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To cite this version:

Valentin A Zagrebnov, Bruno Iochum. A characterisation of Gibbs semigroups. 2023. hal-04177061

HAL Id: hal-04177061

https://amu.hal.science/hal-04177061

Preprint submitted on 3 Aug 2023

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A characterisation of Gibbs semigroups

August 3, 2023

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In memory of Academician Vasiliĭ Sergeevich Vladimirov
on the 100th anniversary of his birthday

Abstract

In the present paper we propose a new characterisation of Gibbs semigroups, which is extension of a similar characterisation for compact semigroups.

Key words: Semigroup theory, compact semigroups, Gibbs semigroups, trace asymptotics

1 Introduction

It is known that to produce a variety of special classes of one-parameter in $\mathbb{R}_0^+$ operator-values *semigroups* on normed (Hilbert or Banach) spaces, one either modifies (relaxes, or reinforces) the condition of continuity at the origin $t = 0$, or changes conditions *away* from the origin in $\mathbb{R}^+ = \mathbb{R}_0^+ \setminus \{0\}$. Finally, one can also combine the both of them.

The right continuity assumption at $t = 0$ is crucial for a *nontrivial* semigroup theory. That is, the *strong continuity* for $t \to +0$ seems to be, in this respect, the most appropriate restriction to make this theory so significant, see discussions in [HiPh57, Chapter X], [Dav07, Section 6.2], and [Zag19, Chapter 1]. The conclusion is that any of the *basic variety* of semigroups reduces to the class C_0 (strongly continuous at $t = +0$) semigroup under natural condition of the semigroup operator-norm boundedness for t in some neighborhood of zero. This property makes the C_0-semigroups, in a certain sense, *exceptional* and the most important among the one-parameter operator semigroups.

Alongside the earlier discussion around the *continuity* at $t = +0$, yet another source of variety of semigroups is possible and related to their behaviour *away* from $t = 0$. The simplest example is the *self-adjoint* C_0-semigroups on a Hilbert spaces.

Let $A \geq 0$ be a non-negative self-adjoint operator on a separable Hilbert space $\mathfrak{H}$. Then it is the generator of a contraction C_0-semigroup $\{U_A(t) = \mathrm{e}^{-tA}\}_{t\geq 0}$, which is in turn a family of bounded self-adjoint operators. Owing to the fact that the *spectral theorem* for A provides a canonical *functional calculus* for (unbounded) self-adjoint operators, we obtain the representation:

$$U_A(t) = \int_{\mathbb{R}_0^+} E_A(\mathrm{d}\lambda)\, \mathrm{e}^{-t\lambda} . \tag{1.1}$$

Here $E_A(\mathrm{d}\lambda)$ is the unique *spectral measure* such that it has the spectral decomposition: $A = \int_{\mathbb{R}_0^+} E_A(\mathrm{d}\lambda)\, \lambda$. Let $t > 0$ and $\sigma \geq 0$. Then on account of (1.1)

$$\begin{aligned}\|U_A(t+\sigma)-U_A(t)\| &= \left\|\int_{\mathbb{R}_0^+} E_A(\mathrm{d}\lambda)\,\mathrm{e}^{-t\lambda}(\mathrm{e}^{-\sigma\lambda}-1)\right\| \\ &\le \sup_{\lambda\in\mathbb{R}_0^+}|\,\mathrm{e}^{-t\lambda}(\mathrm{e}^{-\sigma\lambda}-1)\,| \le \frac{\sigma}{t+\sigma}\left(1+\frac{\sigma}{t}\right)^{-t/\sigma}\,.\end{aligned} \tag{1.2}$$

The estimate (1.2) yields that the C_0-semigroup $\{U_A(t)\}_{t\ge 0}$ with values in the subset $\mathcal{L}^{sa}(\mathfrak{H})$ of bounded self-adjoint operators is (immediately) *operator-norm* continuous for $t>0$.

This example shows that a stronger *regularity* of the range of the map: $t\mapsto U(t)$, *away* from zero, may also provide a *continuity* of the C_0-semigroup for a stronger topology. Besides that, this continuity may to start *eventually* with delay: $t\ge t_0$, for some $t_0>0$.

To *enhance* a topology of the range (i.e., to strengthen *regularity*) one can suppose, for example, that a C_0-semigroup $\{U_A(t)\}_{t\ge 0}$ with generator A, is a subset of *compact* operators $\mathscr{C}_\infty(\mathfrak{H})$ on a Hilbert space $\mathfrak{H}$, as long as $t>0$. So, $\{U_A(t)\}_{t\ge 0}$ is an *immediately* compact C_0-semigroup. Then similarly to the *self-adjoint* case the strong continuity of $\{U_A(t)\}_{t\ge 0}$ is *uplifted* to the *operator-norm* continuity, although now it might happen with delay ($t\ge t_0$) for some $t_0>0$.

In this paper, we study the C_0-semigroups (called *Gibbs semigroups*) such that $U_A(t)\in\mathscr{C}_1(\mathfrak{H})$ for $t_0>0$, and the von Neumann–Schatten class $\mathscr{C}_1(\mathfrak{H})\subset\mathscr{C}_\infty(\mathfrak{H})$ is a two-sided $*$-*ideal* in the algebra of *compact* operators $\mathscr{C}_\infty(\mathfrak{H})\subset\mathcal{L}(\mathfrak{H})$. This ideal endowed with the trace-norm topology is the Banach space of *trace-class* operators.

The characterisation of compact semigroups (relation between semigroup and resolvent of generator, see Proposition 2.2) shows that the topologies used for the semigroup continuity away from t_0 and for the range regularity coincide with the operator-norm topology. A similar observation is known for the Gibbs semigroups and even in a more general setting [Zag19, Chapter 4].

The aim of the present paper is to give a characterisation of Gibbs semigroups, which is analogous to that for compact semigroups. The next Section 2 contains indispensable preliminaries. The main result, Theorem 3.2, is proven in Section 3. Section 4 is devoted to concluding remarks and comments.

2 Preliminaries and Gibbs semigroups

First we remind a statement about regularity and continuity of compact semigroups.

Proposition 2.1. *Let $\{U_A(t)\}_{t\ge 0}$ be a (quasi-bounded) C_0-semigroup such that the operator $U_A(t_0)$ is compact for some $t_0>0$. Then $\{U_A(t)\}_{t\ge t_0}\subset\mathscr{C}_\infty(\mathfrak{H})$ and the map: $t\mapsto U_A(t)$ is operator-norm continuous in (t_0,∞), including the operator-norm right-continuity at $t=t_0+0$.*

Correspondingly, any immediately compact C_0-semigroup $\{U_A(t)\}_{t\ge 0}$ is operator-norm continuous in $\mathbb{R}^+$.

We note that (although it is plausible) the relationship between *compactness* of a C_0-semigroup with generator A and *compactness* of resolvents $\{R_A(z)\}_{z\in\rho(A)}$ is not straightforward. For example, a resolvent corresponding to a (eventually) compact C_0-semigroup is not necessarily compact, see Example 4.3 (f).

The following statement, which is a *characterisation* of compact semigroups, elucidates this relationship, see for example, [Zag19, Proposition 4.17].

Proposition 2.2. *A C_0-semigroup $\{U_A(t)\}_{t\geq 0}$ on $\mathfrak{H}$ with generator A is immediately compact if and only if* :
(*i*) *the mapping*: $t \mapsto U_A(t) \in \mathcal{L}(\mathfrak{H})$, *is operator-norm continuous in* $\mathbb{R}^+$
and
(*ii*) *the resolvent* $R_A(z) = (A - z\,\mathbb{1})^{-1}$ *is compact for some* (*and thus for any*) $z \in \rho(A)$.

Next we introduce the Gibbs semigroups and recall some principal assertions, which are indispensable for the rest.

Definition 2.3. A strongly continuous semigroup $\{G(t)\}_{t\geq 0}$ on a Hilbert space $\mathfrak{H}$ is called an immediately Gibbs semigroup if $\mathbb{R}^+ \ni t \mapsto G(t) \in \mathscr{C}_1(\mathfrak{H})$. We call a strongly continuous semigroup $\{G(t)\}_{t\geq 0}$ an eventually Gibbs semigroup when there exists a threshold $t_0 > 0$ such that $t \mapsto G(t) \in \mathscr{C}_1(\mathfrak{H})$ for $t \geq t_0$.

A semigroup $\{G(t)\}_{t\geq 0}$ is called a self-adjoint Gibbs semigroup if $G^*(t) = G(t)$ for $t \in \mathbb{R}_0^+$.

Example 2.4. *Let $\{e_n\}_{n\geq 1}$ be an orthonormal basis in the Hilbert space $\mathfrak{H}$. Then the one-parameter family of operators $\{G(t)\}_{t\geq 0}$ defined for $u \in \mathfrak{H}$ by*

$$u \mapsto G(t)u := \sum_{n=1}^{\infty} \mathrm{e}^{-t\,\ln(n+1)}\,(u,\,e_n)\,e_n\,, \quad t \geq 0\,, \tag{2.1}$$

is a self-adjoint strongly continuous contraction semigroup with the unbounded self-adjoint generator defined by $Au := \sum_{n=1}^{\infty} \ln(n+1)\,(u,\,e_n)\,e_n$.

By definition (2.1) *the eigenvalues are $\{\lambda_n(G(t)) = 1/(n+1)^t\}_{n\geq 1}$ and, consequently, the C_0-semigroup* (2.1) *is immediately compact*: $\{G(t)\}_{t>0} \subset \mathscr{C}_\infty(\mathfrak{H})$. *In addition, we infer by* (2.1) *that*

$$\|G(t)\|_1 = \sum_{n=1}^{\infty} \frac{1}{(n+1)^{\,t}} < \infty\,, \quad \text{for} \quad t > 1\,. \tag{2.2}$$

Therefore, by Definition 2.3 *and by* (2.1), (2.2), *the immediately compact C_0-semigroup* (2.1) *is an eventually Gibbs semigroup because $\{G(t)\}_{t>0} \subset \mathscr{C}_1(\mathfrak{H})$ only for $t > t_0 = 1$.*

The analysis of continuity of C_0-semigroups with values in $\mathscr{C}_1$ or $\mathscr{C}_\infty$, is simplified by the following statement about *continuity of multiplication* on these ideals.

Proposition 2.5. *Let $\{B_k\}_k$, $\{C_k\}_k$ be two families of bounded operators such that* $\text{s-lim}_{k\to\infty} B_k = B \in \mathcal{L}(\mathfrak{H})$ *and* $\|\cdot\|_p\text{-lim}_{k\to\infty} C_k = C \in \mathscr{C}_p(\mathfrak{H})$ *for $p \in \{1, \infty\}$. Then*

$$\|\cdot\|_p\text{-}\lim_{k\to\infty} B_k C_k = B\,C\,. \tag{2.3}$$

If, in addition, $\text{s-}\lim_{k\to\infty} B_k^* = B^*$*, then also*

$$\|\cdot\|_p\text{-}\lim_{k\to\infty} C_k B_k = C\,B\,. \tag{2.4}$$

Note that $\|\cdot\|_{p=\infty}$ *coincides with the operator norm* $\|\cdot\|$ *on* $\mathcal{L}(\mathfrak{H})$*.*

Proof. Writing

$$\|B_k C_k - B\,C\|_p \le \|B_k\|\,\|C_k - C\|_p + \|(B_k - B)\,C\|_p\,, \tag{2.5}$$

and using the uniform boundedness principle for a strongly convergent sequence $\{B_k\}_{k\ge1}$,

$$\sup_{k\ge1}\|B_k\| \le L < \infty\,,$$

it is sufficient to check that in (2.5), $\lim_{k\to\infty}\|(B_k - B)\,C\|_p = 0$.

Since $C \in \mathscr{C}_p(\mathfrak{H})$, where $p = \{1,\infty\}$, we can find for C and any $\varepsilon > 0$ a finite-rank orthogonal projection P such that $\|(\mathbb{1} - P)\,C\|_p < \varepsilon$. Then the last term in (2.5) can be estimated as

$$\|(B_k - B)\,C\|_p \le \|(B_k - B)\,P\|\,\|C\|_p + (\|B_k\| + \|B\|)\,\|(\mathbb{1} - P)\,C\|_p\,. \tag{2.6}$$

The strong convergence of $\{B_k\}_{k\ge1}$ implies $\lim_{k\to\infty}\|(B_k - B)\,P\| = 0$, and (2.6) gives

$$\limsup_{k\to\infty}\|(B_k - B)\,C\|_p \le (L + \|B\|)\,\varepsilon\,.$$

Since ε is arbitrary, the result (2.3) is proven owing to (2.5).

The proof of (2.4) follows along the same reasoning, with the property of the norm $\|\cdot\|_p$, $p \in \{1,\infty\}$, taken into account. □

Corollary 2.6. (a) *By Definition* 2.3 *and Proposition* 2.5 *we conclude that an immediately Gibbs semigroup* $\{G(t)\}_{t\ge0}$ *is immediately trace-norm continuous for* $t > 0$*, whereas an eventually Gibbs semigroup* (*with threshold* $t_0 > 0$) *is trace-norm continuous only for* $t > t_0$ (*or* $t \ge t_0$)*.*
(b) *Similarly, an immediately compact* C_0*-semigroup* $\{U(t)\}_{t\ge0}$ *is immediately operator-norm continuous for* $t > 0$*, whereas an eventually compact* C_0*-semigroup* (*with a threshold* $t_0 > 0$) *is* $\|\cdot\|$*-continuous only for* $t > t_0$ (*or* $t \ge t_0$)*.*

Proof. (a) Assume a threshold $t_0 > 0$. Then for any $t \ge t_0$ there is $\delta \in \mathbb{R}$ such that $(t - t_0)/2 + \delta > 0$. The semigroup law gives $G(t+\delta) = G((t-t_0)/2+\delta)\,G((t+t_0)/2)$, where $G((t - t_0)/2 + \delta) \in \mathcal{L}(\mathfrak{H})$ and $G((t + t_0)/2) \in \mathscr{C}_1(\mathfrak{H})$. Since the semigroup $\{G(t)\}_{t\ge0}$ is strongly continuous, $\text{s-}\lim_{\delta\to0} G((t - t_0)/2 + \delta) = G((t - t_0)/2)$. Hence, for $t \ge t_0$ one infers the trace-norm limit:

$$\begin{aligned}\|\cdot\|_1 - \lim_{\delta\to0} G(t+\delta) &= \|\cdot\|_1 - \lim_{\delta\to0} G((t-t_0)/2+\delta)\,G((t+t_0)/2)\\ &= \Big(\text{s-}\lim_{\delta\to0} G((t-t_0)/2+\delta)\Big)\,G((t+t_0)/2) = G(t)\,,\end{aligned} \tag{2.7}$$

by virtue of the $\|\cdot\|_1$-continuity of multiplication in $\mathscr{C}_1(\mathfrak{H})$ obtained in Proposition 2.5.

If $t_0 = 0$, then proof of the immediate trace-norm continuity of the Gibbs semigroup $\{G(t)\}_{t\geq 0}$ for $t > 0$ is *verbatim* the above arguments and (2.7).
(b) The proof follows *mutatis mutandis* the line of reasoning in (a) for the C_0-semigroup $\{U(t)\}_{t\geq 0}$ and for the operator norm $\|\cdot\|$ in (2.7). □

On account of Example 2.4 (or in general by Corollary 2.6) we infer that a C_0-semigroup $\{U(t)\}_{t\geq 0}$ may be continuous in different topologies because of rather different properties of *regularity* of the *range* of the map: $t \mapsto U(t)$, *away* from zero. For example the semigroup (2.1) is strongly continuous at $t = 0$, then operator-norm continuous for $t \in (0, 1]$ and trace-norm continuous for $t > 1$. This corresponds to the three different types of the semigroup range regularity (the bound, compact and trace-norm class of operators) in three subsets of $\mathbb{R}_0^+$.

3 Main result

Now we are in position to prove an extension of Proposition 2.2 for compact semigroups to the case of the trace-class ideal $\mathscr{C}_1(\mathfrak{H})$, that is, for the Gibbs semigroups. To this aim we first remind a fundamental statement, which characterises the Bochner integrable functions. Let $I \subset \mathbb{R}$ be an interval *(*bounded or unbounded*)* in $\mathbb{R}$ and $\mathfrak{X}$ be a Banach space. Recall that a vector-valued function $f : I \to \mathfrak{X}$ is called Bochner integrable if it is strongly measurable and the function: $t \mapsto \|f(t)\|_{\mathfrak{X}}$ is Lebesgue integrable.

Proposition 3.1. (Bochner) *If the function f is $\|\cdot\|_{\mathfrak{X}}$-Bochner integrable in the normed space $\mathfrak{X}$, then one has the estimate*

$$\Big\| \int_I \mathrm{d}t\, f(t) \Big\|_{\mathfrak{X}} \leq \int_I \mathrm{d}t\, \|f(t)\|_{\mathfrak{X}} \,. \tag{3.1}$$

In particular, this is the case if the function f is continuous in the topology of $\mathfrak{X}$.

The following statement is a *characterisation* of Gibbs semigroups, which is the analog to Proposition 2.2 as characterisation of compact semigroups.

Theorem 3.2. *Let $\{\mathrm{e}^{-tA}\}_{t\geq 0}$ be a C_0-semigroup on Hilbert space $\mathfrak{H}$ with invertible generator A, that is,* $\ker(A) = \{0\}$. *If $p \geq 1$, then the following assertions are equivalent* :

(a) *The C_0-semigroup $\{\mathrm{e}^{-tA}\}_{t\geq 0}$ is an immediately Gibbs semigroup $\{G_A(t)\}_{t\geq 0}$, with asymptotics* : $\|G_A(t)\|_1 = O(t^{-p})$ *for* $t \downarrow 0$.

(b) *For any $q \in \mathbb{N}$, such that $q > p$* :

(i) *The map* : $\mathbb{R}^+ \ni t \mapsto \mathrm{e}^{-tA}$ *is $\|\cdot\|_q$-continuous.*

(ii) *The resolvent family $\{R_A(z)\}_{z\in\rho(A)}$ is defined using the Laplace transform of $\{\mathrm{e}^{-tA}\}_{t\geq 0}$ by means of the $\|\cdot\|_q$-Bochner integral and $(R_A(z))^q \in \mathscr{C}_1(\mathfrak{H})$.*

Remark 3.3. It is worth to warn that the condition $\ker(A) = \{0\}$ (or equivalently, $0 \notin \sigma(A) = \{\lambda_n(A)\}_{n\in\mathcal{N}}$, eigenvalues of A) does not imply that the trace-norm of semigroup $\{e^{-tA}\}_{t\geq 0}$ is *bounded*, as well as, that assumption (i) in (b) does not yield $U_A(t) \in \mathscr{C}_q(\mathfrak{H})$, neither that this semigroup is *holomorphic*. Note that on account of (3.7), (3.9) we cannot relax in our proof the condition $\ker(A) = \{0\}$.

Proof. $(a) \implies (b)$: Suppose that the C_0-semigroup $\{e^{-tA}\}_{t\geq 0}$ is a Gibbs semigroup. Then it is trace-norm continuous for $t \in \mathbb{R}^+$ and thus immediately continuous in the topology of ideal $\mathscr{C}_q(\mathfrak{H}) \supset \mathscr{C}_1(\mathfrak{H})$ for $q > p \geq 1$. This proves (i).

To prove (ii) we note that the Gibbs semigroup $\{\mathrm{e}^{-tA}\}_{t\geq 0}$ is strongly continuous, quasi-bounded and compact (for $t > 0$) since $\mathscr{C}_1(\mathfrak{H}) \subset \mathscr{C}_\infty(\mathfrak{H})$. Consequently, there exists ω_0 such that $(-\infty, -\omega_0) \subset \rho(A)$ and choosing $\lambda > \omega_0$ (that is, $-\lambda \in \rho(A)$) we obtain $\|\mathrm{e}^{-\lambda t}\,\mathrm{e}^{-tA}\| \leq M\,\mathrm{e}^{(\omega_0-\lambda)t}$. Then by virtue of Proposition 3.1 the resolvent of generator A at $z = -\lambda$ has the operator-norm Bochner integral representation (*Laplace* transform):

$$R_A(-\lambda) = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-\lambda t}\,\mathrm{e}^{-tA}\,, \quad \omega_0 < \lambda\,. \tag{3.2}$$

Hence, by compactness of the family $\{\mathrm{e}^{-tA}\}_{t>0}$ and by construction of the *operator-norm* convergent integral equation (3.2) yields in the left-hand side a compact operator. On account of the resolvent identity, the fact: $R_A(-\lambda) \in \mathscr{C}_\infty(\mathfrak{H})$ for some $-\lambda \in \rho(A)$, implies that $R_A(z)$ is compact for any $z \in \rho(A)$.

Due to the representation (3.2) and to Fubini's theorem we infer that powers $q \in \mathbb{N}$ of the resolvent have the representation:

$$(R_A(z))^q = \Gamma(q)^{-1} \int_0^\infty \mathrm{d}t\, t^{q-1}\,\mathrm{e}^{-t(A-z\mathbb{1})}\,, \quad \omega_0 < \mathrm{Re}\,(-z)\,, \tag{3.3}$$

which similarly to (3.2) is the operator-norm convergent Bochner integral. Note that the Gibbs semigroup is $\|\cdot\|_1$-continuous (so $\|\cdot\|_1$-measurable) on a closed $I \subset \mathbb{R}^+$, cf. Proposition 3.1. Next, by conditions: $q > p$ and $\|\mathrm{e}^{-tA}\|_1 = O(t^{-p})$ for $t \downarrow 0$, the function $\|t^{q-1}\,\mathrm{e}^{-t(A-z\mathbb{1})}\|_1$ is integrable in a (positive) vicinity of zero. Moreover, for $t > T > 0$, by estimate:

$$\|\mathrm{e}^{-t(A-z\mathbb{1})}\|_1 \leq \|\mathrm{e}^{-T(A-z\mathbb{1})}\|_1\, M\,\mathrm{e}^{(t-T)(\omega_0+\mathrm{Re}\,z)} \leq M_{z,T}\,\mathrm{e}^{-t|\omega_0+\mathrm{Re}\,z|}\,, \quad \mathrm{Re}\,z < -\omega_0\,,$$

this function is also integrable at infinity. Hence, the right-hand side of (3.3) is the $\|\cdot\|_1$-Bochner integral and by Proposition 3.1 for $\mathfrak{X} = \mathscr{C}_1(\mathfrak{H})$, together with (3.1), one obtains estimate

$$\|(R_A(z))^q\|_1 \leq \Gamma(q)^{-1} \int_0^\infty \mathrm{d}t\, t^{q-1}\,\|\mathrm{e}^{-tA}\|_1\,\mathrm{e}^{t\,\mathrm{Re}\,(z)} < \infty\,, \quad \omega_0 < \mathrm{Re}\,(-z)\,. \tag{3.4}$$

This proves that $(R_A(z))^q \in \mathscr{C}_1(\mathfrak{H})$, for $z \in \rho(A)$, and hence the assertion (ii).

$(b) \implies (a)$: The converse is much harder since we have to construct a Gibbs semigroup $\{G_A(t)\}_{t\geq 0}$ staring essentially from condition (ii) for resolvent, cf. (3.3).

By hypothesis, the set $\{U_A(t) := \mathrm{e}^{-tA}\}_{t\geq 0}$ is a (quasi-bounded) C_0-semigroup, which by virtue of (i) is also $\mathscr{C}_q(\mathfrak{H})$-continuous in $\mathbb{R}^+$. Hence, there exists ω_0 such that $(-\infty, -\omega_0) \subset \rho(A)$ and $\{\mathrm{e}^{-tA}\}_{t\in\mathbb{R}^+}$ is by $\mathscr{C}_q(\mathfrak{H}) \subset \mathcal{L}(\mathfrak{H})$ an operator-norm continuous *family* of operators. To proceed we fix $\lambda > \omega_0$ and define by the operator-norm Bochner integral (Proposition 3.1 for $\mathfrak{X} = \mathcal{L}(\mathfrak{H})$) a family $\{F(t)\}_{t\geq 0}$ of bounded operators:

$$F(t) := \int_0^t \mathrm{d}\tau\, \mathrm{e}^{-\lambda\tau}\, U_A(\tau)\,, \quad t \geq 0\,. \tag{3.5}$$

Now after integrating the equation

$$\partial_t(\mathrm{e}^{-\lambda t} U_A(t)\, u) = -\,\mathrm{e}^{-\lambda t}\, U_A(t)\, (\lambda \mathbb{1} + A)\, u\,, \quad u \in \mathrm{dom}\,(A)\,,$$

we obtain

$$(\mathbb{1} - \mathrm{e}^{-\lambda t} U_A(t))\, u = \int_0^t \mathrm{d}\tau\, \mathrm{e}^{-\lambda\tau} U_A(\tau)\, (\lambda\, \mathbb{1} + A)\, u\,,$$

or for $w := (\lambda \mathbb{1} + A)\, u$, the equation

$$(\mathbb{1} - e^{-\lambda t} U_A(t))\, (A + \lambda \mathbb{1})^{-1}\, w = \int_0^t \mathrm{d}\tau\, e^{-\lambda\tau} U_A(\tau)\, w\,. \tag{3.6}$$

Since $-\lambda \in \rho(A)$, the range $\mathrm{ran}\,(\lambda\, \mathbb{1} + A) = \mathfrak{H}$. Therefore, equation (3.6) yields that the family of operators (3.5) belongs to the ideal $\mathscr{C}_q(\mathfrak{H})$:

$$F(t) = (\mathbb{1} - \mathrm{e}^{-\lambda t} U_A(t))\, (A + \lambda\, \mathbb{1})^{-1} \in \mathscr{C}_q(\mathfrak{H})\,, \quad t \geq 0\,, \tag{3.7}$$

because by condition (*ii*), the resolvent $R_{-\lambda}(A) \in \mathscr{C}_q(\mathfrak{H})$ and $F(0) = 0 \cdot \mathbb{1}$. As a consequence

$$F(t+\delta) - F(t) = \int_t^{t+\delta} \mathrm{d}\tau\, e^{-\lambda\tau} U_A(\tau) \in \mathscr{C}_q(\mathfrak{H})\,, \quad t > 0\,,\, \delta > 0\,. \tag{3.8}$$

In condition (*i*), the $\|\cdot\|_q$-continuity of the family $\{U_A(\tau)\}_{\tau>0}$ yields, for $t > 0$ and $\delta > 0$, the estimate

$$\Big\|\frac{1}{\delta}\int_t^{t+\delta} \mathrm{d}\tau\; U_A(\tau) - U_A(t)\Big\|_q \leq \frac{1}{\delta}\int_t^{t+\delta} \mathrm{d}\tau\; \|U_A(\tau) - U_A(t)\|_q \leq \varepsilon_t(\delta)\,, \tag{3.9}$$

where $\varepsilon_t(\delta) := \sup_{\tau\in[t,t+\delta]}\; \|U_A(\tau) - U_A(t)\|_q$ and $\lim_{\delta\downarrow 0}\; \varepsilon_t(\delta) = 0$. Therefore, (3.9) implies that

$$U_A(t) = \;\|\cdot\|_q - \lim_{\delta\downarrow 0} \frac{1}{\delta}\int_t^{t+\delta} \mathrm{d}\tau\; U_A(\tau)\,, \quad t > 0\,. \tag{3.10}$$

Note that by invertibility of generator A ($\ker(A) = \{0\}$) and by virtue of (3.7), (3.8) for $\lambda = 0$ the integral in (3.10) coincides with the operator $F(t+\delta) - F(t)$, which belongs to $\mathscr{C}_q(\mathfrak{H})$. Because the limit (3.10) exists and also holds in $\|\cdot\|_q$-norm for the $\|\cdot\|_q$-valued integrals, in the left-hand side of (3.10) the operator $U_A(t) \in \mathscr{C}_q(\mathfrak{H})$ for

any $t > 0$. By the reason of the semigroup law: $U_A(t) = (U_A(t/q))^q$ and properties of ideals $\{\mathscr{C}_q(\mathfrak{H})\}_{1\le q<\infty}$, this implies that $U_A(t) \in \mathscr{C}_1(\mathfrak{H})$ for $t > 0$ and consequently the C_0-semigroup $\{U_A(t) =: G_A(t)\}_{t\ge 0}$ is a Gibbs semigroup.

Seeing that by (3.10) the family $\{U_A(t)\}_{t>0}$ belongs to $\mathscr{C}_q(\mathfrak{H})$ and that it is $\mathscr{C}_q(\mathfrak{H})$-continuous function by condition (i), the integrand in (3.5) is $\mathscr{C}_q(\mathfrak{H})$-continuous. Then for $\omega_0 < \lambda$, we can construct the representation of resolvent $R_A(z = -\lambda)$ as the $\mathscr{C}_q(\mathfrak{H})$-Bochner integral (Laplace transform). To this end it is sufficient to take in (3.5) and (3.7) the $\|\cdot\|_q$-$\lim_{t\to\infty} F(t)$, which then can be extended to $z \in \rho(A)$. As a result we obtain that

$$\|R_A(z)\|_q = \Big\| \int_0^\infty \mathrm{d}t\, \mathrm{e}^{\,t\,\mathrm{Re}\,z}\, \mathrm{e}^{\,-tA} \Big\|_q\,, \quad \mathrm{Re}\,z < -\omega_0\,, \tag{3.11}$$

where the left-hand side of (3.11) is finite by condition (ii). By the same condition, similarly to (3.3) and (3.4), we also obtain the $\mathscr{C}_1(\mathfrak{H})$-Bochner integral for the power q of Laplace transform, which by Fubini's theorem gives

$$\|(R_A(z))^q\|_1 = \Big\|\Gamma(q)^{-1} \int_0^\infty \mathrm{d}t\, t^{q-1}\, \mathrm{e}^{\,-tA}\, e^{t\,\mathrm{Re}\,z} \Big\|_1 < \infty\,, \quad \text{when } \mathrm{Re}\,z < -\omega_0\,. \tag{3.12}$$

Since by Proposition 3.1 the integrand in (3.12) is $\|\cdot\|_1$-integrable, one gets

$$\Gamma(q)^{-1} \int_0^\infty \mathrm{d}t\, t^{q-1}\, \|\,\mathrm{e}^{\,-tA}\|_1\, \mathrm{e}^{\,t\,\mathrm{Re}\,z} < \infty\,, \quad \text{when } \mathrm{Re}\,z < -\omega_0\,, \tag{3.13}$$

and hence, the asymptotics: $\|\,\mathrm{e}^{\,-tA} = G_A(t)\,\|_1 = O(t^{-p})$, for $t \downarrow 0$ where $p < q$. □

Corollary 3.4. *The assertion of Theorem* 3.2 *holds for any real $q > p$, where $p \ge 1$.*

Proof. The key point is to show that fractional power of a resolvent can be defined by the right-hand side of representation (3.3) for any real $q > p$. First we introduce the function

$$F_z:\ (p,\infty) \ni q\ \mapsto\ \Gamma(q)^{-1} \int_0^\infty \mathrm{d}t\, t^{q-1}\, e^{-t\,(A-z\mathbb{1})}\,, \quad \mathrm{Re}\,z < -\omega_0\,, \tag{3.14}$$

where $t \mapsto t^{q-1} := e^{(q-1)\,\ln(t)}$ is the first branch of the fractional power function corresponding to the branch $t \mapsto \ln(t)$ of logarithm on $\mathbb{R}^+$. On account of (3.12) and (3.13) the function (3.14) is well-defined and $\|\cdot\|_1$-continuous on $(p, +\infty)$.

For an integer $q > p$, definitions (3.3) and (3.14) yield $R_A(z)^q = F_z(q)$ with composition law for integers q_1, q_2 : $R_A(z)^{q_1} R_A(z)^{q_2} = R_A(z)^{q_1+q_2}$. To ensure that the representation (3.14) defines the *fractional powers* of the resolvent, one has to show that F_z satisfies the *composition law*:

$$F_z(q_1)\, F_z(q_2) = F_z(q_1 + q_2)\,, \quad \mathrm{Re}\,z < -\omega_0\,, \text{ for any real } q_i > p. \tag{3.15}$$

After a change of variables: $t_1 = x^2$, $t_2 = y^2$ and then $x = r\cos\theta$, $y = r\sin\theta$, we obtain:

$$F_z(q_1)\, F_z(q_2) = \Gamma(q_1)^{-1}\Gamma(q_2)^{-1} \int_0^\infty \mathrm{d}t_1 \int_0^\infty \mathrm{d}t_2\; t_1^{q_1-1}\, t_2^{q_2-1}\, e^{-(t_1+t_2)\,(A-z\mathbb{1})} \tag{3.16}$$
$$= \frac{2}{\Gamma(q_1)\Gamma(q_2)} \int_0^\infty \mathrm{d}r^2\;\; r^{2\,(q_1+q_2)-2}\, e^{-\,r^2\,(A-z\mathbb{1})} \int_0^{\pi/2} \mathrm{d}\theta\, (\cos\theta)^{2\,q_1-1}\, (\sin\theta)^{2\,q_2-1}\,.$$

The last θ-integral is equal to $B(q_1, q_2)/2$, where $B(q_1, q_2) = \Gamma(q_1)\,\Gamma(q_2)/\Gamma(q_1 + q_2)$ is the beta function. Thus, (3.16) and representation (3.14) prove (3.15). □

Remark 3.5. *See [Eck18, Example 2.11].* In Theorem 3.2, we assumed that the generator A is *invertible*, that is, $\ker\,(A) = \{0\}$. Note that if it is *not* the case, the *Mellin transform* $\mathcal{M}$ of $f(t) = \mathrm{Tr}\, e^{-tA}$ that we used previously, cf. (3.13), namely:

$$\mathcal{M}[f](s) = \Gamma(s)^{-1} \int_0^\infty dt\, t^{s-1}\, f(t)\,,$$

does not exist even for a *positive* generator A. Indeed, seeing that for finite dimensional $\ker\,(A)$, we have $\lim_{t\to\infty} \mathrm{Tr}\, e^{-tA} = \dim\ker\,(A)$, the integrand of $\mathcal{M}[f]$ with $f(t) = \mathrm{Tr}\, e^{-tA}$ converges at infinity only for $\mathrm{Re}\, s < 0$, whereas if $\mathrm{Tr}\, e^{-tA} = O_0(t^{-p})$, then the convergence at zero requires that $\mathrm{Re}\, s > p$.
Another way to see this is the following: Let $A' := A + P$ where P is the projector on the (*finite dimensional*) kernel of A. Then operator A' is *invertible* and we obtain

$$\mathrm{Tr}\, e^{-tA} = \mathrm{Tr}\, e^{-tA'} - (e^{-t} - 1)\dim\ker(A)\,. \tag{3.17}$$

Consequently, $\mathrm{Tr}\, e^{-tA} = O_0(t^{-p})$ if and only if $\mathrm{Tr}\, e^{-tA'} = O_0(t^{-p})$ for $p \geq 0$. Owing to the right-hand side by of representation (3.17), we infer that $\mathrm{Tr}\, e^{-tA'}$ has a *Mellin* transform for $\mathrm{Re}\, s > p$, while $\mathcal{M}[t \to e^{-t} - 1](s)$ exists only for $\mathrm{Re}\, s \in (-1, 0)$ (and in this case it is equal to $\Gamma(s)$). This also shows that $\mathcal{M}\,[t > 0 \to \mathrm{Tr}\, e^{-tA}]$ exists *nowhere* in $\mathbb{C}$.

Remark 3.6. To illustrate Theorem 3.2 (and also Proposition 2.2) we return to Example 2.4 of the *eventually* Gibbs (for $t > 1$) C_0-semigroup.
Note that by construction (2.1) the C_0-semigroup $\{G(t)\}_{t\geq 0}$ is an eventually $\mathscr{C}_p$-semigroup for $t > t_p := 1/p$, where $p \in \mathbb{N}$, that is,

$$\{G(t)\}_{t>0} \subset \bigcup_{p\geq 1}^{\infty} \mathscr{C}_p(\mathfrak{H})\,. \tag{3.18}$$

Therefore, $\{G(t)\}_{t\geq 0}$ is a self-adjoint *immediately compact* semigroup, which does not satisfy condition (b) (i) of Theorem 3.2.
As a consequence, the Bochner integral in the Laplace transform (3.2) converges only in the operator-norm topology and defines in the left-hand side a *compact* operator with spectrum: $\sigma(R_A(-\lambda)) = \{(\lambda + \ln(n + 1))^{-1}\}_{n\geq 1}$, here $\lambda \geq 0$. So, condition (b) (ii) of Theorem 3.2 does not hold either.
All together, this bolsters by assertion (a) of Theorem 3.2 that the self-adjoint C_0-semigroup $\{G(t)\}_{t\geq 0}$ defined by (2.1) can not be an *immediately* Gibbs semigroup. On the other hand, as a strongly continuous compact semigroup, $\{G(t)\}_{t\geq 0}$ satisfies

conditions (i) and (ii) of Proposition 2.2, which confirms that it is immediately compact.

For more details about different aspects of the *trace asymptotics* for Gibbs semigroups we refer to our paper [IocZag23]. We conclude by some additional comments concerning Gibbs semigroup construction and Theorem 3.2 together with illustrating examples/contrexamples, which are collected in Section 4.

4 Concluding remarks and comments

We note that a warning similar to that before Proposition 2.2 is also relevant to Theorem 3.2. Indeed, a C_0-semigroup with generator A, which has a trace-class resolvent $\{R_A(z)\}_{z\in\rho(A)}$ on the resolvent set $\rho(A)$, cf. Theorem 3.2 (b) (ii), is not necessarily a Gibbs semigroup. On the other hand, a resolvent of a (eventually) Gibbs semigroup is not necessarily even compact. In fact, to avoid these discrepancies, or triviality, one needs certain supplementary conditions. In our Theorem 3.2 they are expressed, e.g., by condition in (a) for the Gibbs semigroups, or in (b) (i) for the C_0-semigroup. For illustration see Example 4.3 (e) and (f).

(I) A well-known way for construction of the Gibbs semigroups is based on the notion of p-generators [Zag19, Definition 4.26].

Definition 4.1. *An m-sectorial operator A with vertex $\gamma = 0$ and with semi-angle α is called a p-generator if for some $\zeta_0 \in \rho(A)$, the resolvent $R_A(\zeta_0)$ is an element of the von Neumann–Schatten class $\mathscr{C}_p(\mathfrak{H})$ for any finite $p \geq 1$.*

A choice of the vertex $\gamma = 0$ is irrelevant because of a possible shift: $A + \beta\,\mathbb{1}$ for $\beta \in \mathbb{R}$.

A positive self-adjoint operator A with $R_A(\zeta_0) \in \mathscr{C}_{p\geq 1}(\mathfrak{H})$ for $\operatorname{Im}\zeta_0 \neq 0$ is a p-generator for semi-angle $\alpha = 0$.

Proposition 4.2. *Any p-generator A is the generator of a $\|\cdot\|_1$-holomorphic Gibbs semigroup.*

For the proof we refer to [Zag19, Proposition 4.27].

Here we indicate only that an m-sectorial operator A generates a holomorphic semigroup $\{U_A(z)\}_{z\in S_{\pi/2-\alpha}}$ in the open sector $S_{\pi/2-\alpha} = \{z \in \mathbb{C} : |\arg z| < \pi/2 - \alpha\}$. Indeed, thanks to the Riesz–Dunford formula, this semigroup has representation

$$U_A(z) = \frac{1}{2\pi i}\int_\Gamma \mathrm{d}\zeta\, \mathrm{e}^{-z\zeta}\,(\zeta\,\mathbb{1} - A)^{-1}\,. \tag{4.1}$$

Here Γ is a positively oriented contour enclosing the sector $S_\alpha \supset W(A)$, which contains the numerical range $W(A)$ of A, that is, $\Gamma \subset \rho(A) \subset \mathbb{C} \setminus S_\alpha$ and ensures the convergence of integral. Then strong analyticity of (4.1) and Proposition 2.5 imply the $\|\cdot\|_p$- and thus the $\|\cdot\|_1$-analyticity of the Gibbs semigroup $\{U_A(z)\}_{z\in S_{\pi/2-\alpha}} \subset \mathscr{C}_1(\mathfrak{H})$ constructed by (4.1).

The representation (4.1) also allows to establish a semigroup analyticity criterion for an m-sectorial generator A with semi-angle $\alpha < \pi/2$: There exists $M(\alpha) > 0$ such that

$$\|A\, U_A(t)\| \leq M(\alpha)\, t^{-1}\,, \quad t > 0\,. \tag{4.2}$$

(II) We note that the p-generator condition (4.1) is in many cases too restrictive since it is based on analyticity. To bolster characterisation (criterion) of Gibbs semigroups by Theorem 3.2, we give below an example of a infinitely $\|\cdot\|_1$-differentiable non-holomorphic Gibbs semigroup.

Example 4.3. We consider here a list of examples/counterexamples to ilustrate certain assertions of this paper that we started in Remark 3.6.

(a) Consider the Hilbert space $\mathfrak{H} = l^2(\mathbb{C})$. We define in $l^2(\mathbb{C})$ a linear densely defined multiplication operator

$$M_q : u \mapsto q \cdot u = \{q_k\, u_k\}_{k\geq 1}\,, \quad q_k \in \mathbb{C}\,, \tag{4.3}$$

with domain $\mathrm{dom}\,(M_q) = \{u \in l^2(\mathbb{C}) : M_q u \in l^2(\mathbb{C})\}$ for a given vector $q = \{q_k\}_{k\geq 1}$.

(b) By definition (4.3) the spectrum $\sigma(M_q)$ of M_q coincides with the set $\{q_k\}_{k\geq 1}$. If $\sigma(M_q) \subset \mathbb{C}_+$, the closed operator M_q generates the C_0-semigroup $\{U_q(t) = e^{-tM_q}\}_{t\geq 0}$ on the Hilbert space $l^2(\mathbb{C})$:

$$e^{-tM_q}\, u := \mathrm{e}^{-t\,q} \cdot u\,, \quad t \geq 0\,, \quad u \in l^2(\mathbb{C})\,. \tag{4.4}$$

(c) Let the C_0-semigroup generated on $l^2(\mathbb{C})$ by the multiplication operator (4.3) defined by

$$q_\alpha := \{q_k(\alpha) = \mathrm{e}^{\,i\,\alpha}\, k\,\}_{k\geq 1}\,, \quad |\alpha| < \pi/2\,. \tag{4.5}$$

Then the unbounded operator M_{q_α} generates the C_0-semigroup $\{U_{q_\alpha}(t) = \mathrm{e}^{\,-t\,M_{q_\alpha}}\}_{t\geq 0}$ such that for $\|u\| = 1$ and $t > 0$ one gets

$$\|M_{q_\alpha}\, U_{q_\alpha}(t)\, u\|^2 = \sum_{k\geq 1} \mathrm{e}^{\,-\,2\,kt\,\cos\alpha}\, k^2\, |u_k|^2 \leq\; M(\alpha)^2\, t^{-2}\,, \quad M(\alpha) := (e\;\cos\alpha)^{-1}\,.$$

As a consequence, $\|M_{q_\alpha}\, U_{q_\alpha}(t)\| \leq M(\alpha)\, t^{-1}$. Then by (4.2) the semigroup $\{U_{q_\alpha}(t)\}_{t\geq 0}$ may be analytically continued to a bounded $\|\cdot\|$-holomorphic semigroup $\{U_{q_\alpha}(z)\}_{z\in S_\theta}$ in the open sector S_θ, where $\theta = \arcsin\,(e\, M(\alpha))^{-1}$ with $\theta = \pi/2 - \alpha$.

Moreover, for $t > 0$ and $|\alpha| < \pi/2$ the trace-norm

$$\|U_{q_\alpha}(t)\|_1 = \mathrm{Tr}\,(U^*_{q_\alpha}(t)\, U_{q_\alpha}(t))^{1/2} = \sum_{k\geq 1} \mathrm{e}^{\,-\,k\cos\alpha} < \infty\,, \tag{4.6}$$

is finite, that is $U_{q_\alpha}(t) \in \mathscr{C}_1(l^2(\mathbb{C}))$. Then $\{U_{q_\alpha}(z)\}_{z\in S_\theta}$ is a $\|\cdot\|_1$-holomorphic Gibbs semigroup by virtue of Proposition 2.5.

Another way around, we note that by the definition (4.3) and (4.5), the spectrum $\sigma(M_{q_\alpha}) = \{\mathrm{e}^{\,i\,\alpha}\, k\,\}_{k\geq 1}$ for $|\alpha| < \pi/2$, belongs to the closed sector $\overline{S}_\alpha \subset \mathbb{C}_+$. Therefore,

the resolvent $R_{M_{q_\alpha}}(z) \in \mathscr{C}_p(l^2(\mathbb{C}))$ for any $p > 1$, $z \in \rho(M_{q_\alpha})$ and M_{q_α} is an m-sectorial operator. As a consequence, the Riesz–Dunford representation (4.1) holds for the $\|\cdot\|_1$-holomorphic Gibbs semigroup $\{U_{q_\alpha}(z)\}_{z\in S_\theta}$.

(d) Now let

$$q_\beta := \{q_k(\beta) = k + i\,k^\beta\}_{k\geq 1}\,, \quad \beta \geq 0\,, \tag{4.7}$$

that is, the spectrum $\sigma(M_{q_\beta}) = \{k + i\,k^\beta\}_{k\geq 1}$, cf. (b). Then the unbounded operator M_{q_β} generating the C_0-semigroup $\{U_{q_\beta}(t) = \mathrm{e}^{-t\,M_{q_\beta}}\}_{t\geq 0}$ is such that for $u \in l^2(\mathbb{C})$

$$\|M_{q_\beta} U_{q_\beta}(t)\,u\|^2 = \sum_{k\geq 1} \mathrm{e}^{-2\,k\,t}\,(k^2 + k^{2\beta})\,|u_k|^2\,. \tag{4.8}$$

(i) If $\beta \leq 1$, then $\sigma(M_{q_\beta}) \subset \overline{S}_{\alpha=\pi/4}$ and (4.8) yields for some $M > 0$ that

$$\|M_{q_\beta} U_{q_\beta}(t)\| \;\leq\; M\,t^{-1}\,, \quad t > 0\,.$$

Consequently, by (4.2) the semigroup $\{U_{q_\beta}(t)\}_{t\geq 0}$ allows an analytic continuation to a bounded $\|\cdot\|$-holomorphic semigroup $\{U_{q_\alpha}(z)\}_{z\in S_\theta}$ in the open sector $S_{\theta=\pi/2-\alpha}$, where $\alpha = \pi/4$. Since $U_{q_\alpha}(t) \in \mathscr{C}_1(l^2(\mathbb{C}))$, this semigroup is a $\|\cdot\|_1$-holomorphic Gibbs semigroup on account of Proposition 2.5.

Again, similarly to (c) the Riesz–Dunford representation (4.1) also holds for the $\|\cdot\|_1$-holomorphic Gibbs semigroup $\{U_{q_\alpha}(z)\}_{z\in S_\theta}$ for $\beta \leq 1$.

(ii) If $\beta > 1$, then the spectrum $\sigma(M_{q_\beta})$ does not belong to any sector $S_{\alpha<\pi/2}$ and (4.8) yields that operator norm

$$\|M_{q_\beta} U_{q_\beta}(t)\| \;\leq\; M(\beta)\,t^{-\beta}\,, \quad t > 0\,. \tag{4.9}$$

for $M(\beta) > 0$. Hence, although for any $n \in \mathbb{N}$ and $t > 0$ owing to (4.8) one has $\|M^n_{q_\beta} U_{q_\beta}(t)\| < \infty$ (the semigroup $\{U_{q_\beta}(t)\}_{t\geq 0}$ is infinitely operator-norm differentiable for $t > 0$), the inequality (4.9) does not allow any operator-norm analytic continuation of this semigroup to $\mathbb{C}_+$ for the case $\beta > 1$, cf. (4.2) and the case $\beta \leq 1$.

Also, by virtue of Proposition 2.5 the Gibbs semigroup $\{U_{q_\beta}(t)\}_{t\geq 0}$ can not be extended to a holomorphic one in a subset of $\mathbb{C}_+$.

(iii) Note that although for $p > 1$, $z \in \rho(M_{q_\beta})$ and $\beta > 1$ the resolvent $R_{M_{q_\beta}}(z) \in \mathscr{C}_p(l^2(\mathbb{C}))$, the spectrum $\sigma(M_{q_\beta})$ does not belong to any sector $S_{\alpha<\pi/2}$. In particular, the Riesz–Dunford representation (4.1) does not hold either, which bolsters that Gibbs semigroup $\{U_{q_\beta}(t)\}_{t\geq 0}$ has no analytic extension [Zag19, Proposition 4.27].

As a consequence, to construct a holomorphic Gibbs semigroup, one has to ensure, besides the trace-norm differentiability for $t > 0$, the inequality (4.2). Whereas to set up a generic Gibbs semigroup one needs Proposition 2.5 and Theorem 3.2.

(e) For the next example in $\mathfrak{H} = l^2(\mathbb{C})$, we suppose that the closed densely defined multiplication operator M_{q_2} corresponds to

$$q_2 = \{q_k = i\,k^2\}_{k\geq 1}\,, \quad k \in \mathbb{N}\,, \tag{4.10}$$

with spectrum $\sigma(M_{q_2}) = \{i\,k^2\}_{k\geq 1}$, see (b),

(i) Since $\|(\mu\,\mathbb{1} + M_{q_2})^{-1}\| \leq |\mathrm{Re}\,\mu|^{-1}$ for $\mu \in \mathbb{C}$, the operator M_{q_2} satisfies the hypotheses of the Hille–Yosida theorem [HiPh57, Section 12.3]. Therefore, $\{U_{q_2}(t) := \mathrm{e}^{-t\,M_{q_\beta}}\}_{t\geq 0}$ is a *strongly* continuous contraction semigroup. Note that for $t \gtrless 0$ it gives two branches of a unitary group, which is *nowhere* continuous in *operator-norm* topology for $t \in \mathbb{R}$.

(ii) The resolvent $R_{M_{q_2}}(-\mu)$, for $-\mu \in \rho(M_{q_2})$, is a bounded operator with discrete spectrum $\sigma(R_{M_{q_2}}(-\mu)) = \{(\mu + i\,k^2)^{-1}\}_{k\geq 1}$. For $\mu_0 \in \mathbf{C}$ with $\mathrm{Im}\,\mu_0 = 0$, the singular values are

$$s_k(R_{M_{q_2}}(\mu_0)) = (\mu_0^2 + k^4)^{-1/2}\,, \quad k \in \mathbb{N}\,, \tag{4.11}$$

and consequently $R_{M_{q_2}}(\mu_0) \in \mathscr{C}_1(\mathfrak{H})$, which by the first resolvent equation yields $R_{M_{q_2}}(\mu) \in \mathscr{C}_1(\mathfrak{H})$ for any $\mu \in \rho(M_{q_2})$.

(iii) As a result, although the resolvent of generator M_{q_2} consists of trace-norm operators, the semigroup $\{U_{q_2}(t)\}_{t\geq 0}$ is not a Gibbs semigroup since it is only strongly continuous. We note that for this semigroup the Bochner integral representation (Laplace transform, cf. (3.2)):

$$R_A(-\lambda) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\lambda\,t}\, \mathrm{e}^{-t\,A}\,, \quad \mathrm{Re}\,\lambda > 0\,, \tag{4.12}$$

is defined only in the *strong operator topology*, which implies that the left-hand side of (4.12) is simply a bounded operator.

On that account, the condition (b) (i) in Theorem 3.2 serves for improving topology in definition (4.12) of Bochner integral with the aim to exclude this kind of insufficiency.

(f) Now let Hilbert space $\mathfrak{H} = L^2(0,1)$. For $t \geq 0$ and $f \in \mathfrak{H}$, define $U(t) \in \mathcal{L}(\mathfrak{H})$ by

$$(U(t)f)(s) = \begin{cases} f(s+t) & \text{if } \ 0 \leq s+t \leq 1, \\ 0 & \text{if } \ t > 1\,. \end{cases}$$

Then $\{U(t)\}_{t\geq 0}$ is a C_0-semigroup, which is nilpotent, hence an eventually (compact) Gibbs semigroup with $\mathrm{Tr}\,U(t > 1) = 0$. Since the Laplace transform (4.12) defines a resolvent of generator of semigroup by the strongly convergent Bochner integral, we obtain

$$R(-z) = \int_0^1 \mathrm{d}t\, \mathrm{e}^{-z\,t}\, \mathrm{e}^{-t\,A}\,, \quad z \in \mathbb{C}\,. \tag{4.13}$$

As a consequence, the resolvent (4.13) of operator A is entire function of $z \in \mathbb{C}$, that is, the spectrum of generator is *empty*. So, for any $z \in \mathbb{C}$ the operator $R(-z)$ is not in the von Neumann–Schatten ideals.

It is instructive to compare this illustration with Example 2.4 of eventually Gibbs semigroup on a Hilbert space and with discussion in Remark 3.6.

Acknowledgments
This paper was motivated by a lecture given by V.A. Z. on the International Conference dedicated to the 100th anniversary of the birthday of Academician V.S. Vladimirov, that held on the 9th–14th January 2023, at Steklov Mathematical Institute, Moscow. He is very thankful to the Organising Committee and the Chair Professor Igor Volovich for invitation to deliver this talk.

Both of the authors are also grateful for encouragement to submit the contribution based on the lecture for publication in the journal of ”Theoretical and Mathematical Physics”.

Conflict of Interest: The authors declare that they have no conflicts of interest.

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