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► **To cite this version:**

F. Blasques, Christian Francq, Sébastien Laurent. Quasi score-driven models. *Journal of Econometrics*, 2023, 234 (1), pp.251-275. 10.1016/j.jeconom.2021.12.005 . hal-04069143

HAL Id: hal-04069143

<https://amu.hal.science/hal-04069143>

Submitted on 14 Apr 2023

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Quasi score-driven models*

F. Blasques[†], Christian Francq[‡] and Sébastien Laurent[§]

April 13, 2022

Abstract

This paper introduces the class of quasi score-driven (QSD) models. This new class inherits and extends the basic ideas behind the development of score-driven (SD) models and addresses a number of unsolved issues in the score literature. In particular, the new class of models (i) generalizes many existing models, including SD models, (ii) disconnects the updating equation from the log-likelihood implied by the conditional density of the observations, (iii) allows testing of the assumptions behind SD models that link the updating equation of the conditional moment to the conditional density, (iv) allows QML estimation of SD models, (v) and allows explanatory variables to enter the updating equation. We establish the asymptotic properties of the QLE, QMLE and MLE of the proposed QSD model as well as the likelihood ratio and Lagrange multiplier test statistics. The finite sample properties are studied by means of an extensive Monte Carlo study. Finally, we show the empirical relevance of QSD models to estimate the conditional variance of 400 US stocks.

JEL Classification: C13, C32, C53, C58.

Keywords: Score-driven models, GARCH, fat-tails, asymmetry, QLE, QMLE.

*The authors gratefully acknowledge the editor Torben Andersen, the associate editor and two anonymous referees for insightful comments and suggestions, Kris Boudt, Paul Embrechts, Patrick Gagliardini, Peter Reinhard Hansen, Andrew Harvey, Elvezio Ronchetti, Bilel Sanhaji and Olivier Scaillet for helpful discussions as well as the participants of the GFRI seminar in Geneva, the Econometrics seminar at KULeuven, the Quantact seminar in Montreal, the Econometrics and Business Statistics seminar at CREATES in Aarhus and the 22nd Dynamic Econometrics Conference in Oxford. Sébastien acknowledges research support by the French National Research Agency Grant ANR-17-EURE-0020, and by the Excellence Initiative of Aix-Marseille University - A*MIDEX. Christian and Sébastien also acknowledge research support by the French National Research Agency Grant ANR-21-CE26-0007-01. Christian also thanks the ECODEC labex.

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1 Introduction

Score-driven (*SD*) models, also known as generalized autoregressive score (*GAS*) models or dynamic conditional score (*DCS*) models, have been proposed independently by Harvey and Chakravarty (2008) and Creal, Koopman and Lucas (2012). They provide a general modelling strategy for time series data.

Consider a time series $\{y_t\}_{t \in \mathbb{Z}}$ with conditional density indexed by a time-varying parameter $\{f_t\}_{t \in \mathbb{Z}}$,

$$p_t(y_t, \theta) = p(y_t | f_t, \theta) \quad \forall t \in \mathbb{Z}.$$

The conditional distribution $p(y_t | f_t, \theta)$ is often defined by means of a time series model of the type

$$y_t = g(f_t, \epsilon_t), \tag{1}$$

where ϵ_t is an *i.i.d.* random variable that can be interpreted as an error term and g is a differentiable measurable function. $g(f_t, \epsilon_t) := f_t + \epsilon_t$ describes a location model while $g(f_t, \epsilon_t) := \sqrt{f_t} \epsilon_t$ corresponds to a volatility or duration model.

An *SD* model for f_t is a model of the form

$$f_{t+1} = \omega + \alpha S(f_t) \frac{\partial \log p(y_t | f_t, \theta)}{\partial f_t} + \beta f_t, \tag{2}$$

where $S(f_t)$ is a scaling function for the score, e.g., the inverse of an information matrix.

There are more than 220 papers referenced on the `gasmodel.com` website that build upon this modelling strategy and have applications in various areas in financial econometrics (such as default and credit risk modelling, stock volatility and correlation modelling, modelling time-varying dependence structures, CDS spread modelling, systemic risk, and high-frequency data) but also in macroeconomics or public health.

The success of *SD* models is attributable to the following features: (i) these models nest and extend existing observation-driven models such as *GARCH*, (ii) their estimation does not require sophisticated techniques (the conditional likelihood is readily available), (iii) they constitute a natural way to achieve robustness in the presence of fat-tailed innovations, (iv) statistical inference is standard, and (v) the models' predictive performance is outstanding.

They have received considerable attention in the literature on volatility modelling because when $p_t(y_t, \theta)$ has fatter tails than the Gaussian distribution (e.g., a Student's t distribution with a finite degree of freedom), $\frac{\partial \log p_t(y_t, \theta)}{\partial f_t}$ downweights and even bounds the effect of large shocks on the conditional variance; by contrast, in *GARCH* models, the squared residual is the main driver of the dynamics, irrespective of the choice of the density. This is in line with the empirical literature, which suggests that *GARCH* models tend to overestimate the conditional volatility for several days or even weeks following very large unexpected shocks (see Lecourt, Laurent and Palm, 2016, among others).

However, it is also clear from (2) that *SD* models impose a strong link between the conditional distribution of y_t , i.e., $p(y_t | f_t, \theta)$, and the updating equation of f_t , which is not always desirable. Testing the relevance of these restrictions and eventually relaxing them if they are rejected by the data can therefore be advantageous.

In this paper, we keep the downweighting mechanism (2) of *SD* models but allow the updating equation of f_t to be disconnected from the density of the innovations if needed. Our family of models, called quasi score-driven (or *QSD*) models, therefore encompasses not only *SD* models but also many other existing models. For instance, Banulescu-Radu, Hansen, Huang and Matei (2018) use a volatility model whose dynamic is derived from a *QSD* model obtained from a Student's t log-likelihood to bound the effect of large shocks, although the model is estimated by Gaussian ML.

We study the statistical properties of *QSD* models in the case where the conditional moment of interest depends on some covariates. We also study three estimation methods for this model. Since *QSD* models disconnect the dynamics in f_t and the density of the innovations, unlike *SD* models, they permit the consideration of the estimation of the parameters by QML or by ML with a score function in (2) taken with respect to a conditional density other than the one of y_t . We also study the estimation of *QSD* models using the quasi-likelihood estimator (QLE), which encompasses the QMLE. QLE can actually be seen as an extension of QMLE where the mean-variance relationship of the exponential family is relaxed.

In addition to the fact that these estimators are consistent and asymptotically normal, we also show that likelihood ratio and Lagrange multiplier tests of linear restrictions have the usual χ^2 distribution, which offers a strategy to test some restrictions implied by standard *SD* models.

We study in detail a *QSD* volatility model extending the β_T *GARCH* model of Harvey and Chakravarty (2008). This model, called *QSD_T GARCH – T*, relies on a Student's t density for the innovations and the score of a Student's t log-density in the updating equation of the conditional variance but does not restrict the degrees of freedom to be the same. The additional flexibility of this model (over the β_T *GARCH*) is found to be significant at the 5% significance level in more than 50% of the cases out of 400 US stocks.

We also show that when an asymmetric density is used instead of a symmetric Student's t density, the *SD* model is rejected in all cases in favour of a *QSD* model with different parameters for the innovation density and the updating function.

The rest of the paper is structured as follows. Section 2 presents the quasi score filtering equation and the properties of this model. The estimation of this model is studied in Section 3. Section 4 studies in more detail the *QSD_T GARCH – T* model, i.e., the stationarity, invertibility and positivity conditions, its estimation, some hypothesis tests as well as the finite sample properties of the QMLE and MLE via a Monte Carlo simulation. The empirical application is presented in Section 5. Finally, Section 6 concludes. All proofs are given in the online supplement.

2 The *QSD* filtering equation

We propose a new class of observation-driven models for f_t with an updating equation given by

$$f_{t+1} = \omega + \alpha\psi(y_t, X_t, f_t, \theta) + \beta f_t, \quad (3)$$

where ω, α, β are real parameters, $\theta \in \Theta \subset \mathbb{R}^p$ is a parameter vector (possibly containing ω, α, β among other parameters), ψ is a differentiable measurable function and X_t is a vector of exogenous random variables.

2.1 Some examples

The class of models defined by (1) and (3) is very general. For instance, when $p(y_t|f_t, \theta)$ is a Student's t density with conditional variance f_t and $\psi(y_t, X_t, f_t, \theta) := y_t^2$, we obtain the *GARCH – T* model of Bollerslev (1987) and a *GARCH – T* model with additive explanatory variables X_t in the conditional variance when $\psi(y_t, X_t, f_t, \theta) := y_t^2 + \varpi^\top X_t$, where ϖ is a parameter vector.

While the QSD filtering equation (3) is very general, the parallel with *SD* models is clearer when specifying ψ as follows

$$\psi(y_t, X_t, f_t, \theta) := \frac{\partial \rho(y_t, X_t, f_t, \theta)}{\partial f_t} S(f_t). \quad (4)$$

When the updating function $\rho(y_t, X_t, f_t, \theta)$ is the log-likelihood function of y_t , i.e., $\rho(y_t, X_t, f_t, \theta) = \log p(y_t|f_t, \theta)$, we recover the class of score models described in (2).¹ Importantly, when $\rho(y_t, f_t, \theta)$ is a log-likelihood function but $\rho(y_t, f_t, \theta) \neq \log p(y_t|f_t, \theta)$, ψ is no longer proportional to $\frac{\partial \log p(y_t, \theta)}{\partial f_t}$ but is proportional to $\frac{\partial \rho(y_t, f_t, \theta)}{\partial f_t}$. Consequently, the model is called the quasi score-driven (*QSD*) model.

The *GARCH – T* model presented above is an example of a *QSD* model, where $p(y_t|f_t, \theta)$ is a Student's t density with conditional variance f_t but $\rho(y_t, f_t, \theta)$ is a Gaussian log-likelihood with conditional variance f_t while $S(f_t) = -\mathbb{E} \left[\frac{\partial^2 \rho(y_t, f_t, \theta)}{\partial f_t \partial f_t^\top} \right]^{-1} = 2f_t^2$, i.e., the inverse of the information matrix of ρ .

In this formulation, QMLE also becomes a naturally viable alternative to MLE. Indeed, in a volatility model, large shocks can be downweighted by an appropriate choice of the updating function ρ , e.g., a Student's t log-likelihood, without imposing y_t to follow the same conditional distribution; e.g, y_t can be assumed to be conditionally Gaussian as in Banulescu-Radu, Hansen, Huang and Matei (2018). This is in contrast to *SD* models where there is a strong link between the innovation density and the updating function, which makes it unnatural to use QMLE.

We also note that, in our new model formulation, we can formulate updating equations that Winsorize or censor outliers, regardless of the conditional distribution $p(y_t|f_t, \theta)$. More generally, *QSD* models yield filtering equations that employ many popular loss functions used in robust statistics. These include the Cauchy–Lorentzian, the Geman–McClure and the Welsch–Leclerc criteria, as well as the generalized Charbonnier and pseudo Huber–Charbonnier loss functions.

Additionally, in empirical applications, we can define updating equations for volatility models that incorporate leverage effects even if the conditional density of y_t is symmetric or left-skewed. This stands in sharp contrast to “pure” *SD* models that are unable

¹Notice that, when exogenous variables X_t are not present in ψ and ρ , we simply write $\psi(y_t, f_t, \theta)$ and $\rho(y_t, f_t, \theta)$ instead of $\psi(y_t, X_t, f_t, \theta)$ and $\rho(y_t, X_t, f_t, \theta)$.

to deliver an updating equation with a leverage effect when the innovation density is left-skewed (as shown in Example 2 below). One can also have an asymmetric updating equation that gives greater penalty to over-prediction of conditional means or under-prediction of conditional volatilities (as is common in macro and financial policy), regardless of the conditional distributions of y_t . This is impossible in the more restrictive class of score models since $\rho(y_t, f_t, \theta)$ must be equal to $\log p(y_t|f_t, \theta)$.

Examples 1 and 2 cover cases of location and volatility filtering involving nonlinear asymmetric criteria as well as fat-tailed and skewed innovations.

EXAMPLE 1. (Asymmetric forecast with symmetric innovations) *Consider a location model where $y_t = f_t + \epsilon_t$ and ϵ_t is i.i.d.. When ϵ_t is also assumed to follow a $N(0, \sigma^2)$ distribution, the score is symmetric in ϵ_t , and the resulting SD model is given by*

$$f_{t+1} = \omega + \alpha \frac{\epsilon_t}{\sigma^2} + \beta f_t.$$

In many applications it is desirable to weight differently positive and negative shocks, irrespectively of the shape of the distribution of ϵ_t .

As an example, consider the negative linex loss function introduced by Varian (1975), i.e.,

$$\rho(y_t, f_t, \theta) = 1 + \delta \epsilon_t - \exp(\delta \epsilon_t).$$

The QSD filtering equation obtained with this loss function is

$$f_{t+1} = \omega + \alpha \delta (\exp(\delta \epsilon_t) - 1) + \beta f_t$$

and can therefore provide asymmetric forecasts even when the density of the innovations is symmetric.

EXAMPLE 2. (Volatility model with leverage effect and left-skewed innovations) *Stock returns are typically heavy tailed and left-skewed (see Giot and Laurent, 2003 among others). As such, SD models of the conditional volatility $y_t = \sqrt{f_t} \epsilon_t$ employing skewed distributions such as the skewed Gaussian or skewed Student's t distribution may define an updating equation:*

$$f_{t+1} = \omega + \alpha s(y_t, f_t, \theta) + \beta f_t,$$

where the score $s(y_t, f_t, \theta)$ is an asymmetric function of the returns y_t that produces higher volatility for positive returns (i.e., $y_t > 0$) and is more conservative for negative returns (i.e., $y_t < 0$). Unfortunately, this is contrary to the empirical evidence for the leverage effect, which predicts higher volatility after negative returns. Depending on the skewed Student's t distribution that is adopted, pure SD models may thus be unable to capture the leverage effect. This issue does not affect the larger class of QSD models

$$f_{t+1} = \omega + \alpha \psi(y_t, f_t, \theta) + \beta f_t,$$

since the ψ function can adopt nonlinear functional forms independently of the density of the innovations ϵ_t . This model will be discussed in more details in Section 5.2.

2.2 Stationarity and invertibility of QSD models

We now give general conditions for stationarity and invertibility of the QSD model defined by (1) and (3). These general conditions will be illustrated on the QSD_T $GARCH - T$ model presented in Section 4. In an earlier version of this paper (Blasques et al., 2020), the assumptions are also made more explicit on various specific examples.

Let $z_t = (\epsilon_t, X_t^\top)^\top \in \mathbb{R}^d$. Note that the time-varying parameter f_t satisfies a stochastic recurrence equation (SRE) of the form

$$f_{t+1} = \varphi(z_t, f_t), \quad (5)$$

where $\varphi : E \times F \rightarrow F$ is measurable. We assume that E is a convex subspace of \mathbb{R}^d and F is an interval; see Bougerol (1993) and Straumann and Mikosch (2006) for major references on SRE theory.

The results given in this section can be considered a direct application of the general theory developed in Bougerol (1993), but we will provide explicit and self-contained proofs. Our results are also closely in line with those of Straumann and Mikosch (2006), which focus on volatility models, or those reported for SD models in Blasques et al. (2020), which do not include exogenous variables.

Lemma 1 details conditions for the QSD model to generate stationary sequences as a data generating process.

LEMMA 1. (Existence of a DGP) *Assume that (z_t) is stationary and ergodic. Suppose that*

- (i) $\mathbb{E} \log^+ |\psi(g(f^0, \epsilon_t), X_t, f^0, \theta)| < \infty$ for some constant $f^0 \in F \subset \mathbb{R}$;
- (ii) $\mathbb{E} \log \Lambda_t < 0$ with $\Lambda_t = \sup_f \left| \alpha \frac{\partial \psi(g(f, \epsilon_t), X_t, f, \theta)}{\partial f} + \beta \right| < 0$.

Then, there exist unique strictly stationary and ergodic solutions $\{f_t\}_{t \in \mathbb{Z}}$ and $\{y_t\}_{t \in \mathbb{Z}}$ to Equations (1)-(3).

Lemma 2 gives conditions for the existence of bounded unconditional moments.

LEMMA 2. (Existence of a marginal moment) *Under the assumptions of Lemma 1, if the sequence (Λ_t) is *i.i.d.*,*

$$\mathbb{E} |\psi(g(f^0, \epsilon_t), X_t, f^0, \theta)|^r < \infty \quad \text{and} \quad \mathbb{E} \sup_f \left| \frac{\partial \psi(g(f, \epsilon_t), X_t, f, \theta)}{\partial f} \right|^r < \infty$$

for some $r > 0$, then the stationary solution to Equations (1)-(3) satisfies $\mathbb{E} |f_t|^s < \infty$ for some $s > 0$.

Note that if the exogenous variables appear additively, i.e., if $\psi(g(f, \epsilon_t), X_t, f, \theta) = \psi(g(f, \epsilon_t), f, \theta) + \varpi^\top X_t$, then Λ_t only depends on ϵ_t , and therefore the assumption that (Λ_t) is *i.i.d.* is satisfied without having to assume X_t to be *i.i.d.* itself.

Assume that, for some $\theta = \theta_0$ satisfying the assumptions of Lemma 1, (y_t) is the stationary solution to (1)-(3), and recall that the time-varying parameter f_t depends on the true but unknown parameter θ_0 . For all θ , let us investigate the solutions of the filter

$$f_{t+1}(\theta) = \omega + \alpha\psi(y_t, X_t, f_t(\theta), \theta) + \beta f_t(\theta), \quad t \in \mathbb{Z}, \quad (6)$$

so that $f_t(\theta_0) = f_t$. Note, however, that when $\beta \neq 0$, $f_t(\theta)$ is not computable from a finite number of past observations y_1, \dots, y_{t-1} and X_1, \dots, X_{t-1} . We thus approximate $f_t(\theta)$ by the statistics

$$\widehat{f}_{t+1}(\theta) = \omega + \alpha\psi(y_t, X_t, \widehat{f}_t(\theta), \theta) + \beta \widehat{f}_t(\theta), \quad t \geq 1, \quad (7)$$

with a starting value $\widehat{f}_1(\theta) \in \mathbb{C}(\Theta, F)$, where $\mathbb{C}(\Theta, F)$ denotes the space of the continuous functions from Θ to F .

Lemma 3 gives sufficient conditions for the invertibility of the QSD filter.

LEMMA 3. (Properties of the filter) *Let $\{y_t, X_t\}_{t \in \mathbb{Z}}$ be stationary and ergodic, and suppose that*

- (i) *for all $\theta \in \Theta$, there exists $f^0 \in F$ such that $\mathbb{E} \log^+ |\psi(y_t, X_t, f^0, \theta)| < \infty$;*
- (ii) $\mathbb{E} \log \sup_{f \in \mathbb{R}} \sup_{\theta \in \Theta} \left| \alpha \frac{\partial \psi(y_t, X_t, f, \theta)}{\partial f} + \beta \right| < 0$.

Then, for all $\theta \in \Theta$, there exists a unique strictly stationary and ergodic solution $\{f_t(\theta)\}_{t \in \mathbb{Z}}$ to (6). Furthermore, for all starting functions $\widehat{f}_1(\cdot) \in \mathbb{C}(\Theta, F)$, there exists $\varrho \in (0, 1)$ such that

$$\varrho^{-t} \sup_{\theta \in \Theta} \left| \widehat{f}_t(\theta) - f_t(\theta) \right| \rightarrow 0 \quad \text{a.s. as } t \rightarrow \infty. \quad (8)$$

When (8) holds, the model is said to be uniformly invertible. This property will be essential to find a consistent estimator of θ_0 and to approximate the time-varying parameter f_t .

It is necessary to study the first and second derivatives of the filter (6):

$$f'_{t+1}(\theta) := \frac{\partial f_{t+1}(\theta)}{\partial \theta} = A_t + b_t f'_t(\theta), \quad (9)$$

$$f''_{t+1}(\theta) := \text{vec} \left(\frac{\partial^2 f_{t+1}(\theta)}{\partial \theta \partial \theta^\top} \right) = C_t + b_t f''_t(\theta), \quad (10)$$

where

$$\begin{aligned} A_t &= \frac{\partial \omega}{\partial \theta} + \psi_t \frac{\partial \alpha}{\partial \theta} + \alpha \frac{\partial \psi_t}{\partial \theta} + f_t(\theta) \frac{\partial \beta}{\partial \theta}, & b_t &= \alpha \frac{\partial \psi_t}{\partial f} + \beta, \\ C_t &= \text{vec} \left(\frac{\partial^2 \omega}{\partial \theta \partial \theta^\top} + \psi_t \frac{\partial^2 \alpha}{\partial \theta \partial \theta^\top} + \frac{\partial \alpha}{\partial \theta} \frac{\partial \psi_t}{\partial \theta^\top} + \frac{\partial \psi_t}{\partial f} \frac{\partial \alpha}{\partial \theta} (f'_t)^\top \right. \\ &\quad + \frac{\partial \psi_t}{\partial \theta} \frac{\partial \alpha}{\partial \theta^\top} + \alpha \frac{\partial^2 \psi_t}{\partial f \partial \theta} (f'_t)^\top + \alpha \frac{\partial^2 \psi_t}{\partial \theta \partial \theta^\top} + f_t \frac{\partial^2 \beta}{\partial \theta \partial \theta^\top} + \frac{\partial \beta}{\partial \theta} (f'_t)^\top \\ &\quad \left. + \frac{\partial \psi_t}{\partial f} f'_t \frac{\partial \alpha}{\partial \theta^\top} + \alpha f'_t \frac{\partial^2 \psi_t}{\partial f \partial \theta^\top} + f_t \frac{\partial \beta}{\partial \theta^\top} + \alpha \frac{\partial^2 \psi_t}{\partial f^2} f'_t (f'_t)^\top \right), \end{aligned}$$

with $\psi_t = \psi(y_t, X_t, f_t(\theta), \theta)$ and, using Leibniz's notation,

$$\begin{aligned}\frac{\partial \psi_t}{\partial \theta} &= \left. \frac{\partial \psi(y, X, f, \theta)}{\partial \theta} \right|_{(y, X, f, \theta) = (y_t, X_t, f_t(\theta), \theta)}, \\ \frac{\partial \psi_t}{\partial f} &= \left. \frac{\partial \psi(y, X, f, \theta)}{\partial f} \right|_{(y, X, f, \theta) = (y_t, X_t, f_t(\theta), \theta)}\end{aligned}$$

and similar notations for the other derivatives. Assume that Θ is a compact subspace of \mathbb{R}^p with $p \geq 3$. Without loss of generality, assume that $\theta = (\theta_1, \dots, \theta_p)^\top$ with $\theta_1 = \omega$, $\theta_2 = \alpha$ and $\theta_3 = \beta$. Note that the expressions of A_t and C_t then become more explicit because, for instance, $\partial \omega / \partial \theta = (1, 0, \dots, 0)$. As in (7), we approximate $f'_t(\theta)$ by

$$\widehat{f}'_{t+1}(\theta) = \widehat{A}_t + \widehat{b}_t \widehat{f}'_t(\theta), \quad t \geq 1, \quad (11)$$

where $\widehat{f}'_1(\theta) \in \mathbb{C}(\Theta, \mathbb{R}^p)$ is a starting value, \widehat{A}_t and \widehat{b}_t are obtained by substituting $\widehat{f}_t(\theta)$ for $f_t(\theta)$ in A_t and b_t . With similar notations and assumptions, let

$$\widehat{f}''_{t+1}(\theta) = \widehat{C}_t + \widehat{b}_t \widehat{f}''_t(\theta), \quad t \geq 1. \quad (12)$$

Lemma 4 establishes stationarity and invertibility properties for the derivatives of the filter.

LEMMA 4. (Derivatives of the filter) *Let the conditions of Lemma 3 hold, assume that ψ admits continuous second-order derivatives with respect to its last two components, and suppose that*

$$(i) \text{ for all } \theta \in \Theta, \quad \mathbb{E} \left\{ \log^+ |\psi_t| + \log^+ \left\| \frac{\partial \psi_t}{\partial \theta} \right\| + \log^+ \left| \frac{\partial \psi_t}{\partial f} \right| + \log^+ |f_t(\theta)| \right\} < \infty.$$

Then, for all $\theta \in \Theta$, there exists a unique strictly stationary and ergodic solution $\{f'_t(\theta)\}_{t \in \mathbb{Z}}$ to (9). If in addition

$$(ii) \mathbb{E} \left\{ \log^+ \left(\sup_f \left| \frac{\partial \psi_t}{\partial f} \right| + \sup_{f, \theta} \left\| \frac{\partial^2 \psi_t}{\partial \theta \partial f} \right\| + \sup_f \left| \frac{\partial^2 \psi_t}{\partial f^2} \right| + \sup_\theta \|f'_t(\theta)\| \right) \right\} < \infty,$$

then, for all starting functions $\widehat{f}_1(\cdot) \in \mathbb{C}(\Theta, F)$ and $\widehat{f}'_1(\cdot) \in \mathbb{C}(\Theta, \mathbb{R}^p)$, there exists $\varrho \in (0, 1)$ such that

$$\varrho^{-t} \sup_{\theta \in \Theta} \left\| \widehat{f}'_t(\theta) - f'_t(\theta) \right\| \rightarrow 0 \text{ a.s. as } t \rightarrow \infty. \quad (13)$$

If we further assume

$$(iii) \text{ for all } \theta \in \Theta, \quad \mathbb{E} \left\{ \log^+ \left\| \frac{\partial^2 \psi_t}{\partial \theta \partial \theta^\top} \right\| + \log^+ \left\| \frac{\partial^2 \psi_t}{\partial \theta \partial f} \right\| + \log^+ \left| \frac{\partial^2 \psi_t}{\partial f^2} \right| \right\} < \infty,$$

then, for all $\theta \in \Theta$, there exists a unique strictly stationary and ergodic solution $\{f''_t(\theta)\}_{t \in \mathbb{Z}}$ to (10). Under the additional assumption

$$(iv) \mathbb{E} \left\{ \log^+ \left(\sup_{f, \theta_i, \theta_j} \left| \frac{\partial^3 \psi_t}{\partial \theta_i \partial \theta_j \partial f} \right| + \sup_{f, \theta_i} \left\| \frac{\partial^3 \psi_t}{\partial \theta_i \partial f^2} \right\| + \sup_f \left| \frac{\partial^3 \psi_t}{\partial f^3} \right| \right) \right\} < \infty,$$

then, for all starting functions $\widehat{f}_1(\cdot) \in \mathbb{C}(\Theta, F)$, $\widehat{f}'_1(\cdot) \in \mathbb{C}(\Theta, \mathbb{R}^p)$ and $\widehat{f}''_1(\cdot) \in \mathbb{C}(\Theta, \mathbb{R}^{p^2})$, there exists $\varrho \in (0, 1)$ such that

$$\varrho^{-t} \sup_{\theta \in \Theta} \left\| \widehat{f}''_t(\theta) - f''_t(\theta) \right\| \rightarrow 0 \text{ a.s. as } t \rightarrow \infty.$$

3 Estimating the QSD models

In contrast to SD models, QSD models disentangle the parameters involved in f_t of those involved in the conditional distribution. We first consider the case where the time-varying parameter of interest is $f_t = f_t(\theta_0)$, where θ_0 is a p -dimensional vector. It makes sense to estimate θ_0 , trying to be as agnostic as possible on the distribution of the innovations. In Section 3.1, we consider the class of the so-called quasi-likelihood estimators (QLEs), which encompasses the usual QMLEs. Section 3.2 will consider the MLE, which is the natural estimator for SD and QSD models, for which the conditional distribution is entirely specified by θ_0 . In that case, θ_0 contains the parameters governing the dynamic of f_t and possibly the shape parameters of the conditional distribution $p(\cdot | f, \theta_0)$.

3.1 The QLE approach

Assume that (y_t) is a stationary process satisfying the QSD models (1)-(3) with $f_t = f_t(\theta_0)$.

To estimate θ_0 using very weak assumptions, the estimating functions theory can be used. This is a general estimation method introduced in the seminal papers of Durbin (1960) and Godambe (1960) and that encompasses moment, likelihood and quasi-likelihood-based techniques (see Chandra and Taniguchi, 2001, Bera and Biliias, 2002, Heyde, 2008 and the references therein). By extending the Gauss-Markov theorem, Godambe (1960, 1985) developed a concept of an optimal estimating function that applies in finite *i.i.d.* samples, as well as for stochastic processes. In econometrics, Hansen (1985) and Chamberlain (1987) constitute early examples of the literature on the estimating equations approach, estimation of parameters characterized by moment conditions, as well as the literature on optimal instruments for conditional moments. See also Newey (1990) for an efficient instrumental variables estimation of nonlinear models and more recently, Ai and Chen (2003) for an estimation method for models of conditional moment restrictions, which contain finite dimensional unknown parameters and infinite dimensional unknown functions.

In time series models, there generally exists an “unbiased estimating function” $h_t = h_t(\theta_0) \in \mathbb{R}$, depending on y_t and $f_t = f_t(\theta_0)$, such that

$$\mathbb{E}_{t-1}(h_t) = 0,$$

where \mathbb{E}_{t-1} denotes the conditional expectation given the sigma-field \mathcal{F}_{t-1} generated by $\{y_s, X_s; s < t\}$. For a location model of the form $y_t = f_t + \epsilon_t$, where $\{\epsilon_t\}_{t \in \mathbb{Z}}$ is *i.i.d.* with $\mathbb{E}(\epsilon_t) = 0$, one can take $h_t(\theta) = y_t - f_t(\theta)$. For a volatility model $y_t = \sqrt{f_t} \epsilon_t$, with standard notation, we can set $h_t(\theta) = y_t^2 - f_t(\theta)$. Obviously, under standard regularity conditions, the score $\partial \log p_t(y_t, \theta_0) / \partial \theta$ is also an unbiased estimating function. An estimator of θ_0 can be obtained by solving an “estimating equation” of the form

$$\sum_{t=1}^T h_t(\hat{\theta}) a_{t-1} = 0_p, \tag{14}$$

where $a_t = a_t(\theta)$ is a p -dimensional vector of \mathcal{F}_t -measurable weights. Godambe (1985) shows that, within the class of the estimating functions of this form and under mild assumptions, the optimal choice of a_t is

$$a_{t-1} = \frac{1}{\sigma_t^2(\theta)} \mathbb{E}_{t-1} f'_t(\theta), \quad (15)$$

where $\sigma_t^2(\theta) = \mathbb{E}_{t-1} h_t^2(\theta)$ (possibly multiplied by an unimportant non-zero constant).

According to the terminology of the estimating functions theory, a solution to (14)–(15) is called the quasi-likelihood estimator (QLE).

3.1.1 Conditional moment estimation

We now consider the case where $f_t = \mathbb{E}_{t-1}(y_t^k)$ for some $k > 0$. Location models correspond to $k = 1$ and volatility models to $k = 2$. We set $h_t(\theta) = y_t^k - f_t(\theta)$.

Of course, the estimating function $h_t(\theta)$ is generally not computable because it depends on the unknown values $\{y_t, X_t; t \leq 0\}$. We thus approximate $f_t(\theta)$ by $\widehat{f}_t(\theta)$ in (7) and $f'_t(\theta)$ by $\widehat{f}'_t(\theta)$ in (11). Let $\widehat{h}_t(\theta) = y_t^k - \widehat{f}_t(\theta)$. Under the assumptions of Lemma 4, we have seen that there exists $\varrho \in (0, 1)$ such that

$$\varrho^{-t} \sup_{\theta \in \Theta} \left\{ \left| \widehat{f}_t(\theta) - f_t(\theta) \right| + \left\| \widehat{f}'_t(\theta) - f'_t(\theta) \right\| \right\} \rightarrow 0 \text{ a.s. as } t \rightarrow \infty. \quad (16)$$

In general, $\sigma_t^2(\theta)$ also depends on the unknown values $\{y_t, X_t; t \leq 0\}$, but we assume that there exists a sequence $\{\widehat{\sigma}_t^2(\theta)\}_{t \in \mathbb{N}}$ computable from y_1, \dots, y_t and X_1, \dots, X_t such that

$$\varrho^{-t} \sup_{\theta \in \Theta} \left| \widehat{\sigma}_t^2(\theta) - \sigma_t^2(\theta) \right| \rightarrow 0 \text{ a.s. as } t \rightarrow \infty. \quad (17)$$

Moreover, assume that there exists a constant $\underline{\sigma}^2 > 0$ such that

$$\inf_{\theta \in \Theta} \left| \sigma_t^2(\theta) \right| > \underline{\sigma}^2 \text{ a.s.} \quad (18)$$

As an approximation of (14)–(15), it seems natural to consider the solutions of

$$\widehat{G}_T(\widehat{\theta}) = 0_p, \quad \widehat{G}_T(\theta) = \frac{1}{T} \sum_{t=t_0+1}^T \frac{\widehat{h}_t(\theta)}{\widehat{\sigma}_t^2(\theta)} \widehat{f}'_t(\theta). \quad (19)$$

If $\widehat{\sigma}_t^2(\theta) \leq 0$, which is impossible for large t under (17)–(18), the corresponding term can be omitted in (19). The integer t_0 is fixed and does not matter for the asymptotic behaviour of the estimator but is expected to attenuate the effect of the (arbitrary) choice of the initial values $\widehat{f}_1(\theta)$ and $\widehat{f}'_1(\theta)$. In practice, one could take $t_0 = 5$ (one week for most daily series), $\widehat{f}_1(\theta) = \sum_{t=1}^{t_0} y_t^k / t_0$ and $\widehat{f}'_1(\theta) = 0_p$.

3.1.2 Existence of the estimator

Note that the existence of a solution $\hat{\theta} \in \Theta$ to (19) is not guaranteed. For instance, consider a location model $y_t = f_t + \epsilon_t$, where (ϵ_t) is *i.i.d.* with a mean of 0 and variance σ_ϵ^2 . If $f_t(\theta) = \omega + \alpha y_{t-1}$ with $\theta_0 = (\omega_0, 0)$ and $\Theta = [\underline{\omega}, \bar{\omega}] \times [0, \bar{\alpha}]$, then with non-zero probability, we have

$$\widehat{G}_T(\theta) = \frac{1}{T} \sum_{t=t_0+1}^T \frac{y_t - \omega - \alpha y_{t-1}}{\sigma_\epsilon^2} \begin{pmatrix} 1 \\ y_{t-1} \end{pmatrix} \neq 0_2, \quad \forall \theta \in \Theta.$$

More precisely, when the first component of $\widehat{G}_T(\theta)$ is null and $\sum_t (y_t - \bar{y})(y_{t-1} - \bar{y}) < 0$ (which should be the case with probability of approximately 1/2 when $\alpha_0 = 0$), the second component of $\widehat{G}_T(\theta)$ is strictly negative for any value of $\alpha \geq 0$. Instead of (19), we thus define a QLE as a measurable solution of

$$\widehat{\theta}_T = \arg \min_{\theta \in \Theta} \left\| \widehat{G}_T(\theta) \right\| \quad (20)$$

with

$$\widehat{G}_T(\theta) = \frac{1}{T} \sum_{t=t_0+1}^T \widehat{g}_t(\theta), \quad \widehat{g}_t(\theta) = \frac{\widehat{h}_t(\theta)}{\widehat{\sigma}_t^2(\theta)} \widehat{f}'_t(\theta).$$

Since Θ is a compact set and f_t is assumed to be of class C^1 , a solution of (20) always exists but may not be unique. We will see that the asymptotic value of $\widehat{\theta}_T$ does not depend on the norm taken in (20).

3.1.3 Moment and identifiability assumptions

Let the rescaled innovations $\eta_t(\theta) = h_t(\theta)/\sigma_t(\theta)$. Consider the moment conditions

$$\mathbb{E} \sup_{\theta \in \Theta} |\eta_t(\theta)|^r < \infty, \quad \mathbb{E} \sup_{\theta \in \Theta} \left\| \frac{1}{\sigma_t(\theta)} f'_t(\theta) \right\|^r < \infty. \quad (21)$$

Under (21) with $r = 2$, let

$$G(\theta) = \mathbb{E} g_1(\theta), \quad g_t(\theta) = \frac{h_t(\theta)}{\sigma_t^2(\theta)} f'_t(\theta).$$

Since $\mathbb{E}_{t-1}(y_t^k) = f_t(\theta_0)$, we obviously have $G(\theta_0) = 0$. Assume that the equality holds at only $\theta = \theta_0$:

$$\theta_0 \in \Theta \quad \text{and} \quad G(\theta) = 0 \text{ for } \theta \in \Theta \text{ if and only if } \theta = \theta_0. \quad (22)$$

To show the asymptotic normality of the QLEs, we need to consider the extra moment conditions

$$\mathbb{E} \sup_{\theta \in V(\theta_0)} \left\| \frac{1}{\sigma_t^3(\theta)} f'_t(\theta) \frac{\partial \sigma_t^2(\theta)}{\partial \theta^\top} \right\|^r < \infty, \quad \mathbb{E} \sup_{\theta \in V(\theta_0)} \left\| \frac{1}{\sigma_t(\theta)} f''_t(\theta) \right\|^r < \infty, \quad (23)$$

for some neighborhood $V(\theta_0)$ of θ_0 . To deal with the effect of the initial values, we also need

$$\rho^{-t} \sup_{\theta \in \Theta} \left\| \frac{\partial \widehat{\sigma}_t^2(\theta)}{\partial \theta} - \frac{\partial \sigma_t^2(\theta)}{\partial \theta} \right\| \rightarrow 0 \text{ a.s. as } t \rightarrow \infty. \quad (24)$$

3.1.4 Asymptotic behaviour of the QLE

Under (21) with $r = 2$, let $\mathcal{J} = \mathbb{E} \frac{1}{\sigma_t^2(\theta_0)} f'_t(\theta_0) f'_t(\theta_0)^\top$. Assume that

$$\mathcal{J} \text{ is invertible.} \quad (25)$$

Theorem 1 establishes the consistency and asymptotic normality of the QLE for conditional moment models when the sample size diverges, i.e., when $T \rightarrow \infty$.

THEOREM 1. (CAN of the QLE for conditional moment models) *Let $\{y_t\}_{t \in \mathbb{Z}}$ be generated by (1)-(3) with θ replaced by θ_0 and $\mathbb{E}_{t-1}(y_t^k) = f_t(\theta_0)$ for some $\theta_0 \in \Theta$ and $k > 0$. Let the conditions of Lemma 1 hold at $\theta = \theta_0$ and the conditions of Lemma 4 hold. Assume that $\mathbb{E}(\log^+ |y_t|^k) < \infty$, $\mathbb{E}(\log^+ \sup_{\theta \in \Theta} |f_t(\theta)|) < \infty$ and $\mathbb{E}(\log^+ \sup_{\theta \in \Theta} \|f'_t(\theta)\|) < \infty$. Suppose further (17), (18), (21) with $r = 2$, (22), Θ is a compact subset of \mathbb{R}^p and $\theta_0 \in \Theta$. Then, for any sequence $\hat{\theta}_T$ satisfying (20) and for T large enough, we have $\hat{\theta}_T \xrightarrow{a.s.} \theta_0$ as $T \rightarrow \infty$.*

Moreover, if θ_0 belongs to the interior of Θ , (23) holds with $r = 2$, (24) and (25), then

$$\sqrt{T}(\hat{\theta}_T - \theta_0) = \mathcal{J}^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^T \frac{\eta_t(\theta_0)}{\sigma_t(\theta_0)} f'_t(\theta_0) + o_P(1) \xrightarrow{d} N(0, \mathcal{J}^{-1})$$

with the usual notation. A strongly consistent estimator of \mathcal{J} is $\hat{\mathcal{J}}_T = \frac{\partial \hat{G}_T(\hat{\theta}_T)}{\partial \theta^\top}$.

As an application of Theorem 1, let us test that some components of θ_0 are zero, for instance the last p_2 components. The null writes $H_0 : \theta_0^{(2)} = 0_{p_2}$, where $\theta_0^{(2)} = K\theta_0$ with $K = (0_{p_2 \times p_1}, I_{p_2})$ and I_{p_2} is the identity matrix of size p_2 . The vector of the first p_1 components of θ_0 is $\theta_0^{(1)} = \tilde{K}\theta_0$ with $\tilde{K} = (I_{p_1}, 0_{p_1 \times p_2})$. Under H_0 , the QLE of $\theta_0^{(1)}$ is define as a measurable solution of

$$\hat{G}_T^{(1)}(\hat{\theta}_T^{(1)}) = 0_{p_1}, \quad \hat{G}_T^{(1)}(\theta^{(1)}) = \tilde{K} \hat{G}_T \begin{pmatrix} \theta^{(1)} \\ 0_{p_2} \end{pmatrix}. \quad (26)$$

In the appendix (available in the online supplement), we show that (26) has a solution, at least for T large enough. Let $\hat{\theta}_{T|2} = \left(\hat{\theta}_T^{(1)\top}, 0_{p_2}^\top \right)^\top$. Denote by $\chi_{p_2}^2$ the χ^2 distribution with p_2 degrees of freedom.

COROLLARY 1. (Wald and score-like tests based on the QLE) *Let the assumptions of Theorem 1 hold. Under $H_0 : \theta_0^{(2)} = 0_{p_2}$, we have*

$$\begin{aligned} W_T &:= T \hat{\theta}_T^\top K^\top \left(K \hat{\mathcal{J}}_T^{-1} K' \right)^{-1} K \hat{\theta}_T \xrightarrow{d} \chi_{p_2}^2, \\ R_T &:= T \hat{G}_T^\top(\hat{\theta}_{T|2}) \hat{\mathcal{J}}_{T|2}^{-1} \hat{G}_T(\hat{\theta}_{T|2}) \xrightarrow{d} \chi_{p_2}^2 \quad \text{as } T \rightarrow \infty, \end{aligned}$$

where $\hat{\mathcal{J}}_{T|2}$ is any weakly consistent estimator of \mathcal{J} under H_0 .

Note that Corollary 1 can be extended to test more general linear restrictions of the form $H_0 : R\theta_0 = \mathbf{r}$, where R is a full rank $p_2 \times (p - p_2)$ matrix and \mathbf{r} is a p_2 -dimensional vector. Indeed, it is always possible to reparametrize by $\phi = A^{-1}\theta$, where A is a nonsingular matrix such that the null hypothesis is written $H_0 : K\phi = 0_{p_2}$, $\tilde{K}\phi$ unrestricted (see Engle, 1984).

The following remark discusses the link between the QLE in Theorem 1 and the QMLE.

REMARK 1. (Link with the QMLEs) *Since the works of Wedderburn (1974) and Gouriéroux, Monfort and Trognon (1984), it is known that in some location models of the form $y_t = f_t(\theta) + \epsilon_t$, the parameter θ can be estimated consistently by a quasi-maximum likelihood estimator (QMLE), which does not assume a particular distribution for ϵ_t but coincides with the MLE when the distribution of ϵ_t belongs to the linear exponential family, i.e., when, with respect to some σ -finite measure, ϵ_t admits a density of the form*

$$p_{f_t}(x) = \exp\{A(f_t) + B(x) + C(f_t)x\}.$$

Since $A'(f_t) + C'(f_t)f_t = 0$ and $C'(f_t(\theta)) = 1/s_t^2(\theta)$, where $s_t^2(\theta)$ is the variance of the density $p_{f_t(\theta)}$, the quasi-score of this QMLE is

$$s(\theta) = \frac{1}{T} \sum_{t=1}^T \frac{y_t - f_t(\theta)}{s_t^2(\theta)} f_t'(\theta).$$

For instance, the Poisson QMLE

$$\hat{\theta}_T = \arg \max_{\theta \in \Theta} \sum_{t=1}^T \{-f_t(\theta) + y_t \log f_t(\theta)\}$$

is the QLE obtained by assuming $\sigma_t^2 = f_t(\theta)$ in $g_t(\theta)$. The only-but essential-difference between QLE and QMLE is that the QMLE is based on a quasi-score with a variance constrained to be that of a linear exponential distribution. When the true density of ϵ_t does not belong to that family, the QLE and QMLE are generally consistent, but the QLE may be more efficient.

The following examples compare the QLE and QMLE in volatility models.

EXAMPLE 3. (Standard volatility models and link with the Gaussian QMLE) *Consider the case where (1) is of the form $y_t = \sqrt{f_t}\epsilon_t$, where ϵ_t is i.i.d. with a mean of 0 and variance of 1. The usual QMLE of the volatility parameter θ_0 is*

$$\hat{\theta}_T = \arg \min_{\theta \in \Theta} \sum_{t=t_0+1}^T \frac{y_t^2}{\hat{f}_t(\theta)} + \log \hat{f}_t(\theta).$$

Writing the first-order conditions and noting that $\sigma_t^2 = f_t^2 [\mathbb{E}(\epsilon_t^4) - 1]$, it is easy to see that in this case, the QMLE coincides with the optimal QLE.

EXAMPLE 4. (A non-multiplicative volatility model) Let Z_t be a random variable whose distribution, conditional on \mathcal{F}_{t-1} , is a Gamma law of shape parameter $k_t = f_t^2/\sigma_t^2$ and rate parameter $\theta_t = \sigma_t^2/f_t$ (so that $\mathbb{E}_{t-1}(Z_t) = f_t$ and $\text{var}_{t-1}(Z_t) = \sigma_t^2$). Let $y_t = s_t\sqrt{Z_t}$, where s_t is uniformly distributed on $\{-1, 1\}$. We thus have $\mathbb{E}_{t-1}(y_t^2) = f_t$ and $\text{var}_{t-1}(y_t^2) = \sigma_t^2$. When σ_t^2 is not proportional to f_t^2 , the sequence (y_t) does not follow the standard volatility model of Example 3, and the QMLE is not the optimal QLE of θ_0 involved in $f_t = f_t(\theta_0)$.

For the sake of illustration of Example 4, we run a Monte Carlo simulation in which we simulate $T = 4,000$ observations using the above model where f_t is specified as a *GARCH*(1, 1) model, i.e., $f_t = \omega + \alpha Z_t + \beta f_{t-1}$ with $\omega = 0.03, \alpha = 0.13$ and $\beta = 0.84$ and $\sigma_t^2 = 2$ so that σ_t^2 is not proportional to f_t^2 . A *GARCH*(1, 1) model is then estimated by Gaussian QMLE (i.e., $\sigma_t^2 = 2f_t^2$) and optimal QLE (i.e., $\sigma_t^2 = 2$). The biases (over 1,000 simulations) are found to be marginal for both methods. The RMSE of the three parameters, i.e., ω, α and β , are 0.0117, 0.0217, 0.0291 and 0.0073, 0.0093, 0.0128, respectively, for the Gaussian QMLE and optimal QLE, so that, on average, the optimal QLE is two times more efficient than the Gaussian QMLE.

3.2 The MLE approach

When the conditional distribution $p(y_t|f_t, \theta)$ of the observations is entirely specified, the MLE is the benchmark estimator. It results in simultaneous estimation of the parameters involved in the time-varying parameter f_t and the extra parameters involved in $p(\cdot|f, \theta)$. To estimate the p parameters of the model, the MLE is often much more efficient than the QMLE and QLE when the conditional distribution $p(y_t|f_t, \theta)$ is well specified but is likely to be inconsistent when this distribution is misspecified. We will therefore study the asymptotic behaviour of the MLE in both situations.

3.2.1 Strong consistency

Theorem 2 establishes the consistency of the MLE $\hat{\theta}_T$ for *QSD* models satisfying the stationarity and invertibility conditions stated in Section 2.2. This theorem allows model misspecification and ensures only the convergence of the MLE $\hat{\theta}_T$ to the pseudo-true parameter θ_0^* that maximizes the limit log-likelihood and minimizes the limit Kullback-Leibler divergence between the true conditional density of the data and the model-implied conditional density; see, e.g., White (1994, Chapter 3) for details. Below, $\ell(y_t, \hat{f}_t(\theta), \theta)$ denotes the logarithm of the conditional density of y_t given \hat{f}_t , i.e., $\ell(y_t, \hat{f}_t(\theta), \theta) = \log p(y_t|\hat{f}_t, \theta)$, and $\hat{\theta}_T$ is the MLE defined as

$$\hat{\theta}_T = \arg \max_{\theta \in \Theta} \hat{\mathcal{l}}_T(\theta), \quad \hat{\mathcal{l}}_T(\theta) = \frac{1}{T} \sum_{t=2}^T \ell(y_t, \hat{f}_t(\theta), \theta).$$

Let $\ell_t(\theta) = \ell(y_t, f_t(\theta), \theta)$ and $\hat{\ell}_t(\theta) = \ell(y_t, \hat{f}_t(\theta), \theta)$.

THEOREM 2. (Consistency of MLE under misspecification) *Let the conditions of Lemma 3 hold. Suppose further that the parameter space Θ is compact, ℓ is continuous, $\ell(y_t, \cdot, \theta)$ is*

differentiable with $E \sup_{\theta \in \Theta} \sup_f \left| \frac{\partial \ell(y_t, f, \theta)}{\partial f} \right|^s < \infty$ for some $s > 0$, $\mathbb{E} \sup_{\theta \in \Theta} |\ell_t(\theta)| < \infty$ and there exists $\theta_0^* \in \Theta$ such that $\mathbb{E} \ell_t(\theta) < \mathbb{E} \ell_t(\theta_0^*)$ for every $\theta \neq \theta_0^*$, $\theta \in \Theta$. Then, $\hat{\theta}_T \xrightarrow{a.s.} \theta_0^* \in \Theta$ for every $\hat{f}_1 \in \mathbb{C}(\Theta, \mathbb{R})$, as $T \rightarrow \infty$, and

$$\theta_0^* := \arg \min \mathbb{E} \text{KL} \left(p_t^0(y_t), p(y_t | f_t(\theta), \theta) \right).$$

When the *QSD* model is misspecified, the assumption of a unique maximizer of the limit log-likelihood $\theta_0^* \in \Theta$ may be too restrictive. Freedman and Diaconis (1982) show that uniqueness fails in a simple location problem with *i.i.d.* data. Kabaila (1983) provides similar results for ARMA models. Lemma 5 below follows Pötscher and Prucha (1997, Lemma 4.2) and highlights that when the uniqueness fails, the estimator can still be consistent with the argmax set of the limit log-likelihood as long as the *level sets* of the limit log-likelihood function are *regular* (see Definition 4.1 in Pötscher and Prucha, 1997).

LEMMA 5. (Set consistency of MLE under possible misspecification) *Let the conditions of Lemma 3 hold. Suppose further that ℓ is continuous, Θ is compact, and $\mathbb{E} \sup_{\theta \in \Theta} |\ell_t(\theta)| < \infty$. Then, $\hat{\theta}_T \xrightarrow{a.s.} \theta_0^* \in \Theta$ for every $\hat{f}_1 \in \mathbb{C}(\Theta, \mathbb{R})$, as $T \rightarrow \infty$, and*

$$\Theta_0^* := \arg \min \mathbb{E} \text{KL} \left(p_t^0(y_t), p(y_t | f_t(\theta), \theta) \right).$$

In Theorem 2, we imposed high-level conditions on the data $\{y_t\}_{t \in \mathbb{Z}}$ since the data-generating process was left unspecified. Corollary 2 highlights that if the *QSD* model is assumed to be well specified, then we can derive the properties of the data, and the convergence of the MLE $\hat{\theta}_T$ to the vector of true parameters θ_0 can be obtained under the additional conditions of Lemma 1 (ensuring that the data are well behaved).

COROLLARY 2. (Consistency of MLE under correct specification) *Let $\{y_t\}_{t \in \mathbb{Z}}$ be generated by (1) and (3) under some $\theta_0 \in \Theta$, and let the conditions of Lemma 1 hold at $\theta_0 \in \Theta$ and the conditions of Lemma 3 hold on Θ . Suppose further that Θ is compact and $\mathbb{E} \sup_{\theta \in \Theta} |\ell_t(\theta)| < \infty$. Finally, assume $\mathbb{E} \log^+ |f_t| < \infty$ and $\mathbb{E} \ell_t(\theta_0) > \mathbb{E} \ell_t(\theta) \forall \theta \neq \theta_0$. Then, $\hat{\theta}_T \xrightarrow{a.s.} \theta_0 \in \Theta$.*

3.2.2 Asymptotic normality

We now turn to the asymptotic normality of the ML and QML estimators for the static parameters of *QSD* models.

When the *QSD* model is correctly specified, we can use the martingale difference sequence property of the score at θ_0 to obtain a central limit theorem. However, when the model is misspecified, the score will generally fail to be a martingale difference sequence; see White (1994). Lemma 7 ensures that the MLE's score is near epoch dependent (NED) on an underlying (strong mixing) sequence; see e.g., Davidson (1994) and Pötscher and Prucha (1997, Definition 6.3). This lemma is written for robust *QSD* models with

bounded updates delivered by a uniformly bounded ψ function with uniformly bounded derivatives. The NED property gives us sufficient fading memory for establishing the asymptotic normality of the score when the model is misspecified and the score fails to be a martingale difference sequence (Pötscher and Prucha, Chapter 10).²

Let $\widehat{\ell}'_t(\theta_0)$ denote the score evaluated at θ_0 and defined as follows:

$$\widehat{\ell}'_t(\theta) = \frac{\partial \ell(y_t, \widehat{f}_t(\theta), \theta)}{\partial \theta} + \frac{\partial \ell(y_t, \widehat{f}_t(\theta), \theta)}{\partial f} \widehat{f}'_t(\theta) =: \ell'(y_t, \widehat{f}_t(\theta), \widehat{f}'_t(\theta), \theta).$$

Notice that a hat is used in the notation $\widehat{\ell}'_t$ to highlight the fact that the score depends on the filtered values $(\widehat{f}_t, \widehat{f}'_t)$. Define similarly the second-order derivatives $\widehat{\ell}''_t(\theta)$. Let $\ell'_t(\theta)$ and $\ell''_t(\theta)$ be obtained by replacing $\widehat{f}_t(\theta)$ with $f_t(\theta)$ in $\widehat{\ell}'_t(\theta)$ and $\widehat{\ell}''_t(\theta)$.

The next lemma will be used to show that the initial values needed to compute the MLE are irrelevant for its asymptotic distribution.

LEMMA 6. (Irrelevance of the initial values) *Suppose $\ell'(y_t, \cdot, \cdot, \theta)$ is differentiable with $E \sup_{\theta \in \Theta} \sup_{f, f'} \left\| \frac{\partial \ell'(y_t, f, f', \theta)}{\partial f} \right\|^s < \infty$ and $E \sup_{\theta \in \Theta} \sup_{f, f'} \left\| \frac{\partial \ell'(y_t, f, f', \theta)}{\partial f'} \right\|^s < \infty$ for some $s > 0$. Under conditions entailing (8) of Lemma 3 and (13) of Lemma 4, we have*

$$\sum_{t=1}^{\infty} \sup_{\theta \in \Theta} \left\| \widehat{\ell}'_t(\theta) - \ell'_t(\theta) \right\| < \infty \quad a.s.$$

LEMMA 7. (Near epoch dependent score) *Let $\{y_t\}$ have two bounded moments $\sup_t \mathbb{E}|y_t|^2 < \infty$ and be NED of size $-q$ on some process $\{e_t\}_{t \in \mathbb{Z}}$ and suppose that*

$$(i) \sup_{y, X, f} \left| \frac{\partial \psi(y, X, f, \theta_0)}{\partial y} \right| < \infty; \quad (ii) \sup_{y, X, f} \left| \alpha_0 \frac{\partial \psi(y, X, f, \theta_0)}{\partial f} + \beta_0 \right| < 1.$$

Then, $\{\widehat{f}_t(\theta_0)\}_{t \in \mathbb{N}}$ is NED of size $-q$ on $\{e_t\}_{t \in \mathbb{Z}}$. Additionally, if $|\beta_0| < 1$ and

$$(iii) \sup_{y, X, f} |\psi(y, X, f, \theta_0)| < \infty; \quad (iv) \sup_{y, X, f} \left\| \frac{\partial \psi(y, X, f, \theta_0)}{\partial \theta} \right\| < \infty;$$

$$(v) \sup_{y, X, f} \left\| \frac{\partial^2 \psi(y, X, f, \theta_0)}{\partial \theta \partial y} \right\| < \infty; \quad (vi) \sup_{y, X, f} \left\| \frac{\partial^2 \psi(y, X, f, \theta_0)}{\partial \theta \partial f} \right\| < \infty;$$

$$(vii) \sup_{y, X, f} \left| \frac{\partial^2 \psi(y, X, f, \theta_0)}{\partial f \partial y} \right| < \infty; \quad (viii) \sup_{y, X, f} \left| \frac{\partial^2 \psi(y, X, f, \theta_0)}{\partial f^2} \right| < \infty$$

then the derivative process $\{\widehat{f}'_t(\theta_0)\}_{t \in \mathbb{N}}$ is NED of size $-q$ on $\{e_t\}_{t \in \mathbb{Z}}$. Finally, assume the score $\widehat{\ell}'_t(\theta_0)$ is Lipschitz on $(y_t, \widehat{f}_t, \widehat{f}'_t)$,

$$(ix) \sup_{y, f} \left\| \frac{\partial^2 \ell(y, f, \theta_0)}{\partial \theta \partial y} \right\| < \infty; \quad (x) \sup_{y, f} \left\| \frac{\partial^2 \ell(y, f, \theta_0)}{\partial \theta \partial f} \right\| < \infty;$$

$$(xi) \sup_{y, f} \left| \frac{\partial^2 \ell(y, f, \theta_0)}{\partial f^2} \right| < \infty; \quad (xii) \sup_{y, f} \left| \frac{\partial^2 \ell(y, f, \theta_0)}{\partial f \partial y} \right| < \infty;$$

²In general, the concept of mixing sequences is mostly suitable for linear processes; see e.g. pages 65-68 in Pötscher and Prucha (1997) for a detailed discussion of this point. As such, NED offers a way forward to describe fading memory in nonlinear autoregressive processes, which mixing does not.

(xiii) $\sup_{y,f} \left| \frac{\partial \ell(y,f,\theta_0)}{\partial f} \right| < \infty$.

Then, $\{\widehat{\ell}'_t(\theta_0)\}_{t \in \mathbb{N}}$ is also NED of size $-q$ on $\{e_t\}_{t \in \mathbb{Z}}$.

Theorem 3 uses the stochastic properties discussed in Lemmas 3, 4 and 7 to obtain the asymptotic normality of the MLE in a setting where the model is allowed to be misspecified; see White (1982), Domowitz and White (1982), White (1994), and Pötscher and Prucha (1997). In this theorem, we assume that the data are NED on an underlying ϕ -mixing sequence of size $-r/(r-1)$. The same result can, however, be obtained for α -mixing sequences of size $-2r/(r-2)$. We assume that the asymptotic variance exists and is bounded. This condition requires that the partial sum of the covariances of the score sequence converges. This can be verified by appealing to the stationarity and ergodicity and bounded moments of the score, together with appropriate sizes for NED and underlying mixing sequence; see e.g., Theorem 6.4.6 in Davidson (2000) or Lemma 3 in De Jong (1997). Such conditions implicitly impose restrictions on the unknown DGP given the setting of model misspecification.

THEOREM 3. (Asymptotic normality of MLE under possible misspecification) *Assume that the conditions in Theorem 2 and Lemmas 4 and 7 are satisfied. Suppose further that $\theta_0^* \in \text{int}(\Theta)$ and $\{y_t\}_{t \in \mathbb{Z}}$ is NED of size -1 on a ϕ -mixing sequence of size $-r/(r-1)$ for some $r > 2$, and that*

$$\mathbb{E} \|\ell'_t(\theta_0)\|^r < \infty \quad \text{and} \quad \mathbb{E} \sup_{\theta \in \Theta} \|\ell''_t(\theta)\| < \infty.$$

Suppose further that the limit

$$V(\theta_0^*) = \lim_{T \rightarrow \infty} T^{-1} \mathbb{E} \left(\sum_{t=1}^T \ell'_t(\theta_0^*) \right) \left(\sum_{t=1}^T \ell'_t(\theta_0^*)^\top \right)$$

exists and is bounded, and $\mathbb{E} \ell''_t(\theta_0^*)$ is invertible. Then $\sqrt{T}(\widehat{\theta}_T - \theta_0^*) \xrightarrow{d} N(0, \Sigma(\theta_0^*))$ as $T \rightarrow \infty$, where

$$\Sigma(\theta_0^*) = \left(\mathbb{E} \ell''_t(\theta_0^*) \right)^{-1} V(\theta_0^*) \left(\mathbb{E} \ell''_t(\theta_0^*) \right)^{-1}.$$

COROLLARY 3. (Asymptotic normality of MLE under correct specification) *Let $\{y_t\}_{t \in \mathbb{Z}}$ be generated by (1) and (3) under some $\theta_0 \in \text{int}(\Theta)$, and let the conditions of Corollary 2, Lemma 4 and Lemma 6 hold. Suppose further that*

$$\mathbb{E} \|\ell'_t(\theta_0)\|^2 < \infty, \quad \mathbb{E} \sup_{\theta \in \Theta} \|\ell'_t(\theta)\| < \infty \quad \text{and} \quad \mathbb{E} \sup_{\theta \in \Theta} \|\ell''_t(\theta)\| < \infty.$$

Then, $\sqrt{T}(\widehat{\theta}_T - \theta_0) \xrightarrow{d} N(0, \mathcal{I}^{-1})$, when the Fisher information matrix \mathcal{I} is invertible

As a final result, we obtain the asymptotic distribution of the score (also called the Lagrange Multiplier) and log-likelihood ratio tests for testing p_0 linear restrictions on the $p > p_0$ dimensional parameter vector θ_0 . The null hypothesis of interest is $H_0 : R\theta_0 = \mathbf{r}$, where R is a given full rank $p_0 \times p$ matrix and \mathbf{r} is a given p_0 -dimensional vector. Below, $\widehat{\theta}_T^{p_0}$ denotes the MLE of θ_0 in the model constrained by the null.

THEOREM 4. (Score and Likelihood Ratio tests) *Let the conditions of Corollary 3 hold. Then, under $H_0 : R\theta_0 = \mathbf{r}$, we have that*

$$\begin{aligned} \text{LM}_T &= T\widehat{\mathbf{l}}_T^{\prime\top}(\widehat{\theta}_T^{p_0})\widehat{\mathcal{I}}^{-1}\widehat{\mathbf{l}}_T^{\prime}(\widehat{\theta}_T^{p_0}) \xrightarrow{d} \chi_{p_0}^2, \\ \text{LR}_T &= 2T\left(\widehat{\mathbf{l}}_T(\widehat{\theta}_T) - \widehat{\mathbf{l}}_T(\widehat{\theta}_T^{p_0})\right) \xrightarrow{d} \chi_{p_0}^2 \quad \text{as } T \rightarrow \infty, \end{aligned}$$

where $\widehat{\mathcal{I}}$ is a weakly consistent estimator of \mathcal{I} . One can take, for instance,

$$\widehat{\mathcal{I}} = -\widehat{\mathbf{l}}_T^{\prime\prime}(\widehat{\theta}_T^{p_0}) \quad \text{or} \quad \widehat{\mathcal{I}} = \frac{1}{T} \sum_{t=1}^T \widehat{\ell}_t(\widehat{\theta}_T^{p_0})\widehat{\ell}_t^{\top}(\widehat{\theta}_T^{p_0}). \quad (27)$$

In the latter case, we have $\text{LM}_T = \mathbf{1}^{\top}\widehat{L}^{\top}\left(\widehat{L}^{\top}\widehat{L}\right)^{-1}\widehat{L}\mathbf{1}$, where \widehat{L} is a $p \times T$ matrix whose row t is $\widehat{\ell}_t(\widehat{\theta}_T^{p_0})$ and $\mathbf{1}^{\top} = (1, \dots, 1) \in \mathbb{R}^T$. Note that $\text{LM}_T = T \times R^2$, where R^2 denotes the coefficient of determination in the regression of 1 on $\widehat{\ell}_t(\widehat{\theta}_T^{p_0})$.

4 The QSD_T GARCH(1, 1) – T model

In this section, we illustrate our general results on an extension of one of the most popular SD volatility models, namely, the β_T GARCH(1, 1) of Harvey and Chakravarty (2008).

4.1 An extension of the β_T GARCH(1, 1)

Assume the volatility model $y_t = \sqrt{f_t}\epsilon_t$, where

$$f_{t+1} = \omega + \alpha \frac{\nu + 1}{\nu - 2 + \epsilon_t^2} \epsilon_t^2 f_t + \beta f_t \quad (28)$$

with $\omega > 0$, $\alpha > 0$ and $\beta \geq 0$ to ensure positivity and avoid triviality. Harvey and Chakravarty (2008) show that (28) is the updating equation of an SD model when the *i.i.d.* sequence (ϵ_t) follows a standardized Student's t distribution T_{ν} with $\nu > 2$ degrees of freedom (and variance 1). Note that ν plays two roles in this model: it determines the shape of the density of the innovations ϵ_t and bounds the effects of large shocks ϵ_t on future values of the conditional variance (i.e., f_{t+1}) when $\nu < \infty$. It is convenient to reparametrize (28) in terms of $\xi = 1/\nu$, i.e.,

$$f_{t+1} = \omega + \alpha \frac{1 + \xi}{1 - 2\xi + \xi\epsilon_t^2} \epsilon_t^2 f_t + \beta f_t \quad (29)$$

with $0 \leq \xi < 1/2$ so that the GARCH(1, 1) appears as a special case of (29) when $\xi = 0$.

In the sequel, we keep the downweighting mechanism of the above β_T GARCH(1, 1) model but disconnect the updating equation of the conditional variance and the density of the innovations. To do so, we also assume $\epsilon_t \sim T_{1/\xi}$, but we introduce an additional parameter ζ in the updating equation that is not related to ξ . We also consider the

possibility of introducing a vector X_t of positive exogenous variables. The model, called the $QSD_T GARCH(1, 1) - T$ model, is parameterized as follows:³

$$f_{t+1} = \omega + \varpi^\top X_t + \alpha \frac{1 + \zeta}{1 - 2\zeta + \zeta \epsilon_t^2} \epsilon_t^2 f_t + \beta f_t, \quad (30)$$

where $\epsilon_t \stackrel{i.i.d.}{\sim} T_{1/\xi}$ with $0 \leq \xi < 1/2$ and $\varpi \geq 0$ is a parameter vector of the same dimension as X_t . In the absence of exogenous variables, this model is identical to a $\beta_T GARCH(1, 1)$ model when $\xi = \zeta$ and a standard $GARCH(1, 1)$ model with standardized Student's t innovations when $\zeta = 0$.

When $\zeta < 0$ or $\zeta > 1/2$, Equation (30) does not define a proper volatility model because when $\epsilon_t^2 \simeq 2 - 1/\zeta$, f_{t+1} in (30) can be infinite or negative.

Note also that when $\zeta = 1/2$ and $\varpi = 0$, the volatility model is degenerate since $f_{t+1} = \omega + 3\alpha f_t + \beta f_t$ is then constant. To ensure positivity and non-degeneracy of the volatility equation, we could impose $0 \leq \zeta < 1/2$. However, to avoid ζ on the boundary of the parameter space when testing the null hypothesis $\zeta = 0$ (i.e., that the true model is a $GARCH(1, 1) - T$ model), we also consider the alternative specification

$$f_{t+1} = \omega + \varpi^\top X_t + \alpha \Psi \left(\frac{1 + \zeta}{1 - 2\zeta + \zeta \epsilon_t^2} \right) \epsilon_t^2 f_t + \beta f_t \quad (31)$$

with $-1 < \zeta < 1/2$ and $\Psi : \mathbb{R} \rightarrow [0, \infty)$ of class C^2 . To approximately recover (30) when $\zeta \geq 0$, one can choose for Ψ a smooth approximation of the absolute value function. For instance, one can set $\Psi(x) = \sqrt{x^2 + c}$ for some small $c > 0$ or

$$\Psi(x) = x \frac{1 - e^{-cx}}{1 + e^{-cx}} \quad (32)$$

for some large $c > 0$. The latter function is equivalent to $|x|$ when $|x|$ or c is large and is equivalent to $cx^2/2$ when $|x|$ is small. More generally, assume that

$$\Psi(x) \leq c_1(|x| + 1), \quad \Psi(x) \geq c_2|x|^{c_3}, \quad |\Psi'(x)| \leq c_4 \quad (33)$$

for some positive constants c_i , $i = 1, \dots, 4$. In the simulations and the empirical application, we rely on (32) with $c = 1,000$. Note that in the empirical application we do not find a single series (out of 400 stocks) for which ζ is significantly negative.

4.1.1 Stationarity and positivity conditions

Let us consider the stationarity of the general $QSD_T GARCH(1, 1) - T$ model (31) without assuming a particular distribution for (ϵ_t) . For the moment, we just assume that (ϵ_t, X_t) is stationary and ergodic with $\mathbb{E}(\epsilon_t^2) = 1$ and $E\|X_t\|^s < \infty$ for some $s > 0$. By

³The name $QSD_T GARCH - T$ refers to the fact that the model involves two Student's t distributions. Indeed, the conditional log-density $\log p(y_t | f_t, \theta)$ is a Student's t log-density with $1/\xi$ degrees of freedom while $\rho(y_t, f_t, \theta)$ is another Student's t log-density with $1/\zeta$ degrees of freedom. Consequently, the updating equation of f_t depends on ζ and not on ξ .

the Cauchy root test, it is easy to show that there exists a stationary (ergodic) solution to this model, explicitly given by

$$f_t = \omega_t + \sum_{i=1}^{\infty} a(\epsilon_{t-1}) \cdots a(\epsilon_{t-i}) \omega_{t-i-1}, \quad a(z) = \alpha \Psi \left(\frac{1 + \zeta}{1 - 2\zeta + \zeta z^2} \right) z^2 + \beta,$$

$\omega_t = \omega + \varpi^\top X_t$, when

$$\mathbb{E} \log (\alpha \Psi_t \epsilon_t^2 + \beta) < 0, \quad \Psi_t = \Psi \left(\frac{1 + \zeta}{1 - 2\zeta + \zeta \epsilon_t^2} \right). \quad (34)$$

Note that, since $\alpha \neq 0$, (31) corresponds to (3) with

$$\psi(g(f, \epsilon_t), X, f, \theta) = \frac{\varpi^\top X}{\alpha} + \Psi_t \epsilon_t^2 f, \quad \frac{\partial \psi(g(f, \epsilon_t), X, f, \theta)}{\partial f} = \Psi_t \epsilon_t^2.$$

Using the first inequality of (33), it can be seen that condition (i) of Lemma 1 is satisfied when

$$\mathbb{E} \log^- |1 - 2\zeta + \zeta \epsilon_t^2| < \infty \quad (35)$$

and that (ii) is equivalent to (34). Note that the moment condition (35) is very mild. Indeed, (35) is always satisfied when $\zeta \geq 0$ or when the distribution of ϵ_t^2 has a bounded density. On the other hand, (35) for $\zeta < 0$ precludes a distribution of ϵ_t^2 with a mass at $2 - 1/\zeta$. Note that (34) is also a necessary condition for stationarity when (ϵ_t) is *i.i.d.*, which shows that Lemma 1 provides sharp stationarity conditions, at least in this framework.

4.1.2 Invertibility of the filter

We now assume that (ϵ_t) is an *i.i.d.* sequence and that conditions (33)-(34) hold true. When $E\|X_t\|^r < \infty$ for some $r > 0$, all conditions of Lemma 2 are satisfied, which shows that the stationary solution of the *QSD_T GARCH(1,1) - T* model is such that $\mathbb{E}|y_t|^s < \infty$ for some $s > 0$. Assume also that

$$\theta = (\omega, \alpha, \beta, \zeta, \varpi)^\top \in \Theta \subset [\underline{\omega}, \bar{\omega}] \times [\underline{\alpha}, \bar{\alpha}] \times [\underline{\beta}, \bar{\beta}] \times [\underline{\zeta}, \bar{\zeta}] \times [\underline{\varpi}, \bar{\varpi}] \quad (36)$$

with $0 < \underline{\omega} < \bar{\omega}$, $0 < \underline{\alpha} < \bar{\alpha}$, $0 \leq \underline{\beta} \leq \bar{\beta} < 1$, $0 \leq \underline{\zeta} \leq \bar{\zeta} < 1/2$ and $0 \leq \underline{\varpi} \leq \bar{\varpi}$ (componentwise). Since we have

$$\psi(y_t, X_t, f_t, \theta) = \frac{\varpi^\top X_t}{\alpha} + \Psi \left(\frac{1 + \zeta}{1 - 2\zeta + \zeta \frac{y_t^2}{f_t}} \right) y_t^2,$$

condition (i) of Lemma 3 is satisfied when $\|X_t\|$ and y_t admit a small-order moment. Since

$$\frac{\partial \psi(y_t, X_t, f_t, \theta)}{\partial f} = \Psi' \left(\frac{1 + \zeta}{1 - 2\zeta + \zeta \frac{y_t^2}{f_t}} \right) \frac{1 + \zeta}{\left(1 - 2\zeta + \zeta \frac{y_t^2}{f_t}\right)^2} \zeta \frac{y_t^4}{f_t^2},$$

the uniform invertibility condition (ii) is satisfied when

$$\mathbb{E} \log \left(c_4 \bar{\alpha} \frac{1 + \bar{\zeta}}{\left(1 - 2\bar{\zeta} + \bar{\zeta} \frac{y_t^2}{\underline{f}}\right)^2} \bar{\zeta} \frac{y_t^4}{\underline{f}} + \bar{\beta} \right) < 0, \quad (37)$$

where $\underline{f} = \underline{\omega}/(1 - \underline{\beta})$ is a lower bound for the time-varying volatility. Note that the expectation of the left-hand side of (37) cannot be computed exactly because the stationary distribution of (y_t) is generally unknown, but it can be easily evaluated by means of simulations. To relax the constraint $\zeta \geq 0$ or to obtain a more stringent identifiability condition (in particular, to account for a non-cubic parameter space Θ), the supremum involved in condition (ii) of Lemma 3 can be computed numerically.

4.1.3 Derivatives of the filter

With the notation (36), (9) holds with

$$A_t = \begin{pmatrix} 1 \\ \psi_t \\ f_t \\ \alpha \frac{\partial \psi_t}{\partial \zeta} \\ X_t \end{pmatrix}, \quad \frac{\partial \psi_t}{\partial \zeta} = y_t^2 \Psi' \left(\frac{1 + \zeta}{1 - 2\zeta + \zeta \frac{y_t^2}{f_t}} \right) \frac{3 - \frac{y_t^2}{f_t}}{\left(1 - 2\zeta + \zeta \frac{y_t^2}{f_t}\right)^2}.$$

Assume $0 \leq \zeta \leq \bar{\zeta} < 1/2$. We thus have $1 - 2\zeta + \zeta \frac{y_t^2}{f_t} \geq 1 - 2\bar{\zeta} > 0$. Since $f_t \geq \omega > 0$, it can be seen that $\mathbb{E} \|A_t\|^s < \infty$ for some $s > 0$ when $\|X_t\|$ and y_t admit a small-order moment. Therefore, $\mathbb{E} \log^+ \|A_t\| < \infty$, and (i) of Lemma 4 is satisfied. Similarly, it can be seen that the other conditions of that lemma hold true.

4.2 Statistical inference

We now consider the estimation of the QSD_T $GARCH - T$ model. We thus assume the standardized Student's t conditional distribution

$$p(y | f, \theta) = \frac{1}{\sqrt{f}} p_\nu \left(\frac{y}{\sqrt{f}} \right), \quad (38)$$

$$p_\nu(\epsilon) = \frac{1}{\sqrt{\pi(\nu - 2)}} \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})} \left(1 + \frac{\epsilon^2}{\nu - 2} \right)^{-\frac{\nu+1}{2}}, \quad (39)$$

where $p_\nu(\epsilon)$ is the standardized Student's t distribution (with mean 0, unit variance and $\nu > 2$) and denoted as T_ν in short. Setting $\xi = 1/\nu$ and imposing $0 \leq \xi < 1/2$, the Gaussian conditional distribution case corresponds to $\xi = 0$. Let $\theta = (\omega, \alpha, \beta, \zeta, \xi, \varpi^\top)^\top$, Θ a compact subset of $(0, \infty)^2 \times [0, 1] \times (-1, 1/2) \times [0, 1/2) \times [0, \infty)^{d-1}$ and the MLE

$$\hat{\theta}_T = \arg \max_{\theta \in \Theta} \frac{1}{T} \sum_{t=t_0+1}^T \ell(y_t, \hat{f}_t(\theta), \theta),$$

where $\ell(y, f, \theta) = \log p(y | f, \theta)$ and

$$\hat{f}_{t+1}(\theta) = \omega + \varpi^\top X_t + \alpha \Psi \left(\frac{1 + \zeta}{1 - 2\zeta + \zeta \frac{y_t^2}{\hat{f}_t(\theta)}} \right) y_t^2 + \beta \hat{f}_t(\theta),$$

with the initial value $\hat{f}_1(\theta) = \sum_{t=1}^{t_0} y_t^2 / t_0$ and $t_0 = 5$, for instance.

4.2.1 Testing the β_T GARCH(1, 1) restriction

The standard β_T GARCH(1, 1) is obtained when $0 < \zeta = \xi < 1/2$ and $\Psi(x) = x$. It is thus of interest to test the null hypothesis $H_0 : \xi_0 = \zeta_0$. This hypothesis can be written as $H_0 : K\theta_0 = 0$ with $K = (0_3^\top, 1, -1, 0_{d-1}^\top)$. Let the Wald test statistic be

$$W_T^{\zeta, \xi} = T \hat{\theta}_T^\top K^\top \left(K \hat{\Sigma} K^\top \right)^{-1} K \hat{\theta}_T,$$

where $\hat{\Sigma}$ is a consistent estimator of the matrix $\mathcal{I}^{-1}(\theta_0)$ defined in Corollary 3. A direct consequence of that corollary is that $W_T^{\zeta, \xi}$ asymptotically follows a χ_1^2 under H_0 . The test of critical region $\{W_T^{\zeta, \xi} > \chi_1^2(1 - \alpha^*)\}$ thus has the asymptotic level α^* .

Alternatively, one can use Theorem 4 and replace the Wald statistic by the score and likelihood ratio (LR) test statistics

$$LM_T^{\zeta, \xi} = T \hat{\mathbf{l}}_T^{\prime\top}(\hat{\theta}_T^{p_0}) \hat{\mathcal{I}}^{-1} \hat{\mathbf{l}}_T(\hat{\theta}_T^{p_0}), \quad LR_T^{\zeta, \xi} = 2T \left(\hat{\mathbf{l}}_T(\hat{\theta}_T) - \hat{\mathbf{l}}_T(\hat{\theta}_T^{p_0}) \right).$$

For the Wald statistics, it is natural to take $\hat{\Sigma} = \hat{\mathcal{I}}^{-1}$, where $\hat{\mathcal{I}}$ is defined by (27), replacing $\hat{\theta}_T^{p_0}$ by $\hat{\theta}_T$.

4.2.2 Testing the GARCH – T restriction

The GARCH(1, 1) – T volatility model is obtained when $\zeta = 0$ and $\Psi(x) = x$. It is thus of interest to test the null $H_0 : \zeta_0 = 0$ in the QSD_T GARCH – T model defined by (38) and (31), with $-1 < \zeta < 1/2$ and Ψ satisfying (33). Another possibility would be to test $\zeta_0 = 0$ in the model defined by (38) and (31) constrained by $0 \leq \zeta < 1/2$. The drawback of the latter test is that because the parameter stands at the boundary of the parameter space under the null, the asymptotic distribution of the Wald statistic is non-standard (see Pedersen and Rahbek, 2019 and the reference therein).

By considering model (31), we afford to have $-1 < \zeta < 1/2$, and thus the parameter belongs to the interior of Θ under $H_0 : \zeta_0 = 0$. Corollary 3 then entails that the Wald test of critical region $\{W_T^\zeta > \chi_1^2(1 - \alpha^*)\}$ with

$$W_T^\zeta = T \hat{\theta}_T^\top \mathbf{e}_4 \left(\mathbf{e}_4^\top \hat{\Sigma} \mathbf{e}_4 \right)^{-1} \mathbf{e}_4^\top \hat{\theta}_T, \quad \mathbf{e}_4^\top = (0_3^\top, 1, 0_d^\top),$$

has asymptotic level α^* .

4.2.3 Testing the standard *GARCH*

The parameter of interest is often the volatility $f_t = f(\theta_0)$ with $\theta_0 = (\omega_0, \alpha_0, \beta_0, \zeta_0, \varpi^\top)^\top$ and Θ changed accordingly. It is then desirable to estimate θ_0 without assuming (38) or any other particular conditional distribution.

The benchmark estimator in this framework is the QMLE

$$\hat{\theta}_{QMLE} = \arg \min_{\theta \in \Theta} \frac{1}{T} \sum_{t=t_0+1}^T \frac{y_t^2}{\hat{f}_t(\theta)} + \log \hat{f}_t(\theta). \quad (40)$$

As discussed in Section 3.1, one can also use an alternative QLE based on the estimating functions theory:

$$\hat{\theta}_T = \arg \min_{\theta \in \Theta} \left\| \frac{1}{T} \sum_{t=t_0+1}^T \frac{y_t^2 - \hat{f}_t(\theta)}{\hat{\sigma}_t^2(\theta)} \hat{f}_t(\theta) \right\| \quad (41)$$

for some function $\hat{\sigma}_t^2(\theta) > 0 \in \mathcal{F}_{t-1}$. If $\hat{\sigma}_t^2(\theta)$ is chosen proportional to $\hat{f}_t^2(\theta)$, then the two estimators (40) and (41) are equivalent. However, they are not equivalent if, for instance, one takes $\hat{\sigma}_t^2(\theta) = \hat{f}_t(\theta)$.

4.3 Monte Carlo simulation

In this section, we present a Monte Carlo experiment that studies the finite-sample properties of the QSD_T *GARCH*(1,1) – T model as well as some models nested in this specification.

In the simulation study, we consider three data-generating processes (DGPs) corresponding to particular cases of the following QSD_T *GARCH*(1,1) – T model:

$$y_t = \mu + \sqrt{f_t} \epsilon_t \quad (42)$$

$$\epsilon_t \sim T_{1/\xi} \quad (43)$$

$$f_{t+1} = \omega + \varpi X_t + \alpha \frac{1 + \zeta}{1 - 2\zeta + \zeta \epsilon_t^2} \epsilon_t^2 f_t + \beta f_t \quad (44)$$

with $0 < \xi < 1/2$ and $-1 < \zeta < 1/2$ and where $T_{1/\xi}$ is defined in (39). Recall that this model is a pure β_T *GARCH*(1,1) in the absence of an explanatory variable and when $\xi = \zeta$, and it is a *GARCH*(1,1) – T model when $\zeta = 0$. The explanatory variable X_t is taken from the empirical application and corresponds to historical data of the square of the VIX index converted from an annual to a daily horizon.

In all simulations, we set the parameters to a rounded value of the average (over all stocks) of the estimated parameters obtained in Section 5.1.⁴ More specifically, we set $\mu = 0.06, \omega = 0.08, \varpi = 0.13, \alpha = 0.10$ and $\beta = 0.83$. In the first simulation (i.e., Table 1), we set $\xi = \zeta = 0.2$ so that the model is a β_T *GARCH*(1,1) model with a degree of freedom of $\xi^{-1} = 5$. In the second simulation (i.e., Table 2), we set $\xi = 0.2$ and $\zeta = 0.1$ so that the model is a QSD_T *GARCH*(1,1) – T with a higher degree of freedom

⁴Unlike in the empirical application, we do not consider an *AR*(1) specification in the simulations.

in the conditional variance equation than for the density of the innovations. Finally, in the third simulation (i.e., Table 3), $\xi = 0.2$ while $\zeta = 0$, so that the true model is a $GARCH(1, 1) - T$ model.

In all cases, four models are estimated. Three models (i.e., $GARCH(1, 1) - T$, $\beta_T GARCH(1, 1)$ and $QSD_T GARCH(1, 1) - T$) are estimated by ML. The fourth model is the $QSD_T GARCH(1, 1)$ estimated by Gaussian QML (ξ is therefore not estimated). Note that during the optimization, the positivity of the conditional variance of the $QSD_T GARCH(1, 1) - T$ models is imposed by replacing (44) by (31), as discussed in Section 4.

In all cases considered in this section, σ_t^2 is proportional to f_t^2 so that the optimal QLE corresponds to the Gaussian QMLE, which is the reason why specific results for the QLE are not reported below.

Each of the three tables is divided into two major parts. The top panels correspond to the results for a sample size of 3,000 observations, while the bottom panels are for 4,000 observations.⁵ Each panel is again divided into two parts. The first one contains summary statistics on the estimated parameters, while the second reports rejection frequencies of two LRTs. Figures at the right of the name of the models are the empirical biases over 1,000 replications. Figures in parentheses correspond to RMSEs, while those in squared brackets are the 95% coverage probabilities (i.e., percentage of 95% confidence intervals drawn from the asymptotic distribution containing the true parameter). The second part contains rejection frequencies of two LRTs computed from the ML estimates. The first one is for the null hypothesis that the true model is a $\beta_T GARCH(1, 1)$, i.e., $\xi = \zeta$, while the second test is for the null hypothesis that the model is a $GARCH(1, 1) - T$, i.e., $\xi = 0$. Note that some of the figures reported in this part correspond to empirical sizes or powers depending on the DGP. For this reason, for ease of reading of the results, an asterisk is added after the name of the models that do not nest the DGP.

Some comments are in order.

- The most important result is that the bias of the MLEs of the $QSD_T GARCH(1, 1) - T$ is negligible for the two considered sample sizes and the three DGPs.
- When the true model is a $\beta_T GARCH(1, 1)$ (see Table 1), the inverse of the degree of freedom of innovations ξ is slightly more precisely estimated with the $\beta_T GARCH(1, 1)$ model than with the $QSD_T GARCH(1, 1) - T$, but the difference is marginal. Indeed, the RMSE is only 0.001 higher for the latter when $T = 3,000$ and almost identical when $T = 4,000$. Furthermore, while the biases of ξ and ζ are small, the RMSE of ζ is between three and four times higher than for ξ . This is a consequence of the fact that the identification of ζ is only possible from the

⁵A minimum of 1,500 to 2,000 observations is often required to estimate accurately the parameters of standard GARCH models by (Q)ML and to have a satisfactory statistical inference by means of the asymptotic distribution of the estimator. QSD GARCH are more flexible than GARCH models. In particular, the ζ parameter is identified only by the large observations that are underweighted in the updating equation for f_t . It is therefore natural that more observations are required than for standard GARCH. SD models impose $\zeta = \xi$ and therefore also require less observations than QSD models because ζ is also identified through the full conditional density of the data.

observations for which the shocks are truncated, whereas all observations can be used to identify ξ . Testing the null hypothesis that $\xi = \zeta$ is therefore desirable to gain efficiency by imposing this restriction when the null hypothesis is not rejected.

- As expected, some of the parameters of the $GARCH(1, 1) - T$ model (especially α) are biased when wrongly imposing the assumption that $\zeta = 0$, as shown in Tables 1 and 2.
- Similarly, some of the parameters of the $\beta_T GARCH(1, 1)$ model are biased when the true model is a $QSD_T GARCH(1, 1) - T$ with $\xi \neq \zeta$, as shown in Tables 2 and 3.
- As expected again, the QML of the $QSD_T GARCH(1, 1)$ is less precise than its ML version. The biases are higher than for the ML, while the RMSEs are approximately 20-25% higher.
- The coverage probabilities of the parameters of the $QSD_T GARCH(1, 1) - T$ are satisfactory, except for ζ . For a sample size of 3,000 observations, the true value of ζ belongs to the 95% confidence interval drawn from the asymptotic distribution in approximately 83 to 84% (resp. 79 to 83%) of the cases for the MLE (resp. QMLE). The results are slightly better for a sample size of 4,000 observations. Unreported simulation results suggest that a sample size of at least 15,000 observations is needed to perform correct statistical inference on ζ on the basis of t-tests and confidence intervals relying on the asymptotic distribution. For the sample sizes considered in Tables 1 to 3, standard errors of ζ are on average too small compared to the RMSE of the estimated ζ parameter.
- While statistical inference on ζ relying on its standard error (e.g., t-tests and Wald tests) requires a very large sample, the LRT on ζ has good finite sample properties. Indeed, when the sample size is 4,000, the rejection frequencies of the null hypothesis $H_0 : \xi = \zeta$ in Table 1 (where $\xi = \zeta = 0.2$ in the DGP) and of the null hypothesis $H_0 : \zeta = 0$ in Table 3 are close to their nominal sizes. The rejection frequencies for the other tests correspond to empirical powers. Interestingly, the LRT of the null hypothesis $H_0 : \zeta = 0$ has very high power to reject the $GARCH - T$ for which the squared shocks drive the dynamic of the conditional variance (see Tables 1 and 2), while the LRT of the null hypothesis $H_0 : \xi = \zeta$ has very high power when the true model is a $GARCH(1, 1) - T$ (see Table 3) and decent power when the true model is a $QSD_T GARCH(1, 1) - T$ with $\xi = 0.2$ and $\zeta = 0.1$ (approximately 30% for a nominal size of 5% and a sample size of 4,000 observations). The power of course increases with the distance between ξ and ζ .

Finally, to help visualize the impact of a misspecification of the conditional variance, Figure 1 plots 50 observations around a large shock. The DGP is a $QSD_T GARCH(1, 1) - T$ with $\xi = 0.2$ and $\zeta = 0.1$ (as in Table 2), and the models are estimated by ML on 4,000 observations. This figure plots the absolute value of the simulated log-returns (thin solid red line) as well as the estimated conditional volatilities of the QSD_T

$GARCH(1,1) - T$ (thick solid pink line), $\beta_T GARCH(1,1)$ (thin green dashed line), $GARCH(1,1) - T$ (thin blue line with long dashes) and the true conditional volatility (black solid line). It is clear from this graph that unlike the $QSD_T GARCH(1,1) - T$, the $GARCH - T$ model overestimates the volatility during approximately two weeks (i.e., approximately 15 observations) following the large shock (occurring at observation 1475), while the $\beta_T GARCH(1,1)$ underestimates the volatility during the same period.

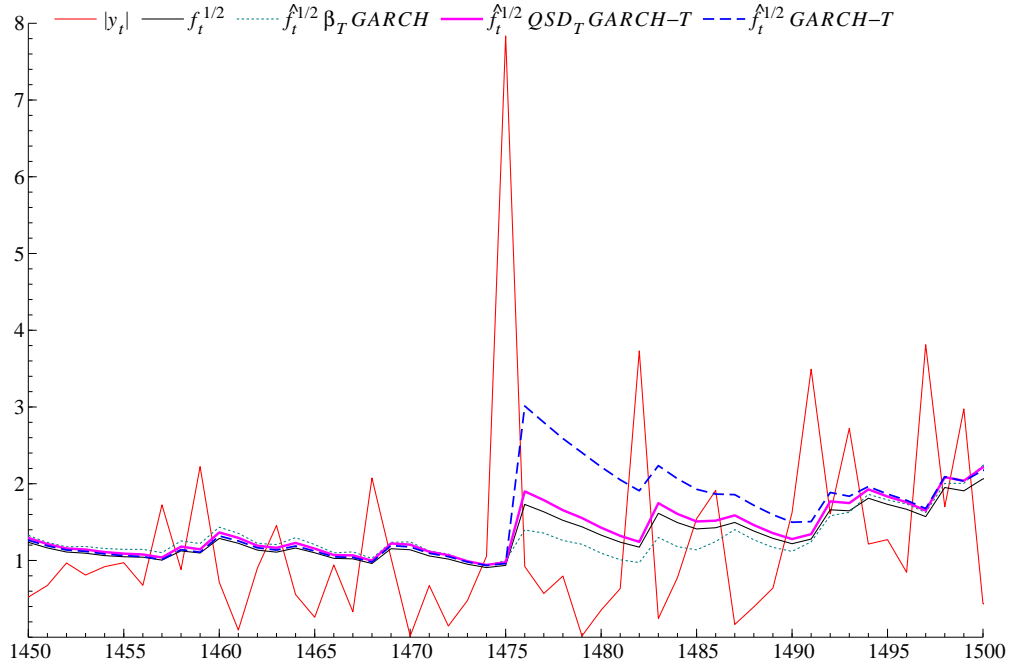


Figure 1: Fifty observations around a large shock for a DGP corresponding to a $QSD_T GARCH(1,1) - T$ with $\xi = 0.2$ and $\zeta = 0.1$ (as in Table 2). The $GARCH(1,1) - T$, $\beta_T GARCH(1,1)$ and $QSD_T GARCH(1,1) - T$ models are estimated by ML on 4,000 observations.

5 Empirical Application

In the empirical application, we consider all stocks belonging to the S&P500 index for the period spanning from 03-01-1995 (or later) to 28-02-2019. All stocks for which less than 4,000 observations are available have been discarded, as well as a few stocks for which one of the competing models encountered convergence problems. We are left with 400 stocks.

The application is divided in two. We first consider volatility models based on symmetric Student's t densities and then extend our analysis by considering models with skewed Student's t densities.

Table 1: Bias, RMSE and 95% coverage probabilities and LRT. The DGP is a β_T *GARCH*(1, 1). Sample size $T = 3,000$ or $4,000$.

$T = 3,000$								
	μ	ω	ϖ	α	β	ξ	ζ	
	0.06	0.08	0.13	0.10	0.83	0.2	0.2	
<i>GARCH</i> (1, 1) – T ML*	0.000 (0.026) [0.961]	0.022 (0.032) [0.937]	0.031 (0.045) [0.918]	-0.038 (0.019) [0.289]	0.022 (0.035) [0.737]	0.002 (0.017) [0.947]		
β_T <i>GARCH</i> (1, 1)	0.000 (0.026) [0.964]	0.008 (0.032) [0.960]	0.011 (0.045) [0.949]	-0.000 (0.019) [0.945]	-0.006 (0.035) [0.947]	-0.001 (0.017) [0.956]		
QSD_T <i>GARCH</i> (1, 1) QML	0.001 (0.031) [0.960]	0.009 (0.048) [0.922]	0.014 (0.087) [0.913]	0.001 (0.029) [0.913]	-0.018 (0.066) [0.909]		0.025 (0.111) [0.793]	
QSD_T <i>GARCH</i> (1, 1) – T ML	0.000 (0.026) [0.963]	0.007 (0.031) [0.962]	0.011 (0.045) [0.944]	-0.002 (0.020) [0.934]	-0.009 (0.039) [0.950]	-0.001 (0.018) [0.957]	0.008 (0.089) [0.852]	
	1%	5%	10%					
$H_0 : \xi = \zeta$ (size)	0.901	6.607	12.713					
$H_0 : \zeta = 0$ (power)	81.481	92.793	95.195					
$T = 4,000$								
	μ	ω	ϖ	α	β	ξ	ζ	
	0.06	0.08	0.13	0.10	0.83	0.2	0.2	
<i>GARCH</i> (1, 1) – T ML*	0.000 (0.021) [0.970]	0.017 (0.027) [0.925]	0.024 (0.035) [0.942]	-0.039 (0.017) [0.196]	0.027 (0.029) [0.684]	0.003 (0.016) [0.950]		
β_T <i>GARCH</i> (1, 1)	0.000 (0.021) [0.966]	0.005 (0.027) [0.939]	0.007 (0.035) [0.955]	-0.000 (0.017) [0.938]	-0.004 (0.029) [0.959]	-0.000 (0.016) [0.953]		
QSD_T <i>GARCH</i> (1, 1) QML	0.001 (0.025) [0.967]	0.007 (0.039) [0.940]	0.009 (0.051) [0.935]	-0.000 (0.025) [0.927]	-0.013 (0.049) [0.936]		0.021 (0.100) [0.804]	
QSD_T <i>GARCH</i> (1, 1) – T ML	0.000 (0.021) [0.967]	0.005 (0.027) [0.941]	0.007 (0.036) [0.949]	-0.001 (0.018) [0.933]	-0.006 (0.033) [0.942]	-0.001 (0.016) [0.955]	0.008 (0.078) [0.864]	
	1%	5%	10%					
$H_0 : \xi = \zeta$ (size)	1.100	6.600	12.900					
$H_0 : \zeta = 0$ (power)	93.700	97.600	98.200					

Note: Monte Carlo simulation results for $T = 3,000$ (top panel) and $T = 4,000$ (bottom panel). Each panel is divided into two parts. The first part is for the estimated parameters of the four models. The figures at the right of the name of the models are the empirical biases over 1,000 replications. The figures in parentheses correspond to RMSEs, while those in squared brackets are the 95% coverage probabilities. The second part contains rejection frequencies of two LR tests computed from the ML estimates. Some of the figures reported in this part correspond to empirical sizes or powers depending on the DGP. Models highlighted with an asterisk after their name do not nest the DGP.

Table 2: Bias, RMSE and 95% coverage probabilities and LRT. The DGP is a $QSD_T GARCH(1, 1) - T$. Sample size $T = 3,000$ or $4,000$.

$T = 3,000$								
	μ	ω	ϖ	α	β	ξ	ζ	
	0.06	0.08	0.13	0.10	0.83	0.2	0.1	
$GARCH(1, 1) - T$ ML*	0.000 (0.025) [0.961]	0.017 (0.030) [0.939]	0.025 (0.042) [0.926]	-0.028 (0.021) [0.506]	0.007 (0.041) [0.872]	0.001 (0.018) [0.950]		
$\beta_T GARCH(1, 1)^*$	0.000 (0.025) [0.966]	0.006 (0.030) [0.964]	0.009 (0.042) [0.946]	0.008 (0.021) [0.954]	-0.021 (0.041) [0.944]	-0.004 (0.018) [0.944]		
$QSD_T GARCH(1, 1)$ QML	0.001 (0.029) [0.958]	0.008 (0.045) [0.914]	0.013 (0.079) [0.909]	0.002 (0.031) [0.910]	-0.016 (0.059) [0.901]		0.035 (0.103) [0.817]	
$QSD_T GARCH(1, 1) - T$ ML	0.000 (0.025) [0.965]	0.007 (0.031) [0.962]	0.011 (0.044) [0.939]	-0.001 (0.021) [0.921]	-0.009 (0.039) [0.939]	-0.001 (0.018) [0.957]	0.013 (0.075) [0.841]	
	1%	5%	10%					
$H_0 : \xi = \zeta$ (power)	10.010	25.125	36.637					
$H_0 : \zeta = 0$ (power)	56.356	73.674	80.981					
$T = 4,000$								
	μ	ω	ϖ	α	β	ξ	ζ	
	0.06	0.08	0.13	0.10	0.83	0.2	0.1	
$GARCH(1, 1) - T$ ML*	0.000 (0.020) [0.969]	0.014 (0.025) [0.932]	0.019 (0.034) [0.944]	-0.029 (0.019) [0.405]	0.012 (0.035) [0.843]	0.001 (0.016) [0.953]		
$\beta_T GARCH(1, 1)^*$	0.000 (0.020) [0.965]	0.004 (0.025) [0.935]	0.005 (0.034) [0.944]	0.007 (0.019) [0.946]	-0.018 (0.035) [0.950]	-0.004 (0.016) [0.933]		
$QSD_T GARCH(1, 1)$ QML	0.001 (0.023) [0.964]	0.008 (0.039) [0.944]	0.010 (0.055) [0.942]	0.001 (0.026) [0.928]	-0.013 (0.047) [0.942]		0.031 (0.092) [0.851]	
$QSD_T GARCH(1, 1) - T$ ML	0.000 (0.020) [0.966]	0.005 (0.025) [0.945]	0.006 (0.034) [0.943]	-0.001 (0.019) [0.931]	-0.006 (0.031) [0.958]	-0.001 (0.016) [0.956]	0.012 (0.063) [0.879]	
	1%	5%	10%					
$H_0 : \xi = \zeta$ (power)	13.300	29.800	40.100					
$H_0 : \zeta = 0$ (power)	71.600	85.500	90.500					

Note: see Table 1.

Table 3: Bias, RMSE and 95% coverage probabilities and LRT. The DGP is a $GARCH(1, 1) - T$. Sample size $T = 3, 000$ or $4, 000$.

$T = 3, 000$							
	μ	ω	ϖ	α	β	ξ	ζ
	0.06	0.08	0.13	0.10	0.83	0.2	0
$GARCH(1, 1) - T$ ML	0.000 (0.025) [0.965]	0.006 (0.027) [0.957]	0.008 (0.037) [0.944]	0.000 (0.046) [0.949]	-0.005 (0.056) [0.938]	-0.001 (0.020) [0.956]	
β_T $GARCH(1, 1)^*$	0.000 (0.025) [0.966]	0.004 (0.027) [0.954]	0.005 (0.037) [0.935]	0.040 (0.046) [0.518]	-0.046 (0.056) [0.730]	-0.010 (0.020) [0.903]	
QSD_T $GARCH(1, 1)$ QML	0.001 (0.029) [0.953]	0.006 (0.038) [0.924]	0.008 (0.052) [0.924]	0.004 (0.030) [0.921]	-0.009 (0.043) [0.911]		0.013 (0.038) [0.834]
QSD_T $GARCH(1, 1) - T$ ML	0.000 (0.025) [0.967]	0.005 (0.026) [0.958]	0.007 (0.036) [0.945]	0.000 (0.019) [0.940]	-0.005 (0.029) [0.936]	-0.001 (0.018) [0.953]	0.004 (0.019) [0.842]
	1%	5%	10%				
$H_0 : \xi = \zeta$ (power)	92.893	97.598	98.599				
$H_0 : \zeta = 0$ (size)	1.702	8.008	14.915				
$T = 4, 000$							
	μ	ω	ϖ	α	β	ξ	ζ
	0.06	0.08	0.13	0.10	0.83	0.2	0
$GARCH(1, 1) - T$ ML	0.000 (0.020) [0.970]	0.004 (0.023) [0.946]	0.004 (0.029) [0.956]	-0.000 (0.044) [0.945]	-0.002 (0.052) [0.948]	-0.001 (0.019) [0.955]	
β_T $GARCH(1, 1)^*$	0.000 (0.020) [0.970]	0.002 (0.023) [0.926]	0.002 (0.029) [0.942]	0.039 (0.044) [0.395]	-0.044 (0.052) [0.646]	-0.010 (0.019) [0.877]	
QSD_T $GARCH(1, 1)$ QML	0.001 (0.023) [0.968]	0.006 (0.032) [0.937]	0.007 (0.041) [0.945]	0.003 (0.025) [0.926]	-0.007 (0.034) [0.945]		0.008 (0.025) [0.866]
QSD_T $GARCH(1, 1) - T$ ML	0.000 (0.020) [0.971]	0.003 (0.022) [0.944]	0.004 (0.029) [0.956]	-0.000 (0.017) [0.947]	-0.002 (0.023) [0.961]	-0.001 (0.016) [0.954]	0.003 (0.015) [0.870]
	1%	5%	10%				
$H_0 : \xi = \zeta$ (power)	98.300	99.500	99.700				
$H_0 : \zeta = 0$ (size)	1.500	5.700	10.900				

Note: see Table 1.

5.1 Symmetric densities

A $GARCH(1, 1) - T$, a $\beta_T GARCH(1, 1)$ and a $QSD_T GARCH(1, 1) - T$ are estimated by ML on all series of log-returns.⁶ The explanatory variable X_t used in the conditional variance equation f_{t+1} of all models is the square of the VIX index converted from an annual to a daily horizon.

The stationarity and invertibility conditions seem to be satisfied for all series according to conditions (ii) of Lemma 1 and (ii) of Lemma 3 evaluated at the MLE estimates of the parameters.

The first two columns of Table 4 contain the rejection frequencies at the 5% nominal level (over the 400 stocks) of two LRTs. Interestingly, the null hypothesis $H_0 : \zeta = 0$ is rejected in approximately 90% of the cases, suggesting that downweighting large shocks in the conditional variance of these US stocks is empirically relevant.

These results naturally call for the use of an SD model rather than a $GARCH$ dynamic. However, the null hypothesis $H_0 : \xi = \zeta$ is rejected in more than 50% of the cases (again at the 5% nominal level) suggesting that the additional flexibility of the $QSD_T GARCH(1, 1) - T$ over the $\beta_T GARCH(1, 1)$ is empirically relevant. Furthermore, all the estimated ζ values are positive, and thus specification (30) can be used instead of (31).

Table 4: Rejection frequencies (at 5%) of two LRTs and a goodness-of-fit test for the models with symmetric Student's t densities

LRT		PIT test		
$H_0 : \xi = \zeta$	$H_0 : \zeta = 0$	$GARCH - T$	$\beta_T GARCH$	$QSD_T GARCH - T$
53.5	89.5	40.0	46.0	45.7

The figures reported in the table are rejection frequencies over the 400 stocks and a nominal size of 5%. The figures in the two columns under LRT are for the LRT whose null hypothesis is reported just above. These hypothesis are tested on the MLEs of the QSD model. The figures reported in the three columns under the PIT test χ^2 test are for the PIT test applied on the residuals of the corresponding model.

The difference $\hat{\xi} - \hat{\zeta}$ is plotted in Figure 2 for the 400 US stocks (sorted in alphabetical order of the ticker name). A full (resp. empty) circle corresponds to a significant (resp. insignificant) difference (according to an LRT at the 5% nominal level). For all (but one) stocks for which $\hat{\xi} \neq \hat{\zeta}$, $\hat{\xi} > \hat{\zeta}$, suggesting that the downweighting of the $\beta_T GARCH(1, 1)$ is too strong.

To visualize the added value of the $QSD_T GARCH(1, 1) - T$ model over the $GARCH - T$ and $\beta_T GARCH(1, 1)$ models, we randomly selected a stock for which both the null hypotheses $H_0 : \xi = \zeta$ and $H_0 : \zeta = 0$ are rejected. We choose Centene (whose ticker is CNC), a multi-line healthcare enterprise.

The MLEs of parameters entering the conditional variance equation of the three models obtained for daily log-returns of CNC during the period spanning from December 2001 to the end of February 2019 (i.e., 4330 observations) are reported in Table 5 together

⁶To account for possible serial correlation in the data, an AR(1) specification is used for all series.

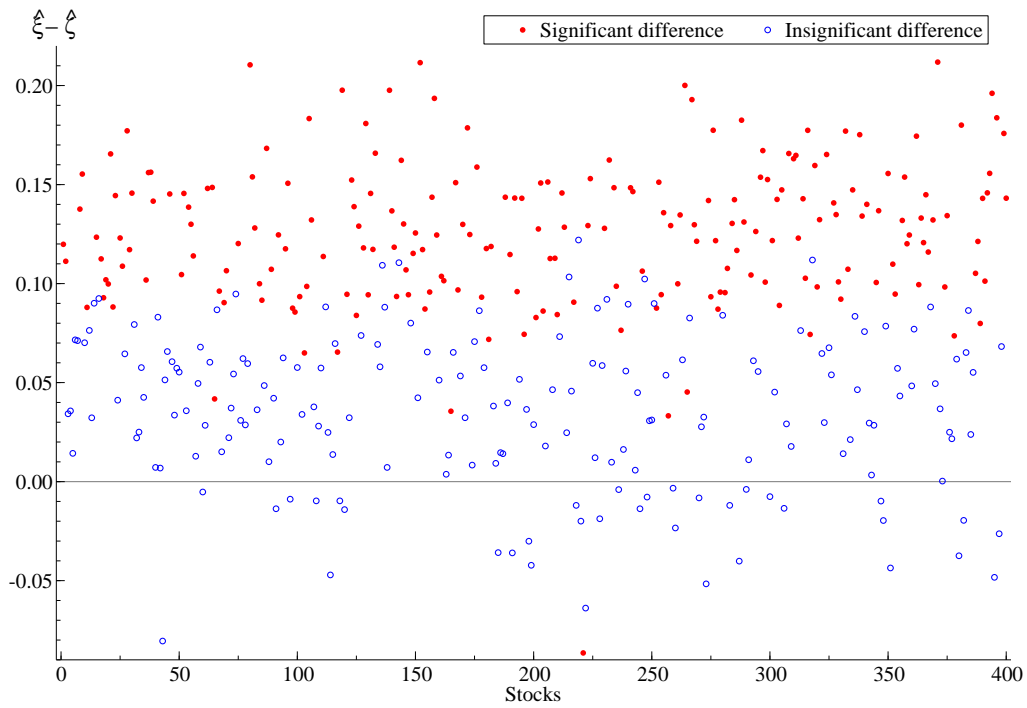


Figure 2: $\hat{\xi} - \hat{\zeta}$ for the QSD_T $GARCH(1, 1) - T$ estimated on the 400 US stocks. A full (resp. empty) circle corresponds to a significant (resp. insignificant) difference (according to an LRT at the 5% nominal level).

with the log-likelihood values and the two LRT statistics of the tests presented above (with the corresponding p-values in squared brackets). Interestingly, the estimated ξ parameters of the $GARCH(1,1) - T$ and $\beta_T GARCH(1,1)$ do not differ much and are approximately equal to 0.25, which corresponds to a degree of freedom of the Student's t distribution close to 4. The log-likelihood of the $\beta_T GARCH(1,1)$ is very close to the one of the $GARCH(1,1) - T$ model but slightly lower. Given that the two models are not nested, LRTs cannot be used to discriminate between these two models. Importantly, while $\hat{\xi}$ is also close to 0.25 for the $QSD_T GARCH(1,1) - T$ model, $\hat{\zeta}$ is approximately 0.05 (and therefore $1/\hat{\zeta}$ is close to 20), suggesting that the $\beta_T GARCH(1,1)$ downweights the large shocks far too much.

To help visualize the differences between the three models, news impact curves (NICs) are plotted in Figure 3. An NIC measures how new information is incorporated into the conditional variance. Since the $QSD_T GARCH(1,1) - T$ nests the other two models, we can write the NIC of the three models as the function mapping the shocks ϵ_t to $\frac{1+\zeta}{1-2\zeta+\zeta\epsilon_t^2}\epsilon_t^2$, where $\zeta = 0$ for the $GARCH(1,1) - T$ model and $\zeta = \xi$ for the $\beta_T GARCH(1,1)$. The NIC of the $GARCH(1,1) - T$ model corresponds to the green dashed line. For the $\beta_T GARCH(1,1)$ and $QSD_T GARCH(1,1) - T$ models, three NICs are plotted, evaluated at the 5, 50 and 95% quantiles of the 400 estimated values of ζ . For these two models, the median of $1/\hat{\zeta}$ are respectively 5.0702 and 9.4805. The NICs of the $QSD_T GARCH(1,1) - T$ (resp. $\beta_T GARCH(1,1)$) model correspond to the blue solid (resp. red dotted) lines. The NICs evaluated at the median are highlighted in bold. We see from Figure 3 that the NICs of the $QSD_T GARCH(1,1) - T$ stock lie between the NICs of the other two models and that there is more heterogeneity in the NICs of the $QSD_T GARCH(1,1) - T$ model.

To see the impact of different NICs on the estimated conditional volatilities, the absolute value of the daily log-returns of CNC (thin solid red line) as well as the estimated conditional volatilities of the $GARCH(1,1) - T$ (thin blue line with long dashes), $\beta_T GARCH(1,1)$ (thin green dashed line) and $QSD_T GARCH(1,1) - T$ (thick solid pink line) estimated by ML (on the full period) are plotted for the sub-period spanning from the beginning of March 2008 to the end of April 2008 in Figure 4. The box highlights a period of several days around a large shock (i.e., an absolute return of approximately 25 %) and for which we see large differences between the estimated conditional standard deviations of the three models. In line with what we observed in Figure 1 for simulated data, the conditional volatility of the $GARCH(1,1) - T$ model is much higher than that of the $\beta_T GARCH(1,1)$ for several days after the shock while the conditional volatility of the $QSD_T GARCH(1,1) - T$ lies between the two.

As pointed out by a referee, the ability of the $QSD_T GARCH(1,1) - T$ to outperform the other two models in most cases does not mean this model adequately captures the main features of the data. Indeed, it may be that the $QSD_T GARCH(1,1) - T$ model describes the data better than the $\beta_T GARCH(1,1)$ and $GARCH(1,1) - T$ models but that these three models do not capture well the dynamics in the conditional variance or the shape of the distribution for some quantiles (such as the tails or the degree of asymmetry of the distribution). Several tests are available to test a specific type of misspecification. To avoid the problem of multiple tests we prefer a global misspecification

Table 5: MLEs of the parameters of the conditional variance equation of the $GARCH(1,1) - T$, $\beta_T GARCH(1,1)$ and $QSD_T GARCH(1,1) - T$ for CNC during the period spanning from December 2001 to the end of February 2019 (i.e., 4,330 observations).

	$GARCH(1,1) - T$	$\beta_T GARCH(1,1)$	$QSD_T GARCH(1,1) - T$
ω	1.1278 (0.1932)	0.9857 (0.1850)	0.9833 (0.1842)
ϖ	0.8074 (0.1573)	0.6570 (0.1469)	0.6890 (0.1493)
α	0.1574 (0.0266)	0.2256 (0.0283)	0.2015 (0.0320)
β	0.5394 (0.0603)	0.5076 (0.0621)	0.5577 (0.0590)
ξ	0.2593 (0.0136)	0.2568 (0.0135)	0.2584 (0.0137)
ζ			0.0479 (0.0270)
Log-Likelihood	-9630.6	-9630.8	-9626.4
$H_0 : \xi = \zeta$	8.812 [0.003]		
$H_0 : \zeta = 0$	8.4169 [0.004]		

Note: The figures at the right of $H_0 : \xi = \zeta$ and $H_0 : \zeta = 0$ are the values of the LRT corresponding to the specified null hypothesis (with the p-value below in squared brackets).

test, namely the goodness-of-fit test proposed by Diebold, Gunther and Tay (1998).⁷ They show that under the null hypothesis of correct specification, the probability integral transform (PIT) series $\{u_t\}_{t=1}^T \stackrel{i.i.d.}{\sim} U(0,1)$ so that $u_t \stackrel{i.i.d.}{\sim} (1/2, 1/12)$ and therefore $\sum_{t=1}^T u_t \sim N(T/2, T/12)$ as $T \rightarrow \infty$, which suggests rejecting the null of correct specification at the 5% nominal level when $|(\sum_{t=1}^T u_t - T/2)|/\sqrt{T/12} > 1.96$. This result is only true when the PIT series is observed without error, while in practice, it is computed from the cumulative distribution function of the residuals. Since the MLEs of the models considered in this section are asymptotically Gaussian under the assumption of correct specification, we can apply Pierce's (1982) theorem to derive the asymptotic distribution of $\sum_{t=1}^T u_t$ conditional on the MLEs of the model fitted on the data. More details on the PIT test as well as the results of a small Monte Carlo simulation study that shows its

⁷Let us assume the true model is a $QSD_T GARCH - T$ model as described in Section 4.1, with $\zeta \neq \xi$, but one estimates a $\beta_T GARCH$ model that imposes $\zeta = \xi$. Wrongly imposing $\zeta = \xi$ may have various consequences on the residuals: standardized residuals might not follow the assumed distribution, standardized residuals might not be *i.i.d.* or both. It is precisely because we do not know in advance whether the potential misspecification of the SD models will be seen in the unconditional density of the standardized residuals or in the presence of serial correlation in these standardized residuals (or their powers) that we use a global test of misspecification on both aspects.

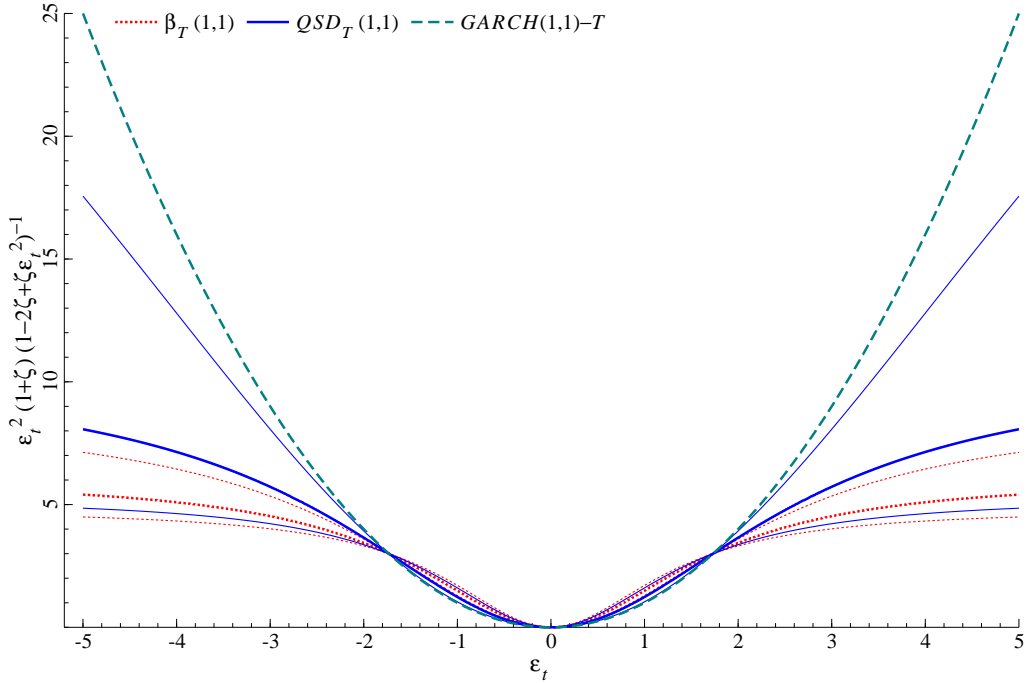


Figure 3: News impact curves for the MLEs of the $QSD_T GARCH(1,1) - T$ (blue solid lines), $\beta_T GARCH(1,1)$ (red dotted lines) and $GARCH(1,1) - T$ (green dashed line) evaluated at the 5, 50 and 95% quantiles of the 400 values of $\hat{\zeta}$. The NICs evaluated at the median are highlighted in bold.

good finite sample properties are reported in Appendix C in the online supplement.

The rejection frequencies (over the 400 stocks and for a nominal size of 5%) of the above PIT test (accounting for estimation error) are reported in the last three columns of Table 4 (under *PIT*) for the three models considered in this section. The results clearly suggest that the three models are rejected in more than 40% of the cases, which calls for the use of a different model.

5.2 Skewed densities

The distribution of daily stock returns is known to have heavy tails (as illustrated in the previous section) and is also often found to be left-skewed, as shown, for instance, by Giot and Laurent (2003) and Harvey and Lange (2016). It is also known that negative shocks have a deeper impact on the volatility than positive shocks of the same magnitude (so-called leverage effect). Hansen and Lunde (2005) find that the leverage is actually empirically more relevant than tails for forecasting volatility.

Harvey and Lange (2016) are the first to consider a skewed distribution in the context of *SD* volatility models. More specifically, they propose an *SD* model in which the innovations are assumed to follow an asymmetric generalized t distribution. In this section,

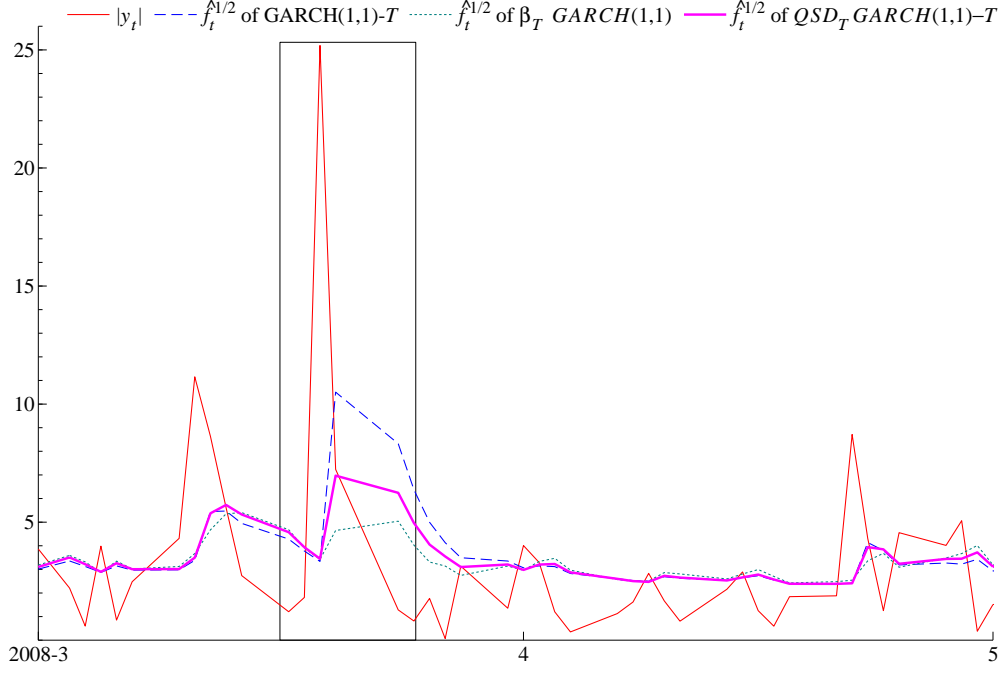


Figure 4: Absolute value of log-returns and estimated conditional volatility of the $GARCH(1, 1) - T$, $\beta_T GARCH(1, 1)$ and $QSD_T GARCH(1, 1) - T$ estimated by ML for the ticker CNC (Centene). The graph only shows the sub-period spanning from the beginning of March 2008 to the end of April 2008.

we consider a constrained version of this density corresponding to the non-standardized skewed Student's t (ST) of Zhu and Galbraith (2010) with degree of freedom $1/\xi$ and asymmetry parameter κ , which we denote as $ST_{1/\xi, \kappa}$. If $y_t = \sqrt{f_t} \epsilon_t$ and ϵ_t follows a $ST_{1/\xi, \kappa}$ distribution, the log-likelihood can be written as

$$\log p(y_t | f_t, \theta) = \ln(\delta K_{1/\xi}) - \frac{1}{2} \ln(f_t) - \left(\frac{1/\xi + 1}{2} \right) \ln \left(1 + \frac{\xi \epsilon_t^2}{4\kappa_t^2} \right), \quad (45)$$

where $K_v \equiv \Gamma((v + 1)/2) / [\sqrt{\pi v} \Gamma(v/2)]$ (with $\Gamma(\cdot)$ the gamma function), $\kappa_t = \kappa$ (resp. $1 - \kappa$) if $\epsilon_t \leq 0$ (resp. $\epsilon_t > 0$) and

$$m = \frac{4}{1-\xi} \left[(1-\kappa)^2 - \kappa^2 \right] K_{1/\xi}$$

$$\delta^2 = \frac{4}{1-2\xi} \left[\kappa^3 + (1-\kappa)^3 \right] - m^2.$$

Importantly, the ST density is symmetric when $\kappa = 1/2$ and left-skewed (resp. right skewed) when $\kappa > 1/2$ (resp. $\kappa < 1/2$). Interestingly, Zhu and Galbraith (2010) show that the skewed Student's t distributions of Hansen (1994) and Fernandez and Steel (1998) can be recovered as special cases of the above ST density.

While Harvey and Lange (2016) consider the asymmetric generalized t distribution in the context of an SD model for the log of the conditional variance (i.e., in the spirit of an EGARCH model), we report below the specification of an SD model for the conditional variance when the innovations follow the above $ST_{1/\xi,\kappa}$ distribution.⁸ This model, denoted $\beta_{ST} GARCH(1,1)$, takes the form

$$f_{t+1} = \omega + \alpha \frac{(1 + \xi)}{4\kappa_t^2 + \xi \epsilon_t^2} \epsilon_t^2 f_t + \beta f_t. \quad (46)$$

When $\kappa > 1/2$, the ST density is left-skewed, and in this case, the $\beta_{ST} GARCH(1,1)$ produces volatilities that are smaller after negative shocks than after positive shocks of the same magnitude. Indeed, when $\kappa = 0.6$, $\kappa_t^2 = 0.6^2$ when $\epsilon_t \leq 0$ while $\kappa_t^2 = 0.4^2$ when $\epsilon_t > 0$. Unfortunately, this contradicts the leverage effect. This issue does not affect the larger class of QSD models because the updating function ρ can be disconnected from $\log p(y_t | f_t, \theta)$.⁹ To preserve the flexibility and generality of the ST density, an $ST_{1/\xi,\kappa}$ distribution can be assumed for the innovations ϵ_t , while $\rho(y_t, X_t, f_t, \theta)$ can be chosen to be the log-likelihood of an ST density with different parameters, i.e., a $ST_{1/\zeta,\tau}$ distribution, which leads to the following $QSD_{ST} GARCH(1,1) - ST$ model:

$$f_{t+1} = \omega + \alpha \frac{(1 + \zeta)}{4\tau_t^2 + \zeta \epsilon_t^2} \epsilon_t^2 f_t + \beta f_t, \quad (47)$$

where τ_t is defined similarly to κ_t . A value of $\tau < 0.5$ can therefore produce a leverage effect, irrespective of the value taken by κ .

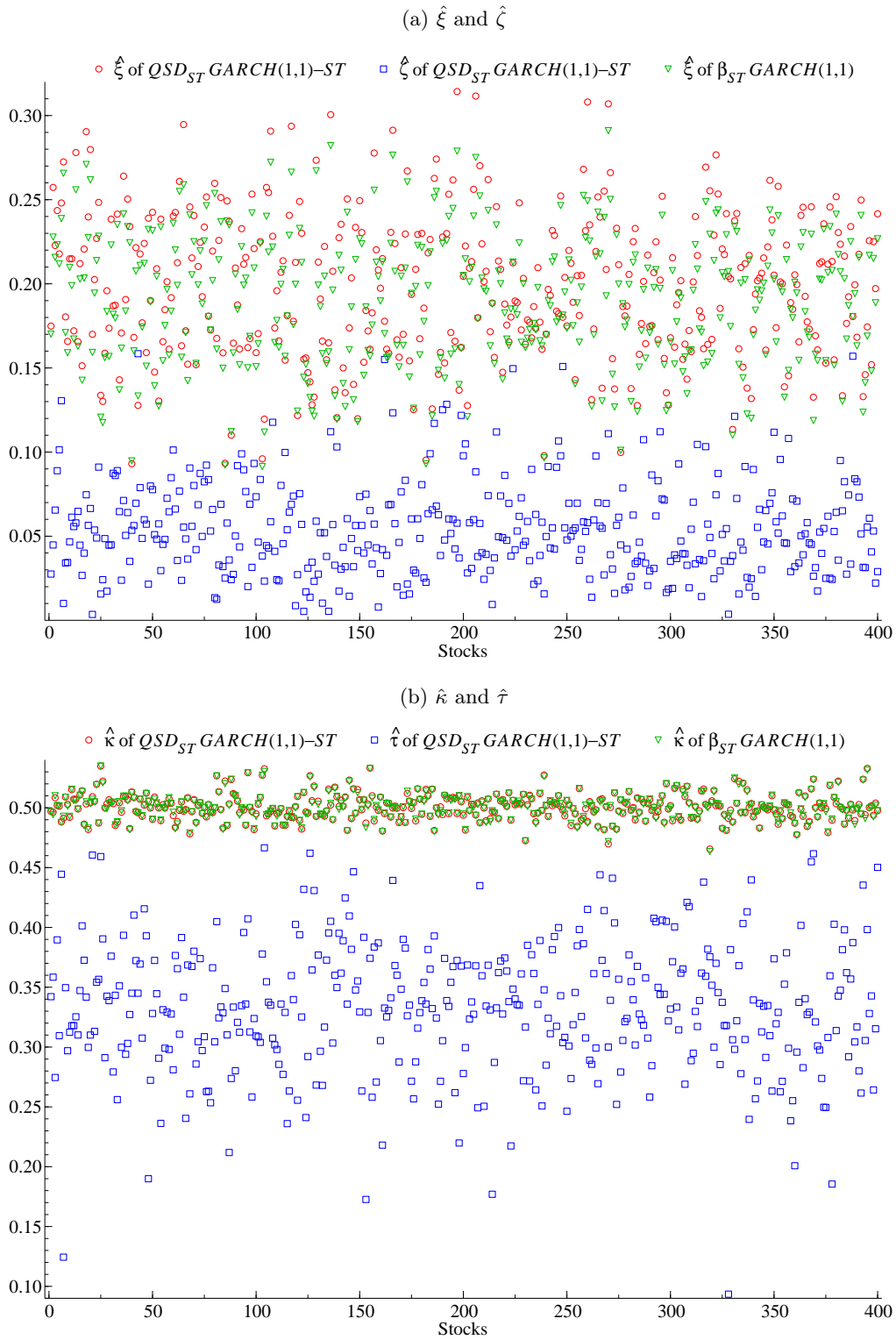
A $GARCH(1,1) - ST$, a $\beta_{ST} GARCH(1,1)$ and a $QSD_{ST} GARCH(1,1) - ST$ are estimated on the same 400 stocks. To help visualize the difference between the last two models, their MLEs of the tail parameters ξ and ζ are plotted in Panel (a) of Figure 5, while the asymmetry parameters κ and τ are plotted in Panel (b). Two conclusions emerge from this graph. First, the estimated values of ξ and κ of the $\beta_{ST} GARCH$ are systematically very close to those of the $QSD_{ST} GARCH - ST$. Second, in most cases, $\hat{\xi}$ and $\hat{\zeta}$ are quite different. More importantly, $\hat{\tau}$ is far below 0.5 in all cases, while $\hat{\kappa}$ is on average close to 0.5 (although very often significantly different from 0.5). This suggests that the additional flexibility of the $QSD_{ST} GARCH - ST$ is needed in all cases.

As in Figure 3, NICs of each estimated model with a skewed Student's t distributions are plotted in Figure 6. Since the $QSD_{ST} GARCH(1,1) - T$ nests the other two models, we can write the NIC of the three models as the function mapping the shocks ϵ_t to $\frac{(1+\zeta)}{4\tau_t^2 + \zeta \epsilon_t^2} \epsilon_t^2$, where $\zeta = 0$ for the $GARCH(1,1) - T$ model while $\zeta = \xi$ and $\tau = \kappa$ for

⁸Harvey and Lange (2016) rely on a non-standardized ST density because a study of the NIC of a $\beta_{ST} GARCH(1,1)$ derived from a standardized ST distribution shows that small shocks can lead to negative values of f_t even when the usual positivity constraints are imposed. We therefore also consider a non-standardized ST distribution for the sake of comparison.

⁹Similarly but in a different context, durations are strictly positive and their conditional distribution is known to be right skewed and fat-tailed. Modeling durations with a SD model based on a skewed and fat-tailed distribution necessarily imposes shocks to have a bounded and asymmetric impact on future values of the conditional durations. This might be restrictive for some series and could be tested using a QSD version of these models.

Figure 5: Estimates of the shape parameters of the $QSD_{ST} GARCH - ST$ and $\beta_{ST} GARCH$ models



the $\beta_{ST} GARCH(1, 1)$. The NIC of the $GARCH(1, 1) - ST$ model corresponds to the green dashed line. For the other two models, three NICs are plotted, evaluated at the 5, 50 and 95% quantiles of $\hat{\zeta}$ and $\hat{\tau}$. The NICs of the $QSD_{ST} GARCH(1, 1) - ST$ (resp. $\beta_{ST} GARCH(1, 1)$) model correspond to the blue solid (resp. red dotted) lines. The NICs evaluated at the median are highlighted in bold, i.e. $1/\zeta = 5.2383$ and $\tau = 0.5002$ for the $\beta_{ST} GARCH(1, 1)$ and $1/\zeta = 19.8381$ and $\tau = 0.33267$ for the $QSD_{ST} GARCH(1, 1) - ST$. This figure clearly suggests that the $QSD_{ST} GARCH(1, 1) - ST$ model bounds the effect of large shocks while allowing negative shocks to have a deeper impact on future values of the volatility.

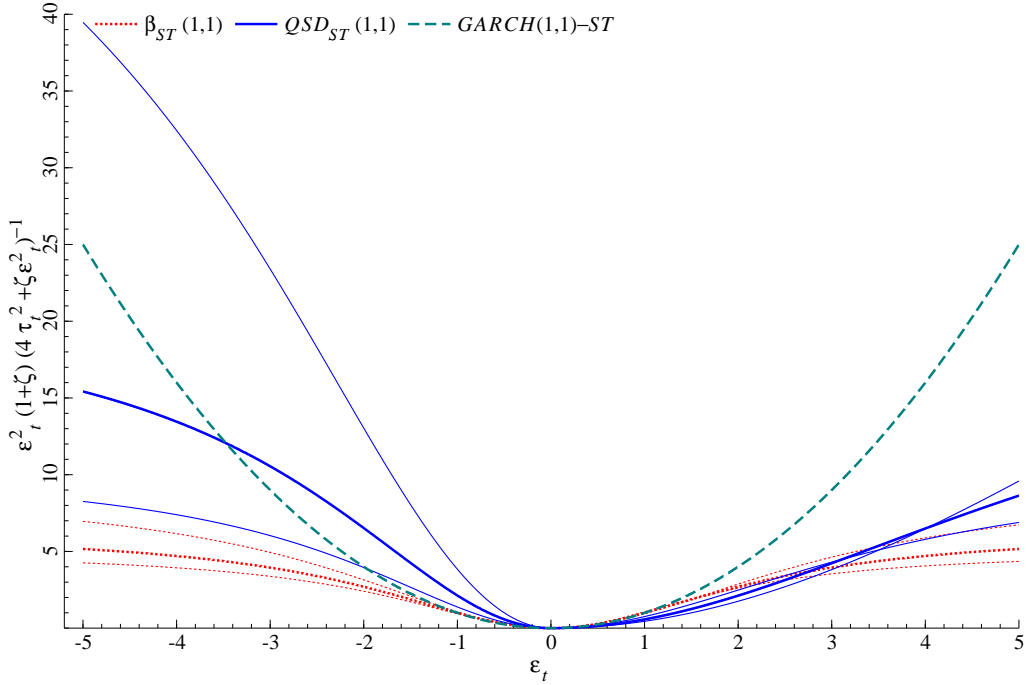


Figure 6: News impact curve for the averaged values (over the 400 stocks) of the MLEs of the $GARCH(1, 1) - ST$, $\beta_{ST} GARCH(1, 1)$ and $QSD_{ST} GARCH(1, 1) - ST$ models.

To complement the above analysis, the rejection frequencies of the LRTs of the null hypotheses $H_0 : \xi = \zeta; \kappa = \tau$ (i.e., $QSD_{ST} GARCH(1, 1) - ST = \beta_{ST} GARCH(1, 1)$) and $H_0 : \zeta = 0$ (i.e., $QSD_{ST} GARCH(1, 1) - ST = GARCH(1, 1) - ST$) are reported in the first two columns of Table 6, while the rejection frequencies of the PIT test presented in the previous section are reported in the last three columns (under PIT test) for the three models considered in this section.

The results suggest that the restriction in the $\beta_{ST} GARCH$ that imposes $\xi = \zeta$ and $\kappa = \tau$ is rejected in all cases, while the GARCH restriction that imposes $\zeta = 0$ (i.e., no bounding on the effect of large shocks) is rejected in 98.7% of the cases. Interestingly, although the skewed Student's t distribution is more flexible than the symmetric Student's

t distribution, the β_{ST} *GARCH* is rejected more often than the symmetric Student's t density considered in Section 5.1 (i.e., in 72.2% of the cases compared to 46% for the symmetric Student's t distribution). Importantly, the QSD_{ST} *GARCH* – *ST* model is rejected in only 11.3% of the cases (recall that we expect already 5% of rejections because of the multiple tests and the type I errors).

Table 6: Rejection frequencies (at 5%) of two LRTs and a goodness-of-fit test for the models with skewed Student's t densities

LRT		PIT test		
$H_0 : \xi = \zeta$	$H_0 : \zeta = 0$	<i>GARCH</i> – <i>ST</i>	β_{ST} <i>GARCH</i>	QSD_{ST} <i>GARCH</i> – <i>ST</i>
$\kappa = \tau$				
100.0	98.8	33.1	72.2	11.3

Note: see Table 4.

6 Conclusion

SD models have received considerable attention in the time series literature. In this paper, we relax a restriction imposed by this general class of models. Specifically, we break the strict link between the shape of the conditional distribution of y_t and the loss function used to design the updating equation of f_t . We thus arrive at a more general family of models called *QSD*. This class of models allows researchers to design parameter updating equations that are guided by a multitude of statistical loss functions beyond the log-likelihood function.

We study the statistical properties of the *QSD* filter as well as the QLE, QMLE and MLE of the parameters of this model. We show how to test the relevance of some of the constraints in the *SD* models, linking f_t to $p_t(y_t|f_t, \theta)$.

We study in detail the QSD_T *GARCH*(1, 1) – *T* model, a volatility model extending the β_T *GARCH*(1, 1) model of Harvey and Chakravarty (2008). This model relies on a standardized Student's t density for the innovations and the score of a standardized Student's t density in the updating equation of the conditional variance but does not restrict the degrees of freedom to be the same. The additional flexibility of this model (compared with the β_T *GARCH*(1, 1)) is found to be significant at the 5% significance level using a standard LRT in more than 50% of the cases (out of 400 stocks). However, we find that this model is rejected in more than 40% of the cases when using the PIT test of Diebold, Gunther and Tay (1998) extended to account for the estimation uncertainty.

Finally, we also propose a volatility model derived from the *ST* density of Zhu and Galbraith (2010), denoted as QSD_{ST} *GARCH*(1, 1) – *ST*. Unlike pure *SD* models, this volatility model allows us to introduce the leverage effect in a *QSD* model even when the series are left-skewed. We show that, according to the PIT test, this model captures the most important features of more than 88% of the stocks considered in the empirical application.

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