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Targeting in networks under costly agreements [☆]

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Abstract

We consider agents organized in an undirected network of local complementarities. A principal with a fixed budget offers costly bilateral contracts in order to increase the sum of agents' effort. We study contracts rewarding effort exceeding the effort made in the absence of the principal. First, targeting a subgroup of the whole society becomes optimal under substantial contracting costs, which significantly increases the computational complexity of the principal's problem. In particular, under sufficiently low intensity of complementarities, a correspondence is established between optimal targeting and an NP-hard problem. Second, for any intensities of complementarities, the optimal unit returns offered to all targeted agents are positive for all contracting costs and in general heterogeneous, even though networks are undirected. Yet, heterogeneity never leads to negative returns, which implies that, with these linear payment schemes, coordination is never an issue for the principal.

JEL classification: C72 D85

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1. Introduction

In many economic situations, institutions trade agents' effort against transfers in a context of networked synergies and positive externalities. A common feature of such contracts is that they can involve high administration and enforcement costs.¹ For instance, a frequent criticism of conditional cash transfer (CCT) programs and other social safety net programs is that a large proportion of their budgets never reaches the intended beneficiaries, but is absorbed by administration costs (see Grosh (1994), Caldés and Maluccio (2005)). R&D subsidies also involve high administration costs (see GAO study of 1989 by U.S. General Accounting Office, Stoffregen (1995), and Hall and Van Reenen (2000)). Another costly contract for firms in terms of administration, monitoring, and enforcement is organizing a performance appraisal system (see Murphy and Cleveland (1995), and Aggarwal and Thakur (2013)). Given the prevalence of such high contracting costs, institutions may only contract with a subset of economic agents. This paper adds to the literature on contracting in networks (for example Belhaj and Deroïan (2019)) by allowing the principal to exclude some agents and only contract with a subset of

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¹ Such costs are, at least partially, increasing with the number of contracts. For instance, with respect to administration costs, the cost of actually making/writing contracts is clearly increasing with the number of contracts. Similarly, as the number of contracts increases, an increasing number of agents is needed to monitor/enforce them, which increases overall enforcement costs.

agents. In particular, we investigate how contracting costs affect optimal group selection and the design of optimal contracts when there are synergies and positive externalities between agents.

To analyze the relationship between the network of synergies among agents, targeting, and contracting costs, we build a principal / multi-agent model where the principal has a fixed budget and offers costly linear bilateral payment schemes in order to increase the sum of agents' effort.² We consider an environment where bilateral synergies are perfectly reciprocal, and where non-contracting agents (hereafter termed *outsiders* for convenience) exert effort and interact with contracting agents, so that reservation utilities are endogenous to offered contracts.³ Utilities are linear quadratic in effort with local synergies and positive externalities, and contracts are 'excess-effort' linear payment schemes – rewarding effort in excess of the effort made in the absence of the principal. The timing of the game is as follows. First, a principal offers to a set of agents contracts involving either everyone or only a subset of agents that the principal wishes to target. These contracts take the form of a transfer per unit of additional effort (hereafter termed unit return); they are offered simultaneously and made public. Second, agents decide whether to accept or reject their respective offers. Finally, all agents, including outsiders, simultaneously exert effort and transfers are realized.

Before we proceed further, it is important to explain why we focus on 'excess-effort' linear payment schemes. First, for any given set of executed contracts, the utility specification generates a linear system of best-response effort, which leads to a unique Nash effort profile. Second, due to the presence of complementarities, the principal may worry about coordination among agents in the participation game. However, we show that the optimal unit return offered in any contract is always positive, which avoids all potential coordination issues for any level of contracting costs; indeed, this scheme provides incentives to all targeted agents, and higher effort enhances agents' utilities in this game with positive externalities. This allows us to study optimal excess-effort linear payment schemes, without having to worry about such coordination issues, and to examine the relationship between contracting costs and optimal contracts.⁴

Two main messages are delivered by our analysis. First, *when contracting costs are substantial, their shape deeply affects the level of computational complexity associated with the optimal group selection problem of the principal*. Actually, contracting with central agents involves a trade-off between centrality and budget spending - central agents generate synergies but are also budget-consuming. This is because they receive substantial synergies, which makes them highly responsive to rewards. In the absence of contracting costs, the principal finds it optimal to contract with every agent, and all agents get the same unit return, irrespective of their position on the network. Indeed, by the symmetry of bilateral interactions, the benefit from larger synergies emanating from central agents is exactly counterbalanced by their propensity to overstretch the budget.⁵ In contrast, when contracting costs are sufficiently high, the optimal group selection problem can be NP-hard. In particular, we provide an instance of the game where optimal targeting is isomorphic to the densest k -subgraph problem, which is known to be NP-hard.

Second, *the optimal unit returns offered to all targeted agents are positive for all contracting costs, and these unit returns can be heterogeneous*, even though networks are undirected. It is the presence of outsiders that allows heterogeneity. In short, the sensitivity of the behavior of a targeted agent to the behavior of another targeted agent decreases with the number of connections with outsiders. This creates asymmetric interaction between contracting agents. Now, the presence of heterogeneous unit returns never leads to negative returns, which ultimately explains the absence of coordination problems in excess-effort linear payment contracting. This finding of positiveness is one of the main insights of this paper. Indeed, the principal may well-want to tax an agent and reallocate the tax in order to subsidize a more central agent and exploit his strong influence on the network. While this can happen in directed networks, our positiveness finding rules it out when links are reciprocal.

This paper contributes to two strands of the growing literature on networks: optimal intervention in presence of synergies, and optimal linear pricing with interdependent consumers.⁶ The first considers optimal targeting. Demange (2017) studies the optimal targeting strategies of a planner aiming to increase the aggregate action of agents embedded in a social network, allowing for non-linear interactions. Galeotti et al. (2020) study optimal targeting in networks under similar utility specifications, where a principal aims at maximizing utilitarian welfare or minimizing the volatility of aggregate activity

² Our results extend to any principal's objective being a weighted sum of efforts with positive weights.

³ In our model, agents exert effort even in the absence of intervention by the institution, which implies synergies not only between contracting agents, but also between contracting and non-contracting agents. To cite a few examples: in CCT programs, students not receiving the grant still interact with those who obtain the grant; in R&D networks where a public fund provider allocates subsidies, a non-subsidized firm still spends on R&D and interacts with partner firms; in performance appraisal systems, firms offer workers a salary increase (or a bonus), and a worker not receiving any increase in salary still interacts with other workers. This sort of problem is distinct from cases where agents' exert no effort without the intervention of the institution. For the latter, see for instance pricing with interdependent consumers.

⁴ From the principal's point of view, more sophisticated contracts might perform better - for instance, contracts offered sequentially based on a "divide-and-conquer" strategy. Such contracts will moreover have no coordination issue (full implementation) by construction (see Bernstein and Winter (2012)). However, in the current setting with a fixed budget, it is non-trivial to determine an optimal divide-and-conquer strategy, which remains an open topic for future research.

⁵ This result is tightly linked to the linearity of the system of best-responses and it holds under reciprocal interactions only.

⁶ This paper is also related to a strand of the moral hazard literature, which, by putting forward the optimality of linear contracts, helps explain the widespread use of linear contracts in the real world. See for instance Laffont and Tirole (1986), Holmstrom and Milgrom (1987), McAfee and McMillan (1987), Edmans and Gabaix (2011). Linear contracts are also natural in environments with synergies, including franchising arrangements (Lafontaine (1992)) and the rental of space in malls (Gould et al. (2005)).

by using a costly technology to modify agents' private returns; in an online appendix, they extend the analysis to a planner maximizing the sum of efforts in a context where the interaction matrix is row-stochastic (see their Assumption OA1 therein). In contrast, we consider contracting costs, which means that the principal targets an optimal group before establishing optimal contacts. A series of papers studies principal/multi-agent contracting in presence of synergies, taking into account participation constraints not addressed in the above papers. Bernstein and Winter (2012) and Sakovics and Steiner (2012) consider coordination issues with binary actions. In particular, Bernstein and Winter (2012) study a costly participation game where participants receive positive and heterogeneous externalities from other participants, and they characterize the contracts inducing full participation while minimizing total subsidies. In Sakovics and Steiner (2012), a principal subsidizes agents facing a coordination problem akin to the adoption of a network technology. Optimal subsidies target agents imposing high externalities on others and on whom others impose low externalities. Belhaj and Deroian (2019), focusing on situations where the principal's objective is to maximize aggregate effort on the network, study optimal contracting, thus taking into account participation constraints. In their paper, targeting can be optimal because the principal can exploit outsiders to discipline contracting agents, whereas in the present paper, targeting is optimal solely due to high contracting costs.

Next, our paper complements the recent literature on optimal linear pricing with interdependent consumers (Candogan et al. (2012), Bloch and Quérou (2013)). Under perfect price discrimination, network topology has no effect on optimal unit price when the network is undirected. The intuition is that there is a perfect balance between incentives to charging central agents more as their probability of consuming increases, and incentives to charging them less to exploit their influence power. In Bloch & Quérou, this is confirmed in a model with unit demand and incomplete information about neighbors' decisions. In Candogan et al. (2012), the same logic obtains from a slightly different model. Here, the monopolist sells a divisible good, and sets linear prices. In both contexts, the problem turns out to be one of maximizing a quadratic function of the vector of demand, once the firm allows for the fact that the demand reacts linearly to prices. Our finding that, under sufficiently low contracting costs, the unit return rewarding each agent does not depend on position on the network echoes theirs, but the intuition is different: in our model, while central agents are more influential, they are also budget-consuming, and the two effects compensate at equilibrium. However, under high contracting costs, our framework differs substantially from theirs by the presence of outsiders (those agents not receiving an offer) who exert effort, whereas consumers do not consume without the firm. Outsiders play a key role in our model, generating heterogeneous optimal unit returns even on undirected networks.

This paper also makes a technical contribution to games with interdependent agents. The existing literature has shown the emergence of NP hard problems in such games – in a monopoly setting with two prices by Candogan et al. (2012), and in a sequential game by Zhou and Chen (2015).⁷ Both papers involve a Max-Cut problem, whereas our paper shows the emergence of a distinct NP-hard problem, the densest k -subgraph problem.⁸

The paper is organized as follows. Section 2 presents the model. The case of null contracting cost is studied in Section 3, while contracting cost is incorporated in Section 4. Section 5 concludes. All proofs are relegated to Appendix A, with algorithmic considerations in Appendix B.

2. Model

We consider a three-stage game of complete information between one principal and a finite set of agents. In the first stage, the principal offers bilateral contracts. Each contract is a linear payment scheme. In the second stage, agents simultaneously decide whether to accept or reject their respective offers. In the third stage, agents exert effort and transfers are realized. Effort profile, contracts, and network are assumed to be publicly observable. We study the Subgame Perfect Nash Equilibria of this game.

There is a finite set $\mathcal{N} = \{1, 2, \dots, n\}$ of agents who exert effort and interact on a network. We let $x_i \in \mathbb{R}_+$ represent agent i 's effort, and $\mathbf{x} = (x_i)_{i \in \mathcal{N}}$ an effort profile. We let the $n \times n$ matrix $\mathbf{G} = [g_{ij}]$, with $g_{ij} \in \mathbb{R}_+$, represent the network of interaction between agents, where $g_{ij} > 0$ when agent i is influenced by agent j and $g_{ij} = 0$ otherwise. By convention, $g_{ii} = 0$ for all $i \in \mathcal{N}$. Links can be either binary or weighted. Throughout the paper, we will refer to network \mathbf{G} (implicitly assuming a number of agents equal to n). The network is undirected, i.e. $\mathbf{G}^T = \mathbf{G}$ where superscript T quotes for the transpose operator (we discuss directed networks in Remark 1 in Section 3). Symbol $\mathbf{1}$ represents the n -dimensional vector of ones, \mathbf{I} the n -dimensional identity matrix, d_i the degree of consumer i , $\mathbf{d} = \mathbf{G}\mathbf{1}$ the profile of consumers' degrees. We let $\mu(\mathbf{G})$ represent the largest (real) eigenvalue of network \mathbf{G} . We let \mathbf{J} be the n -square matrix of ones, and we let $\bar{\mathbf{G}}$ be the complementary network of \mathbf{G} , so that $\bar{\mathbf{G}} = \mathbf{J} - \mathbf{I} - \mathbf{G}$ (all active links in \mathbf{G} are inactive in $\bar{\mathbf{G}}$ and vice-versa).

⁷ Algorithm researchers are also often interested in whether it's hard even to approximate a decent solution to a problem. See Li et al. (2022) for a recent paper that considers this.

⁸ This problem is different from the k -core problems, that emerge in minimum effort games (see for instance Harkins (2016)), and in market versus network-based activities (see Gagnon and Goyal (2017)).

In the absence of the principal, agents play an effort game on a social network, exhibiting both positive local externalities and local synergies. We focus on linear quadratic utilities.⁹ The utility that agent i derives from exerting effort x_i on network \mathbf{G} is given by:

$$u(x_i, \mathbf{x}_{-i}) = a_i x_i - \frac{1}{2} x_i^2 + \delta \sum_{j \in \mathcal{N}} g_{ij} x_i x_j \quad (1)$$

where $a_i > 0$ represents the private preference of agent i , and $\mathbf{a} = (a_i)_{i \in \mathcal{N}}$ the profile of private preferences. The last term represents the utility that consumer i derives from neighbors' effort. Parameter $\delta > 0$ implies positive externalities and local complementarities: incentives increase with neighbors' effort.¹⁰

In this game, Nash equilibrium efforts are shaped by Bonacich centralities, which we formally define (see Bonacich (1987)). We let the n -dimensional square matrix $\mathbf{M} = (\mathbf{I} - \delta \mathbf{G})^{-1}$. The condition $\delta \mu(\mathbf{G}) < 1$ assumed throughout the paper guarantees $\mathbf{M} \geq 0$. We let the n -dimensional vector $\mathbf{b} = \mathbf{M} \mathbf{1}$, with entry i called b_i , denote the vector of *Bonacich centralities* of the network weighted by parameter δ (avoiding references to network \mathbf{G} and parameter δ for convenience). Then b_i is the number of walks (in our terminology, walks can contain loops) from agent i to others, where the weight of a walk of length k from agent i to agent j is δ^k . Similarly, we define the vector $\mathbf{b}_a = (b_{a,i})_{i \in \mathcal{N}}$, such that $\mathbf{b}_a = \mathbf{M} \mathbf{a}$ represents the weighted (by \mathbf{a}) Bonacich centrality. We define $b = \mathbf{1}^T \mathbf{b}$ (resp. $b_a = \mathbf{1}^T \mathbf{b}_a$) as the sum of centralities (resp. centralities weighted by \mathbf{a}). The condition $\delta \mu(\mathbf{G}) < 1$ guarantees the existence of a unique Nash equilibrium effort profile in the absence of a principal. In this equilibrium, any agent $i \in \mathcal{N}$ exerts an effort equal to her Bonacich centrality $b_{a,i}$ and obtains a utility level equal to $\frac{1}{2} b_{a,i}^2$.

A principal offers individual payment schemes with the aim of increasing aggregate effort.¹¹ For instance, this objective is natural when effort concerns education, protection, work, R&D, etc. The principal has a fixed budget t to enhance effort. Offers are excess-effort linear contracts, rewarding increased effort with respect to effort level in the absence of the principal's intervention.

Formally, agent i is offered a transfer function $t_i(x_i) = w_i(x_i - b_{a,i})$, with $w_i \in \mathbb{R}$. The variable w_i represents the return per unit of excess-effort; for convenience, w_i is considered a *unit return* throughout the paper.

We assume a costly enforcement environment. These costs can have diverse sources, for instance administrative or enforcement costs. We consider general cost functions $C(\mathcal{S})$, increasing in group size (in set inclusion terms); formally, we impose that for all groups $\mathcal{S}, \mathcal{S}'$ with $\mathcal{S} \subset \mathcal{S}'$, $C(\mathcal{S}) \leq C(\mathcal{S}')$. The supermodularity (vs. submodularity) property of cost functions will play a crucial role in the paper. With supermodular cost functions, an agent's contribution to the cost is non-decreasing with group size. Formally, for all groups \mathcal{S}, \mathcal{T} with $\mathcal{S} \subset \mathcal{T}$, for all $i \notin \mathcal{T}$, $C(\mathcal{T} \cup \{i\}) - C(\mathcal{T}) \geq C(\mathcal{S} \cup \{i\}) - C(\mathcal{S})$. For instance, when contracting cost depends solely on number of contracts $s = |\mathcal{S}|$, then $C(\mathcal{S}) = C(s)$, and the supermodularity assumption boils down to linearity or convexity of the function $C(s)$. This paper will focus on two canonical examples of supermodular cost functions:

Example 1 (linear cost function). Consider a fixed cost per contract $c \geq 0$. Then for a given group \mathcal{S} of size $s = |\mathcal{S}|$,

$$C(\mathcal{S}) = c \cdot s \quad (2)$$

For instance, this constant cost per contract can represent monitoring costs, which increase with the number of contracted agents.

Example 2 (step function). Consider a step function with zero cost up to group size $k < n$, and then prohibitive cost for larger sizes. Formally,

$$\begin{cases} C(\mathcal{S}) = 0 & \text{if } s \leq k \\ C(\mathcal{S}) = +\infty & \text{if } s > k \end{cases} \quad (3)$$

For instance, in many economic situations, there are upper bounds to the number of possible contracts. Such rationing can apply to the number of grants under a CCP program or to the number of R&D subsidies.

By convention, the principal suffers no cost under the default offer $w_i = 0$. Throughout the paper, by abuse of language, we will refer to a "contract" when it is not the default offer. The principal's offers are compiled in an n -dimensional vector $\mathbf{w} = (w_i)_{i \in \mathcal{N}}$, where a zero entry means no offer in practice. Hence, when the principal targets group \mathcal{S} , $i \in \mathcal{S}$ if and only if

⁹ The model can be generalized to any utility function generating the same first-order conditions in effort, and generating positive externalities with respect to neighbors' effort. Indeed, the cardinal amount of externalities between utilities plays no role in the analysis.

¹⁰ This linear-quadratic utility specification applied to network games in economics was introduced by Ballester et al. (2006).

¹¹ The main results of the paper extend to an objective that is a weighted sum of efforts with positive weights. The proof of Theorem 1 is presented in this enlarged setting.

$w_i \neq 0$. We let \mathbf{w}_{-i} represent the set of contracts in \mathcal{S} but setting $w_i = 0$. For convenience, we will abuse the notation and write $\mathbf{w} = (w_i)_{i \in \mathcal{S}}$ (excluding default transfers in this latter notation) when there is no confusion.

In the third stage of the game, we denote by $\mathbf{x}^*(\mathbf{w})$ the equilibrium effort associated with the set of accepted contracts \mathbf{w} . Profile $\mathbf{x}^*(\mathbf{w})$ takes into account both the change in effort of the contracting agent and the induced change in others' effort on the network. The condition $\delta\mu(\mathbf{G}) < 1$ still guarantees a unique Nash equilibrium effort profile for any set of accepted contracts, one which satisfies $\mathbf{x}^*(\mathbf{w}) = \mathbf{M}(\mathbf{a} + \mathbf{w})$. That is, the Nash effort of agent i under any set \mathcal{S} of accepted contracts $\mathbf{w} = (w_j)_{j \in \mathcal{S}}$ is given by

$$\forall i \in \mathcal{N}, x_i^*(\mathbf{w}) = b_{\mathbf{a},i} + \sum_{j \in \mathcal{S}} m_{ij} w_j \quad (4)$$

where m_{ij} is the entry (i, j) of matrix \mathbf{M} . That is, the excess effort of any agent, whether this agent contracts with the principal or not, is given by the sum of returns offered to agents under contract weighted by the number of (weighted) walks to them.

We now turn to the principal's program. The principal maximizes aggregate effort minus contracting costs under the budget constraint and under individual participation constraints.¹² For convenience, let $\mathcal{S}(\mathbf{w}) = \{i \in \mathcal{N}, w_i \neq 0\}$ represent the set of contracts offered by the principal through payment scheme profile \mathbf{w} . We let \mathbf{w}^* denote the optimal excess-effort linear contract, and $\mathbf{x}^* = \mathbf{x}^*(\mathbf{w}^*)$ for convenience when there is no confusion. The optimal contract \mathbf{w}^* solves:

$$\begin{aligned} \max_{\mathbf{w} \in \mathbb{R}^n} \quad & \sum_{i=1}^n x_i^*(\mathbf{w}) - C(\mathcal{S}(\mathbf{w})) \\ \text{s.t.} \quad & \begin{cases} \sum_{i \in \mathcal{S}} w_i (x_i^*(\mathbf{w}) - b_{\mathbf{a},i}) \leq t \\ \forall i \in \mathcal{S}, u(x_i^*(\mathbf{w}), \mathbf{x}_{-i}^*(\mathbf{w})) + w_i (x_i^*(\mathbf{w}) - b_{\mathbf{a},i}) \geq u(x_i^*(\mathbf{w}_{-i}), \mathbf{x}_{-i}^*(\mathbf{w}_{-i})). \end{cases} \end{aligned}$$

The first constraint is the standard budget constraint requiring that the sum of transfers is not larger than the available resource. The set of individual participation constraints means that any agent receiving an offer is better-off accepting the contract when other agents accept. Note that other agents modify their effort level when the agent rejects the offer.

The presence of complementarities entails a potential coordination issue in participation; however, when unit returns are positive, there are no such coordination issues, so the principal does not need explicitly to take account of the participation constraints (when unit returns are positive). The next lemma makes this point clear:

Lemma 1. *For any set of contracts $(w_i)_{i \in \mathcal{S}}$ such that $w_i > 0$ for all $i \in \mathcal{S}$, the configuration where all agents accept their offer is the unique Nash equilibrium of the participation game.*

Lemma 1 rests on a structure of utilities that exhibits complementarities and positive externalities.

3. No contracting costs

In this section, we assume that there are no contracting costs; i.e. $C(\mathcal{S}) = 0$ for all \mathcal{S} . The objective function in the principal's program is here equal to the aggregate effort.

At least two questions arise. First, are optimal returns higher for central agents? On the one hand, the principal may devote more resource to central agents so as to exploit their greater influence power, making savings on the budget allocating to promoting others' effort. On the other hand, central agents themselves are more responsive to unit returns, which may lead to overstretching the budget. Second, is there targeting? I.e., is it optimal for the principal to contract with a strict subset of the society, allocating full budget to those agents who are more productive with respect to the principal's objective? The next proposition gives a clear answer to both questions:

Proposition 1. *Under zero contracting cost, there is a unique equilibrium of the two-stage game, where the principal contracts with all agents. The optimal unit returns, $w_i^* = \sqrt{\frac{t}{b}}$ for all $i \in \mathcal{N}$, are homogeneous across agents and the aggregate effort change is equal to \sqrt{tb} .*

Proposition 1 calls for several remarks. First, all agents receive an offer, i.e. there is no targeting. Second, since optimal returns are positive, by Lemma 1 there is no coordination problem in the participation game. Third, agents' positions do

¹² An alternative formulation would be to include contracting costs in the budget constraint. Results keep unchanged qualitatively in that formulation. Furthermore, note that there is no equivalence between high contracting costs and low budget: as will be clear later on, for all budgets there is a cost threshold above which targeting becomes optimal. Below this threshold, costs are positive but targeting does not occur.

not affect the optimal unit return¹³: network influence is exactly counterbalanced by budget-spending effect.¹⁴ Note that private returns' heterogeneity does not affect the optimal unit returns here (whereas it shapes unit prices under monopoly pricing). Thus, neither transfers nor aggregate effort changes are related to preferences. Fourth, the variation of aggregate effort is equal to $x^* - b_a = \sqrt{tb}$; hence, the networks maximizing the impact of the principal's intervention are also those maximizing the sum of centralities (whatever the size of the budget). An important implication of this finding is that, given two networks such that all the links of the first are included in the second, the impact of the principal's intervention is greater on the denser network.

Remark 1. The assumption that $\mathbf{G}^T = \mathbf{G}$ is crucial to establish Proposition 1. If not, optimal unit returns become heterogeneous and targeting becomes possible.¹⁵

4. Costly contracting

In this section, we incorporate contracting costs, which can be generated for instance by enforcement considerations. The objective function of the principal is then given by the aggregate effort minus the cost of contracting with all agents in the targeted group. Under sufficiently high contracting cost, it is not optimal for the principal to contract with the whole society. Thus, the principal contracts with a strict subgroup of the society. Generally speaking, the principal finds the best target for each given group, and then selects the best group to contract with. Importantly, the separability of contracting costs within the principal's objective means that the cost structure only affects group selection, not the shape of optimal contracts.

We first study optimal contracting when the principal targets a given arbitrary group, establishing notably that optimal unit returns are always positive, and characterizing the performance of the group. Then, we explore group selection and the related NP-hardness issue. Finally we illustrate optimal targeting on specific network structures.

4.1. Optimal contracts with a given targeted group

Under costly contracting, a rational principal should target agents maximizing the performance of each contractor per dollar invested, i.e., the ratio of the variation of aggregate effort induced by contracting with the agent over the amount of resource transferred to this agent (what we will call agent i 's productivity). Centrality has an ambiguous effect on productivity: high centrality has a great influence on others, but it also induces a large transfer to such an agent because that agent is highly incentivized through the contract. Overall, centralities allow the productivity of an agent to be non-monotonic.

Optimal targeting takes this trade-off into account. Let us consider a given group S , of size s . We let the $s \times s$ matrix $\mathbf{M}_{[S]}$ represent the submatrix of \mathbf{M} restricted to agents in S . Let $\mathbf{b}_{[S]} = (b_i)_{i \in S}$ with b_i the un-weighted Bonacich centrality on network \mathbf{G} . Understanding whether optimal unit returns are positive or not is not a trivial task.¹⁶ Indeed, the principal may want to tax an agent and reallocate the tax as a subsidy for another agent, which can happen when links are directed.¹⁷ However, when links are reciprocal, any connected agent exerts an influence on others. The question is whether reallocating a local tax as an extra-subsidy for another agent can still be good for the principal. The next theorem provides a clear-cut answer to this question:

Theorem 1. *When the principal contracts with group S , optimal unit returns are positive for all agents in S . The optimal unit return vector \mathbf{w}^* satisfies*

$$\mathbf{w}^* = \left(\sqrt{\frac{t}{\mathbf{b}_{[S]}^T \mathbf{M}_{[S]}^{-1} \mathbf{b}_{[S]}}} \right) \mathbf{M}_{[S]}^{-1} \mathbf{b}_{[S]}$$

and the optimal objective is given by

$$F(S) = b_a + \sqrt{t \mathbf{b}_{[S]}^T \mathbf{M}_{[S]}^{-1} \mathbf{b}_{[S]}} \quad (5)$$

¹³ This result is easily generalized even further to payment schemes of the sort $t(x_i) = w_i(x_i - \alpha b_{a,i})$ where $\alpha > 0$ (proof omitted).

¹⁴ Still, agent i 's payment can be computed as $t_i^* = \frac{b_i}{b} \cdot t$; i.e., the transfer received is proportional to relative centrality.

¹⁵ For example, in a 3-agent directed star with two links, both originating from peripherals (meaning that the central agent receives all synergies while providing none), it can be optimal to target only the peripheral agents, as illustrated numerically in Example 4 later on. For the general case of directed networks, we define $\mathbf{M} = (\mathbf{I} - \delta \mathbf{G})^{-1}$, $\mathbf{M}^T = (\mathbf{I} - \delta \mathbf{G}^T)^{-1}$, $\bar{\mathbf{M}} = \left(\mathbf{I} - \delta \left(\frac{\mathbf{G}^T + \mathbf{G}}{2} \right) \right)^{-1}$. We obtain $\mathbf{w}^* = \frac{2}{\lambda} \left(\mathbf{1} + \left(\frac{\mathbf{G}^T - \mathbf{G}}{2} \right) \bar{\mathbf{b}} \right)$, where $\bar{\mathbf{b}} = \bar{\mathbf{M}} \mathbf{1}$ represents the Bonacich centrality of the average interaction matrix, and $\lambda = \sqrt{\frac{\|(\mathbf{I} + \mathbf{M}^T \mathbf{M})^{-1} \mathbf{1}\|_{\mathbf{M}}}{t}}$.

¹⁶ Note that the off-diagonal entries of matrix $\mathbf{M}_{[S]}^{-1}$ are non-positive. Indeed, any principal submatrix of an inverse M-matrix is an inverse M-matrix (see Corollary 3 in Johnson (1982, p. 202)). Therefore, matrix $\mathbf{M}_{[S]}^{-1}$ is an M-matrix.

¹⁷ In Example 4 presented below, the taxed agent exerts no influence on others, meaning that taxing that agent entails no indirect effect on others' effort. Since other agents are influential, the principal finds it better off using the tax as an extra-subsidy for an agent in order to benefit from induced (free of charge) complementarities.

Intuitively, the reciprocity of the bilateral interaction means that the negative influence of a taxed agent always outweighs the gain in positive influence of the subsidized agent (this is because the influence of the subsidized agent uses paths initiated by the taxed agent). To prove positiveness of optimal unit returns, we express them in a way that allows us to isolate the contribution of each outsider. Then, using inverse M-matrix theory,¹⁸ we can show the positiveness of all outsiders' contributions, and thus of optimal unit returns.

Positiveness of optimal unit returns implies that, departing from any cost consideration, the optimal (gross) objective of the principal is increasing in group size, and the second-stage participation game does not face any coordination problem (using Lemma 1). That is, there is no strict subgroup of the targeted set that constitutes a proper equilibrium of the participation game; any non-member of that subgroup is better-off accepting the offer.¹⁹

Outsiders generate asymmetric interaction between members of the targeted group. This asymmetry generates heterogeneous but still positive unit returns.

By Theorem 1, optimal returns are generically heterogeneous inside the targeted group. In our model, heterogeneity arises when the members of the targeted group have distinct numbers of connections with outsiders, which induces asymmetric interactions inside the group. To see this formally, we write the system of interaction between members of the targeted group as a function of their interaction with outsiders as follows. Let \mathbf{I}_q be the identity matrix of size q , $\mathbf{G}_{[\mathcal{S}]}$ the submatrix of \mathbf{G} restricted to agents in \mathcal{S} , and $\tilde{\mathbf{G}}$ the $s \times n - s$ submatrix where g_{ij} represents the link between agent $i \in \mathcal{S}$ and agent $j \in \mathcal{N} \setminus \mathcal{S}$:

$$(\mathbf{I}_s - \delta \mathbf{G}_{[\mathcal{S}]}) \mathbf{x}_{[\mathcal{S}]} = \mathbf{1} + \mathbf{w} + \underbrace{\delta \tilde{\mathbf{G}} (\mathbf{I}_{n-s} + \delta \mathbf{G}_{[\mathcal{N} \setminus \mathcal{S}]} + \delta \tilde{\mathbf{G}}^T \mathbf{x}_{[\mathcal{S}]})}_{=\mathbf{x}_{[\mathcal{N} \setminus \mathcal{S}]}}$$

That is, the system of bilateral interactions among members of set \mathcal{S} is thus given by $(\mathbf{I}_s - \delta \mathbf{H}_{[\mathcal{S}]}) \mathbf{x}_{[\mathcal{S}]} = \mathbf{k}$, where (\mathbf{k} is a heterogeneous constant, and) matrix $\mathbf{H} = \mathbf{G}_{[\mathcal{S}]} + \delta \tilde{\mathbf{G}} \tilde{\mathbf{G}}^T$. We observe that $\mathbf{H}^T = \mathbf{H}$, but now the diagonal is generically (not zero and) heterogeneous; Letting $d_i^{\mathcal{N} \setminus \mathcal{S}} = |\{j \in \mathcal{N} \setminus \mathcal{S}, g_{ij} = 1\}|$ represent the number of neighbors of agent i among outsiders, we get $h_{ii} = \delta d_i^{\mathcal{N} \setminus \mathcal{S}}$. This heterogeneity of the diagonal entries of the matrix of interaction creates asymmetric interaction (the sensitivity of agent i to agent j 's move is not equal to the reciprocal sensitivity, even if $h_{ij} = h_{ji}$). Note that if all members have the same number of neighbors among outsiders (i.e. if $d_i^{\mathcal{N} \setminus \mathcal{S}} = d_j^{\mathcal{N} \setminus \mathcal{S}}$ for all $i, j \in \mathcal{S}$), then optimal returns are homogeneous in the targeted group, irrespective of the organization of those links.

The result on positiveness of optimal unit returns extends to any principal's objective being a weighted sum of efforts with positive weights; Formally, it holds for any objective $F = \sum_{i \in \mathcal{N}} e_i x_i$, where the exogenous weight $e_i > 0$ for all i (see the proof of Theorem 1).²⁰ For instance, a firm may be more interested in fostering effort in a given subgroup, a teacher may focus more on certain children (because of their bad results, poor health, etc).

Remark 2 (Heterogeneity). The heterogeneity of private returns does not affect the aggregate effort variation, $\sum_{i \in \mathcal{N}} x_i^* - b_a$, that is induced by the principal's intervention.

We now consider the limitations of Theorem 1. Firstly, the very nature of the objective function matters. While Theorem 1 extends to a weighted sum of efforts, it does not extend to a principal maximizing the sum of agents' utilities, as shown in the following numerical five-agent example:

Example 3. Assume that the principal targets a given group $\mathcal{S} = \{1, 2, \dots, s\} \subset \mathcal{N}$ (choosing these labels without loss of generality). Recall that equilibrium utilities, for given unit returns $\mathbf{w} = (w_1, \dots, w_s)$, are given by $u_i^* = \frac{1}{2}(x_i^*)^2 - w_i b_{a,i}$ for all $i \in \mathcal{S}$ and $u_i^* = \frac{1}{2}(x_i^*)^2$ for all $i \in \mathcal{N} \setminus \mathcal{S}$; and for all $i \in \mathcal{N}$, $x_i^* = \mathbf{b}_{A,i} + \sum_{j \in \mathcal{S}} m_{ij} w_j$. The objective function F is:

¹⁸ Every outsider's contribution turns out to be proportional to an almost principal minor of matrix \mathbf{M} . We show that all almost principal minors associated with outsiders have the same sign, which depends on whether the cardinal of the targeted group is even or odd. To prove this, we use a result that relates the sign of an almost principal minor to the labels of the switching columns (i.e. those columns whose switch makes a principal minor become almost principal).

¹⁹ The positiveness of optimal unit returns implies that individual participation constraints play no role, and the effective principal's program boils down to the program maximizing the principal's objective under the sole (binding) budget constraint:

$$\max_{\substack{\mathbf{w} \in \mathbb{R}^n \\ \sum_{i \in \mathcal{S}} w_i (x_i^*(\mathbf{w}) - b_{a,i}) = t}} \sum_{i=1}^n x_i^*(\mathbf{w}) - C(\mathcal{S}(\mathbf{w})).$$

²⁰ Moreover, allowing for null weights in the objective, for a targeted group \mathcal{S} and $i \in \mathcal{S}$, we can show that $w_i^* = 0$ iff $e_i = e_k = 0$ for all $k \in \mathcal{N} \setminus \mathcal{S}$, otherwise $w_i^* > 0$.

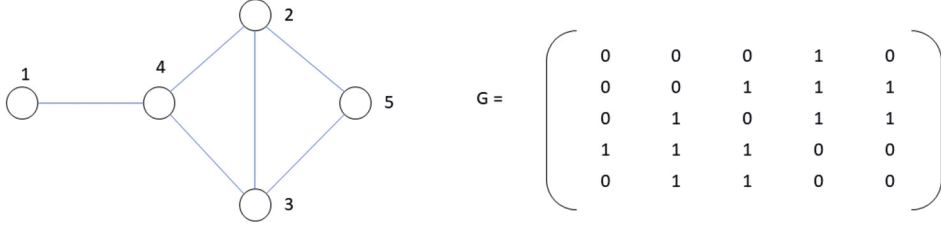


Fig. 1. A five-agent network and its adjacency matrix.

$$F = \frac{1}{2} \sum_{i \in \mathcal{N}} \left(b_{A,i} + \sum_{j \in \mathcal{S}} m_{ij} w_j \right)^2 - \sum_{j \in \mathcal{S}} w_j b_{A,j}$$

I.e.,

$$F = \frac{1}{2} \sum_{i \in \mathcal{N}} b_{A,i}^2 + \sum_{j \in \mathcal{S}} w_j \left(\sum_{i \in \mathcal{N}} b_{A,i} m_{ij} - b_{A,j} \right) + \frac{1}{2} \sum_{i \in \mathcal{N}} \left(\sum_{j \in \mathcal{S}} m_{ij} w_j \right)^2$$

Define, for all subgroups \mathcal{S} , all n -dimensional vector \mathbf{z} , $\mathbf{z}_{[\mathcal{S}]} = (\mathbf{z}_i)_{i \in \mathcal{S}}$ as the vector restricted to agents in \mathcal{S} . Denote the n -dimensional vector $\mathbf{q} = \mathbf{M}^2 \mathbf{a} - \mathbf{M} \mathbf{a} = (\mathbf{M} - \mathbf{I}) \mathbf{M} \mathbf{a} = \delta \mathbf{G} \mathbf{M}^2 \mathbf{a}$. Also, denote by \mathbf{M}_{ns} the matrix of size $n \times s$ that contains the first columns of matrix \mathbf{M} , and call the $s \times s$ matrix $\mathbf{R} = \mathbf{M}_{ns}^T \mathbf{M}_{ns}$. Then,

$$F = \frac{1}{2} \mathbf{a}^T \mathbf{M}^2 \mathbf{a} + (\mathbf{q}_{[\mathcal{S}]})^T \mathbf{w} + \frac{1}{2} \mathbf{w}^T \mathbf{R} \mathbf{w}$$

The optimal weights solve:

$$\begin{aligned} \max_{\mathbf{w} \in \mathbb{R}^s} \quad & (\mathbf{q}_{[\mathcal{S}]})^T \mathbf{w} + \frac{1}{2} \mathbf{w}^T \mathbf{R} \mathbf{w} \\ \text{s.t.} \quad & \mathbf{w}^T \mathbf{M}_{[\mathcal{S}]} \mathbf{w} = t \end{aligned}$$

provided weights are non-negative. The Lagrangian with weight $\lambda > 0$ is given by:

$$\mathcal{L} = (\mathbf{q}_{[\mathcal{S}]})^T \mathbf{w} - \frac{1}{2} \mathbf{w}^T \left(2\lambda \mathbf{M}_{[\mathcal{S}]} - \mathbf{R} \right) \mathbf{w} + \lambda t$$

As long as the matrix $2\lambda \mathbf{M}_{[\mathcal{S}]} - \mathbf{R}$ is a P-matrix (i.e. all eigenvalues are strictly positive), the Hessian matrix of the Lagrangian is negative semi-definite, so first-order conditions are necessary and sufficient. The optimal positive weights are then given by

$$\mathbf{w}^* = (2\lambda^* \mathbf{M}_{[\mathcal{S}]} - \mathbf{R})^{-1} \mathbf{q}_{[\mathcal{S}]}$$

where λ^* satisfies:

$$(\mathbf{q}_{[\mathcal{S}]})^T (2\lambda^* \mathbf{M}_{[\mathcal{S}]} - \mathbf{R})^{-1} \mathbf{M}_{[\mathcal{S}]} (2\lambda^* \mathbf{M}_{[\mathcal{S}]} - \mathbf{R})^{-1} \mathbf{q}_{[\mathcal{S}]} = t$$

provided that all eigenvalues of the matrix $2\lambda^* \mathbf{M}_{[\mathcal{S}]} - \mathbf{R}$ are strictly positive.

We provide a numerical example in which not all optimal weights are positive, meaning that Theorem 1 is violated here. Take $n = 5$, $\mathbf{a} = \mathbf{1}$, $\delta = 0.2$, $t = 100$, and consider the network depicted in Fig. 1: When the targeted group is $\mathcal{S} = \{2, 4, 5\}$, the optimal weight vector is around $\mathbf{w}^* \simeq (-0.28, 1.44, 1.39)$, and the value of the optimal objective is about 146.23. Hence, targeting $\mathcal{S} = \{2, 4, 5\}$, it is best to contract with subgroup $\{4, 5\}$ and offer $\mathbf{w}^* = (0, 2.76, 8.67)$, leading to an optimal objective around 102.13 (which is better than selecting $\{2, 4\}$ or $\{2, 5\}$, both leading to an objective around 98).

Secondly, breaking link reciprocity in our model would generate negative unit returns.²¹ This means that undirectedness of interaction is key to guaranteeing positive weights. To illustrate, we consider the following example:

Example 4. Consider the three-player *undirected* network presented in Fig. 2-Right, which represents a three-star undirected network in which agent 2 is the central agent. When $\mathcal{S} = \mathcal{N}$, there is no outsider and optimal unit returns are homogeneous across agents. When $\mathcal{S} = \{1, 2\}$, however, agent 3 is an outsider and we find $w_2^* > w_1^* > 0$ for all values of δ .²² Now consider

²¹ In Candogan et al. (2012)'s setting, link directedness generates heterogeneous pricing, possibly below marginal cost for some consumers, which is the equivalent in their setting to having negative unit returns in our model – see Example 1 in Candogan et al. (2012).

²² E.g., for $\delta = 0.7$, we find $w_1^* = 1.7$, $w_2^* = 1$.



Fig. 2. Left: In the 3-agent *directed* network, targeting the whole society can entail a negative unit return for agent 3, leading the principal to target agents 1 and 2 alone. Right: In the 3-agent *undirected* network, targeting agents 1 and 2 leads to $w_2^* > w_1^* > 0$ for all values of δ .

the *directed* network depicted in Fig. 2-Left, where agent 3 receives synergies from other agents but does not deliver any synergy to others. The optimal unit returns associated with any group \mathcal{S} satisfy $\mathbf{w}^* = \frac{1}{2\lambda^*} (\mathbf{M}_{[\mathcal{S}]} + \mathbf{M}_{[\mathcal{S}]}^T)^{-1} \mathbf{b}_{[\mathcal{S}]}$, provided that transfers are nonnegative (an agent receiving a negative transfer is never better off accepting the contract). For $\mathcal{S} = \mathcal{N}$ and sufficiently high intensities of interaction, the above formula provides a negative unit return to agent 3 and a negative transfer,²³ and it is optimal to target agents 1 and 2 alone.

Remark 3 (*Optimal scheme when the planner uses a uniform excess-effort transfer scheme*). If the principal cannot discriminate among agents, every agent in the targeted group receives the same positive unit return (again, there is no coordination issue in the participation game). The optimal target then maximizes a specific group-index. Letting $\mathbf{1}_s$ denote the vector of ones of size s , and index $I(\mathcal{S}) = \frac{1}{\mathbf{1}_s^T \mathbf{M}_{[\mathcal{S}]} \mathbf{1}_s} (\mathbf{1}_s^T \mathbf{b}_{[\mathcal{S}]})^2$, the group performance (gross of contracting cost) is proportional to index $I(\mathcal{S})$.

4.2. Group selection

In this subsection, we establish that the shape of the contracting cost function is a key determinant of the level of complexity of the principal's problem. The potential complexity of the group selection problem can generate an NP-hard problem. In particular, we find a precise correspondence with the densest k-subgraph problem, a known NP-hard problem - the crux of the NP-hardness issue is observed in the subproblem involving the ranking of groups of fixed size. Thus, we turn to algorithmic considerations.

We start with the case where the intensity of interaction tends to zero, requiring the following notation. For any group $\mathcal{S} \subset \mathcal{N}$ and any agent $i \in \mathcal{S}$, we let $d_i^{\mathcal{S}}$ (resp. $d_i^{\mathcal{N} \setminus \mathcal{S}}$) represent the number of linked agents in set \mathcal{S} (resp. in the set $\mathcal{N} \setminus \mathcal{S}$). Then $d_i = d_i^{\mathcal{S}} + d_i^{\mathcal{N} \setminus \mathcal{S}}$. We let $L_{\mathcal{S}}$ represent the number of links among agents in \mathcal{S} , and we let $L_{\mathcal{S}, \mathcal{N} \setminus \mathcal{S}}$ be the number of links between group \mathcal{S} and group $\mathcal{N} \setminus \mathcal{S}$ (i.e. the number of links between agents of distinct groups). Then $\sum_{i \in \mathcal{S}} d_i^{\mathcal{S}} = 2L_{\mathcal{S}}$ and $\sum_{i \in \mathcal{S}} d_i^{\mathcal{N} \setminus \mathcal{S}} = L_{\mathcal{S}, \mathcal{N} \setminus \mathcal{S}}$. Then,

Proposition 2. *When the intensity of interaction tends to zero, the principal's optimal target is given by²⁴*

$$\max_{\mathcal{S}} F(\mathcal{S}) = \sqrt{t(s + 2\delta(L_{\mathcal{S}} + L_{\mathcal{S}, \mathcal{N} \setminus \mathcal{S}})) - C(\mathcal{S})} \quad (6)$$

When parameter δ tends to zero, larger sizes can be considered where contracting costs are low, whereas high contracting costs call for targeting. Moreover, for a given size s , the best group of given size s maximizes the sum of internal links plus the sum of cross-links. In a word, the effort change of agents in $\mathcal{N} \setminus \mathcal{S}$ comes from cross links, and the effort change of agents in \mathcal{S} comes from both internal links and cross links because when agent i receives a reward, increased effort takes into account his synergies with agents in $\mathcal{N} \setminus \mathcal{S}$. For instance, on the circle, the best group of size s is $\mathcal{S} = \{1, 3, 5, \dots, 2s+1\}$; this is because this group maximizes the number of cross links, and by grouping agents we lose cross links without doubling the number of internal links. On the star, the best group of given size s contains the central agent.

The shape of the contracting cost function plays a crucial role in determining how to target, and can lead to an NP-hardness issue. A striking example is given by the case where the cost function is a step function:

Theorem 2. *Assume that the cost function is described by system (3). Under sufficiently low intensity of interaction, the optimal group-selection problem is NP-hard.*

²³ E.g., for $\delta = 0.7$, we find $w_1^* = w_2^* \simeq 0.89$, $w_3^* \simeq -0.12$; agent 3's transfer is also around -0.12 .

²⁴ We disregard the term b_a in the maximization problem since this does not depend on the target.

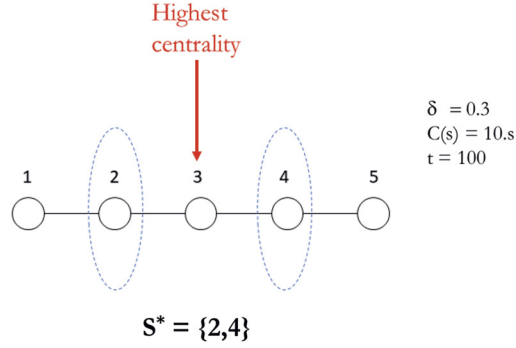


Fig. 3. Optimal targeting on the five-agent line network (here $\mathbf{a} = \mathbf{1}$); the agent with highest centrality is not selected.

Table 1

Optimal groups as a function of contracting cost for $\delta = 0.2$. For all costs, there is no optimal group of size 2.

c	$[0, 1.6[$	$[1.6, 1.7[$	$[1.7, 1.8[$	$[1.8, 1.9[$	$[1.9, 3.9[$	$[3.9, 4[$	$[4, +\infty[$
S^*	\mathcal{N}	$\{1, 2, 3, 4, 5, 6\}$	$\{1, 2, 3, 4, 5\}$	$\{1, 2, 3, 4\}$	$\{2, 3, 4\}$	$\{1\}$	\emptyset

Intuitively, absent any contracting cost, the principal is always better off enlarging the targeted set by Theorem 1. Given the structure of the cost function, this directly implies that the optimal group size is equal to k (it cannot be of smaller size, and any target of larger size would entail prohibitive contracting costs). That is, the problem amounts to ranking groups of same size k in terms of number of internal links plus number of cross-links. Theorem 2 then shows that this problem reduces to the densest $(n - k)$ -subgraph problem associated with the complementary network $\bar{\mathbf{G}}$, which is a well-known NP-hard problem.^{25,26}

We now describe qualitatively how network structure impacts optimal contracts, for all intensities of interaction.

Not contracting with central agents. Centrality has an ambiguous effect on optimal targeting. On the one hand, agents with large centralities have a huge influence on others, i.e. centrality has a social multiplier effect allowing the principal to save on budget; on the other hand, central agents themselves are resource consuming: first by responding to their own contract, and second by responding more to the others' contracts. This can lead the principal to allocate contracts to agents located at a large distance from each other. To illustrate, Fig. 3 shows on a five-agent line network that this trade off between centrality and budget effect can lead to an optimal target that excludes the most central agent in the context of a linear cost function as in (2).

Optimal scheme when the targeted group is a singleton. Define the intercentrality index of agent i in network \mathbf{G} as $c_i = \frac{b_i^2}{m_{ii}}$ (this is an immediate implication of Remark 3). When the principal targets only one agent, *the optimal target maximizes the intercentrality index*. The intercentrality index is familiar from key-player analysis (Ballester et al. (2006)). We see here that linear contracting has the same qualitative effect on optimal targeting as dropping an agent. In particular, by Theorem 1, the optimal return is given by $w_i^* = \frac{b_i}{m_{ii}}$. This ratio represents the aggregate effort change following the increase of a one-unit increase in agent i 's effort (see Belhaj and Deroian (2018)).

Some group sizes are never optimal for any cost value. Increasing the cost function induces a clear restriction on the optimal number of contracts all else being equal. However, as cost increases, the reduction in size may be more than one unit. To illustrate, we consider the following example:

Example 5. We consider a linear cost function as in (2). We set $\mathbf{a} = \mathbf{1}$, and we consider the network depicted in Fig. 4. In this network, there are three classes of agents: the central agent (labeled agent 1), three intermediaries (labeled 2, 3, 4), and three peripheral agents (labeled 5, 6, 7). Table 1 presents the optimal groups as a function of contracting cost (there may be multiple optimal groups due to symmetric positions; for convenience we only label one representative optimal group when multiplicity arises). In this example, there is no optimal group of size 2. This is due to the composition effect arising when optimal group size changes, which puts a premium on larger groups. Here, at threshold cost where the performances of the respective groups $\{1\}$ and $\{1, 2\}$ are the same, the size-3 group $\{2, 3, 4\}$ has a higher value, and this group is therefore

²⁵ See Feige et al. (2001); see also Faragó and Mojaveri (2019) for a recent survey on densest subgraph problems.

²⁶ It can be shown that the objective of the principal, gross of contracting costs, is submodular (proof available upon request). Then, under supermodular contracting costs, the objective is submodular which raises NP-hardness as a potential issue.

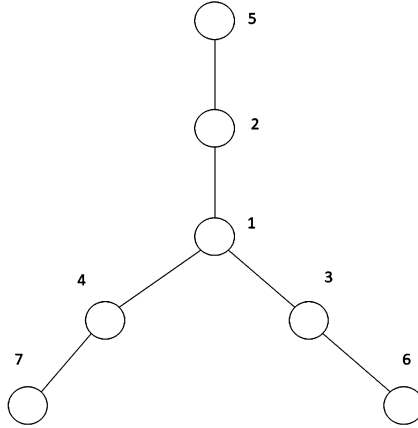


Fig. 4. A 7-agent network.

Table 2

$n = 16$; Targeting through the greedy algorithm: Average over 1000 random networks for relative error of approximation as a percentage of optimal performance.

δ	$c = 0.1$	$c = 0.2$	$c = 0.3$	$c = 0.4$	$c = 0.5$	$c = 0.6$
$\delta = 0.01$	100	99.85	99.94	99.97	100	100
$\delta = 0.06$	99.97	99.68	99.78	99.65	99.84	99.82

optimal.²⁷ This prevents any group of size 2 from being optimal for all costs. This *jump* in group size results from the following composition effect: the best group of size 2 is $\{1, 2\}$, whereas the best group of size 3 is $\{2, 3, 4\}$.²⁸

Algorithmic considerations. Since the program of the principal is possibly NP-hard under low intensity of interaction, it would be useful to know whether algorithms generate good approximations of the solution. By submodularity of the objective function, the greedy algorithm performs rather well for problems with fixed group size, the theoretical lower bound to efficiency being $100 \cdot (1 - 1/e) \sim 63$ percent of the optimal objective. However, for the unconstrained problem with endogenous contracting cost, the best algorithms give a lower bound of 50 percent inefficiency. We performed the greedy algorithm on Erdős-Renyi random networks for various linear contracting cost functions where the cost of each contract is a constant c (see more details in Appendix B). The greedy algorithm works as follows. In step 1, it determines the best singleton; in step 2 it determines the best pair containing the best singleton and stops if the singleton performs better than the pair; in step 3 it determines the best triplet containing the best pair and stops if the pair performs better than the triplet; etc, until the best group is reached. This algorithm therefore converges in no more than $n!$ steps. We found that the greedy algorithm performs very well for $n \leq 30$.²⁹ To illustrate, the average relative error of approximation is less than 1 percent in the simulations presented in Table 2 for $n = 16$, $t = 1$ and various values of parameters δ and c .

We also tested network characteristics. There was a positive correlation between the global clustering coefficient and the average performance of the greedy algorithm, and a negative correlation between the Pearson degree assortativity coefficient and the average performance. The matlab programs are available upon request.

4.3. Specific network structures

In this subsection, we illustrate the impact of network structure on optimal targeting, examining two polar network structures in terms of distribution of centralities: the star network and regular networks.

The star network. In the star network, the trade off between centrality and budget constraint effects is extreme. The following corollary shows that the centrality effect systematically dominates the budget effect:

Corollary 1. *In the star network, the optimal target always contains the central agent.*

²⁷ Group $\{2, 3, 4\}$ is optimal below the threshold cost where both groups $\{1\}$ and $\{2, 3, 4\}$ have the same value, and above the threshold cost that equates its performance with that of group $\{1, 2, 3, 4\}$.

²⁸ The objective function being submodular, the marginal individual contribution to group value is decreasing with group size. This implies that group size is monotonic with cost. Therefore, this result necessarily stems from the composition effect. Furthermore, a qualitative difference arises from applying the greedy algorithm that will be presented in the next section. Given that this algorithm proceeds by group size, and in contrast with this example, the (approximated) optimal group size found by the greedy algorithm is reduced unit by unit as the unit contracting cost continuously increases.

²⁹ The performance of the algorithms should be lower for larger network size.

Table 3

Optimal group size for the star network for $n = 10$, $t = 1$, $c = 0.1$ and for various values of δ .

δ	0.09	0.12	0.15	0.18	0.21	0.24	0.27	0.3
s^*	10	10	10	10	10	5	1	1

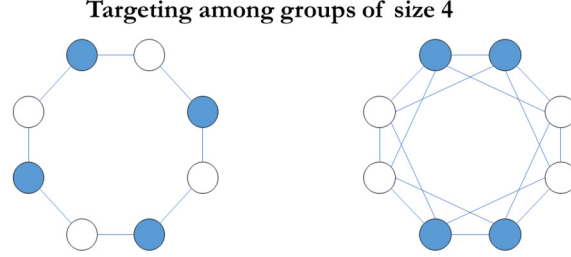


Fig. 5. Targeting on regular networks. Blue nodes are the optimal targets among all groups of size 4. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

This result is not trivial. For instance, when the principal contracts with a single agent, it is the agent with maximum intercentrality index $\frac{b_i^2}{m_{ii}}$, as shown before. Interestingly, the optimal unit return $w_i^* = \frac{b_i}{m_{ii}}$ is favorable to peripherals on the star network (of at least four players) for high intensities of interaction. Yet, by Corollary 1 the balance between this effect and centrality favors centrality, i.e. $\frac{b_i}{m_{ii}} \times b_i$ is always favorable to the central agent. Table 3 presents optimal group sizes on the star network for various parameter values (we set $\mathbf{a} = \mathbf{1}$ to isolate network effect). The optimal group size is decreasing in δ .

Regular networks. We only provide a partial insight - even on regular networks, the analysis is difficult. For simplicity, we focus on the sub-problem of targeting among groups of given fixed size s . In regular networks, all agents have the same degrees and the same Bonacich centralities. Therefore, when the principal contracts with group \mathcal{S} on a regular network, the optimal objective (absent contracting cost) is driven by function $F_0(\mathcal{S}) = \mathbf{1}_{\mathcal{S}}^T \mathbf{M}_{[\mathcal{S}]}^{-1} \mathbf{1}_{\mathcal{S}}$. For two groups $\mathcal{S}, \mathcal{S}'$ of equal size, we need to compare $F_0(\mathcal{S})$ with $F_0(\mathcal{S}')$. For that purpose, the following definition is useful:

Definition 1. The intensity of interaction δ is *moderate* whenever, for every group \mathcal{S} , $\mathbf{V} = \mathbf{M}_{[\mathcal{S}]}^{-1}$ is diagonal dominant, i.e. $v_{ii} \geq \sum_{j \neq i} |v_{ij}|$.

Moderate intensity of interaction guarantees the diagonal dominance of the inverse of any principal submatrix $\mathbf{M}_{[\mathcal{S}]}$. By contrast, under very high intensity of interaction, this property may not hold.³⁰ Under moderate intensity of interaction, the respective performances of same-sized groups can be unambiguously compared when their associated M-matrices are themselves unambiguously ranked such that one matrix dominates the other:

Lemma 2. Assume moderate intensity of interaction and consider two groups $\mathcal{S}, \mathcal{S}'$ of equal size. If $\mathbf{M}_{[\mathcal{S}]} < \mathbf{M}_{[\mathcal{S}]}$, then $F_0(\mathcal{S}) > F_0(\mathcal{S}')$.

Let us call entry m_{ii} of matrix \mathbf{M} agent i 's self-loop centrality. While self-loop centralities are homogeneous across agents on the wheel network, this does not apply to all regular networks.³¹ By Lemma 2, and given that entry m_{ij} represents a (weighted) number of walks between agents i and j , under moderate intensity of interaction, optimal targeting among groups of size two is easily identified on regular networks with homogeneous self-loop centralities. Indeed, for $s = 2$, $\mathbf{M}_{[\mathcal{S}]}^{-1}$ is diagonal dominant and thus Lemma 2 holds. Hence, *in a regular network with homogeneous self-loop centralities, a principal establishing two contracts should always maximize the distance between the two contracting agents, for all intensities of interaction.* Is there a maximum separation principle beyond the case where $s = 2$? Fig. 5 illustrates that this is not always true. In this example with $s = 4$, it is optimal to select the configuration in which no pair of agents are connected. However, in the 16-link regular network presented in the same figure, it is optimal to constitute two connected pairs and to separate the pairs.

³⁰ Incidentally, in the pool of inverse M-matrices, ultrametric matrices are such that their inverse matrix is diagonal dominant (see Martinez et al. (1994)).

³¹ Self-loop centralities are homogeneous across agents in the class of distance-based regular graphs, which includes for instance the wheel and strongly regular graphs.

5. Conclusion

In many economic situations, organizations trade agents' effort against transfers where synergies exist not only among contracting agents, but also among contracting and non-contracting agents. This paper has explored a salient aspect of such situations: the high administration and enforcement costs that often accompany contracting. We built a principal-agent model where the principal has a limited budget and offers costly linear bilateral payment schemes in order to increase the sum of agents' effort. The optimal targeting problem involves a trade-off between centrality and budget spending – while central agents are influential, are also more budget-consuming. Our analysis provides two main insights. First, whatever the shape of contracting costs, the optimal unit returns offered to every targeted agent are positive and generically heterogeneous. This heterogeneity is due to the presence of outsiders, who create asymmetric interaction between contracting agents. Proving positiveness of unit returns was the main analytical challenge of the paper. Positiveness prevents any coordination failure of the participation game. Second, when contracting costs are sufficiently high and typically convex (or supermodular in mathematical language), optimal group selection can lead to NP-hard problems.

Future research on this topic could take several directions. Firstly, more sophisticated contracts could be considered. For instance, contracts offered sequentially based on a divide and conquer strategy would also avoid coordination issues by construction and perform better from the principal's perspective. However, under a fixed budget, determining an optimal divide-and-conquer strategy is challenging. Secondly, it might be interesting to further analyze the problem of a principal maximizing the sum of agents' utilities, going beyond this paper's initial exploration. Finally, policy mechanisms open to the principal as alternatives to transfers would be worth exploring.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A. Proofs

Proof of Lemma 1. Under positive unit returns, all agents increase their effort level (i.e., both contracting agents and outsiders) by complementarities and positive externalities. This entails an increase in the utilities of contracting agents, and adding a positive transfer further increases the utility derived from acceptance. Thus, not only are individual participation constraints satisfied at the optimum, but there is also no strict subset, say \mathcal{T} , of the targeted group, say \mathcal{S} , that constitutes another equilibrium of the participation game, because every agent in $\mathcal{S} \setminus \mathcal{T}$ would then be better-off accepting the offer (indeed, the above argument showing that the all-accept configuration is Nash still holds for any subset of positive unit returns). \square

Proof of Proposition 1. Agent i 's best-response satisfies $x_i^{BR}(y_i) = a_i + w_i + \delta \sum_j g_{ij} x_j$, which entails the Nash profile $\mathbf{x}^*(\mathbf{w}) = \mathbf{M}(\mathbf{a} + \mathbf{w})$. Thus, the sum of excess-effort at equilibrium is given by $\mathbf{1}^T \mathbf{x}^*(\mathbf{w}) - \mathbf{b}_a^T = \mathbf{b}^T \mathbf{w}$, which does not depend on the vector of preferences. Moreover, the budget constraint is not affected by preferences. Indeed, it is given by $t = \mathbf{w}^T (\mathbf{x}^*(\mathbf{w}) - \mathbf{b}_a)$ and, by $\mathbf{x}^*(\mathbf{w}) - \mathbf{M}\mathbf{a} = \mathbf{M}\mathbf{w}$, we get $\mathbf{w}^T \mathbf{M}\mathbf{w} = t$. In the end, the principal's problem is independent of the vector of private preferences \mathbf{a} , and the principal wants to maximize the quantity $\mathbf{b}^T \mathbf{w}$ over $\mathbf{w} \in \mathbb{R}^n$ under the budget constraint $\mathbf{w}^T \mathbf{M}\mathbf{w} = t$.

The Lagrangian \mathcal{L} is written (we ignore non-negativity constraints $w_i \geq 0$ and check that they are satisfied ex post):

$$\mathcal{L}(\mathbf{w}, \lambda) = \mathbf{b}^T \mathbf{w} + \lambda(t - \mathbf{w}^T \mathbf{M}\mathbf{w})$$

Applying the first order conditions w.r.t. the unit return w_i for all $i \in N$, we obtain $\frac{\mathbf{b}}{\lambda^*} = 2\mathbf{M}\mathbf{w}^*$; that is, given that $\mathbf{b} = \mathbf{M}\mathbf{1}$, $\mathbf{w}^* = \frac{1}{2\lambda^*} \mathbf{1}$. Since $\mathbf{w}^{*T} \mathbf{M}\mathbf{w}^* = t$, and recalling that $\mathbf{1}^T \mathbf{M}\mathbf{1} = b$, we derive $\lambda^* = \frac{1}{2} \sqrt{\frac{b}{t}}$. The vector of optimal unit returns is given by $\mathbf{w}^* = \sqrt{\frac{t}{b}} \cdot \mathbf{1}$. Furthermore, the transfer to agent i is given by $t_i^* = w_i^* \cdot \sum_{j \in \mathcal{N}} m_{ij} w_j^*$. Incorporating the optimal returns, we get $t_i^* = \frac{b_i}{b} \cdot t$. Last, considering aggregate excess-effort, we get $x^* - \mathbf{b}_a = \sum_{i \in \mathcal{N}} w_i^* b_i$. Incorporating the optimal unit returns, we obtain $x^* - \mathbf{b}_a = \sqrt{tb}$. \square

Proof of Theorem 1. We first characterize the optimal returns assuming that they are positive. Second, we show positiveness.

We start by characterizing the optimal returns. Let $\mathbf{w} = (w_i)_{i \in \mathcal{S}}$ be a set of contracts offered to a given group $\mathcal{S} = \{1, 2, \dots, s\}$. We determine the optimal contracts, assuming that $w_i > 0$ for all $i \in \mathcal{S}$. For convenience, we introduce the n -dimensional vector $\bar{\mathbf{w}} = (w_1, w_2, \dots, w_s, 0, \dots, 0)^T$. Let $\mathbf{x}^* = (x_i^*)_{i \in \mathcal{N}}$ be the Nash effort profile, conditional on all offers proposed to agents in set \mathcal{S} being accepted. The Nash equilibrium effort profile satisfies

$$\mathbf{x}^* - \mathbf{M}\mathbf{a} = \mathbf{M}\bar{\mathbf{w}} \quad (7)$$

We deduce the change in aggregate effort induced by the introduction of the contracts (with the notation $b_{\mathbf{a}} = \mathbf{1}^T \mathbf{M}\mathbf{a}$, $\mathbf{b} = \mathbf{M}\mathbf{1}$, $x^* = \mathbf{1}^T \mathbf{x}^*$), which measures the performance of group \mathcal{S} : $x^* - b_{\mathbf{a}} = \mathbf{b}^T \bar{\mathbf{w}}$. Plugging agents equilibrium effort as given by equation (7) into the budget constraint, we get

$$t = \mathbf{w}^T \mathbf{M}_{[\mathcal{S}]} \mathbf{w} \quad (8)$$

The Lagrangian of the program is well defined (linear objective under convex constraint) and given by

$$\mathcal{L} = \mathbf{b}_{[\mathcal{S}]}^T \mathbf{w} + \lambda \left(t - \mathbf{w}^T \mathbf{M}_{[\mathcal{S}]} \mathbf{w} \right)$$

Exploiting the first-order conditions of the Lagrangian with respect to returns, the optimal returns $(w_i^*)_{i \in \mathcal{S}}$ solve the following linear system:

$$\mathbf{w}^* = \frac{1}{2\lambda} \mathbf{M}_{[\mathcal{S}]}^{-1} \mathbf{b}_{[\mathcal{S}]} \quad \text{for all } i \in \mathcal{S} \quad (9)$$

Assume that optimal returns are positive. Plugging equation (9) into the budget constraint equation (8), we get $2\lambda = \sqrt{\frac{\mathbf{b}_{[\mathcal{S}]}^T \mathbf{M}_{[\mathcal{S}]}^{-1} \mathbf{b}_{[\mathcal{S}]}}{t}}$, meaning that

$$\mathbf{w}^* = \left(\sqrt{\frac{t}{\mathbf{b}_{[\mathcal{S}]}^T \mathbf{M}_{[\mathcal{S}]}^{-1} \mathbf{b}_{[\mathcal{S}]}}} \right) \mathbf{M}_{[\mathcal{S}]}^{-1} \mathbf{b}_{[\mathcal{S}]} \quad \text{for all } i \in \mathcal{S}$$

The performance of group \mathcal{S} is therefore given by $F(\mathcal{S}) = \sqrt{t \mathbf{b}_{[\mathcal{S}]}^T \mathbf{M}_{[\mathcal{S}]}^{-1} \mathbf{b}_{[\mathcal{S}]}}$.

Second, we show that optimal unit returns are positive. The proof regarding positiveness holds for any objective function that is a weighted sum of efforts with positive weights (in the paper we assume equal weights). Consider an objective equal to $\sum_{i \in \mathcal{N}} e_i x_i$ with $e_i > 0$ (the benchmark case with homogeneous weights corresponds to $\mathbf{e} = \mathbf{1}$). Let $\mathbf{e} = (e_i)_{i \in \mathcal{N}}$, $\mathbf{b}_{\mathbf{e}} = \mathbf{M}\mathbf{e}$,

$\mathbf{b}_{\mathbf{e}[\mathcal{S}]} = (\mathbf{b}_{\mathbf{e},j})_{j \in \mathcal{S}}$. Then it can be shown that optimal weights are proportional to $\mathbf{M}_{[\mathcal{S}]}^{-1} \mathbf{b}_{\mathbf{e}[\mathcal{S}]}$ (this is a direct extension of the proof given just above). We prove that $w_i^* > 0$ for all $i \in \mathcal{S}$. It is the case that $\mathbf{w}^* > 0$ if and only if

$$\begin{pmatrix} w_1^* \\ w_2^* \\ \dots \\ w_s^* \end{pmatrix} = \mathbf{M}_{[\mathcal{S}]}^{-1} \mathbf{b}_{\mathbf{e}[\mathcal{S}]} = \begin{pmatrix} m_{11} & \dots & m_{1s} \\ m_{21} & \dots & m_{2s} \\ \dots & \dots & \dots \\ m_{s1} & \dots & m_{ss} \end{pmatrix}^{-1} \begin{pmatrix} m_{11}e_1 + \dots + m_{1n}e_n \\ m_{21}e_1 + \dots + m_{2n}e_n \\ \dots \\ m_{s1}e_1 + \dots + m_{sn}e_n \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \end{pmatrix}$$

Consider the return of agent $1 \in \mathcal{S}$ (this is without loss of generality). Then, a few calculations show that

$$w_1^* = e_1 + \frac{(-1)^{s+1}}{\det(\mathbf{M}_{[\mathcal{S}]})} \cdot \sum_{k>s} e_k \begin{vmatrix} m_{12} & \dots & m_{1s} & m_{1k} \\ m_{22} & \dots & m_{2s} & m_{2k} \\ \dots & \dots & \dots & \dots \\ m_{2s} & \dots & m_{ss} & m_{sk} \end{vmatrix} \quad (10)$$

Observe that each outsider k is associated with a proper contribution to the optimal unit return of agent 1, whose sign is related to the sign of the associated almost principal minor. We will show that the sign of every almost principal minor is that of $(-1)^{s+3}$, implying that agent k 's contribution is always positive. To see this, we decompose $w_1^* = \sum_{j \in \mathcal{N}} w_{1j}^*$, where

$$w_{1j}^* = e_j \begin{pmatrix} m_{11} & \dots & m_{1s} \\ m_{21} & \dots & m_{2s} \\ \dots & \dots & \dots \\ m_{s1} & \dots & m_{ss} \end{pmatrix}^{-1} \begin{pmatrix} m_{1j} \\ m_{2j} \\ \dots \\ m_{sj} \end{pmatrix}_1$$

where subscript $_1$ refers to entry 1 of the vector resulting from the above matrix multiplication. By definition of inverse matrices,

$$\begin{pmatrix} m_{11} & \cdots & m_{1s} \\ m_{21} & \cdots & m_{2s} \\ \cdots & & \cdots \\ m_{s1} & \cdots & m_{ss} \end{pmatrix}^{-1} \begin{pmatrix} m_{11} \\ m_{21} \\ \cdots \\ m_{s1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}$$

and for all $i \in \{2, \dots, s\}$,

$$\begin{pmatrix} m_{11} & \cdots & m_{1s} \\ m_{21} & \cdots & m_{2s} \\ \cdots & & \cdots \\ m_{s1} & \cdots & m_{ss} \end{pmatrix}^{-1} \begin{pmatrix} m_{1i} \\ m_{2i} \\ \cdots \\ m_{si} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix}$$

It follows that $w_{11}^* = e_1$ and $w_{1j}^* = 0$ for all $j \in \{2, \dots, s\}$. Turning to outsiders' contributions, we observe that for all $k > s$,

$$w_{1k}^* = e_k \begin{pmatrix} m_{11} & \cdots & m_{1s} \\ m_{21} & \cdots & m_{2s} \\ \cdots & & \cdots \\ m_{s1} & \cdots & m_{ss} \end{pmatrix}^{-1} \begin{pmatrix} m_{1k} \\ m_{2k} \\ \cdots \\ m_{sk} \end{pmatrix}_1 = (-1)^{s+1} e_k \cdot \frac{\begin{vmatrix} m_{12} & \cdots & m_{1s} & m_{1k} \\ m_{22} & \cdots & m_{2s} & m_{2k} \\ \cdots & & \cdots & \cdots \\ m_{2s} & \cdots & m_{ss} & m_{sk} \end{vmatrix}}{\det(\mathbf{M}_{[s]})}$$

This can be checked directly.³² In total, we get $w_1^* = e_1 + \sum_{k>s} w_{1k}^*$, and we obtain equation (10).

We will sign every determinant in equation (10). Recognizing a principal minor and $n - s$ almost principal minors, we will show that, for any given size s all almost principal minors have the same sign, and that this sign alternates according to whether s is odd or even. And taking into account the multiplication by $(-1)^{s+1}$, all contributions will be seen to be positive. Formally, we define the set $\alpha = \{1, 2, \dots, s, k\}$, for any $k \in \{s+1, \dots, n\}$ (we write α rather than $\alpha(s, k)$ when there is no confusion). We use the informal notation $\alpha + i$ (resp. $\alpha - i$) to denote the augmentation of set α by $i \notin \alpha$ (resp. the deletion of $i \in \alpha$ from α). The left determinant is that of the principal submatrix $\mathbf{M}_{[\alpha]} = \mathbf{M}[\alpha - k; \alpha - k]$, it is positive, being a principal minor of an inverse M-matrix. Moreover, every other determinant is that of an almost principal submatrix $\mathbf{M}[\alpha - k; \alpha - 1]$ for $k > s$. We then use the following lemma from inverse M-matrices theory (see for instance Johnson and Smith (2011, Theorem 3.1 p. 961)):

Lemma 3. *The sign of the determinant of this almost principal submatrix $\mathbf{M}[\alpha - k; \alpha - 1]$ is that of $(-1)^{r+t+1}$, where r is the number of indices of α less than or equal to k , and t is the number of indices of α less than or equal to 1.*

Here, we get $r = s + 1, t = 1$, which means that the sign of almost principal minors is the sign of $(-1)^{s+3}$. In total, for any $k > s$, the sign of the contribution of outsider k to w_1^* is $(-1)^{s+1} \times (-1)^{s+3} = (-1)^{2s+4}$, i.e. it is positive. This shows that $w_1^* > 0$. \square

Proof of Proposition 2. We consider that δ is close to zero. We identify the objective function. Suppose that the principal selects group \mathcal{S} . For any agent $i \in \mathcal{S}$, we let $d_i^{\mathcal{S}}$ (resp. $d_i^{\mathcal{N} \setminus \mathcal{S}}$) represent the number of linked agents in set \mathcal{S} (resp. in set $\mathcal{N} \setminus \mathcal{S}$). Then $d_i = d_i^{\mathcal{S}} + d_i^{\mathcal{N} \setminus \mathcal{S}}$. We let $L_{\mathcal{S}}$ represent the number of links among members of \mathcal{S} , and we let $L_{\mathcal{S}, \mathcal{N} \setminus \mathcal{S}}$ be the number of links between \mathcal{S} and $\mathcal{N} \setminus \mathcal{S}$ (i.e. the number of cross links between the two sets). Then $\sum_{i \in \mathcal{S}} d_i^{\mathcal{S}} = 2L_{\mathcal{S}}$ and $\sum_{i \in \mathcal{S}} d_i^{\mathcal{N} \setminus \mathcal{S}} = L_{\mathcal{S}, \mathcal{N} \setminus \mathcal{S}}$. Under low intensity of interaction, we approximate $b_i = 1 + \delta d_i + o(\delta)$. We also have $\mathbf{M}_{[\mathcal{S}]} = \mathbf{I} + \delta \mathbf{G}_{[\mathcal{S}]} + o(\delta)$, which means that $\mathbf{M}_{[\mathcal{S}]}^{-1} = \mathbf{I} - \delta \mathbf{G}_{[\mathcal{S}]} + o(\delta)$. We plug these Taylor approximations into $F(\mathcal{S})$ as defined in equation (5) and we focus on order 1. We get:

$$\mathbf{b}_{[\mathcal{S}]}^T \mathbf{M}_{[\mathcal{S}]}^{-1} \mathbf{b}_{[\mathcal{S}]} \sim \sum_{i \in \mathcal{S}} b_i^2 - \delta \sum_{i \in \mathcal{S}} \sum_{j \in \mathcal{S}} g_{ij} \sim \sum_{i \in \mathcal{S}} (1 + 2\delta d_i) - \delta \sum_{i \in \mathcal{S}} d_i^{\mathcal{S}}$$

That is,

$$\mathbf{b}_{[\mathcal{S}]}^T \mathbf{M}_{[\mathcal{S}]}^{-1} \mathbf{b}_{[\mathcal{S}]} \sim s + 2\delta \sum_{i \in \mathcal{S}} d_i - \delta \sum_{i \in \mathcal{S}} d_i^{\mathcal{S}}$$

³² Call $\mathbf{A}_k = \begin{pmatrix} m_{12} & \cdots & m_{1s} & m_{1k} \\ m_{22} & \cdots & m_{2s} & m_{2k} \\ \cdots & & \cdots & \cdots \\ m_{2s} & \cdots & m_{ss} & m_{sk} \end{pmatrix}$. Developing the determinant of matrix \mathbf{A}_k by its last column (this is column s , which corresponds to entries with index k), we get $\det(\mathbf{A}_k) = \sum_{j=1, \dots, s} (-1)^{s+j} m_{jk} \det(\mathbf{A}_k)_{j,s} = \sum_{j=1, \dots, s} (-1)^{s+1} m_{jk} (\text{com } \mathbf{M}_{[s]})_{1,j}$ (where $\text{com } \mathbf{M}_{[s]}$ represents the co-matrix of $\mathbf{M}_{[s]}$).

Taking into account that $d_i = d_i^{\mathcal{S}} + d_i^{\mathcal{S}, \mathcal{N} \setminus \mathcal{S}}$,

$$\mathbf{b}_{[\mathcal{S}]}^T \mathbf{M}_{[\mathcal{S}]}^{-1} \mathbf{b}_{[\mathcal{S}]} \sim s + \delta \sum_{i \in \mathcal{S}} d_i^{\mathcal{S}} + 2 \sum_{i \in \mathcal{S}} \delta d_i^{\mathcal{S}, \mathcal{N} \setminus \mathcal{S}} = s + 2\delta L_{\mathcal{S}} + 2\delta L_{\mathcal{S}, \mathcal{N} \setminus \mathcal{S}}$$

In total, $F(\mathcal{S}) = \sqrt{t} \sqrt{s + 2\delta(L_{\mathcal{S}} + L_{\mathcal{S}, \mathcal{N} \setminus \mathcal{S}})} - C(\mathcal{S})$. \square

Proof of Theorem 2. We show that, when the cost function is a step function as in system (3), the problem reduces to finding the optimal group among groups of size k . To see this, suppose that the set of best groups contains \mathcal{S}^* , of size $s^* < k$. Then, for any group of larger size and such that $\mathcal{S}^* \subset \mathcal{S}'$, the optimal weights of agents in set $\mathcal{S}' \setminus \mathcal{S}^*$ should be equal to zero by construction. But by Theorem 1 we know that for any target, all optimal weights are positive inside the group, which contradicts the idea that \mathcal{S}^* is an optimal group. This shows that the principal should only care about groups of size k .

Then, we show NP-hardness formally, observing that the subproblem of finding the best target among all groups of fixed size boils down to a densest $(n - k)$ -subgraph problem. The next lemma establishes the reduction formally:

Lemma 4. *The problem of finding the group maximizing the objective function $L_{\mathcal{S}} + L_{\mathcal{S}, \mathcal{N} \setminus \mathcal{S}}$ among all groups of size k is isomorphic with a densest $(n - k)$ -subgraph problem, which is NP-hard.*

Proof of Lemma 4. We show that the problem of finding the group of size k maximizing the number of internal links plus the number of cross-links in network \mathbf{G} reduces to finding the group of size $n - k$ maximizing the number of internal links of the complementary graph $\bar{\mathbf{G}}$. Indeed, the optimum, which we call $F(\mathbf{G})$ for convenience, is given by

$$F(\mathbf{G}) = \max_{\mathcal{S}, |\mathcal{S}|=k} L_{\mathcal{S}}(\mathbf{G}) + L_{\mathcal{S}, \mathcal{N} \setminus \mathcal{S}}(\mathbf{G})$$

i.e., defining $L_{\mathcal{N} \setminus \mathcal{S}}(\mathbf{G})$ as the number of links in set $\mathcal{N} \setminus \mathcal{S}$ in network \mathbf{G} , and noting that $L_{\mathcal{S}}(\mathbf{G}) + L_{\mathcal{S}, \mathcal{N} \setminus \mathcal{S}}(\mathbf{G}) = L - L_{\mathcal{N} \setminus \mathcal{S}}(\mathbf{G})$,

$$F(\mathbf{G}) = \min_{\mathcal{S}, |\mathcal{S}|=n-k} L_{\mathcal{S}}(\mathbf{G})$$

We let \mathbf{J} be the n -square matrix of ones, and we let $\bar{\mathbf{G}}$ be the complementary network of \mathbf{G} , so that $\mathbf{G} + \bar{\mathbf{G}} = \mathbf{J} - \mathbf{I}$. Then

$$\min_{\mathcal{S}, |\mathcal{S}|=n-k} L_{\mathcal{S}}(\mathbf{G}) = \max_{\mathcal{S}, |\mathcal{S}|=n-k} L_{\mathcal{S}}(\bar{\mathbf{G}})$$

This last problem is the (densest $n - k$)-subgraph problem, which boils down to seeking the group of size $n - s$ having the largest number of internal links. This problem is known to be NP-hard. \square \square

Proof of Corollary 1. We compute the performance of a group of size s with and without the central agent, and then we show that including the central agent is always better.

In the star network, the matrix $\mathbf{M} = (\mathbf{I} - \delta \mathbf{G})^{-1}$ is given by

$$\mathbf{M} = \frac{1}{1 - (n-1)\delta^2} \begin{pmatrix} 1 & \delta & \delta & \dots & \delta \\ \delta & 1 - (n-2)\delta^2 & \delta^2 & \dots & \delta^2 \\ \delta & \delta^2 & 1 - (n-2)\delta^2 & \dots & \delta^2 \\ \dots & \dots & \dots & \dots & \dots \\ \delta & \delta^2 & \dots & \delta^2 & 1 - (n-2)\delta^2 \end{pmatrix}$$

Consider first a group of s peripherals. The inverse of matrix $\mathbf{M}_{\mathcal{S}}$ has homogeneous diagonal entries $1 - (n-2)\delta^2$ and homogeneous off-diagonal entries δ^2 . Hence, defining $\mathbf{V} = \mathbf{M}_{[\mathcal{S}]}^{-1}$ for convenience, matrix \mathbf{V} has homogeneous diagonal entries $\frac{1 - (n-s)\delta^2}{1 - (n-s-1)\delta^2}$ and homogeneous off-diagonal entries $\frac{-\delta^2}{1 - (n-s-1)\delta^2}$. Then the objective is given by $b_{\mathbf{a}} + \sqrt{t} \sqrt{f_p(s)} - cs$, with

$$f_p(s) = s \left(\frac{1 + \delta}{1 - (n-1)\delta^2} \right)^2$$

Second consider a group of size s containing the central agent. Then $\mathbf{M}_{\mathcal{S}}$ is a s -square matrix structurally equivalent to \mathbf{M} . We get the corresponding matrix

$$\mathbf{V} = \begin{pmatrix} 1 - (n-s)\delta^2 & -\delta & -\delta & \dots & -\delta \\ -\delta & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\delta & 0 & \dots & 0 & 1 \end{pmatrix}$$

from which we deduce that the objective function is equal to $b_{\mathbf{a}} + \sqrt{t} \sqrt{f_c(s)} - cs$, with

$$f_c(s) = s + \frac{(n-1)\delta(2+n\delta - 2(n-1)\delta^2 - n(n-1)\delta^3)}{(1 - (n-1)\delta^2)^2}$$

Then, few computations indicate that $f_p(s) \leq f_c(s)$ whenever

$$\delta^2 s^2 - (2 + (2n-1)\delta)\delta s + (n-1)\delta(2+n\delta) \geq 0$$

which holds for all $s \leq n-1$. Thus for all group sizes, it is always better to include the central agent. \square

Proof of Lemma 2. We apply the Sherman-Morrison formula to examine the impact of a small impulse in one entry of the matrix $\mathbf{M}_{\mathcal{I}_{S^1}}$ on the sum of entries of inverse matrix. Suppose \mathbf{Q} is an invertible n -square matrix with real entries and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are column vectors. Then $\mathbf{Q} + \mathbf{u}\mathbf{v}^T$ is invertible iff $1 + \mathbf{v}^T \mathbf{Q}^{-1} \mathbf{u} \neq 0$, in which case, by the Sherman-Morrison formula,

$$(\mathbf{Q} + \mathbf{u}\mathbf{v}^T)^{-1} = \mathbf{Q}^{-1} - \frac{\mathbf{Q}^{-1} \mathbf{u}\mathbf{v}^T \mathbf{Q}^{-1}}{1 + \mathbf{v}^T \mathbf{Q}^{-1} \mathbf{u}}$$

We apply this formula with $\mathbf{Q} = \mathbf{M}$, $\mathbf{u} = (0, \dots, 0, \epsilon_{ij}, 0, \dots, 0)^T$ with ϵ_{ij} at entry i , $\mathbf{v}^T = (0, \dots, 0, 1, 0, \dots, 0)^T$ with 1 at entry j . Hence, $\mathbf{E} = \mathbf{u}\mathbf{v}^T$ s.t. $\mathbf{E} = [e_{pq}]$ is such that $e_{ij} = \epsilon_{ij}$, $e_{pq} = 0$ otherwise. Setting $\mathbf{V} = \mathbf{M}^{-1}$, we get $[(\mathbf{M} + \mathbf{E})^{-1}] = \mathbf{M}^{-1} + \zeta_{ij} \epsilon_{ij} \mathbf{W}$, with $\zeta_{ij} = \frac{-1}{1 + \epsilon_{ij} h_{ji}}$, and $w_{pq} = h_{pi} h_{jq}$. Hence, denoting $\mathbf{M}' = \mathbf{M} + \mathbf{E}$:

$$\mathbf{1}^T \mathbf{M}'^{-1} \mathbf{1} - \mathbf{1}^T \mathbf{M}^{-1} \mathbf{1} = \zeta_{ij} \epsilon_{ij} \left(\sum_{k \in N} h_{jk} \right) \left(\sum_{k \in N} h_{ki} \right)$$

For $j = i$, $\zeta_{ii} < 0$, and given the symmetry of matrix \mathbf{V} , we obtain $\mathbf{1}^T \mathbf{M}'^{-1} \mathbf{1} - \mathbf{1}^T \mathbf{M}^{-1} \mathbf{1} = \zeta_{ii} \epsilon_{ii} \left(\sum_{k \in N} h_{ik} \right)^2 \leq 0$. And for $j \neq i$, we have $\zeta_{ij} < 0$ for ϵ_{ij} small enough, which implies that the diagonal dominance of $\mathbf{V} = \mathbf{M}^{-1}$ entails a decrease in the sum of entries.

The proof is completed by starting from matrix $\mathbf{M}_{\mathcal{I}_{S^1}}$ and adding impulses iteratively in the direction of matrix $\mathbf{M}_{\mathcal{I}_{S^1}}$, the preceding argument showing a monotonic response to the sum of entries of inverse matrices along the direction. \square

Appendix B. Greedy algorithm

In this Appendix, we introduce the greedy algorithm and describe our computations and their performances; all matlab programs are available upon request.

Since the problem faced by the principal may be NP-hard under low intensity of interaction, it can be useful to know whether algorithms generate good approximations of the solution. When the cost function is a step function as in (3), the problem reduces to the densest $(n-k)$ -subgraph problem on the complementary network, which ensures that the so-called greedy algorithm performs satisfactorily (no less than $1 - 1/e \sim 63$ percent of maximum efficiency for large societies - for instance see details in Ballester et al. (2010, Appendix A, Proposition 10). However, the unconstrained problem with contracting cost and endogenous group size involves greater complexity, and faster algorithms give a lower bound to the ratio of inefficiency of one half.

The greedy algorithm works as follows. In step 1, it determines the best singleton; in step 2 it determines the best pair containing the best singleton and stops if the singleton performs better than the pair; in step 3 it determines the best triplet containing the best pair and stops if the pair performs better than the triplet; and so on, until the best group is reached. This algorithm therefore converges in no more than $n!$ steps.

We performed the greedy algorithm on Erdö-Renyi random networks under various linear contracting costs. It was implemented with the following initial parameters. We fixed $t = 1$ in all simulations and examined its performance in proportion to the optimal objective attained with the optimal target, which requires computing the optimal target. We started with $n = 16$. Indeed, combinatorial concerns regarding the computation of the optimal target become very important for larger values of n , since all possible groups have to be browsed. We selected low versus high intensities of interaction $\delta \in \{0.01, 0.06\}$. We tested $c \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6\}$ (for these parameter sets, larger costs induce trivial solutions). In each scenario, we performed a simulation generating 1000 Erdö-Renyi random networks. For the uniform probability p of link existence, we set $p = 0.5$.³³

On each network generated, we identified the optimal group from all possible groups (at $n = 16$, the program searches through $2^{10} - 1 = 65535$ groups) and obtained its performance. Then, we ran the greedy algorithm, computing the algorithm's relative error of approximation as a percentage of the optimal performance. These results are presented in Table 4.

³³ We also tested alternative values of $p \in \{0.25, 0.75\}$, to test the impact of network density. They do not qualitatively affect results.

Table 4
 $n = 16$; Targeting through greedy algorithm: Average over 1000 random networks of the relative error of approximation as a percentage of the optimal performance.

δ	$c = 0.1$	$c = 0.2$	$c = 0.3$	$c = 0.4$	$c = 0.5$	$c = 0.6$
$\delta = 0.01$	100	99.85	99.94	99.97	100	100
$\delta = 0.06$	99.97	99.68	99.78	99.65	99.84	99.82

The numbers in the table are the average relative error of approximation (in percentage) over the 1000 networks in each scenario. Roughly speaking, the greedy algorithm performed very well for these values overall, the average relative error of approximation being less than 1 percent in all cases.³⁴

We also explored the impact of clustering coefficient and degree assortativity on the algorithm performance. We found that clustering tends to increase performance, while degree assortativity tends to decrease performance. The correlation coefficient was computed between all statistics and average algorithm performance. The statistics are given as follows. We define $g = \mathbf{1}^T \mathbf{G} \mathbf{1}$, $\mathbf{D} = \mathbf{G} \mathbf{1}$, \mathbf{J} the n -square matrix of ones, and $\mathbf{H} = \frac{\mathbf{G} \mathbf{G}}{g}$:

Global clustering coefficient $C(\mathbf{G})$:

$$C(\mathbf{G}) = \frac{\text{trace}(\mathbf{G}^3)}{\mathbf{1}^T (\mathbf{G}^2 - \mathbf{G}) \mathbf{1}}$$

Degree assortativity Pearson coefficient $DA(\mathbf{G})$ ³⁵:

$$DA(\mathbf{G}) = g \frac{\mathbf{D}^T (\mathbf{G} - \mathbf{H}) \mathbf{D}}{\mathbf{D}^T \mathbf{D} - \frac{g^2}{n}}$$

For $n = 30$, we computed the global clustering coefficient, the degree assortativity, and the performance of the greedy algorithm on 100 randomly generated networks, from which we deduced the following correlation coefficients for $c \in \{0.2, 0.3, 0.4\}$:

- For $c = 0.2$,
 - . correlation coefficient clustering/greedy: 0.8977
 - . correlation coefficient assortativity/greedy: -0.2213
- For $c = 0.3$,
 - . correlation coefficient clustering/greedy: 0.9149
 - . correlation coefficient assortativity/greedy: -0.2193
- For $c = 0.4$,
 - . correlation coefficient clustering/greedy: 0.8941
 - . correlation coefficient assortativity/greedy: -0.1519

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³⁴ Algorithm performance should be lower for larger network sizes.

³⁵ The formula given below was proved in the paper entitled 'The value of network information: assortative mixing makes the difference' (Belhaj and Deroïan (2021)).

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