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ABSTRACT This paper deals with statistical inference for lifetime data in presence of imperfect maintenance. For the maintenance model, the Sheu and Griffith model is considered. The lifetime distribution belongs to exponential distribution class. The maximum likelihood estimation procedure of the model parameters is discussed, and confidence intervals are provided using the asymptotic likelihood theory and bootstrap approach. Based on conjugate and discrete priors, Bayesian estimators of the model parameters are developed under symmetric and asymmetric loss functions. The proposed methodologies are applied to simulated data and sensitivity analysis to different parameters and data characteristics is carried out. The effect of model misspecification is also assessed within this class of distributions through a Monte Carlo simulation study. Finally, two datasets are analyzed for demonstrative aims.

Keywords: Bayesian inference; Confidence interval; Exponential-based class; Imperfect repair; Maximum likelihood estimation; Model misspecification.

1 Introduction

A repairable system is a system which, after failing to perform one or more of its functions satisfactorily it can be restored an acceptable but necessarily new condition [1] . Analysis of the reliability of these kinds of systems must consider the effects of successive repair actions.

In general, maintenance tasks are classified into two extreme cases: minimal repair and perfect repair. Maintenance actions that restore a system such that its failure rate remains unchanged after maintenance is called minimal repair (as bad as old). Perfect repair refers to maintenance actions,

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which restore a system to a new one (as good as new). The corresponding stochastic models for the failure process are respectively the Non-Homogeneous Poisson Processes (NHPP) and the Renewal Processes (RP). Chaudhuri and Sahu [2] introduced a more general maintenance which includes these two extreme cases, called as "imperfect repair".

Among the first studies dealing with the imperfect repair concept, we can recall the model of Brown and Proschan [3] (BP model). In this model, two types of failures can occur: catastrophic failure (type-II failure) and minor failure (type-I failure). The state after type-II failure is as good as new with probability p and after a type-I failure is as bad as old with probability $1-p$ where p is in [0, 1]. In extension of BP model, Block et al. [4] proposed the age-dependent imperfect repair model where p is allowed to depend on t and t is the age of the item in use at the failure time (the time since the last perfect repair). They considered a continuous lifetime distribution. Pham and Wang [5] summarized and discussed various methods and optimal policies for imperfect maintenance. Sheu and Griffith [6] were concerned with modeling systems with dependent components having specific multivariate distributions, and undergoing imperfect repair.

A very important class of models proposed by Kijima [7], discussed the concept of virtual age models and applied it to repairable systems. There are many papers addressing this issue in the literature, refer to Langseth and Lindqvist [8], Doyen and Gaudoin [9] and Nguyen et al. [10]. An extension of the BP model was proposed by Sheu and Griffith [6] as a generalized model for determining the optimal number of minimal repair before replacement of a system subject to shocks. In this model, the system is replaced at first type-II failure, or at nth type-I failure. They also supposed that shocks occur according to a Non-Homogeneous Poisson process (NHPP) and based on an optimal replacement policy, the optimal number of n is defined. This optimal maintenance policy depends on the parameters of the system lifetime. Most of the authors suppose that the reliability parameters and repair effects are known.

However, in practice, these parameters are often unknown. Estimation of these parameters is essential for maintenance planning and optimization. In fact, for a repairable system, the time to failure depends on both the lifetime distribution and the impact of maintenance actions performed on the system. Generally, for reliable systems, only a few failures occur in practice. On the other hand, the engineers' knowledge about the failure process could be useful to improve the estimations of the model parameters. Thus, a Bayesian analysis of this model is an interesting alternative to frequentist methods. Among the studies dealing with Bayesian approaches for imperfect maintenance, we can name Nguyen et al. [11] and Moneim et al. [12].

Recently, Kamranfar et al. [13] have dealt with Sheu and Griffith [6] imperfect repair model (SG model) from a statistical point of view. They have considered Weibull distribution as inter-arrival time distribution and presented both frequentist and Bayesian approaches to estimate the model parameters. This paper focuses on different inferential methods for the SG model in both directions: frequentist and Bayesian. The lifetime distribution of the system is assumed to belong to a class of univariate distributions generated from the exponential distribution. Special distributions of this class are Burr-XII, Gompertz, Weibull, and bathtub-shaped lifetime distributions all of which are commonly used to model lifetime data. More details on this class are given in the next section and also one can refer to Wang and Shi [14] and Ahmadi et al. [15]. The main contributions of this paper are as follows.

1. The SG imperfect repair model is considered based on a general class of univariate distributions generated from the exponential distribution. No researcher has studied the SG model for this class so far. Also among the specific members of the class, to the best of our knowledge, no study has been conducted on the SG model based on Burr-XII, Gompertz, and bathtub-shaped lifetime distributions which are widely used in reliability engineering.

2. In the frequentist approach, the maximum likelihood (ML) method is used for the parameter estimation, and confidence intervals (CIs) are constructed by using the observed Fisher information matrix, and a parametric bootstrap method.

3. The Bayesian estimators of the parameters are obtained under symmetric as well as asymmetric loss functions.

4. The effect of model misspecification on the estimation of the mean time to prefect repair is investigated through a detailed Monte Carlo simulation study.

5. Using the likelihood-based method, a detailed study of model selection based on Boeing air conditioner data is presented.

The outline of the paper is as follows. Section 2 gives the problem description and associated assumptions. In Section 3 inference problem, including ML estimators (MLEs), CIs, and likelihood ratio test (LRT) is developed. Bayesian analysis is presented in Section 4 under symmetric and asymmetric loss functions. Section 5 is devoted to Monte Carlo simulation study for assessing the performance of the inferential methods. In this section, the results of a study on the effect of model misspecification are also presented. Section 6 provides analyses of two datasets. Finally, in Section 7, the paper ends with some conclusions and perspectives for future works.

2 Model description and assumption

Many authors have considered optimal policies, which often use a long-run average cost rate or availability evaluated from system lifetime distribution and maintenance unit costs, refer to Tsai et al. [16] and Sheut et al. [17] among others. This optimal maintenance policy depends on the parameters of the system lifetime distribution. Often, it is assumed that parameters of lifetime distribution and repair efficiency parameter (p) are known. However, it is not generally the case in practice and statistical inference is needed to estimate model parameters and compute reliability indicators from failure data.

Kamranfar et al. [13] considered a homogeneous case of the SG model with $q = 1 - p$ and p as the probability of minimal repair and replacement, respectively. Based on the applying assumptions in Kamranfar et al. [13, p. 3-4] and the SG model, we can present the following expression as the replacement time

$$Y^* = \sum_{i=1}^{n} T_i I(M \geq i),$$

where, $I(.)$ is the indicator function, M denotes the number of the shocks until the first type-II failure since the last replacement and T_i is the duration of functioning of the system after the $(i-1)$th minimal repair.

There are many distributions that have been suggested for lifetime data modeling. Among the existing distributions, the class of univariate distributions generated from the exponential distribution is one of the most used. The probability density function (PDF) and cumulative distribution function (CDF) of this class can be expressed as follows

$$f(t; \alpha, \lambda) = \lambda \psi(t; \alpha) \exp\{-\lambda \Psi(t; \alpha)\}, \quad t, \alpha, \lambda > 0, \tag{1}$$

$$F(t; \alpha, \lambda) = 1 - \exp\{-\lambda \Psi(t; \alpha)\}, \tag{2}$$

respectively, where α and λ are unknown model parameters, $\psi(t; \alpha) = \frac{\partial}{\partial t}\Psi(t; \alpha)$, $\Psi(t; \alpha)$ is increasing in t with $\Psi(0; \alpha) = 0$ and $\Psi(\infty; \alpha) = \infty$. The reliability function $R(t)$ and the hazard function $H(t)$ of the model (1) at time t can be written as

$$R(t) = \exp\{-\lambda \Psi(t; \alpha)\}, \qquad H(t) = \lambda \psi(t; \alpha).$$

Note that the general form for lifetime model (1) includes some well-known and useful models such as Burr-XII distribution with $\Psi(t; \alpha) = \ln(1 + t^{\alpha})$, Gompertz distribution with $\Psi(t; \alpha) = \frac{1}{\alpha}(e^{\alpha t} - 1)$, Weibull distribution with $\Psi(t; \alpha) = t^{\alpha}$, two parameters bathtub-shaped lifetime distribution(see Chen [18]) with $\Psi(t; \alpha) = e^{t^{\alpha}} - 1$, and so on. Hereafter, the two parameters bathtub-shaped distribution is called Chen distribution.

Let us consider a repairable system under the SG model. It is assumed that the initial lifetime of the system follows from the continuous distribution (2). The system is replaced at the first type-II failure or at the nth type-I failure whichever occurs first. Let $\boldsymbol{x} = (x_1, \ldots, x_m); 1 \leq m \leq n$, be the observed failure times of the system until replacement.

3 Frequentist inference

3.1 Maximum likelihood method of estimation

To apply the ML method, the first step is the develop the likelihood function concerning to available data. Kamranfar et al. [13] assumed that the number of minimal repairs in the SG model is fixed and proposed the likelihood function as

$$
\begin{aligned}
L(p, F; \boldsymbol{x}, m) &= \left(\prod_{j=1}^{n-1} (1-p)^{I(m>j)} p^{I(m=j)} [f(x_j|x_{j-1})]^{I(m\geq j)} \right) [f(x_n|x_{n-1})]^{I(m=n)} \\
&= (1-p)^{m-1} p^{I(m<n)} \prod_{j=1}^{m} \frac{f(x_j)}{\bar{F}(x_{j-1})}, \quad x_1 < \cdots < x_m,
\end{aligned} \tag{3}
$$

where $f(x_i|x_{i-1})$ is the truncated density function of x_i given x_{i-1}, $\bar{F}(.) = 1 - F(.)$ is the survival function of F, and $x_0 = 0$. For a sample of k independent and identically distributed (i.i.d.) systems (or k replacements), the likelihood function can be expressed as

$$
\begin{aligned}
L(p, F; \tilde{\boldsymbol{x}}, \boldsymbol{m}) &= \prod_{i=1}^{k} \prod_{j=1}^{m_i} (1-p)^{m_i - 1} p^{I(m_i<n)} \frac{f(x_{i,j})}{\bar{F}(x_{i,j-1})} \\
&= (1-p)^{m^{(k)}-k} p^{\sum\limits_{i=1}^{k} I(m_i<n)} \prod_{i=1}^{k} \prod_{j=1}^{m_i} \frac{f(x_{i,j})}{\bar{F}(x_{i,j-1})},
\end{aligned} \tag{4}
$$

where $\tilde{\boldsymbol{x}} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k)$, $\boldsymbol{x}_i = (x_{i,1}, x_{i,2}, ..., x_{i,m_i}); 1 \leq i \leq k$, and $m^{(k)} = \sum_{i=1}^{k} m_i$. Hereinafter, for more simplification in notations, instead of $L(p, F; \tilde{\boldsymbol{x}}, \boldsymbol{m})$ and $\ln(L(p, F; \tilde{\boldsymbol{x}}, \boldsymbol{m}))$ are denoted by L^* and ℓ^*.

In the following, under the assumption that the lifetime distribution belongs to the introduced class in Eq. (1), the maximum likelihood estimates are developped. Considering the desirable class, the likelihood function (4) may be simplified as

$$
L^* = (1-p)^{m^{(k)}-k} p^{\sum_{i=1}^{k} I(m_i<n)} \lambda^{m^{(k)}} \left(\prod_{i=1}^{k} \prod_{j=1}^{m_i} \psi(x_{i,j}; \alpha) \right) \exp\left\{ -\lambda \sum_{i=1}^{k} \Psi(x_{i,m_i}; \alpha) \right\}, \tag{5}
$$

which yields

$$
\begin{aligned}
\ell^* &= \left(m^{(k)} - k \right) \ln(1-p) + \sum_{i=1}^{k} I(m_i < n) \ln(p) \\
&\quad + m^{(k)} \ln(\lambda) + \sum_{i=1}^{k} \sum_{j=1}^{m_i} \ln \psi(x_{i,j}; \alpha) - \lambda \sum_{i=1}^{k} \Psi(x_{i,m_i}; \alpha).
\end{aligned} \tag{6}
$$

There are three parameters p, λ, and α that need to be estimated. The likelihood equations for p, λ, and α are respectively

$$\frac{\partial \ell^*}{\partial p} = \frac{k - m^{(k)}}{1-p} + \frac{1}{p}\sum_{i=1}^{k} I(m_i < n) = 0, \tag{7}$$

$$\frac{\partial \ell^*}{\partial \lambda} = \frac{m^{(k)}}{\lambda} - \sum_{i=1}^{k} \Psi\left(x_{i,m_i};\alpha\right) = 0, \tag{8}$$

$$\frac{\partial \ell^*}{\partial \alpha} = \sum_{i=1}^{k}\sum_{j=1}^{m_i} \frac{(\partial/\partial\alpha)\psi\left(x_{i,j};\alpha\right)}{\psi\left(x_{i,j};\alpha\right)} - \lambda \sum_{i=1}^{k} \frac{\partial}{\partial \alpha}\Psi\left(x_{i,m_i};\alpha\right) = 0. \tag{9}$$

From Eqs. (7)-(9) the MLEs of p and λ can be obtained as

$$\hat{p} = \frac{\sum_{i=1}^{k} I(m_i < n)}{m^{(k)} - k + \sum_{i=1}^{k} I(m_i < n)}, \tag{10}$$

$$\hat{\lambda}(\hat{\alpha}) = \frac{m^{(k)}}{\sum_{i=1}^{k} \Psi\left(x_{i,m_i};\hat{\alpha}\right)}, \tag{11}$$

respectively, where $\hat{\alpha}$, the MLE of α, can also be obtained by solving the following equation

$$\sum_{i=1}^{k}\sum_{j=1}^{m_i} \frac{(\partial/\partial\alpha)\psi\left(x_{i,j};\alpha\right)}{\psi\left(x_{i,j};\alpha\right)} - \frac{m^{(k)}\sum_{i=1}^{k}(\partial/\partial\alpha)\Psi\left(x_{i,m_i};\alpha\right)}{\sum_{i=1}^{k}\Psi\left(x_{i,m_i};\alpha\right)} = 0. \tag{12}$$

Clearly, nonlinear equation (12) cannot be solved analytically and mathematical or statistical software should be applied to get a numerical solution via iterative techniques. Here, R package *nleqslv* is used to find the root of Eq. (12) by the Newton-Raphson method. It can be seen that $\hat{p}$ is free from the values of observed failure times i.e. $x_{i,j}$, $1 \le i \le k, 1 \le j \le m_i$. Moreover, it may be noted that $\hat{p}$ is equal to 0 if $m_i = n$ and to 1 if $m_i = 1$ for all $1 \le i \le k$.

3.2 Interval estimation

3.2.1 Asymptotic confidence intervals (ACIs)

In this subsection, we derive ACIs of the parameters p, α, and λ, by applying the property of the asymptotic normality of the MLE $\hat{\boldsymbol{\theta}} = (\hat{p}, \hat{\alpha}, \hat{\lambda})$ of the unknown parameter $\boldsymbol{\theta} = (p, \alpha, \lambda)$. Let the observed Fisher information matrix $J(\boldsymbol{\theta}) = [J_{\ell,s}] = [-\frac{\partial^2 \ell^*}{\partial\theta_\ell \partial\theta_s}]$, $\ell, s = 1, 2, 3$, then we achieve the elements of $J(\boldsymbol{\theta})$ by obtaining the second partial derivatives of function (6) as follows:

$$
\begin{aligned}
J_{11} &= \frac{m^{(k)}-k}{(1-p)^2}+\frac{\sum_{i=1}^{k} I(m_i<n)}{p^2},\\
J_{22} &= \lambda\sum_{i=1}^{k}\frac{\partial^2}{\partial\alpha^2}\Psi\left(x_{i,m_i};\alpha\right)-\sum_{i=1}^{k}\sum_{j=1}^{m_i}\left(\frac{(\partial^2/\partial\alpha^2)\psi\left(x_{i,j};\alpha\right)}{\psi\left(x_{i,j};\alpha\right)}-\frac{\left((\partial/\partial\alpha)\psi\left(x_{i,j};\alpha\right)\right)^2}{\psi^2\left(x_{i,j};\alpha\right)}\right),\\
J_{23} &= J_{32}=\sum_{i=1}^{k}\frac{\partial}{\partial\alpha}\Psi\left(x_{i,m_i};\alpha\right),\\
J_{33} &= \frac{m^{(k)}}{\lambda^2}.
\end{aligned}
$$

Due to the independence of α and λ from p, we have $J_{12}=J_{21}=J_{13}=J_{31}=0$. It should be noted that $J_{11}=\frac{m^{(k)}-k}{1-p}$ if $m_i=n$ and $J_{11}=\frac{k}{p^2}$ if $m_i=1$, for all $1\le i\le k$. If we denote V as the asymptotic variance-covariance matrix for $\boldsymbol{\theta}=(p,\alpha,\lambda)$, then the estimate of V can be obtained as

$$
\hat{V}=\begin{bmatrix}\hat{V}_{11} & 0 & 0\\ 0 & \hat{V}_{22} & \hat{V}_{23}\\ 0 & \hat{V}_{32} & \hat{V}_{33}\end{bmatrix}=J^{-1}(\hat{\boldsymbol{\theta}}),
$$

where $\hat{V}_{11}=\frac{1}{\hat{J}_{11}}$, $\hat{V}_{22}=\frac{\hat{J}_{33}}{\hat{J}_{22}\hat{J}_{33}-\hat{J}_{23}^2}$, $\hat{V}_{33}=\frac{\hat{J}_{22}}{\hat{J}_{22}\hat{J}_{33}-\hat{J}_{23}^2}$, and $\hat{V}_{23}=\hat{V}_{32}=\frac{\hat{J}_{32}}{\hat{J}_{23}^2-\hat{J}_{22}\hat{J}_{33}}$. Therefore, for $0<\gamma<1$, the $100(1-\gamma)\%$ ACIs for p,α, and λ are respectively given by

$$
\hat{p}\pm z_{\gamma/2}\sqrt{\hat{V}_{11}},\quad \hat{\alpha}\pm z_{\gamma/2}\sqrt{\hat{V}_{22}},\quad \text{and}\quad \hat{\lambda}\pm z_{\gamma/2}\sqrt{\hat{V}_{33}},
$$

where $z_{\gamma/2}$ is the upper $\gamma/2$th percentile point of the standard normal distribution.

In the Burr-XII distribution with $\Psi(t;\alpha)=\ln(1+t^\alpha)$, we have

$$
J_{22}=\frac{m^{(k)}}{\alpha^2}+\lambda\sum_{i=1}^{k}\frac{x_{i,m_i}^{\alpha}\left(\ln x_{i,m_i}\right)^2}{\left(1+x_{i,m_i}^{\alpha}\right)^2}+\sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{x_{i,j}^{\alpha}\left(\ln x_{i,j}\right)^2}{\left(1+x_{i,j}^{\alpha}\right)^2}, \tag{13}
$$

$$
J_{23}=J_{32}=\sum_{i=1}^{k}\frac{x_{i,m_i}^{\alpha}\ln x_{i,m_i}}{1+x_{i,m_i}^{\alpha}}. \tag{14}
$$

For Gompertz distribution with $\Psi(t;\alpha)=\frac{1}{\alpha}(e^{\alpha t}-1)$,

$$
J_{22}=\frac{\lambda}{\alpha^3}\left[\sum_{i=1}^{k}\left(\left(\alpha x_{i,m_i}-1\right)^2+1\right)e^{\alpha x_{i,m_i}}-2k\right], \tag{15}
$$

$$
J_{23}=J_{32}=\frac{1}{\alpha^2}\left[k+\sum_{i=1}^{k}\left(\alpha x_{i,m_i}-1\right)e^{\alpha x_{i,m_i}}\right]. \tag{16}
$$

In the case of Weibull distribution with $\Psi(t;\alpha) = t^\alpha$,

$$J_{22} = \frac{m^{(k)}}{\alpha^2} + \lambda \sum_{i=1}^{k} x_{i,m_i}^{\alpha} \left(\ln x_{i,m_i}\right)^2, \tag{17}$$

$$J_{23} = J_{32} = \sum_{i=1}^{k} x_{i,m_i}^{\alpha} \ln x_{i,m_i}. \tag{18}$$

Finally, for Chen distribution with $\Psi(t;\alpha) = e^{t^\alpha} - 1$,

$$J_{22} = \frac{m^{(k)}}{\alpha^2} + \lambda \sum_{i=1}^{k} \left(x_{i,m_i}^{\alpha} + 1\right) (\ln x_{i,m_i})^2 x_{i,m_i}^{\alpha} e^{x_{i,m_i}^{\alpha}} - \sum_{i=1}^{k} \sum_{j=1}^{m_i} \left(\ln x_{i,j}\right)^2 x_{i,j}^{\alpha}, \tag{19}$$

$$J_{23} = J_{32} = \sum_{i=1}^{k} (\ln x_{i,m_i}) x_{i,m_i}^{\alpha} e^{x_{i,m_i}^{\alpha}}. \tag{20}$$

3.2.2 Bootstrap-based confidence intervals

Since the exact distributions of $\hat{p}, \hat{\alpha}, \hat{\lambda}$ are not available, we fail to find the exact CIs for the unknown parameters p, α, and λ. For this reason, the bootstrap method can be used an alternative to construct approximate CIs for p, α, and λ. There are two types of bootstrap method for constructing CIs: non-parametric bootstrap and parametric bootstrap. The difference between non-parametric (re-sampling with replacement) and parametric (re-sampling from the fitted model) bootstrap methods lies in the way of generating bootstrap samples. Since the unknown parameters p, α, and λ can be estimated by ML method and often the parametric bootstrap method is better than the non-parametric bootstrap method (see Dekking et al. [19]). In this section, the parametric bootstrap method used by Dekking et al. [19] and Efron [20] is considered. Here are the main steps of using the parametric bootstrap to compute CIs for the parameters p, α, and λ as follows.

Step 1: Given the original data, calculate $\hat{\boldsymbol{\theta}} = (\hat{p}, \hat{\alpha}, \hat{\lambda})$.

Step 2: Using the MLE $\hat{\boldsymbol{\theta}}$ as the true value of the parameter, within the same sampling framework, generate sample $(\tilde{\boldsymbol{x}}^*, \boldsymbol{m}^*)$ for given n and k.

Step 3: Based on the bootstrap sample obtained above, calculate $\hat{\boldsymbol{\theta}}^* = (\hat{p}^*, \hat{\alpha}^*, \hat{\lambda}^*)$, the MLE for $\boldsymbol{\theta} = (p, \alpha, \lambda)$, in the same way as described in subsection 3.1.

Step 4: Repeat Steps 2 and 3, $B-1$ times. Then denote the MLEs by $\hat{\boldsymbol{\theta}}_1^*, \hat{\boldsymbol{\theta}}_2^*, \ldots, \hat{\boldsymbol{\theta}}_B^*$, where $\hat{\boldsymbol{\theta}}_\ell^* = (\hat{p}_\ell^*, \hat{\alpha}_\ell^*, \hat{\lambda}_\ell^*)$ is the MLE of $\boldsymbol{\theta}$ based on the ℓ-th bootstrap sample, $1 \le \ell \le B$.

Step 5: To construct a bootstrap-p confidence interval(BCI), arrange $\hat{\alpha}_\ell^*, 1 \le \ell \le B$ in an ascending order to obtain the bootstrap samples as $\hat{\alpha}_{(1)}^*, \hat{\alpha}_{(2)}^*, \ldots, \hat{\alpha}_{(B)}^*$.

Then $\left(\hat{\alpha}^*_{(\lfloor B\gamma/2\rfloor)}, \hat{\alpha}^*_{(\lfloor B-B\gamma/2\rfloor)}\right)$ is a two-sided $100(1-\gamma)\%$ BCI for α, where $\lfloor x\rfloor$ is the largest integer less than or equal to x. The BCIs for p and λ are obtained in an analogous manner.

To improve the precision of the percentile bootstrap CI, we can further use the following bootstrap bias correction. For a model parameter, say α, a two-sided $100(1-\gamma)\%$ parametric bias-corrected bootstrap confidence interval ($\mathrm{BCI_a}$) is specified by

$$\hat{\alpha} - b_\alpha \pm z_{\gamma/2}\sqrt{v_\alpha},$$

where b_α and v_α are respectively the bootstrap bias and bootstrap variance for MLE $\hat{\alpha}$ and are defined as

$$b_\alpha = \bar{\hat{\alpha}}^* - \hat{\alpha}, \quad v_\alpha = \frac{1}{B-1}\sum_{\ell=1}^{B}(\hat{\alpha}^*_\ell - \bar{\hat{\alpha}}^*)^2,$$

with $\bar{\hat{\alpha}}^* = \sum_{\ell=1}^{B}\hat{\alpha}^*_\ell/B$. The parametric $\mathrm{BCI_a}$ for p and λ can be constructed in a similar way.

3.3 Likelihood ratio test

A very popular form of statistical test, which is used to compare two nested models, is the likelihood ratio test (LRT). This test examines whether a reduced model provides the same fit as a full model or not. The likelihood ratio test statistic is given by

$$\Lambda(\tilde{\boldsymbol{x}}) = \frac{\sup\limits_{\boldsymbol{\theta}\in\Theta_0} L(\boldsymbol{\theta};\tilde{\boldsymbol{x}})}{\sup\limits_{\boldsymbol{\theta}\in\Theta} L(\boldsymbol{\theta};\tilde{\boldsymbol{x}})} = \frac{L(\tilde{\boldsymbol{x}};\hat{\boldsymbol{\theta}}_0)}{L(\tilde{\boldsymbol{x}};\hat{\boldsymbol{\theta}})},$$

where $\hat{\boldsymbol{\theta}}_0$ is the constrained MLE under hypothesis H_0, and $\hat{\boldsymbol{\theta}}$ is the (unconstrained) MLE of θ. Let $\boldsymbol{\theta}_0 = (p_0, \lambda_0, \alpha_0)$, where $p_0 \in (0,1)$, $\lambda_0 > 0$ and $\alpha_0 > 0$, are known. Now we are interested in testing hypotheses $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0 \quad vs. \quad H_1 : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$. By inserting representation (6), it is readily seen that the test statistic $T_{\mathrm{LR}} = \ln(\Lambda(\tilde{\boldsymbol{x}}))$ of the LR test is given by

$$\begin{aligned} T_{\mathrm{LR}} &= \left(m^{(k)} - k\right)\ln\left(\frac{1-p_0}{1-\hat{p}}\right) + \sum_{i=1}^{k} I(m_i < n)\ln\left(\frac{p_0}{\hat{p}}\right) + m^{(k)}\ln\left(\frac{\lambda_0}{\hat{\lambda}}\right) \\ &\quad + \sum_{i=1}^{k}\sum_{j=1}^{m_i}\ln\left(\frac{\psi\left(x_{i,j};\alpha_0\right)}{\psi\left(x_{i,j};\hat{\alpha}\right)}\right) - \lambda_0\sum_{i=1}^{k}\Psi\left(x_{i,m_i};\alpha_0\right) + \hat{\lambda}\sum_{i=1}^{k}\Psi\left(x_{i,m_i};\hat{\alpha}\right). \end{aligned} \tag{21}$$

For testing $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0 \;\; vs. \;\; H_1 : \boldsymbol{\theta} = \boldsymbol{\theta}_1$, wherein $\boldsymbol{\theta}_1 = (p_1, \alpha_1, \lambda_1)$, the test statistic T_{LR} can be obtained by substituting $\hat{p}, \hat{\alpha}, \hat{\lambda}$ with p_1, α_1, λ_1 in Eq. (21).

4 Bayesian inference

In recent decades, the Bayes viewpoint, as a powerful and valid alternative to traditional statistical perspectives, has received frequent attention for statistical inference. In fact, the use of MLEs is not necessarily suitable, when there are only a few failures for the repairable systems. Furthermore, the engineer's knowledge about the degradation and failure process could be very helpful to reach the more accurate estimations of the model parameters. This is especially true about the repair efficiency parameter p, because usually in real cases, p may not be predetermined, hence, it is reasonable to consider p as a random variable with a prior distribution. Then, a Bayesian approach for estimating the reliability and repair efficiency parameters is an interesting alternative to usual frequentist methods.

In this section, we obtain the Bayesian estimators of model parameters of the SG model under symmetric as well as asymmetric loss functions. The most commonly used loss function is the squared error which is symmetric in the sense that underestimation and overestimation are equally penalized. However, there is no specific procedure in the estimation process to determine which loss function should be used. Thus, we need to consider some asymmetric loss functions as well. Therefore, we consider different loss functions to get a better understanding of Bayesian analysis. Varian [21] introduced the linear-exponential (LINEX) loss function which is asymmetric. Another useful loss function is the general entropy loss function. In this paper, we apply the stated loss functions such as squared error, LINEX, and general entropy to calculate the desired Bayesian estimators which are defined as

$$
\begin{aligned}
L_S\left(\hat{\delta}, \delta\right) &= \left(\hat{\delta} - \delta\right)^2, \\
L_L\left(\hat{\delta}, \delta\right) &= e^{c\left(\hat{\delta}-\delta\right)} - c\left(\hat{\delta} - \delta\right) - 1 \quad c \neq 0, \\
L_E\left(\hat{\delta}, \delta\right) &\propto \left(\frac{\hat{\delta}}{\delta}\right)^w - w \ln\left(\frac{\hat{\delta}}{\delta}\right) - 1 \quad w \neq 0,
\end{aligned}
$$

respectively, where $\hat{\delta}$ is an estimator of δ. The Bayesian estimator of δ concerning to the loss function L_S is the posterior mean of δ, say $\hat{\delta}_{BS}$. Under the loss function L_L, the Bayesian estimator of δ is given by $\hat{\delta}_{BL} = -\frac{1}{c} \ln E\left(e^{-c\delta}|\boldsymbol{x}\right)$, provided $E\left(e^{-c\delta}|\boldsymbol{x}\right)$ exists and is finite. Finally, under the loss function L_E the corresponding estimator is of the form $\hat{\delta}_{BE} = (E(\delta^{-w}|\boldsymbol{x}))^{-\frac{1}{w}}$, provided $E(\delta^{-w}|\boldsymbol{x})$ exists and is finite.

4.1 Prior information

In this subsection, the necessary assumptions about prior distributions are developed. Under the assumption that two parameters α and λ are unknown, specifying a general conjugate joint prior for α and λ, is not an easy task. In this case, we develop the Bayesian set-up by considering the idea of Soland [22] regarding the choice of prior distributions. Suppose that α has a discrete prior and λ has a continuous conditional prior for given α. Thus, the prior distribution of α is of the form

$$P(\alpha = \alpha_\ell) = \xi_\ell, \qquad \ell = 1, 2, ..., N, \tag{22}$$

where $0 \leq \xi_\ell \leq 1$ and $\sum_{\ell=1}^{N} \xi_\ell = 1$. For a given α_ℓ, we use an exponential prior distribution for λ to achieve a close-conjugate family, which in turn makes the computation simple. Then, we have

$$\pi(\lambda|\alpha_\ell) = b_\ell \exp\{-b_\ell \lambda\}, \quad \lambda, b_\ell > 0, \tag{23}$$

where b_ℓs are hyper-parameters. Since the repair efficiency parameter p belongs to $[0, 1]$, we choose the beta distribution, denoted by $B(r, s)$, as a prior distribution with the following probability density function

$$\pi(p) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} p^{r-1} (1-p)^{s-1} I_{(0,1)}(p). \tag{24}$$

4.1.1 The choice of hyper-parameters

The priors specification are completed by specifying α_l, ξ_l and hyper-parameters b_ℓ for $1 \leq \ell \leq N$. The values of α_ℓ and ξ_ℓ are fairly straightforward to specify, but sometimes it is not always possible to know the value of the hyper-parameter b_ℓ, in prior distribution of λ. In practice, the value of b_ℓ is difficulty to know, since it is necessary to condition prior beliefs about λ on each α_ℓ, $1 \leq \ell \leq N$. Therefore, the estimation problem for hyper-parameter b_ℓ is considered. For given α_ℓ, the value of the hyper-parameter b_ℓ can be obtained based on the maximum likelihood Type-II method (see Berger [23, p.99]).

It is worth noting that if random variable X has the PDF (1), then $Y = \Psi(X; \alpha)$ has an exponential distribution with the following PDF

$$f_Y(y; \lambda) = \lambda e^{-\lambda y} \quad y > 0.$$

Therefore, given α_ℓ the marginal PDF and CDF of Y can be written as

$$f_Y(y) = \int_0^\infty \pi(\lambda|\alpha_\ell) f_Y(y; \lambda)\, d\lambda = \frac{b_\ell}{b_\ell + y} \quad y > 0,$$

and $F_Y(y) = 1 - \frac{b_\ell}{(b_\ell + y)^2}$, respectively.

Now, given $\alpha = \alpha_\ell$; $1 \leq \ell \leq N$, let us consider $y_{i,j} = \Psi(x_{i,j}; \alpha_\ell); 1 \leq i \leq k,\ 1 \leq j \leq m_i$. Using (4), the likelihood function can be written as

$$L(p, F_Y; \tilde{\boldsymbol{y}}, \boldsymbol{m}) = (1-p)^{m^{(k)}-k}\ p^{\sum_{i=1}^{k} I(m_i<n)} \prod_{i=1}^{k}\prod_{j=1}^{m_i} \frac{y_{i,j-1}+b_\ell}{(y_{i,j}+b_\ell)^2}$$
$$\propto b_\ell^k \left(\prod_{i=1}^{k} \frac{1}{y_{i,m_i}+b_\ell}\right)\left(\prod_{i=1}^{k}\prod_{j=1}^{m_i} \frac{1}{y_{i,j}+b_\ell}\right), \tag{25}$$

where $\tilde{\boldsymbol{y}} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k)$, $\boldsymbol{y}_i = (y_{i,1}, y_{i,2}, ..., y_{i,m_i})$; $1 \le i \le k$. By differentiating the log-likelihood function with respect to b_ℓ, we immediately have the following nonlinear equation:

$$\frac{k}{b_\ell} - \sum_{i=1}^{k} \frac{1}{y_{i,m_i}+b_\ell} - \sum_{i=1}^{k}\sum_{j=1}^{m_i} \frac{1}{y_{i,j}+b_\ell} = 0. \tag{26}$$

As a consequence, the MLE of b_ℓ, say $\hat{b}_\ell$, can be obtained by solving Eq. (26). It is evident that explicit expression cannot be obtained directly by solving Eq. (26) and a numerical method such as the Newton-Raphson method is used to compute $\hat{b}_l$. Note that, it can be shown that the nonlinear Eq. (26) has a unique solution with respect to b_ℓ (the proof is very similar to the proof of Wang and Shi [14, p.376] and is therefore omitted for the sake of brevity).

4.2 Posterior analysis

Attentive to the likelihood function (5) and the prior distributions (22)-(24), the joint posterior density function of p, α_ℓ and λ can be written as

$$\pi(p, \alpha_\ell, \lambda | \tilde{\boldsymbol{x}}, \boldsymbol{m}) \propto p^{r+\sum_{i=1}^{k} I(m_i<n)-1}(1-p)^{m^{(k)}+s-k-1}\xi_\ell$$
$$\times \lambda^{m^{(k)}} \exp\{-\lambda\beta_\ell\} \prod_{i=1}^{k}\prod_{j=1}^{m_i} \psi\left(x_{i,j};\alpha_\ell\right), \tag{27}$$

where $\beta_\ell = b_\ell + \sum_{i=1}^{k} \Psi\left(x_{i,m_i};\alpha_\ell\right)$. It follows, from Eq. (27), that the marginal posterior distribution of p is specified by

$$\pi\left(p | \tilde{\boldsymbol{x}}, \boldsymbol{m}\right) \propto p^{r+\sum_{i=1}^{k} I(m_i<n)-1}(1-p)^{m^{(k)}+s-k-1}$$
$$\equiv B\left(r + \sum_{i=1}^{k} I\left(m_i < n\right), m^{(k)} + s - k\right). \tag{28}$$

By considering

$$L(\alpha_\ell, \lambda; \tilde{\boldsymbol{x}}, \boldsymbol{m}) = \lambda^{m^{(k)}} \left(\prod_{i=1}^{k}\prod_{j=1}^{m_i} \psi\left(x_{i,j};\alpha_\ell\right)\right) \exp\left\{-\lambda \sum_{i=1}^{k} \Psi\left(x_{i,m_i};\alpha_\ell\right)\right\}, \tag{29}$$

the conditional posterior of λ given α_ℓ can be expressed as

$$\begin{aligned}\pi\left(\lambda|\alpha_\ell,\tilde{\boldsymbol{x}},\boldsymbol{m}\right) &= \frac{\pi(\lambda|\alpha_\ell)L(\alpha_\ell,\lambda;\tilde{\boldsymbol{x}},\boldsymbol{m})}{\int_0^\infty \pi(\lambda|\alpha_\ell)L(\alpha_\ell,\lambda;\tilde{\boldsymbol{x}},\boldsymbol{m})\,d\lambda}\\ &= \frac{1}{\Gamma(m^{(k)}+1)}\beta_\ell^{m^{(k)}+1}\lambda^{m^{(k)}}\exp\{-\lambda\beta_\ell\} \qquad 1\le\ell\le N. \end{aligned} \tag{30}$$

In this case, the conditional posterior of λ given α_ℓ has a gamma distribution with the shape parameter $m^{(k)}+1$ and the scale parameter $1/\beta_\ell$. Utilizing Eqs.(22), (23), (29), and the discrete version of Bayes theorem, the marginal posterior mass function of α_ℓ can be written as

$$\begin{aligned}\eta_\ell &= P(\alpha=\alpha_\ell|\tilde{\boldsymbol{x}},\boldsymbol{m})\\ &= \frac{\int_0^\infty P(\alpha=\alpha_\ell)\pi(\lambda|\alpha_\ell)L(\alpha_\ell,\lambda;\tilde{\boldsymbol{x}},\boldsymbol{m})\,d\lambda}{\sum_{u=1}^N\int_0^\infty P(\alpha=\alpha_u)\pi(\lambda|\alpha_u)L(\alpha_u,\lambda;\tilde{\boldsymbol{x}},\boldsymbol{m})\,d\lambda}\\ &= \frac{\left(\prod_{i=1}^k\prod_{j=1}^{m_i}\psi(x_{i,j};\alpha_\ell)\right)\beta_\ell^{-(m^{(k)}+1)}\xi_\ell b_\ell}{\sum_{u=1}^N\left(\prod_{i=1}^k\prod_{j=1}^{m_i}\psi(x_{i,j};\alpha_u)\right)\beta_u^{-(m^{(k)}+1)}\xi_u b_u}, \qquad 1\le\ell\le N.\end{aligned} \tag{31}$$

From Eqs. (28), (30) and (31), the Bayesian estimators of p, α, and λ under the squared error loss function, are given respectively by

$$\hat{p}_{BS} = \frac{r+\sum_{i=1}^k I\left(m_i<n\right)}{r+\sum_{i=1}^k I\left(m_i<n\right)+m^{(k)}+s-k}, \tag{32}$$

$$\hat{\alpha}_{BS} = \sum_{\ell=1}^N \eta_\ell\alpha_\ell, \tag{33}$$

$$\hat{\lambda}_{BS} = \sum_{\ell=1}^N\left(1+m^{(k)}\right)\frac{\eta_\ell}{\beta_\ell}. \tag{34}$$

Under the LINEX loss function, we get

$$\hat{p}_{BL} = -\frac{1}{c}\ln\left[1+\sum_{v=1}^\infty\left(\prod_{z=0}^{v-1}\frac{r+z+\sum_{i=1}^k I\left(m_i<n\right)}{r+\sum_{i=1}^k I\left(m_i<n\right)+m^{(k)}+s-k+z}\right)\frac{(-c)^v}{v!}\right], \tag{35}$$

$$\hat{\alpha}_{BL} = -\frac{1}{c}\ln\left[\sum_{\ell=1}^N\eta_\ell e^{-c\alpha_\ell}\right], \tag{36}$$

$$\hat{\lambda}_{BL} = -\frac{1}{c}\ln\left[\sum_{\ell=1}^N\eta_\ell\left(1+\frac{c}{\beta_\ell}\right)^{-\left(m^{(k)}+1\right)}\right]. \tag{37}$$

Finally, the Bayesian estimators of p, α, and λ under entropy loss function can be obtained as follows

$$\hat{p}_{BE} = \left[\frac{\Gamma\left(r + \sum_{i=1}^{k} I\left(m_i < n\right) - w\right)}{\Gamma\left(r + \sum_{i=1}^{k} I\left(m_i < n\right)\right)} \times \frac{\Gamma\left(r + \sum_{i=1}^{k} I\left(m_i < n\right) + m^{(k)} + s - k\right)}{\Gamma\left(r + \sum_{i=1}^{k} I\left(m_i < n\right) + m^{(k)} + s - k - w\right)} \right]^{-\frac{1}{w}}, \tag{38}$$

$$\hat{\alpha}_{BE} = \left[\sum_{\ell=1}^{N} \eta_\ell \alpha_\ell^{-w} \right]^{-\frac{1}{w}}, \tag{39}$$

$$\hat{\lambda}_{BE} = \left[\frac{\Gamma\left(m^{(k)} - w + 1\right)}{\Gamma\left(m^{(k)} + 1\right)} \sum_{\ell=1}^{N} \eta_\ell \beta_\ell^{w} \right]^{-\frac{1}{w}}. \tag{40}$$

The Bayesian estimators regarding different members of the desirable class can be obtained by putting various functions for Ψ and ψ in Eqs. (32)-(40). However, it can be seen that, in all cases, the Bayesian estimator of p is free from distribution.

5 Numerical computations

In this section, a simulation study was mainly performed to illustrate the effect of the proposed methodology. All the computations were conducted in R software (R x64 4.0.3) and the R code can be obtained on request from the authors. They were performed at the high-performance computing research center (HPCRC) of Amirkabir University of Technology using a machine equipped with 12 processor cores(2.3 GHZ) and 16 GB RAM. The performance of all estimates has been compared numerically in terms of their biases, mean squared errors (MSEs), and interval estimates in terms of average lengths (ALs) and coverage percentages (CPs) of two-sided CIs.

The simulation study was carried out based on the Weibull and Chen distributions, which are of great interest in the application. The Weibull distribution has been extensively used in many different fields such as reliability engineering and industrial applications. For more details see Murthy et al. [24]. The hazard function of Weibull distribution can be increasing ($\alpha > 1$), decreasing ($\alpha < 1$) or constant ($\alpha = 1$), which makes it suitable for modeling many of lifetime data. The bathtub-shape hazard function provides an appropriate conceptual model, for some electronic and mechanical products as well as the lifetime of humans. For example complex systems usually have a bathtub-shaped failure rate over the life cycle of the product. Thus, the Weibull distribution does not provide a reasonable parametric fit for lifetime data modeling with a bathtub-shaped

Table 1: The hyper-parameter values of prior distribution of α.

Distribution	α	$\ell \rightarrow$	1	2	3	4	5	6	7	8
Weibull	2	$\alpha_\ell \rightarrow$	1.5	1.6	1.7	1.8	1.9	2	2.1	2.2
	4		3.5	3.6	3.7	3.8	3.9	4	4.1	4.2
	6		5.5	5.6	5.7	5.8	5.9	6	6.1	6.2
Chen	0.4		0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
	1		0.7	0.8	0.9	1	1.1	1.2	1.3	1.4
	1.2		0.7	0.8	0.9	1	1.1	1.2	1.3	1.4
		$\xi_\ell \rightarrow$	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.125

hazard function. The Chen distribution exhibits a bathtub-shaped failure rate, which makes it flexible enough for modeling phenomena with both monotonic and non-monotonic failure rates, which are common in reliability and biological studies. The hazard function of Chen distribution has a bathtub-shape when $\alpha < 1$ and is an increasing function when $\alpha \geq 1$.

In the simulation study, it was assumed that $k = 10, n = 5, p = 0.25, 0.4$ and the true values of (α, λ) were (2, 0.4), (2, 0.8), (2, 1.5), (4, 0.4), (4, 0.8), (4, 1.5), (6, 0.4), (6, 0.8), (6, 1.5) for the Weibull distribution. In the case of the Chen distribution, we considered the true values of (α, λ) as (0.4, 0.7), (0.4, 1), (1, 0.7), (1,1), (1.2, 0.7), (1.2,1). To solve the nonlinear Eq. (12) and obtain the estimates of the unknown parameters using the ML method, the *nleqslv* package was applied. Although we employed the true values of alpha as starting values, it is worth mentioning that all starting values that were generated randomly yielded similar results. Moreover, to obtain the bootstrap CIs, we used $B = 5000$ bootstrap samples and follow the procedure described in subsection 3.2.2.

In the Bayesian context, we chose the values of the hyper-parameters of prior distribution of p to be $(r, s) = (2, 3)$. The prior knowledge about the true values of the unknown parameter α is given in Table 1. Using the Newton-Raphson method and Eq.(26), we found the values of the hyper-parameter b_ℓ for given values of α_ℓ, for $1 \leq \ell \leq 8$. To evaluate the Bayesian estimates under the loss functions L_L and L_E we took $c = 1$ and $w = 1$ respectively. With 10000 times simulation, the biases, MSEs, 90% and 95% CPs, and ALs for the CIs of the unknown parameters p, α, and λ were computed. And the simulation results are shown in Tables 1-8. The respective results for biases and MSEs are reported up to 4 decimal places. For the Weibull distribution, the biases and MSEs of the MLEs and Bayesian estimators of p, α and λ are given in Tables 2 and 3. Moreover, the ALs and CPs for the CIs of unknown parameters are shown in Tables 4 and 5. For the Chen distribution, we report the biases and MSEs of the MLEs and different Bayesian estimators in Tables 6 and 7. Table 8 presents ALs and CPs for the 90% and 95% CIs of p, α, and λ, when the Chen distribution has been used.

Table 2: Biases and MSEs of the ML and Bayesian estimators of unknown parameters for the Weibull distribution and $p = 0.25$.

			Bias			MSE		
α	λ	Method	p	α	λ	p	α	λ
2	0.4	MLE	0.0120	0.1100	-0.0057	0.0079	0.1327	0.0253
		BS	0.0320	-0.0785	0.0737	0.0068	0.0184	0.0166
		BL	0.0290	-0.0953	0.0651	0.0065	0.0216	0.0148
		BE	0.0090	-0.0965	0.0398	0.0058	0.0220	0.0116
	0.8	MLE	0.0127	0.1118	-0.0114	0.0081	0.1312	0.0610
		BS	0.0327	-0.0773	0.1061	0.0069	0.0179	0.0449
		BL	0.0297	-0.0941	0.0845	0.0066	0.0211	0.0382
		BE	0.0097	-0.0953	0.0595	0.0059	0.0215	0.0348
	1.5	MLE	0.0128	0.1167	-0.0030	0.0080	0.1350	0.1384
		BS	0.0328	-0.0761	0.1420	0.0068	0.0178	0.1185
		BL	0.0298	-0.0928	0.0895	0.0065	0.0210	0.0957
		BE	0.0098	-0.0940	0.0772	0.0059	0.0213	0.0987
4	0.4	MLE	0.0111	0.2192	-0.0056	0.0078	0.5418	0.0256
		BS	0.0313	-0.1265	0.0555	0.0067	0.0219	0.0104
		BL	0.0283	-0.1495	0.0506	0.0064	0.0282	0.0095
		BE	0.0083	-0.1384	0.0345	0.0057	0.0251	0.0079
	0.8	MLE	0.0115	0.2212	-0.0089	0.0080	0.5316	0.0612
		BS	0.0316	-0.1258	0.0891	0.0068	0.0214	0.0355
		BL	0.0286	-0.1489	0.0734	0.0065	0.0277	0.0312
		BE	0.0086	-0.1378	0.0542	0.0058	0.0246	0.0288
	1.5	MLE	0.0113	0.2150	0.0058	0.0080	0.5187	0.1351
		BS	0.0315	-0.1263	0.1305	0.0068	0.0216	0.1057
		BL	0.0285	-0.1494	0.0843	0.0065	0.0279	0.0862
		BE	0.0085	-0.1383	0.0734	0.0059	0.0248	0.0886
6	0.4	MLE	0.0117	0.3445	-0.0080	0.0079	1.1751	0.0248
		BS	0.0318	-0.1372	0.0457	0.0067	0.0217	0.0090
		BL	0.0288	-0.1619	0.0416	0.0064	0.0291	0.0084
		BE	0.0088	-0.1456	0.0282	0.0058	0.0241	0.0073
	0.8	MLE	0.0111	0.3498	-0.0125	0.0078	1.2278	0.0627
		BS	0.0313	-0.1372	0.0749	0.0066	0.0218	0.0323
		BL	0.0283	-0.1619	0.0611	0.0063	0.0291	0.0288
		BE	0.0083	-0.1456	0.0436	0.0057	0.0242	0.0269
	1.5	MLE	0.0114	0.3506	-0.0047	0.0081	1.2346	0.1353
		BS	0.0315	-0.1372	0.1106	0.0069	0.0218	0.0990
		BL	0.0285	-0.1619	0.0670	0.0066	0.0291	0.0822
		BE	0.0085	-0.1457	0.0562	0.0059	0.0242	0.0844

Table 3: Biases and MSEs of the ML and Bayesian estimators of unknown parameters for the Weibull distribution and $p = 0.4$.

			Bias			MSE		
α	λ	Method	p	α	λ	p	α	λ
2	0.4	MLE	0.0205	0.1436	-0.0063	0.0134	0.1833	0.0282
		BS	0.0136	-0.0847	0.0770	0.0082	0.0183	0.0194
		BL	0.0092	-0.1028	0.0677	0.0080	0.0219	0.0172
		BE	-0.0093	-0.1042	0.0408	0.0082	0.0223	0.0137
	0.8	MLE	0.0175	0.1471	-0.0046	0.0124	0.1873	0.0710
		BS	0.0114	-0.0840	0.1151	0.0077	0.0182	0.0560
		BL	0.0071	-0.1020	0.0907	0.0075	0.0218	0.0471
		BE	-0.0114	-0.1034	0.0628	0.0078	0.0223	0.0432
	1.5	MLE	0.0170	0.1402	0.0323	0.0128	0.1854	0.1673
		BS	0.0109	-0.0859	0.1599	0.0079	0.0186	0.1525
		BL	0.0066	-0.1039	0.0958	0.0077	0.0223	0.1182
		BE	-0.0119	-0.1053	0.0817	0.0080	0.0227	0.1236
4	0.4	MLE	0.0189	0.2775	-0.0048	0.0130	0.7336	0.0285
		BS	0.0124	-0.1292	0.0586	0.0080	0.0216	0.0135
		BL	0.0080	-0.1528	0.0527	0.0078	0.0282	0.0123
		BE	-0.0105	-0.1415	0.0337	0.0080	0.0249	0.0103
	0.8	MLE	0.0167	0.3044	-0.0058	0.0128	0.7682	0.0698
		BS	0.0106	-0.1270	0.0967	0.0079	0.0210	0.0471
		BL	0.0063	-0.1506	0.0771	0.0077	0.0275	0.0405
		BE	-0.0123	-0.1392	0.0538	0.0080	0.0242	0.0375
	1.5	MLE	0.0175	0.2991	0.0240	0.0128	0.7434	0.1681
		BS	0.0113	-0.1269	0.1397	0.0079	0.0208	0.1427
		BL	0.0070	-0.1506	0.0802	0.0077	0.0273	0.1115
		BE	-0.0115	-0.1392	0.0666	0.0080	0.0241	0.1169
6	0.4	MLE	0.0183	0.4196	-0.0044	0.0127	1.6513	0.0279
		BS	0.0119	-0.1393	0.0504	0.0079	0.0218	0.0125
		BL	0.0076	-0.1643	0.0453	0.0076	0.0294	0.0115
		BE	-0.0109	-0.1479	0.0286	0.0079	0.0243	0.0099
	0.8	MLE	0.0174	0.4252	-0.0042	0.0127	1.6523	0.0687
		BS	0.0112	-0.1391	0.0830	0.0078	0.0218	0.0437
		BL	0.0069	-0.1641	0.0650	0.0076	0.0293	0.0380
		BE	-0.0116	-0.1477	0.0429	0.0079	0.0242	0.0355
	1.5	MLE	0.0178	0.4319	0.0275	0.0129	1.6677	0.1705
		BS	0.0115	-0.1388	0.1313	0.0080	0.0217	0.1427
		BL	0.0072	-0.1638	0.0733	0.0078	0.0292	0.1126
		BE	-0.0113	-0.1474	0.0598	0.0081	0.0241	0.1180

Table 4: Average widths and coverage percentages(in parentheses) of 90% and 95% CIs of unknown parameters for the Weibull distribution and $p = 0.25$.

		p						α						λ					
		90%			95%			90%			95%			90%			95%		
α	λ	ACI	BCI	BCI_a	ACI	BCI	BCI_a	ACI	BCI	BCI_a	ACI	BCI	BCI_a	ACI	BCI	BCI_a	ACI	BCI	BCI_a
2	0.4	0.2735 (87.78)	0.2880 (86.23)	0.2924 (90.63)	0.3258 (92.99)	0.3471 (92.43)	0.3481 (93.85)	1.0779 (89.86)	1.2007 (85.19)	1.2289 (93.20)	1.2844 (95.00)	1.4568 (90.54)	1.4644 (96.82)	0.5090 (85.50)	0.5010 (87.13)	0.5078 (85.54)	0.6058 (90.14)	0.5968 (92.70)	0.6044 (90.23)
	0.8	0.2736 (87.30)	0.2883 (86.12)	0.2927 (90.17)	0.3260 (92.83)	0.3476 (92.50)	0.3485 (93.63)	1.0792 (89.97)	1.2024 (85.22)	1.2311 (93.72)	1.2859 (95.14)	1.4584 (90.78)	1.4670 (97.12)	0.7947 (87.22)	0.7991 (87.89)	0.8063 (88.09)	0.9469 (91.73)	0.9525 (93.02)	0.9608 (92.34)
	1.5	0.2741 (87.44)	0.2885 (86.06)	0.2928 (90.58)	0.3265 (93.12)	0.3476 (92.37)	0.3486 (93.88)	1.0817 (89.97)	1.2049 (84.87)	1.2337 (93.94)	1.2890 (95.36)	1.4615 (90.39)	1.4701 (97.14)	1.1696 (87.78)	1.2234 (88.16)	1.2444 (90.54)	1.3937 (93.13)	1.4726 (93.42)	1.4828 (94.38)
4	0.4	0.2729 (87.42)	0.2874 (86.10)	0.2919 (90.31)	0.3251 (92.72)	0.3465 (92.79)	0.3475 (93.66)	2.1551 (89.21)	2.3972 (84.50)	2.4543 (93.28)	2.5680 (94.72)	2.9080 (90.29)	2.9244 (96.61)	0.5092 (85.35)	0.5007 (86.84)	0.5076 (85.40)	0.6060 (89.50)	0.5963 (92.34)	0.6042 (89.60)
	0.8	0.2732 (87.42)	0.2875 (86.44)	0.2920 (90.21)	0.3254 (92.71)	0.3465 (92.58)	0.3477 (93.59)	2.1576 (90.44)	2.3998 (85.67)	2.4572 (93.77)	2.5710 (95.48)	2.9108 (91.10)	2.9279 (97.29)	0.7959 (87.33)	0.8002 (88.05)	0.8075 (88.11)	0.9483 (91.68)	0.9540 (93.14)	0.9621 (92.25)
	1.5	0.2730 (87.53)	0.2876 (86.14)	0.2919 (90.23)	0.3252 (92.74)	0.3465 (92.54)	0.3476 (93.40)	2.1529 (90.20)	2.3961 (85.95)	2.4533 (93.74)	2.5654 (95.45)	2.9059 (91.28)	2.9233 (97.24)	1.1729 (88.73)	1.2272 (88.71)	1.2489 (90.97)	1.3976 (93.44)	1.4770 (93.93)	1.4880 (94.63)
6	0.4	0.2734 (87.98)	0.2879 (86.05)	0.2921 (90.70)	0.3257 (92.86)	0.3469 (92.42)	0.3478 (93.66)	3.2416 (90.13)	3.6086 (85.07)	3.6943 (93.85)	3.8626 (95.41)	4.3776 (90.67)	4.4020 (97.2)	0.5076 (85.06)	0.4989 (86.63)	0.5058 (85.08)	0.6040 (89.66)	0.5942 (92.28)	0.6021 (89.75)
	0.8	0.2730 (87.96)	0.2876 (86.55)	0.2919 (90.84)	0.3252 (93.19)	0.3464 (92.80)	0.3476 (93.95)	3.2395 (89.59)	3.6087 (84.55)	3.6949 (93.45)	3.8601 (94.86)	4.3773 (90.38)	4.4028 (96.83)	0.7933 (86.40)	0.7978 (87.08)	0.8050 (87.46)	0.9452 (91.04)	0.9510 (92.59)	0.9592 (91.69)
	1.5	0.2730 (87.20)	0.2875 (86.27)	0.2919 (90.02)	0.3252 (92.67)	0.3462 (92.23)	0.3475 (93.53)	3.2441 (89.53)	3.6121 (84.69)	3.6978 (93.45)	3.8656 (95.06)	4.3803 (90.24)	4.4062 (96.95)	1.1674 (87.96)	1.2207 (87.93)	1.2416 (90.09)	1.3910 (92.87)	1.4687 (93.55)	1.4793 (94.05)

Table 5: Average lengths and coverage percentages(in parentheses) of 90% and 95% CIs of unknown parameters for the Weibull distribution and $p = 0.4$.

		p						α						λ					
		90%			95%			90%			95%			90%			95%		
α	λ	ACI	BCI	BCI_a	ACI	BCI	BCI_a	ACI	BCI	BCI_a	ACI	BCI	BCI_a	ACI	BCI	BCI_a	ACI	BCI	BCI_a
2	0.4	0.3435	0.3587	0.3634	0.4093	0.4315	0.4330	1.2274	1.4199	1.4691	1.4625	1.7350	1.7506	0.5283	0.5244	0.5333	0.6276	0.6255	0.6339
		(87.77)	(86.57)	(88.85)	(92.68)	(91.87)	(92.89)	(89.81)	(83.86)	(94.55)	(94.93)	(89.46)	(97.47)	(84.97)	(86.58)	(85.56)	(89.32)	(91.86)	(89.81)
	0.8	0.3431	0.3590	0.3633	0.4088	0.4317	0.4328	1.2272	1.4169	1.4655	1.4623	1.7312	1.7463	0.8264	0.8514	0.8678	0.9847	1.0215	1.0339
		(88.23)	(87.98)	(89.18)	(93.54)	(92.94)	(93.74)	(89.66)	(83.45)	(94.51)	(94.74)	(89.03)	(97.34)	(86.29)	(86.76)	(88.44)	(91.17)	(92.20)	(92.10)
	1.5	0.3424	0.3581	0.3627	0.4080	0.4307	0.4322	1.2221	1.4127	1.4615	1.4562	1.7263	1.7414	1.2520	1.3869	1.4750	1.4919	1.7008	1.7538
		(87.48)	(87.06)	(88.66)	(92.66)	(92.12)	(93.07)	(89.26)	(83.99)	(94.21)	(94.91)	(89.39)	(97.29)	(88.08)	(87.75)	(92.14)	(93.15)	(93.05)	(95.10)
4	0.4	0.3431	0.3586	0.3632	0.4087	0.4315	0.4327	2.4455	2.8295	2.9272	2.9139	3.4579	3.4880	0.5293	0.5252	0.5339	0.6288	0.6265	0.6346
		(87.79)	(87.18)	(88.84)	(93.19)	(92.21)	(93.26)	(89.70)	(84.05)	(94.31)	(95.08)	(89.99)	(97.45)	(84.48)	(86.43)	(85.01)	(89.47)	(92.04)	(89.65)
	0.8	0.3425	0.3579	0.3625	0.4081	0.4305	0.4319	2.4591	2.8408	2.9393	2.9301	3.4714	3.5024	0.8252	0.8503	0.8666	0.9833	1.0201	1.0325
		(87.42)	(87.25)	(88.35)	(92.65)	(92.42)	(93.07)	(89.40)	(83.75)	(94.48)	(94.87)	(89.37)	(97.45)	(86.49)	(86.99)	(88.44)	(91.22)	(92.54)	(92.19)
	1.5	0.3428	0.3581	0.3629	0.4084	0.4307	0.4323	2.4576	2.8400	2.9383	2.9284	3.4701	3.5013	1.2484	1.3829	1.4722	1.4875	1.6965	1.7505
		(87.73)	(87.40)	(89.15)	(93.25)	(92.40)	(93.41)	(89.98)	(83.83)	(94.90)	(95.36)	(89.61)	(97.52)	(88.52)	(88.51)	(92.17)	(93.43)	(93.57)	(95.09)
6	0.4	0.3432	0.3587	0.3632	0.4089	0.4315	0.4328	3.6669	4.2432	4.3895	4.3693	5.1853	5.2304	0.5297	0.5259	0.5346	0.6293	0.6273	0.6355
		(87.96)	(87.38)	(89.03)	(93.10)	(92.62)	(93.22)	(90.09)	(84.14)	(94.52)	(95.19)	(89.77)	(97.42)	(85.53)	(87.03)	(86.00)	(89.86)	(92.30)	(90.21)
	0.8	0.3429	0.3583	0.3630	0.4085	0.4311	0.4325	3.6701	4.2417	4.3882	4.3732	5.1836	5.2289	0.8267	0.8517	0.8677	0.9850	1.0214	1.0338
		(88.03)	(87.16)	(89.16)	(92.94)	(92.61)	(93.17)	(89.50)	(83.76)	(94.53)	(95.34)	(89.69)	(97.55)	(86.75)	(87.38)	(88.74)	(91.20)	(93.01)	(92.18)
	1.5	0.3427	0.3583	0.3629	0.4083	0.4312	0.4323	3.6750	4.2502	4.3969	4.3790	5.1937	5.2392	1.2488	1.3849	1.4768	1.4881	1.6983	1.7563
		(87.60)	(86.99)	(88.75)	(92.89)	(92.40)	(93.08)	(89.48)	(83.56)	(94.31)	(94.82)	(89.34)	(97.25)	(88.35)	(87.84)	(91.95)	(93.07)	(93.23)	(94.99)

Table 6: Biases and MSEs of the ML and Bayesian estimators of unknown parameters for the Chen distribution and $p = 0.25$.

			Bias			MSE		
α	λ	Method	p	α	λ	p	α	λ
0.4	0.7	MLE	0.0105	0.0171	-0.0036	0.0079	0.0033	0.0357
		BS	0.0308	0.0047	0.0392	0.0066	0.0018	0.0339
		BL	0.0278	0.0037	0.0229	0.0063	0.0018	0.0308
		BE	0.0078	-0.0009	-0.0040	0.0057	0.0018	0.0303
	1	MLE	0.0112	0.0190	0.0031	0.0082	0.0041	0.0582
		BS	0.0313	0.0041	0.0518	0.0069	0.0019	0.0552
		BL	0.0283	0.0029	0.0264	0.0066	0.0019	0.0494
		BE	0.0083	-0.0023	0.0036	0.0060	0.0019	0.0499
1	0.7	MLE	0.0108	0.0450	-0.0045	0.0078	0.0213	0.0355
		BS	0.0311	0.0292	0.0256	0.0066	0.0150	0.0337
		BL	0.0281	0.0218	0.0093	0.0063	0.0144	0.0312
		BE	0.0081	0.0145	-0.0191	0.0057	0.0142	0.0319
	1	MLE	0.0112	0.0497	0.0036	0.0080	0.0259	0.0583
		BS	0.0314	0.0330	0.0356	0.0068	0.0162	0.0555
		BL	0.0284	0.0245	0.0100	0.0065	0.0155	0.0507
		BE	0.0083	0.0162	-0.0141	0.0058	0.0153	0.0523
1.2	0.7	MLE	0.0121	0.0510	-0.0044	0.0081	0.0301	0.0355
		BS	0.0321	-0.0125	0.0577	0.0068	0.0108	0.0324
		BL	0.0291	-0.0205	0.0420	0.0065	0.0113	0.0289
		BE	0.0091	-0.0268	0.0175	0.0059	0.0120	0.0273
	1	MLE	0.0116	0.0620	-0.0008	0.0082	0.0392	0.0584
		BS	0.0317	-0.0159	0.0715	0.0070	0.0113	0.0531
		BL	0.0287	-0.0246	0.0468	0.0066	0.0120	0.0466
		BE	0.0087	-0.0317	0.0256	0.0060	0.0129	0.0460

From the tabulated values, we can draw the following conclusions:

• For the fixed values of α and λ, the biases and MSEs increase for each method as the true value of p increases. It is quite natural, because our tendency to change the system is greater once a shock occurs. Thus, the large values for p result to small number of samples. Moreover, in this case, the biases and MSEs for ML estimator of p and the MSEs for all the Bayesian estimators of p increase. As the last conclusion in this area, the ALs increase for all unknown parameters.

• For the fixed values of p and λ, as the true value of α increases, the biases and MSEs for all estimators of α increase. Another trend that we can see is related to increasing the AL of CIs, as α increases. The performance of the estimators of α and λ are to some extent similar, based on MSE; since for the fixed values of α and p, as λ increases, MSEs for all estimators of λ increase.

• It is observed that BS, BL, and BE are better than MLE in estimating all three parameters in terms of MSEs, which makes them more attractive to use in practical problems. For parameter p, in most times, BE is the best estimator, followed by the BL, the MLE, and finally the BS estimators in terms of biases. We can also find that the biases of Bayesian estimators are smaller than MLE

Table 7: Biases and MSEs of the ML and Bayesian estimators of unknown parameters for the Chen distribution and $p = 0.4$.

			Bias			MSE		
α	λ	Method	p	α	λ	p	α	λ
0.4	0.7	MLE	0.0198	0.0257	-0.0013	0.0133	0.0055	0.0432
		BS	0.0130	0.0047	0.0472	0.0082	0.0020	0.0402
		BL	0.0087	0.0034	0.0293	0.0079	0.0020	0.0360
		BE	-0.0098	-0.0024	0.0003	0.0082	0.0021	0.0352
	1	MLE	0.0174	0.0288	0.0223	0.0132	0.0067	0.0762
		BS	0.0112	0.0040	0.0721	0.0082	0.0021	0.0705
		BL	0.0068	0.0025	0.0423	0.0079	0.0021	0.0607
		BE	-0.0117	-0.0040	0.0169	0.0083	0.0022	0.0608
1	0.7	MLE	0.0175	0.0634	0.0008	0.0128	0.0336	0.0425
		BS	0.0113	0.0391	0.0316	0.0079	0.0173	0.0397
		BL	0.0069	0.0297	0.0137	0.0077	0.0164	0.0362
		BE	-0.0116	0.0206	-0.0167	0.0080	0.0161	0.0367
	1	MLE	0.0178	0.0767	0.0148	0.0129	0.0449	0.0745
		BS	0.0115	0.0460	0.0479	0.0079	0.0191	0.0704
		BL	0.0072	0.0356	0.0185	0.0077	0.0180	0.0623
		BE	-0.0114	0.0256	-0.0080	0.0080	0.0175	0.0638
1.2	0.7	MLE	0.0174	0.0756	-0.0008	0.0127	0.0496	0.0419
		BS	0.0113	-0.0196	0.0668	0.0078	0.0118	0.0388
		BL	0.0070	-0.0293	0.0494	0.0076	0.0126	0.0343
		BE	-0.0116	-0.0373	0.0224	0.0079	0.0137	0.0322
	1	MLE	0.0168	0.0834	0.0246	0.0127	0.0606	0.0757
		BS	0.0108	-0.0264	0.0924	0.0078	0.0121	0.0701
		BL	0.0065	-0.0369	0.0627	0.0076	0.0131	0.0595
		BE	-0.0121	-0.0458	0.0384	0.0079	0.0145	0.0584

in estimating α. On the other hand, MLE is better those BS, BL, and BE in estimating λ in terms of biases. Results for parameter α, especially in the case of Chen distribution, show that $\mathrm{BCI_a}$ has higher CPs than the other intervals, as well as the biggest ALs can be reached by $\mathrm{BCI_a}$. In this case, the actual CPs of BCI are far below the specified nominal level.

Table 8: Average lengths and coverage percentages(in parentheses) of 90% and 95% CIs of unknown parameters for the Chen distribution.

			p						α						λ					
			90%			95%			90%			95%			90%			95%		
p	α	λ	ACI	BCI	$\mathrm{BCI_a}$	ACI	BCI	$\mathrm{BCI_a}$	ACI	BCI	$\mathrm{BCI_a}$	ACI	BCI	$\mathrm{BCI_a}$	ACI	BCI	$\mathrm{BCI_a}$	ACI	BCI	$\mathrm{BCI_a}$
0.25	0.4	0.7	0.2729 (87.54)	0.2872 (86.48)	0.2915 (90.45)	0.3250 (93.20)	0.3461 (92.41)	0.3471 (93.80)	0.1729 (88.90)	0.1892 (85.12)	0.1928 (92.52)	0.2060 (94.22)	0.2294 (90.85)	0.2297 (96.51)	0.6038 (87.43)	0.6155 (88.49)	0.6238 (89.11)	0.7195 (92.44)	0.7367 (93.53)	0.7432 (93.45)
		1	0.2729 (87.08)	0.2872 (85.87)	0.2916 (90.04)	0.3250 (92.55)	0.3460 (92.45)	0.3472 (93.35)	0.1912 (89.61)	0.2120 (84.82)	0.2170 (93.26)	0.2278 (94.60)	0.2577 (90.82)	0.2586 (96.97)	0.7626 (88.30)	0.8004 (88.72)	0.8194 (91.33)	0.9087 (93.40)	0.9649 (93.82)	0.9761 (94.84)
	1	0.7	0.2729 (87.66)	0.2874 (86.34)	0.2917 (90.83)	0.3251 (93.13)	0.3463 (92.62)	0.3473 (93.76)	0.4335 (88.58)	0.4739 (84.46)	0.4829 (92.15)	0.5165 (94.02)	0.5746 (90.27)	0.5754 (96.38)	0.6033 (87.79)	0.6146 (88.26)	0.6228 (89.40)	0.7188 (92.29)	0.7358 (93.65)	0.7421 (93.24)
		1	0.2731 (87.48)	0.2875 (86.43)	0.2918 (90.37)	0.3253 (92.86)	0.3465 (92.52)	0.3475 (93.66)	0.4791 (89.34)	0.5310 (84.55)	0.5436 (93.38)	0.5709 (94.54)	0.6455 (90.39)	0.6478 (96.99)	0.7631 (88.16)	0.8006 (88.22)	0.8191 (90.68)	0.9093 (92.86)	0.9650 (93.72)	0.9758 (94.43)
	1.2	0.7	0.2736 (87.18)	0.2877 (85.87)	0.2923 (90.23)	0.3259 (92.69)	0.3469 (92.16)	0.3480 (93.53)	0.5188 (88.94)	0.5683 (84.85	0.5790 (92.46	0.6182 (94.06)	0.6890 (90.62)	0.6899 (96.47)	0.6034 (87.52)	0.6149 (87.96)	0.6230 (89.12)	0.7190 (92.45)	0.7360 (93.64)	0.7423 (93.07)
		1	0.2730 (86.92)	0.2873 (85.60)	0.2918 (90.10)	0.3252 (92.49)	0.3462 (92.03)	0.3474 (93.45)	0.5750 (88.67)	0.6376 (83.92)	0.6527 (92.66)	0.6851 (94.07)	0.7749 (90.03)	0.7778 (96.48)	0.7605 (87.81)	0.7980 (88.34)	0.8165 (90.27)	0.9062 (92.79)	0.9615 (93.44)	0.9728 (93.99)
0.4	0.4	0.7	0.3433 (87.45)	0.3588 (86.75)	0.3633 (88.50)	0.4090 (92.77)	0.4317 (92.05)	0.4328 (92.87)	0.2085 (87.86)	0.2402 (82.35)	0.2487 (93.67)	0.2484 (93.50)	0.2947 (88.74)	0.2963 (96.93)	0.6376 (86.98)	0.6764 (87.82)	0.7047 (90.16)	0.7597 (91.86)	0.8194 (93.28)	0.8390 (93.55)
		1	0.3424 (87.36)	0.3578 (86.79)	0.3624 (88.32)	0.4079 (92.39)	0.4305 (91.81)	0.4318 (92.58)	0.2300 (89.22)	0.2694 (82.70)	0.2807 (94.56)	0.2741 (94.67)	0.3317 (89.05)	0.3345 (97.52)	0.8231 (88.41)	0.9280 (87.94)	1.0344 (92.50)	0.9808 (93.29)	1.1452 (93.06)	1.2284 (95.09)
	1	0.7	0.3427 (87.65)	0.3582 (87.26)	0.3629 (88.85)	0.4082 (93.19)	0.4312 (92.42)	0.4323 (93.45)	0.5200 (88.72)	0.5981 (83.15)	0.6190 (93.97)	0.6196 (94.07)	0.7338 (89.21)	0.7376 (97.36)	0.6380 (87.03)	0.6765 (87.74)	0.7043 (90.03)	0.7603 (92.05)	0.8195 (93.20)	0.8384 (93.69)
		1	0.3428 (87.91)	0.3583 (87.38)	0.3629 (89.05)	0.4084 (93.36)	0.4310 (92.49)	0.4323 (93.52)	0.5767 (88.56)	0.6752 (82.08)	0.7032 (94.70)	0.6871 (94.00)	0.8310 (88.44)	0.8379 (97.63)	0.8182 (87.50)	0.9189 (87.26)	0.9939 (91.44)	0.9749 (92.48)	1.1330 (92.96)	1.1807 (94.48)
	1.2	0.7	0.3427 (87.46)	0.3584 (87.28)	0.3630 (88.60)	0.4083 (93.00)	0.4312 (92.35)	0.4325 (92.90)	0.6221 (88.28)	0.7164 (82.17)	0.7414 (93.86)	0.7412 (93.87)	0.8789 (89.02)	0.8834 (97.30)	0.6360 (87.10)	0.6741 (87.71)	0.7004 (90.13)	0.7578 (92.00)	0.8162 (93.10)	0.8339 (93.57)
		1	0.3426 (88.12)	0.3579 (87.77)	0.3627 (89.05)	0.4081 (93.15)	0.4308 (92.66)	0.4321 (93.20)	0.6887 (88.89)	0.8057 (83.02)	0.8395 (94.72)	0.8206 (94.66)	0.9920 (89.13)	1.0003 (97.74)	0.8238 (88.22)	0.9274 (87.93)	1.0035 (92.44)	0.9816 (93.60)	1.1444 (93.35)	1.1919 (95.42)

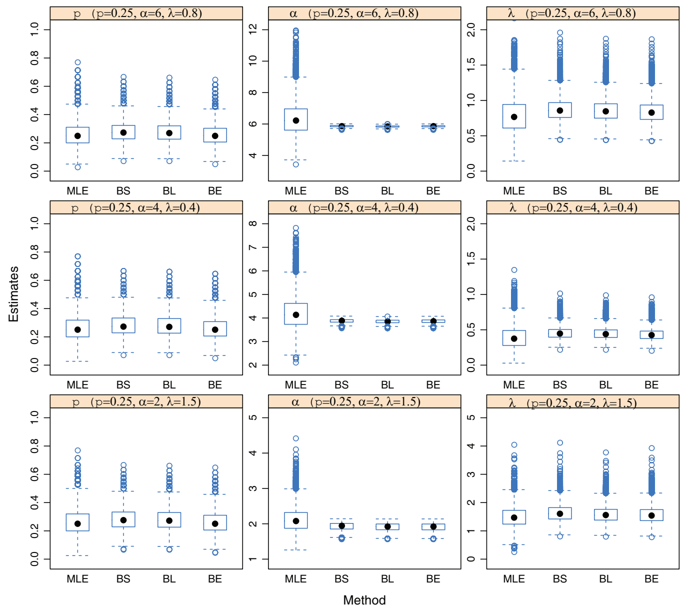

Figure 1: Boxplot for estimates of (p, α, λ) under different methods, for the Weibull distribution.

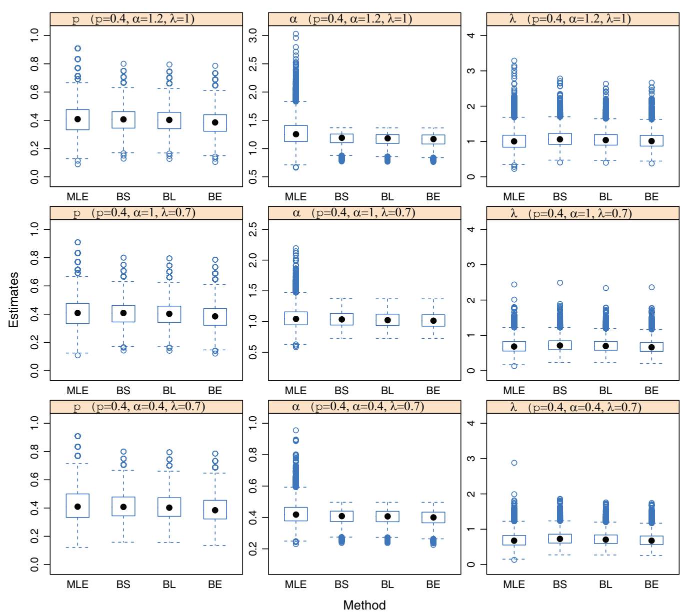

Figure 2: Boxplot for estimates of (p, α, λ) under different methods, for the Chen distribution.

The summary for the 10000 simulation runs for $(p, \alpha, \lambda) = (0.25, 2, 1.5)$,$(0.25, 4, 0.4)$, $(0.25, 6, 0.8)$ in the case of the Weibull model and $(p, \alpha, \lambda) = (0.4, 0.4, 0.7)$,$(0.4, 1, 0.7)$, $(0.4, 1.2, 1)$ in the case of the Chen model is graphically illustrated in Figures 1 and 2. These figures are a confirmation of

Table 9: The simulated critical and power values of T_{LR} for the hypothesis testing (41) at significance level 0.05.

Lifetime model	$(p_0, \alpha_0, \lambda_0)$	$(p_1, \alpha_1, \lambda_1)$	Critical Values			Power Values		
			$n = 3$	$n = 5$	$n = 7$	$n = 3$	$n = 5$	$n = 7$
Weibull	(0.25, 2, 0.8)	(0.25, 2, 0.4)	0.2205	0.9331	1.3627	0.9506	0.9793	0.9876
	(0.25, 2, 0.8)	(0.25, 2, 1.5)	-0.7529	0.0563	0.3940	0.9023	0.9638	0.9768
	(0.25, 2, 0.8)	(0.4, 2, 0.8)	-1.2817	-1.5285	-1.5523	0.4251	0.4561	0.4917
	(0.25, 4, 1.5)	(0.25, 4, 0.8)	-0.2010	0.3680	0.6976	0.9136	0.9584	0.9728
	(0.4, 4, 0.8)	(0.4, 4, 0.4)	-0.1456	0.1514	0.2736	0.9178	0.9461	0.9510
	(0.25, 4, 1.5)	(0.4, 4, 1.5)	-1.2817	-1.5286	-1.5523	0.4263	0.4556	0.4925
	(0.4, 6, 0.4)	(0.4, 6, 0.8)	-0.6623	-0.2952	-0.2375	0.9108	0.9444	0.9508
	(0.25, 6, 0.4)	(0.4, 6, 0.4)	-1.2817	-1.3528	-1.5760	0.4275	0.4637	0.4742
	(0.4, 6, 1.5)	(0.4, 4, 1.5)	-0.7338	-0.3682	-0.2528	0.7905	0.8703	0.8922
	(0.25, 6, 1.5)	(0.4, 2, 1.5)	7.4864	11.0460	12.8390	0.9999	1.0000	1.0000
Chen	(0.25, 0.4, 1)	(0.25, 0.4, 0.7)	-1.2240	-1.1238	-1.1054	0.5520	0.6525	0.6886
	(0.25, 0.4, 1)	(0.4, 0.4, 1)	-1.2817	-1.3529	-1.5523	0.4270	0.4651	0.4892
	(0.25, 0.4, 0.7)	(0.4, 0.4, 0.7)	-1.2817	-1.5286	-1.5760	0.4284	0.4598	0.4785
	(0.25, 1, 0.7)	(0.25, 1.2, 0.7)	-1.4492	-1.5542	-1.4127	0.3455	0.5734	0.7210
	(0.4, 1, 0.7)	(0.4, 1.2, 0.7)	-1.3299	-1.4705	-1.4668	0.2897	0.3982	0.4548
	(0.4, 1, 1)	(0.4, 0.4, 1)	3.2883	4.6902	4.9980	0.9978	0.9996	0.9997
	(0.4, 1, 1)	(0.25, 1, 1)	-1.2440	-1.3723	-1.1491	0.4346	0.5082	0.5796
	(0.25, 1.2, 0.7)	(0.25, 1, 0.7)	-1.0418	-0.8617	-0.6100	0.5156	0.7101	0.8082
	(0.25, 1.2, 0.7)	(0.25, 1.2, 1)	-1.4837	-1.4315	-1.3818	0.4770	0.5855	0.6414
	(0.25, 1.2, 0.7)	(0.4, 1.2, 0.7)	-1.2817	-1.3529	-1.5523	0.4259	0.4606	0.4938

the above results about point estimation. It is observed that the medians of the boxplots are close to the input parameters. From dispersions of the boxplots shown in Figures 1 and 2, it is found the Bayesian estimators provide the most precise results than MLEs in the cases of α and λ. Based on the result given in subsection 3.3, we consider the problem of testing

$$H_0 : (p, \alpha, \lambda) = (p_0, \alpha_0, \lambda_0) \quad \text{vs.} \quad H_1 : (p, \alpha, \lambda) = (p_1, \alpha_1, \lambda_1), \tag{41}$$

about the vector of parameters p, α, and λ. For each of the Weibull and Chen models, we obtain the estimated critical and power values of T_{LR} for the hypothesis testing (41) by a Monte Carlo simulation study. The critical and power values are given in Table 9 for $k = 10,\ n = 3, 5, 7$ and the significance level 0.05. It is worth mentioning that, based on Neyman-Pearson lemma, T_{LR} is the most powerful among all tests at a significance level of 0.05 for the hypothesis testing (41). Table 9 presents when n increases the power of the test increases.

5.1 The Severity of the effect of misspecification

The objective of this subsection focuses on the effect of model misspecification. This is an important problem, because empirical research (see Law and Wong [25]) has demonstrated that measurement model misspecification can bias structural parameter estimates. Especially, misspecification of reliability models related to lifetime data can lead to biased estimators, which in turn can lead to incorrect inference and models. Here, utilizing a simulation study, we show how misinterpretation of model parameters in the presence of model misspecification could be serious in some cases. To examine the effect of model misspecification, we consider the mean time to perfect repair, which is directly affected by the estimation results.

If Y^* denotes the time to the first perfect repair, then from Kamranfar et al. [13], the mean time to perfect repair is given as

$$\mu = E(Y^*) = \sum_{m=1}^{n} E(X_M|M=m)P(M=m), \tag{42}$$

where

$$E(X_M|M=m) = \frac{\lambda^m}{(m-1)!}\int_0^\infty x\psi(x;\alpha)[\Psi(x;\alpha)]^{m-1}\exp\{-\lambda\Psi(x;\alpha)\}\,dx,$$

$$P(M=m) = \begin{cases} (1-p)^{m-1}p & 1 \le m \le n-1, \\ (1-p)^{n-1} & m=n. \end{cases}$$

Utilizing invariance property of the ML estimators, the MLE of μ, say $\hat{\mu}$, can be obtained by substituting $\hat{p}, \hat{\alpha}$, and $\hat{\lambda}$ in Eq. (42). For the Burr-XII model as a special member of the class, we have

$$E(X_M|M=m) = \frac{\lambda^m}{(m-1)!}\int_1^\infty \frac{(x-1)^{\frac{1}{\alpha}}}{x^{1+\lambda}}(\ln x)^{m-1}\,dx. \tag{43}$$

For the Weibull model, we obtain

$$E(X_M|M=m) = \frac{\Gamma(m+\frac{1}{\alpha})}{\lambda^{\frac{1}{\alpha}}(m-1)!}. \tag{44}$$

Finally, in the case of the Chen model,

$$E(X_M|M=m) = \frac{\lambda^m e^{\lambda}}{(m-1)!}\int_1^\infty (x-1)^{m-1}(\ln x)^{\frac{1}{\alpha}}e^{-\lambda x}\,dx. \tag{45}$$

In the following, the effect of model misspecification on μ is assessed through a Monte Carlo simulation study. We consider the cases in which the data are originally from one of two models, Weibull and Chen. For example, suppose data are originally from the Weibull, but wrongly fitted to the Burr-XII or Chen models, then "What is the effect of model misspecification on the estimating μ?" Here, to answer this question, we consider the following steps:

Step 1: Given k, n and the parameters p, α, and λ, generate data from the Weibull model.

Step 2: Based on generated data, obtain the MLE of μ, say $\hat{\mu}^w$, using Eqs. (42) and (44).

Step 3: Wrongly fit the Burr-XII model, and calculate the MLE of μ, say $\hat{\mu}^b$, using Eqs. (42) and (43). Similarly, for the Chen model get the of MLE μ, say $\hat{\mu}^c$, using Eqs. (42) and (45).

Step 4: Repeat Steps 1-3, Q-1 times. Then denote the MLEs by $\hat{\mu}_1^w, \hat{\mu}_2^w, \ldots, \hat{\mu}_Q^w$ for the Weibull, $\hat{\mu}_1^b, \hat{\mu}_2^b, \ldots, \hat{\mu}_Q^b$ in the case of the Burr-XII and $\hat{\mu}_1^c, \hat{\mu}_2^c, \ldots, \hat{\mu}_Q^c$ for the Chen models.

Step 5: Considering true value μ^w (the mean time to perfect repair in the Weibull model), compute the bias and MSE of the estimated means for all three models. For example, in the case of the Burr-XII model the bias is given by $\frac{1}{Q}\sum_{i=1}^{Q}(\hat{\mu}_i^b - \mu^w)$.

Similarly, we also consider the case in which the data are originally generated from the Chen model. Table 10 presents the true value of the means μ^b, μ^w, μ^c as well as the biases and MSEs of the estimates of means for $Q = 50000, p = 0.25, k = 10, n = 3, 5, 7$ and different values of the parameters α and λ.

The results reveal that the model misspecification is not negligible when the Weibull and Chen models are misspecified as the Burr-XII model. We observe that biases and MSEs of the estimates of the mean time to perfect repair are big when the Burr-XII model is wrongly fitted. The effect of model misspecification between the Weibull and Chen models on the estimation of the mean time to perfect repair is not critical. It is observed when the Chen model is misspecified as the Weibull model the biases of the estimates of the mean time to perfect repair become a little larger in comparison with corresponding biases under the true model.

6 Illustrative examples

In this section, we present the analyses of two datasets for the SG model, considering four candidate members of the class of exponential distribution—Weibull, Chen, Gompertz and Burr-XII, all of which are well-known lifetime models.

Example 1. We consider the real data related to the Boeing air conditioner originally discussed by Proschan [26]. The original data contains the intervals between failures of 13 plane systems with numbers: 7907, 7908, 7909, 7910, 7911, 7912, 7913, 7914, 7915, 7916, 8044, 8045. The system of conditioner is embedded such a way that after roughly 2000 hours of service the plants received a major overhaul. Proschan [26] omitted the failure interval immediately following a major overhaul. These values are denoted by the symbol $**$. Presnell et al. [27] assumed that all repairs

Table 10: Biases and MSEs of the estimates of the mean time to perfect repair for all three models.

True model	(p, α, λ)	n	μ(True Value)	$\hat{\mu}$(Burr-XII)		$\hat{\mu}$(Weibull)		$\hat{\mu}$(Chen)	
				Bias	MSE	Bias	MSE	Bias	MSE
Weibull	(0.25, 1.2, 1.5)	3	1.3786	0.0982	0.1246	-0.0046	0.0756	0.0062	0.0773
		5	1.7295	0.1779	0.2441	0.0039	0.1286	0.0079	0.1307
		7	1.9135	0.2719	0.3988	-0.0133	0.1762	-0.0135	0.1789
	(0.25, 1.2, 1)	3	1.9327	0.3060	0.4430	-0.0051	0.1482	0.0102	0.1511
		5	2.4248	0.5518	0.9528	0.0066	0.2525	0.0097	0.2566
		7	2.6828	0.8428	1.8467	-0.0099	0.3518	-0.0059	0.3573
Chen	(0.25, 1.2, 0.7)	3	1.2196	0.0333	0.0268	-0.0116	0.0208	-0.0040	0.0210
		5	1.3522	0.0475	0.0343	-0.0128	0.0257	-0.0055	0.0256
		7	1.4014	0.0617	0.0418	-0.0160	0.0290	-0.0089	0.0288
	(0.25, 0.5, 1.2)	3	1.1504	0.3558	0.5611	0.0322	0.1009	-0.0048	0.0925
		5	1.5527	0.4835	0.7809	0.0248	0.1626	-0.0121	0.1556
		7	1.7573	0.6628	1.2203	0.0216	0.2132	-0.0173	0.2132

Table 11: Intervals between failures of the Boeing air conditioner systems.

Plane number															
7907	194	15	41	29	33	18									
7910	74	57	48	29	502	12	70	21	29	386	59	27	$**$	153	26	326
7911	55	320	56	104	220	239	47	246	176	182	33	$**$	15	104	35	
7915	359	9	12	270	603	3	104	2	438							
7916	50	254	5	283	35	12										
7917	130	493														
8044	487	18	100	7	98	5	85	91	43	230	3	130				

are imperfect and considered the major overhauls and the last observed failure ages of the planes as the times of the first perfect repair. By using a nonparametric procedure, they showed that imperfectly repaired systems are minimally repaired and there is no evidence against the minimal repair assumptions.

To illustrate the application of the SG model, Kamranfar et al. [13] considered plane numbers $7907, 7910, 7911, 7915, 7916, 7917, 8044$, and treated the major overhauls and the last observed failures of the remaining planes as a perfect repair. The intervals between failures are presented in Table 11. Thus, based on the SG model, in which system is replaced at the nth type-I failure or at the first type-II failure whichever occurs first, the main parameters in this model are as $n = 12, k = 7$, and $m = (6, 12, 11, 9, 6, 2, 12)$.

To examine whether theoretical models Weibull, Chen, Gompertz, and Burr-XII are suitable

Table 12: MLEs of α and λ and the measures K-S and AIC.

Fitted Model	$\hat{\alpha}$	$\hat{\lambda}$	K-S	P-value	AIC
Weibull	1.2780	0.0011	0.1963	0.9059	91.0202
Chen	0.2814	0.0102	0.1820	0.9434	91.9430
Gompertz	0.0017	0.0038	0.1796	0.9486	91.2358
Burr-XII	5.2688	0.0386	0.5486	0.0167	109.1632

to describe the air conditioner data using the SG model, or not, the Kolmogorov-Smirnov (K-S) test is adopted. In this part, the tests investigate the first time to failure of all 7 planes in Table 11. The MLEs of unknown parameters α and λ, K-S distances, and the corresponding P-values as well as Akaike's information criterion (AIC) are reported in Table 12. Based on the K-S distances and the p-values, the Weibull, Chen, and Gompertz models fit very well to the first time to failure; while the Burr-XII cannot be addressed as a well fitted model to this real dataset. However, the AIC index offers Weibull as the best-fitted model between the aforementioned models. Since the systems are minimally repaired before replacement and based on the obtained results in Table 12, the first time to failure follows Weibull, Chen or Gompertz distributions. Thus, we can fit these models to the intervals between failures in 7 planes in Table 11.

Now, to illustrate the applicability of the discussed methods, based on all the three fitted models and the real data in Table 11, the point and interval estimates of the unknown parameters p, α, and λ are obtained. To compute the Bayesian estimates, it is assumed that the values of the hyper-parameters of prior distribution of p to be $(r, s) = (2, 3)$. Table 13 gives the prior knowledge about the parameters α and λ, where the hyper-parameters $b_\ell, 1 \leq \ell \leq 8$ are estimated by using the proposed method in subsection 4.1.1. The MLEs and Bayesian estimates as well as 95% approximate and bootstrap confidence intervals of the unknown parameters p, α, and λ are listed in Table 14. As it can be observed the estimates of parameter p for the three models are the same and it gives us a confirmation for the obtained results in subsection 3.1.

A natural question that arises from the above results is: Which model to select? To answer this question, we consider a likelihood-based method used for the complete samples by Marshall et al. [28]. This method tends to select a model that gives the largest maximum likelihood value for the aforesaid dataset. If we show log-likelihoods functions for the Weibull, Chen, and Gompertz models respectively by ℓ^*_W, ℓ^*_C, and ℓ^*_G; thus from the calculated results in Table 14, the maximized log-likelihoods values are $\ell^*_W(\hat{p}, \hat{\alpha}, \hat{\lambda}) = -358.4198$, $\ell^*_C(\hat{p}, \hat{\alpha}, \hat{\lambda}) = -360.5057$, and $\ell^*_G(\hat{p}, \hat{\alpha}, \hat{\lambda}) = -359.0034$ respectively. As it can be seen, the Weibull model, leading to the largest maximum likelihood, seems to be the most appropriate model among these three suggested models for Boeing air conditioner data based on the SG model.

Table 13: The hyper-parameters values of prior distributions of α and λ.

Model	$l \rightarrow$	1	2	3	4	5	6	7	8
Weibull	α_l	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7
	b_l	53.7995	96.2755	171.6359	304.9463	540.1561	954.2036	1681.5856	2957.1133
Chen	α_l	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4
	b_l	0.3566	0.6752	1.4681	3.8529	12.7538	54.7334	315.3978	2658.6118
Gompertz	α_l	0.0005	0.0008	0.0011	0.0014	0.0017	0.0020	0.0023	0.0026
	b_l	61.0729	65.8819	71.0199	76.4818	82.2634	88.3629	94.7813	101.5228
	ξ_l	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.125

Table 14: The point and interval estimates of the unknown parameters ($c = 1, w = 1$).

Fitted Model	parameter	MLE	BS	BL	EP	ACI	BCI	BCI$_\mathrm{a}$
Weibull	p	0.0893	0.1147	0.1139	0.1000	(0.0146, 0.1639)	(0.0294, 0.2121)	(0.0000, 0.1738)
	α	1.1935	1.1959	1.1882	1.1830	(0.9168, 1.4701)	(0.9636, 1.5833)	(0.8422, 1.4640)
	λ	0.0018	0.0026	0.0026	0.0012	(0.0000, 0.0056)	(0.0001, 0.0098)	(0.0000, 0.0067)
Chen	p	0.0893	0.1147	0.1139	0.1000	(0.0146, 0.1639)	(0.0294, 0.2143)	(0.0000, 0.1743)
	α	0.2397	0.2495	0.2495	0.2494	(0.2177, 0.2616)	(0.2187, 0.2674)	(0.2129, 0.2616)
	λ	0.0359	0.0246	0.0246	0.0233	(0.0027, 0.0692)	(0.0107, 0.0830)	(0.0000, 0.0721)
Gompertz	p	0.0893	0.1147	0.1139	0.1000	(0.0146, 0.1639)	(0.0298, 0.2137)	(0.0000, 0.1722)
	α	0.0003	0.0006	0.0006	0.0005	(0.0000, 0.0008)	(0.0000, 0.0011)	(0.0000, 0.0007)
	λ	0.0062	0.0050	0.0050	0.0049	(0.0033, 0.0091)	(0.0035, 0.0084)	(0.0039, 0.0090)

The three-parameter generalized gamma (GG) distribution is commonly used in the reliability literature for modeling real data. It has the PDF

$$f(t;\alpha,\nu,\theta) = \frac{\alpha}{\theta\Gamma(\nu)}\left(\frac{t}{\theta}\right)^{\alpha\nu-1}\exp\left\{-\left(\frac{t}{\theta}\right)^{\alpha}\right\}, \quad t > 0, \tag{46}$$

where $\theta > 0$ is the scale parameter and $\alpha > 0$, and $\nu > 0$ are the shape parameters. Since different values of its parameters provide different forms of the hazard function such as constant, increasing, decreasing, and bathtub, the GG distribution is more flexible and applicable in reliability and lifetime studies. It is noticed that the GG distribution in Eq. (46) reduces to the Weibull distribution with $\Psi(t;\alpha) = t^\alpha$, when $\nu = 1$ and $\lambda = (1/\theta)^\alpha$. For the given dataset in Table 11, we fit the three-parameter GG distribution and statistically test whether the GG model can be reduced to the Weibull model. The MLEs and the maximized log-likelihood determined by fitting the GG distribution are $(\hat{p},\hat{\alpha},\hat{\nu},\hat{\theta}) = (0.0893, 0.9020, 2.2417, 76.2998)$, $\ell^*_{GG}(\hat{p},\hat{\alpha},\hat{\nu},\hat{\theta}) = -358.0490$. Therefore, the LR statistic is $2\ell^*_{GG}(\hat{p},\hat{\alpha},\hat{\nu},\hat{\theta}) - 2\ell^*_{W}(\hat{p},\hat{\alpha},\hat{\lambda}) = 0.7416$. The LR test yields a p-value of 0.3891 by a chi-squared distribution with one degree of freedom. Hence, for any usual significance

Table 15: Failure data observed from five copies of identical repairable systems.

System ID								
1	23.88	32.76	46.69	53.73	66.16			
2	39.50	39.76	41.65	48.71	50.62	55.66	55.85	67.22
3	17.74	30.58	30.98	45.16	55.63	62.18		
4	21.38	21.54	30.26	52.99	68.82			
5	9.15	14.85	47.80	54.69	59.98	62.68		

level, this analysis confirms that the extension from the Weibull distribution to the GG distribution is not statistically significant for modeling the given dataset.

Example 2. The dataset which provided by Liu et al. [29] is considered. The failure dataset is collected from five copies of identical repairable systems which are governed by the imperfect maintenance of the Kijima type I model (virtual age). Liu et al. [29] supposed that the system is discarded at the end of the third preventive maintenance (PM) cycle, which each PM is imperfect, and the failures between any two consecutive PM are minimally repaired. We use the third PM cycle and treat the last observed failure time as a perfect repair (as the first type-II failure). The failure dataset is tabulated in Table 15. It shows that the main parameters in the SG model are as $n = 8, k = 5$, and $m = (5, 8, 6, 5, 6)$.

Similar to **Example 1**, we consider the first time to failure of all five copies to examine whether theoretical models Weibull, Chen, Gompertz, and Burr-XII are suitable to describe the air conditioner data using the SG model, or not. Table 16 lists the MLEs of α and λ from the fitted Weibull, Chen, Gompertz and Burr-XII models and the values of K-S and AIC. The results show based on the K-S distances and the p-values, all four models fit to the first time to failure. However, the AIC index shows that the Weibull model yields a better fit compared with other fitted models. From the dataset, $(\hat{p}, \hat{\alpha}, \hat{\lambda})$, the MLE of (p, α, λ), for the Weibull, Chen, Gompertz and Burr-XII models are respectively as follows: $(0.1379, 1.9718, 0.0016)$, $(0.1379, 0.4121, 0.0222)$, $(0.1379, 0.0322, 0.0267)$ and $(0.1379, 8.8986, 0.1613)$. Here the issue of model choice arises. Using the likelihood-based method, the aim is to select a model which yields the largest maximum likelihood value for the the aforesaid dataset. The log-likelihood values for the Weibull, Chen, Gompertz and Burr-XII models are derived respectively as follows: $\ell^*_W(\hat{p}, \hat{\alpha}, \hat{\lambda}) = -107.6981$, $\ell^*_C(\hat{p}, \hat{\alpha}, \hat{\lambda}) = -108.2408$, $\ell^*_G(\hat{p}, \hat{\alpha}, \hat{\lambda}) = -108.2314$ and $\ell^*_B(\hat{p}, \hat{\alpha}, \hat{\lambda}) = -141.0657$. As it can be noticed the value of the log-likelihood function for the Weibull model is the largest. Hence the Weibull model can be labeled as the best model among these four suggested models for the dataset presented in Table 15. For the sake of brevity, the point and interval estimates of p, α and λ for the Weibull model are presented. Similar to **Example 1**, to compute the Bayesian estimates, it is assumed that the values of the hyper-parameters of prior

Table 16: MLEs of α and λ and the measures K-S and AIC.

Fitted Model	$\hat{\alpha}$	$\hat{\lambda}$	K-S	P-value	AIC
Weibull	2.4202	0.0004	0.2179	0.9288	40.7096
Chen	0.4912	0.0069	0.2505	0.8431	41.2538
Gompertz	0.0774	0.0114	0.2537	0.8330	41.4231
Burr-XII	8.7493	0.0381	0.5218	0.0858	54.9898

Table 17: The hyper-parameters values of prior distributions of α and λ.

Model	$l \rightarrow$	1	2	3	4	5	6	7	8
Weibull	α_l	1.6	1.7	1.8	1.9	2	2.1	2.2	2.3
	b_l	53.9339	76.4457	108.2384	153.0989	216.3468	305.4486	430.8787	607.3207
	ξ_l	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.125

distribution of p to be $(r, s) = (2, 3)$. Table 17 describes the prior knowledge about the parameters α and λ. For $c = 1$ and $w = 1$ the results of the Bayesian estimates of p, α and λ are computed as $(\hat{p}_{BS}, \hat{\alpha}_{BS}, \hat{\lambda}_{BS}) = (0.17647, 1.9411, 0.0026)$, $(\hat{p}_{BL}, \hat{\alpha}_{BL}, \hat{\lambda}_{BL}) = (0.1744, 1.9191, 0.0026)$ and $(\hat{p}_{BE}, \hat{\alpha}_{BE}, \hat{\lambda}_{BE}) = (0.1515, 1.9182, 0.0012)$. According to subsection 3.2, the 95% ACI, BCI and $\mathrm{BCI_a}$ for p are $(0.0124, 0.2634)$, $(0.0312, 0.3571)$ and $(0.0001, 0.2927)$, respectively. In the case of α these interval estimates are as $(1.2683, 2.6752)$, $(1.4719, 3.0653)$ and $(1.0085, 2.6397)$, respectively and for λ they are $(0.0001, 0.0062)$, $(0.0001, 0.0132)$ and $(0.0001, 0.0085)$.

7 Discussions and conclusions

The likelihood function (3) has been obtained based on the observed data $\boldsymbol{x} = (x_1, \ldots, x_m)$; $1 \leq m \leq n$. When m is smaller than n, it is clear that the first $m-1$ failures follow from minimal repair and the type of the mth failure is perfect. However, when $m = n$, again the first $n-1$ failures follow from minimal repair, but in this case, there is no information about the type of the nth failure. In other words, based on the observed data $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, the type of the last failure is ignored. Now, we suppose that when the nth failure is observed, the type of failure is known. This is described by the discrete random variable Z. Let $Z = 1$ if the nth failure is perfect and $Z = 0$ otherwise. Then, for all m; $1 \leq m \leq n$ and $x_1 < \cdots < x_m$, the likelihood function of the observed data can be written as

$$\begin{aligned}L(p, F; \boldsymbol{x}, z, m) &= \left(\prod_{j=1}^{n-1} (1-p)^{I(m>j)} p^{I(m=j)} [f(x_j|x_{j-1})]^{I(m\geq j)} \right) \\ &\qquad \times \left[(1-p)^{I(z=0)} p^{I(z=1)} f(x_n|x_{n-1}) \right]^{I(m=n)} \\ &= (1-p)^{m-1+I(z=0)I(m=n)} p^{I(m<n)+I(z=1)I(m=n)} \prod_{j=1}^{m} \frac{f(x_j)}{\bar{F}(x_{j-1})}. \end{aligned} \tag{47}$$

Due to the structure of the likelihood function (47), it is easy to see that for k systems, the ML and Bayesian estimators of the unknown parameters α and λ are the same as the ML and Bayesian estimators presented in subsections 3.1 and 4.2. It is clear that the ML and Bayesian estimators of p depend on z_i if $m_i = n$; $1 \leq i \leq k$, where z_i describes the type of the nth failure in the ith system. As an example, $\hat{p}$ is equal to 0 if $m_i = n$ and $z_i = 0$ for all $1 \leq i \leq k$.

This paper considers the statistical inference procedures for the SG model based on a class of univariate distributions generated from the exponential distribution. Both frequentist and Bayesian approaches are implemented. Through a Monte Carlo simulation study, the performance of the inferential methods are studied and the results are reported comprehensively in Section 5. The effect of model misspecification on the estimation of the mean time to perfect repair is investigated through a detailed Monte Carlo simulation study. This paper can be extended for more complicated situations requiring more complex models such as the systems with more components or parallel and series systems.

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