\alpha_1}} \right\} \quad (67)$$

From (61), (63) and (65) we obtain:

$$T_k = r^* = \alpha_0 \left[\frac{(1-\alpha_0)\alpha_1}{(1-\alpha_0)\alpha_1 k + (\alpha_0-\alpha_1)g} \right]^{1-\alpha_0} \quad (68)$$

and from (61), (64), (66) and (68):

$$T_y = p^* = \frac{\alpha_0[(1-\alpha_0)\alpha_1]^{1-\alpha_0} [\alpha_0(1-\alpha_1)]^{-(1-\alpha_1)} [(1-\alpha_0)\alpha_1 k + (\alpha_0-\alpha_1)g]^{\alpha_0-\alpha_1}}{\alpha_1} \quad (69)$$

By the derivation of g , we have, for any equilibrium path, the identity $(1 - \alpha_0)\alpha_1 k + (\alpha_0 - \alpha_1)g = \alpha_0(1 - \alpha_1)(g/y)^{1/(1-\alpha_1)}$.

Substituting this into (68) and (69) gives after simplifications:

$$T_k(k, y) = \alpha_0 \left(\frac{(1-\alpha_0)\alpha_1}{\alpha_0(1-\alpha_1)} \right)^{1-\alpha_0} \left(\frac{y}{g} \right)^{\frac{1-\alpha_0}{1-\alpha_1}}$$

$$T_y(k, y) = -\frac{\alpha_1}{\beta_1} \left(\frac{(1-\alpha_0)\alpha_1}{\alpha_0(1-\alpha_1)} \right)^{1-\alpha_0} \left(\frac{y}{g} \right)^{\frac{\alpha_1-\alpha_0}{1-\alpha_1}}$$

$$T_{kk}(k, y) = -T_k(k, y) \frac{g_1}{g}$$

with $g_1 = \partial g(k, y) / \partial k$. A steady state k^* is then defined as $T_k(k^*, k^*) + \beta T_y(k^*, k^*) = 0$. Denote $g^* = g(k^*, k^*)$ and $y^* = k^*$. Using the derivatives of T in the definition of k^* gives:

$$g^* = \beta \alpha_1 k^* \quad (70)$$

Substituting (70) into the definition of g , we find

$$k^* = \frac{\alpha_0(1-\alpha_1)(\beta\alpha_1)^{\frac{1}{1-\alpha_1}}}{\alpha_1[1-\alpha_0+\beta(\alpha_0-\alpha_1)]} \quad (71)$$

Considering (67), we easily derive

$$g_1 = \frac{\beta\alpha_1(1-\alpha_0)(1-\alpha_1)}{1-\alpha_0+\beta(\alpha_0-\alpha_1)} \quad (72)$$

From all these results and (3), we get

$$\begin{aligned} c^* + d^* = T(k^*, k^*) &= \left(\frac{\alpha_0(1-\alpha_1)}{(1-\alpha_0)\alpha_1} \right)^{\alpha_0} \frac{(1-\alpha_0)(1-\beta\alpha_1)(\beta\alpha_1)^{\frac{\alpha_0}{1-\alpha_1}}}{1-\alpha_0+\beta(\alpha_0-\alpha_1)} \\ r^* = T_k(k^*, k^*) &= \alpha_0 \left(\frac{(1-\alpha_0)\alpha_1}{\alpha_0(1-\alpha_1)} \right)^{1-\alpha_0} (\beta\alpha_1)^{-\frac{1-\alpha_0}{1-\alpha_1}} \\ T_{kk}(k^*, k^*) &= -\frac{T_k(k^*, k^*)}{k^*} \frac{(1-\alpha_0)^2}{1-\alpha_0+\beta\alpha_1(\alpha_0-\alpha_1)} \\ b &= \frac{\beta(\alpha_1-\alpha_0)}{1-\alpha_0} \end{aligned}$$

We then easily derive

$$\varepsilon_{ck} = \frac{\alpha_0}{1-\beta\alpha_1} \text{ and } \varepsilon_{rk} = \frac{(1-\alpha_0)^2}{1-\alpha_0+\beta\alpha_1(\alpha_0-\alpha_1)} \quad (73)$$

Considering (29) with $\gamma = 1$, the characteristic polynomial (24) becomes here

$$\mathcal{P}(\lambda) = \left(\lambda + \frac{\varepsilon_{cc}-\varepsilon}{\varepsilon} \right) \left(\lambda + \frac{\varepsilon}{\beta(\varepsilon_{cc}-\varepsilon)} \right) \frac{(\lambda b - 1)(\lambda \beta - b)}{\beta b} \quad (74)$$

and the characteristic roots are

$$\begin{aligned} \lambda_1 &= -\frac{\varepsilon_{cc}-\varepsilon}{\varepsilon} = -\frac{\phi}{1-\phi}, \quad \lambda_2 = -\frac{\varepsilon}{\beta(\varepsilon_{cc}-\varepsilon)} = -\frac{1-\phi}{\beta\phi}, \\ \lambda_3 &= \frac{1}{b} = \frac{1-\alpha_0}{\beta(\alpha_1-\alpha_0)} \text{ and } \lambda_4 = \frac{b}{\beta} = \frac{\alpha_1-\alpha_0}{1-\alpha_0} \end{aligned} \quad (75)$$

Assume that $\alpha_0 > (1 + \alpha_1)/2 > \alpha_1$, then $\lambda_3 \in (-1, 0)$ if and only if $\beta > (1 - \alpha_0)/(\alpha_0 - \alpha_1) \equiv \beta_1 (\in (0, 1))$ while $\lambda_4 < -1$. The critical value β_1 is thus a flip bifurcation value. Moreover, we then have the following cases:

- (a) If $\phi \leq 1/2$ then $\lambda_1 > -1$ and $\lambda_2 < -1$ for any $\beta \in (0, 1)$.
- (d) If $\phi > 1/2$ then $\lambda_2 > -1$ if and only if $\beta > \frac{1-\phi}{\phi} \equiv \beta_2$ while $\lambda_1 < -1$. The critical value β_2 is thus a flip bifurcation value.

We notice that if $\phi > 1/2$, then $\beta_1 \geq \beta_2$ if and only if $\phi \geq \frac{\alpha_0-\alpha_1}{1-\alpha_1} \equiv \phi^* (> 1/2)$. In this case, there are two flip bifurcation values for β implying that two flip bifurcations can occur successively when β is decreased from 1. More importantly, if $\phi = \frac{\alpha_0-\alpha_1}{1-\alpha_1} \equiv \phi^*$, then $\beta_1 = \beta_2 = \beta$ and two characteristic roots are simultaneously equal to -1 when $\beta = \beta$. This corresponds to a co-dimension 2 flip bifurcation with a 1:2 resonance. \square

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