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Différentes méthodes pour obtenir la forme polaire d'une matrice de Lorentz

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***Different methods for obtaining
the polar form of a Lorentz matrix M under the form :***

$$M = \begin{bmatrix} 1 + \frac{\gamma^2}{(1+\gamma)} \beta_1^2 & \frac{\gamma^2}{(1+\gamma)} \beta_1 \beta_2 & \frac{\gamma^2}{(1+\gamma)} \beta_1 \beta_3 \\ \frac{\gamma^2}{(1+\gamma)} \beta_2 \beta_1 & 1 + \frac{\gamma^2}{(1+\gamma)} \beta_2^2 & \frac{\gamma^2}{(1+\gamma)} \beta_2 \beta_3 \\ \frac{\gamma^2}{(1+\gamma)} \beta_3 \beta_1 & \frac{\gamma^2}{(1+\gamma)} \beta_3 \beta_2 & 1 + \frac{\gamma^2}{(1+\gamma)} \beta_3^2 \end{bmatrix} \cdot \Omega$$

*Summary : We present three different methods
to obtain the polar form of a Lorentz matrix.
We provide references for two additional methods.*

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If the matrix $M = (m^j_i)_{0 \leq i \leq n, 0 \leq j \leq n}$ we note ${}^tM = (m^i_j)_{0 \leq i \leq n, 0 \leq j \leq n}$.

A Lorentz matrix M is defined as a square matrix satisfying ${}^tMGM = G$, where $G = (g^i_j)$ with $g^i_j = 0$ si $i \neq j$, $g^1_1 = 1$, $g^i_i = -1$ for $i > 1$.

If M is a 4th-order matrix, it represents the transformation matrix from a spacetime basis of an observer O to another spacetime basis associated with another observer O' , moving relative to O with a uniform rectilinear motion of velocity \vec{V} . We aim to identify the terms of this matrix M as a function of \vec{V} or $\vec{\beta} = \frac{\vec{V}}{c}$, c the lightspeed.

First method :

The first method is a consequence of the theorems found in:
J.-M. Souriau. "Calcul Linéaire" * PUF 1964 * p.378.

Theorem 1 :

Let $X \in \mathbf{R}^n$ such that ${}^tXX = 1$.

Let N be the matrix defined by:

$$\mathbf{O}_{n-1} \text{ being the zero matrix of } \mathbf{M}_{n-1}(\mathbf{R}), N = \begin{bmatrix} 0 & {}^tX \\ X & \mathbf{O}_{n-1} \end{bmatrix}.$$

Then, $\forall \alpha \in \mathbf{R}$:

$$M = \exp(\alpha N) = \begin{bmatrix} ch(\alpha) & sh(\alpha) {}^tX \\ sh(\alpha) X & (Id_{\mathbf{R}^{n-1}} + (ch(\alpha) - 1) X {}^tX) \end{bmatrix}$$

is a Lorentz matrix of determinant equal to 1.

Proof :

We have $N^2 = \begin{bmatrix} 1 & 0 \\ 0^{n-1} & X {}^tX \end{bmatrix}$ by noting 0^{n-1} as the zero column of order $n-1$,

since ${}^tXX = 1$ and then we have $N^3 = N$.

By noting that for any square matrix A and $\forall \alpha \in \mathbf{R}$,

$$\exp(A) = \sum_{n=0}^{+\infty} \frac{A^n}{n!}, \quad sh(\alpha) = \sum_{n=0}^{+\infty} \frac{\alpha^{2n+1}}{(2n+1)!}, \quad ch(\alpha) = \sum_{n=0}^{+\infty} \frac{\alpha^{2n}}{(2n)!}$$

therefore

$$\begin{aligned} \exp(\alpha N) &= Id_{\mathbf{R}^n} + \alpha N + \frac{\alpha^2 N^2}{2!} + \frac{\alpha^3 N^3}{3!} + \frac{\alpha^4 N^4}{4!} + \dots + \\ &+ \frac{\alpha^{2p-1} N^{2p-1}}{(2p-1)!} + \frac{\alpha^{2p} N^{2p}}{(2p)!} + \dots \end{aligned}$$

$$= Id_{\mathbf{R}^1} + sh(\alpha)N + (ch(\alpha) - 1)N^2.$$

Now we can verify if M is indeed a Lorentz transformation: ${}^tMGM = G$?

It is worth noting that M is symmetric: ${}^tM = M$.

$$MG = \begin{bmatrix} ch(\alpha) & sh(\alpha) {}^tX \\ sh(\alpha)X & (Id_{\mathbf{R}^1-1} + (ch(\alpha) - 1)X^tX) \end{bmatrix} \begin{bmatrix} 1 & {}^t0 \\ 0 & -Id_{\mathbf{R}^1-1} \end{bmatrix}$$

$$= \begin{bmatrix} ch(\alpha) & -sh(\alpha) {}^tX \\ sh(\alpha)X & -(Id_{\mathbf{R}^1-1} + (ch(\alpha) - 1)X^tX) \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} ch(\alpha) & -sh(\alpha) {}^tX \\ sh(\alpha)X & -(Id_{\mathbf{R}^1-1} + (ch(\alpha) - 1)X^tX) \end{bmatrix} \begin{bmatrix} ch(\alpha) & sh(\alpha) {}^tX \\ (sh(\alpha))X & (Id_{\mathbf{R}^1-1} + (ch(\alpha) - 1)X^tX) \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad : \quad \text{Let's calculate } a, b, c, \text{ and } d.$$

$$a = ch^2(\alpha) - sh^2(\alpha) {}^tXX = 1, \text{ because } {}^tXX = 1.$$

$$\begin{aligned} b &= ch(\alpha) sh(\alpha) {}^tX - sh(\alpha) {}^tX (Id_{\mathbf{R}^1-1} + (ch(\alpha) - 1)X^tX) \\ &= ch(\alpha) sh(\alpha) {}^tX - sh(\alpha) {}^tX - sh(\alpha)(ch(\alpha) - 1) {}^tXX^tX \\ &= ch(\alpha) sh(\alpha) {}^tX - sh(\alpha) {}^tX - sh(\alpha)ch(\alpha) {}^tX + sh(\alpha) {}^tX = 0, \end{aligned}$$

$$\begin{aligned} c &= ch(\alpha) sh(\alpha) X - (Id_{\mathbf{R}^1-1} + (ch(\alpha) - 1)X^tX)sh(\alpha) X \\ &= ch(\alpha) sh(\alpha) X - sh(\alpha) X - sh(\alpha)(ch(\alpha) - 1)X = 0, \end{aligned}$$

$$d = sh^2(\alpha)X^tX - (Id_{\mathbf{R}^1-1} + (ch(\alpha) - 1)X^tX)^2 \quad \text{but:}$$

$$(Id_{\mathbf{R}^1-1} + (ch(\alpha) - 1)X^tX)(Id_{\mathbf{R}^1-1} + (ch(\alpha) - 1)X^tX)$$

$$= Id_{\mathbf{R}^1-1} + (ch(\alpha) - 1)X^tX + (ch(\alpha) - 1)X^tX + (ch(\alpha) - 1)^2 X^tXX^tX,$$

(It can be observed that as ${}^tXX = 1$: $X({}^tXX) {}^tX = X$ (using matrix multiplication associativity),

$$\text{and } (ch(\alpha) - 1)^2 = ch^2(\alpha) - 2ch(\alpha) + 1)$$

$$= Id_{\mathbf{R}^1-1} + (2(ch(\alpha) - 1) + ch^2(\alpha) - 2ch(\alpha) + 1)X^tX$$

$$= Id_{\mathbf{R}^i-1} + (ch^2(\alpha) - 1)X^tX = Id_{\mathbf{R}^i-1} + sh^2(\alpha)X^tX$$

hence :

$$sh^2(\alpha)X^tX - (Id_{\mathbf{R}^i-1} + (ch(\alpha) - 1)X^tX)^2 = -Id_{\mathbf{R}^i-1}.$$

Consequently
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & {}^t\theta \\ 0 & -Id_{\mathbf{R}^i-1} \end{bmatrix}$$

and M is a Lorentz matrix.

Another proof :

$$\begin{aligned} \exp(\alpha N)G &= \begin{bmatrix} ch(\alpha) & sh(\alpha) {}^tX \\ sh(\alpha)X & (Id_{\mathbf{R}^i-1} + (ch(\alpha) - 1)X^tX) \end{bmatrix} \begin{bmatrix} 1 & {}^t\theta \\ 0 & -Id_{\mathbf{R}^i-1} \end{bmatrix} \\ &= \begin{bmatrix} ch(\alpha) & -sh(\alpha) {}^tX \\ sh(\alpha)X & -(Id_{\mathbf{R}^i-1} + (ch(\alpha) - 1)X^tX) \end{bmatrix} \text{ and} \end{aligned}$$

$$\begin{aligned} G\exp(-\alpha N) &= \begin{bmatrix} 1 & {}^t\theta \\ 0 & -Id_{\mathbf{R}^i-1} \end{bmatrix} \begin{bmatrix} ch(\alpha) & -sh(\alpha) {}^tX \\ -sh(\alpha)X & (Id_{\mathbf{R}^i-1} + (ch(\alpha) - 1)X^tX) \end{bmatrix} \\ &= \begin{bmatrix} ch(\alpha) & -sh(\alpha) {}^tX \\ sh(\alpha)X & -(Id_{\mathbf{R}^i-1} + (ch(\alpha) - 1)X^tX) \end{bmatrix} \end{aligned}$$

hence $\exp(\alpha N)G = G\exp(-\alpha N) \Rightarrow \exp(\alpha N)G\exp(\alpha N) = G$

as $M = \exp(\alpha N)$ is symmetric : ${}^tMGM = G$ and M is a Lorentz matrix.

Lemma :

If M is a Lorentz matrix and its first column K_1 has the form : $K_1 = {}^t(\alpha, 0, \dots, 0)$, then M has the form :

$$M = \begin{bmatrix} \varepsilon & {}^tQ \\ Q & \Omega \end{bmatrix} \quad \text{où } \varepsilon = \pm 1, {}^tQ = (0, 0, 0), {}^t\Omega\Omega = Id_{\mathbf{R}^i-1} \quad (1).$$

Conversely, any matrix of this form (1) is a Lorentz matrix.

Proof :

Let $M = \begin{bmatrix} \alpha & {}^tL \\ 0 & C \end{bmatrix}$ with $C \in \mathbf{M}_3(\mathbf{R})$, ${}^tL = (l_1, l_2, l_3)$.

Since $({}^tMG)M = G$, we have :

$$G = \begin{bmatrix} \alpha & 0 \\ L & -{}^tC \end{bmatrix} \begin{bmatrix} \alpha & {}^tL \\ 0 & C \end{bmatrix} = \begin{bmatrix} \alpha^2 & \alpha {}^tL \\ \alpha L & L {}^tL - {}^tCC \end{bmatrix}.$$

By identification, we obtain $\alpha = \pm 1$, $L = 0$, ${}^tCC = Id_{\mathbf{R}^n - 1}$

If M is of the form (1), an immediate calculation shows that ${}^tMGM = G$.

Theorem 2 :

Any Lorentz matrix M can be written in the form :

$$M = \exp(\alpha N) \cdot \begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix} \text{ where } \alpha \in \mathbf{R}, N = \begin{bmatrix} 0 & {}^tX \\ X & 0 \end{bmatrix}, X \in \mathbf{R}^{n-1}, \varepsilon = \pm 1,$$

such as : ${}^tXX = 1$, ${}^t\Omega\Omega = Id_{\mathbf{R}^n - 1}$.

Proof:

Let M be a Lorentz matrix $M = (m[i, j])$

and $M[I] = (m[i, 1])$ be the first column of M then:

$${}^tM_1GM_1 = m_{1,1}^2 - m_{1,2}^2 - \dots - m_{1,n}^2 = G_{1,1} = 1$$

We can set ${}^tM_1 = (\beta, Y)$ avec $\beta = m_{1,1}$ and ${}^tY = (m_{1,2}, \dots, m_{1,n})$. Thus, we have $\beta^2 - {}^tYY = 1$, so $|\beta| \geq 1$.

We can set $\beta = \varepsilon \cdot \text{ch}(\alpha)$, where $\varepsilon = \pm 1$ and $\alpha \in \mathbf{R}$.

Additionally, ${}^tYY = \beta^2 - 1 = \text{sh}^2(\alpha)$.

If $\alpha \neq 0$, let's set $X = \frac{Y}{\varepsilon \cdot \text{sh}(\alpha)}$, then we have ${}^tXX = 1$.

If $\alpha = 0$, we have ${}^tM_1 = (\varepsilon, 0, 0, 0)$.

In all cases, ${}^tM_1 = \varepsilon(\text{ch}(\alpha), \text{sh}(\alpha)X)$ with ${}^tXX = 1$.

According to lemma, $M[I]$ is the first column of $\varepsilon \cdot \exp(\alpha N)$ where $N = \begin{bmatrix} 0 & {}^tX \\ X & 0 \end{bmatrix}$.

We notice that $\exp(\alpha N) \cdot M$ is a Lorentz matrix since M and $\exp(-\alpha N)$ are.

Let's calculate the first column $K[I]$ of this product :

$$K[I] = \begin{bmatrix} \text{ch}(\alpha) & (\text{sh}(\alpha)) {}^tX \\ (\text{sh}(\alpha))X & Id_{\mathbf{R}^n} + (\text{ch}(\alpha) - 1)X {}^tX \end{bmatrix} \cdot M[I]$$

By noting that ${}^tXX = 1$, $\text{ch}^2(\alpha) - \text{sh}^2(\alpha) = 1$, and

$$(Id_{\mathbf{R}^n} + (\text{ch}(\alpha) - 1)X {}^tX) \cdot X = \text{ch}(\alpha) \cdot X,$$

we obtain : ${}^tK_1 = (\varepsilon, 0, \dots, 0)$.

We can then apply the lemma, and the theorem is proven.

Unicity of the decomposition :

The decomposition of the previous theorem is unique.

Proof : It follows from the classical results that follow :

Lemma :

Any square symmetric matrix A of order $n \geq 1$, with real coefficients, is diagonalizable.

Moreover the eigenspaces of A are pairwise orthogonal. Therefore, we can write $A = PD^tP$, where P is an orthogonal matrix and D is a diagonal matrix.

Lemma :

Let M be an invertible matrix with real coefficients

. Then there exists a unique pair of matrices O (orthogonal) and S (symmetric positive definite) such that $M = SO$.

Proof : The proof of this lemma can be found at the following link : https://en.wikipedia.org/wiki/Polar_decomposition.

This lemma implies that the decomposition in the theorem is unique.

For more details, refer to the following link : https://en.wikipedia.org/wiki/Matrix_exponential.

Now let's apply these lemmas :

It is sufficient to verify that $\exp(\alpha N)$ is positive definite. Since αN is a real symmetric matrix, there exists a real orthogonal matrix U and a real diagonal matrix $D = (d_{i,i})$ such that $\alpha N = {}^tUDU$.

As $\exp(\alpha N) = \exp({}^tUDU) = {}^tU \exp(D) U$, the eigenvalues of $\exp(\alpha N)$ are strictly positive, and $\exp(\alpha N)$ is positive definite. From the formula $\det(e^A) = e^{\text{Tr}(A)}$, we deduce that $\det(\exp(\alpha N)) = 1$,

and $\det(M) = \epsilon \cdot \det(O) = \pm 1$.

Application :

Let M be a Lorentz matrix, we have the following equivalences :

M is symmetric $\Leftrightarrow M = \exp(\alpha N)$

M is orthogonal $\Leftrightarrow M = \begin{bmatrix} \epsilon & 0 \\ 0 & \Omega \end{bmatrix}$

For the direct parts, it is sufficient to note that :

$$M = \exp(\alpha N) \cdot \begin{bmatrix} \epsilon & 0 \\ 0 & \Omega \end{bmatrix} \text{ and } M = M \cdot Id \text{ or } M = Id \cdot M.$$

Consequence :

We can now specify the polar form. By taking the previous notations, we will show that :

If we define \vec{V} as the velocity of the observer O' relative to the observer O , c as the speed of light, and $\vec{\beta} = \vec{V}/c$. We have if $\gamma = \frac{1}{\sqrt{1 - \beta^2}}$:

$$\exp(\alpha N) = \begin{bmatrix} \gamma & {}^t[\vec{\gamma\beta}] \\ [\vec{\gamma\beta}] & C \end{bmatrix} \text{ with } [\vec{\gamma\beta}] \text{ the coordinates of } \vec{\gamma\beta} \text{ in the frame of the observer } \mathbf{O} ,$$

$$\vec{\beta} = th(\alpha)\vec{X}, N = \begin{bmatrix} 0 & {}^tX \\ X & \mathbf{O}_{n-1} \end{bmatrix}, \gamma = ch(\alpha),$$

$$C = \begin{bmatrix} 1 + \frac{\gamma^2}{(1+\gamma)}\beta_1^2 & \frac{\gamma^2}{(1+\gamma)}\beta_1\beta_2 & \frac{\gamma^2}{(1+\gamma)}\beta_1\beta_3 \\ \frac{\gamma^2}{(1+\gamma)}\beta_2\beta_1 & 1 + \frac{\gamma^2}{(1+\gamma)}\beta_2^2 & \frac{\gamma^2}{(1+\gamma)}\beta_2\beta_3 \\ \frac{\gamma^2}{(1+\gamma)}\beta_3\beta_1 & \frac{\gamma^2}{(1+\gamma)}\beta_3\beta_2 & 1 + \frac{\gamma^2}{(1+\gamma)}\beta_3^2 \end{bmatrix}.$$

Proof :

Let us consider $W' = \begin{bmatrix} ct' \\ 0 \\ 0 \\ 0 \end{bmatrix}$ as the coordinates of \mathbf{O}' in the frame of the observer \mathbf{O}' .

Also, let $W = MW' = \exp(\alpha N) \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{\Omega} \end{bmatrix} W'$ be the coordinates of \mathbf{O}' in the frame of the observer \mathbf{O} , since M is the transformation matrix from the frame of the observer \mathbf{O} to the frame of observer \mathbf{O}' ,

which are also equal to $\begin{bmatrix} ct \\ c\beta_1 \\ c\beta_2 \\ c\beta_3 \end{bmatrix}$. We first observe that

$$\begin{bmatrix} 1 & 0 \\ 0 & \mathbf{\Omega} \end{bmatrix} W' = W' \text{ thus } W = \exp(\alpha N) W'. \text{ We are led to :}$$

$$\begin{bmatrix} ch(\alpha) & sh(\alpha) {}^t X \\ sh(\alpha) X & (Id_{\mathbb{R}^3} + (ch(\alpha) - 1) X {}^t X) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \stackrel{t'}{=} \begin{bmatrix} 1 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} t, \text{ and}$$

$$\begin{bmatrix} ch(\alpha) \\ sh(\alpha) X_1 \\ sh(\alpha) X_2 \\ sh(\alpha) X_3 \end{bmatrix} \stackrel{t'}{=} \begin{bmatrix} 1 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} t \text{ and } t = ch(\alpha) t' \text{ and consequently:}$$

$$ch(\alpha) \begin{bmatrix} 1 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} ch(\alpha) \\ sh(\alpha) X_1 \\ sh(\alpha) X_2 \\ sh(\alpha) X_3 \end{bmatrix} \text{ for } t' \neq 0 \Rightarrow ch(\alpha) \vec{\beta} = sh(\alpha) \vec{X},$$

thus $\vec{\beta} = th(\alpha) \vec{X} \Rightarrow \vec{\beta}^2 = th^2(\alpha)$ since $\vec{X}^2 = 1$.

We put $\beta = \sqrt{\vec{\beta}^2}$ et $\gamma = \frac{1}{\sqrt{1 - \vec{\beta}^2}}$.

$$\text{As } 1 - th^2(\alpha) = \frac{1}{ch^2(\alpha)} \Rightarrow ch^2(\alpha) = \frac{1}{1 - th^2(\alpha)} = \frac{1}{1 - \beta^2} = \gamma^2,$$

$$\text{as } ch(\alpha) \geq 1 \quad \gamma = ch(\alpha); \text{ as } sh^2(\alpha) = ch^2(\alpha) - 1 = \gamma^2 - 1 \\ = \frac{1}{1 - \beta^2} - 1 = \frac{\beta^2}{1 - \beta^2} = \gamma^2 \beta^2.$$

For short we have $\gamma = ch(\alpha)$, $\gamma^2 \beta^2 = sh^2(\alpha)$, $\beta^2 = th^2(\alpha)$.

Furthermore:

$$\gamma^2 \beta^2 = \gamma^2 - 1 = (\gamma + 1)(\gamma - 1) \Rightarrow \frac{\gamma^2 \beta^2}{(1 + \gamma)} = (\gamma - 1) = ch(\alpha) - 1$$

$$\text{et } X_i X_j = \frac{(\beta_i \beta_j)}{th^2(\alpha)} = \frac{(\beta_i \beta_j)}{\beta^2} \text{ consequently}$$

$$(ch(\alpha) - 1) X_i X_j = \frac{\gamma^2 \beta^2}{(1 + \gamma)} \frac{(\beta_i \beta_j)}{\beta^2} = \frac{\gamma^2}{(1 + \gamma)} \beta_i \beta_j, \text{ that implies:}$$

$$Id_{\mathbb{R}^3} + (ch(\alpha) - 1)X^tX = \begin{bmatrix} 1 + \frac{\gamma^2}{(1+\gamma)}\beta_1^2 & \frac{\gamma^2}{(1+\gamma)}\beta_1\beta_2 & \frac{\gamma^2}{(1+\gamma)}\beta_1\beta_3 \\ \frac{\gamma^2}{(1+\gamma)}\beta_2\beta_1 & 1 + \frac{\gamma^2}{(1+\gamma)}\beta_2^2 & \frac{\gamma^2}{(1+\gamma)}\beta_2\beta_3 \\ \frac{\gamma^2}{(1+\gamma)}\beta_3\beta_1 & \frac{\gamma^2}{(1+\gamma)}\beta_3\beta_2 & 1 + \frac{\gamma^2}{(1+\gamma)}\beta_3^2 \end{bmatrix} = C$$

As $sh(\alpha)X_i = \frac{sh(\alpha)\beta_i}{th(\alpha)} = ch(\alpha)\beta_i = \gamma\beta_i$, we have finally :

$$\exp(\alpha N) = \begin{bmatrix} \gamma & {}^t[\gamma\vec{\beta}] \\ [{}^t\gamma\vec{\beta}] & C \end{bmatrix} \text{ with } [{}^t\gamma\vec{\beta}] \text{ the coordinates of } \vec{\gamma\beta}.$$

We notice that the inverse of $\exp(\alpha N)$ is $\exp(-\alpha N)$, simply by replacing $\vec{\beta}$ with $-\vec{\beta}$ to obtain the inverse while noting that γ and C remain unchanged, since : $\exp(\alpha N) \cdot \exp(-\alpha N) = \exp(0) = Id$

if $\vec{\beta}$ is parallel to the space coordinate of \vec{i} then only the diagonal term of C are nonzero : 1

$$+ \frac{\gamma^2}{(1+\gamma)}\beta_r^2, 1, 1 \text{ and}$$

$$1 + \frac{\gamma^2}{(1+\gamma)}\beta_1^2 = \frac{(1+\gamma) + \gamma^2\beta_1^2}{(1+\gamma)} = \frac{1+\gamma+\gamma^2-1}{(1+\gamma)} = \gamma \text{ because}$$

$$\gamma^2 - 1 = \frac{1}{1-\beta^2} - 1 = \frac{\beta^2}{1-\beta^2} = \gamma^2\beta^2.$$

Other forms of expression of matrix M :

We can also write after the above;

$$M = \begin{bmatrix} \varepsilon \cdot \gamma & {}^t[{}^t\gamma\vec{\beta}] \cdot \Omega \\ \varepsilon \cdot [{}^t\gamma\vec{\beta}] & C \cdot \Omega \end{bmatrix} = \exp \left(Argch(\gamma) \begin{bmatrix} 0 & \frac{{}^t[{}^t\gamma\vec{\beta}]}{\beta} \\ \frac{[{}^t\gamma\vec{\beta}]}{\beta} & \mathbf{O} \end{bmatrix} \right) \cdot \begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix}.$$

Another form of the polar form :

Furthermore if $\varepsilon = 1$:

Let's write : $\Lambda(\vec{\beta}) = \begin{bmatrix} \gamma & {}^t[\gamma\vec{\beta}] \\ [\gamma\vec{\beta}] & C([\vec{\beta}]) \end{bmatrix}$ and $\tilde{\Omega} = \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}$.

We have

$$M = \Lambda(\vec{\beta}) \cdot \tilde{\Omega} = \tilde{\Omega} \cdot ({}^t\tilde{\Omega} \cdot \Lambda(\vec{\beta}) \cdot \tilde{\Omega}) = \tilde{\Omega} \cdot \Lambda({}^t\Omega[\vec{\beta}]) \text{ because :}$$

$$\begin{aligned} {}^t\tilde{\Omega} \cdot \Lambda(\vec{\beta}) \cdot \tilde{\Omega} &= \begin{bmatrix} 1 & 0 \\ 0 & {}^t\Omega \end{bmatrix} \cdot \begin{bmatrix} \gamma & {}^t[\gamma\vec{\beta}] \\ [\gamma\vec{\beta}] & C([\vec{\beta}]) \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} \\ &= \begin{bmatrix} \gamma & {}^t[\gamma\vec{\beta}] \\ {}^t\Omega \cdot [\gamma\vec{\beta}] & {}^t\Omega \cdot C \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} = \begin{bmatrix} \gamma & {}^t[\gamma\vec{\beta}] \cdot \Omega \\ {}^t\Omega \cdot [\gamma\vec{\beta}] & {}^t\Omega \cdot C \cdot \Omega \end{bmatrix} = \begin{bmatrix} \gamma & \gamma \cdot {}^t({}^t\Omega[\vec{\beta}]) \\ \gamma \cdot {}^t\Omega \cdot [\vec{\beta}] & {}^t\Omega \cdot C \cdot \Omega \end{bmatrix} \end{aligned}$$

as $C = Id + \frac{\gamma^2}{(1+\gamma)} [\vec{\beta}] \cdot {}^t[\vec{\beta}]$ we have :

$${}^t\Omega \cdot C \cdot \Omega = Id + \frac{\gamma^2}{(1+\gamma)} {}^t\Omega \cdot [\vec{\beta}] \cdot {}^t[\vec{\beta}] \cdot \Omega = Id + \frac{\gamma^2}{(1+\gamma)} ({}^t\Omega[\vec{\beta}]) \cdot ({}^t\Omega[\vec{\beta}]).$$

Independence of Ω with respect to $\vec{\beta}$ when $\varepsilon = +1$:

Now let's show that Ω does not depend on $\vec{\beta}$. We assume $\varepsilon = +1$ (non - time inversion).

Let M be a Lorentz matrix of order 4; it represents the transformation from a spacetime basis of an observer O to another spacetime basis associated with another observer O' ,

who is in relative motion with respect to O , moving with a uniform velocity \vec{V} .

More precisely : Let $\mathcal{B}(\vec{c\tau}, \vec{i}, \vec{j}, \vec{k})$ be the basis associated with observer O ,

and $\mathcal{B}'(\vec{c\tau}', \vec{i}', \vec{j}', \vec{k}')$ be the basis associated with observer O' .

$M = \mathcal{P}(\mathcal{B}(\vec{c\tau}, \vec{i}, \vec{j}, \vec{k}), \mathcal{B}'(\vec{c\tau}', \vec{i}', \vec{j}', \vec{k}'))$ is the change - of - basis matrix .

A change - of - basis matrix $\mathcal{P}(\mathcal{B}(\vec{c\tau}, \vec{i}, \vec{j}, \vec{k}), \mathcal{B}'(\vec{c\tau}', \vec{i}', \vec{j}', \vec{k}'))$ is the representation of vectors $(\vec{c\tau}', \vec{i}', \vec{j}', \vec{k}')$ in the old basis $\mathcal{B}(\vec{c\tau}, \vec{i}, \vec{j}, \vec{k})$

For observer O , we consider the basis $\mathcal{B}_1(\vec{c\tau}, \vec{i}_1, \vec{j}_1, \vec{k}_1)$ such that the change - of - basis matrix

$A = \mathcal{P}(\mathcal{B}(\vec{c\tau}, \vec{i}, \vec{j}, \vec{k}), \mathcal{B}_1(\vec{c\tau}, \vec{i}_1, \vec{j}_1, \vec{k}_1))$ is an orthogonal matrix of the form

$$A = \begin{bmatrix} 1 & 0 \\ 0 & R_1 \end{bmatrix} \text{ where } R_1 \text{ an orthogonal operator, transforms the spatial components of } \vec{i}$$

to those of \vec{i}_1 such that they are parallel and have the same direction as $\vec{\beta}$.

For the observer O' , we consider the basis $\mathcal{B}'_1(\vec{c\tau}', \vec{i}'_p, \vec{j}'_p, \vec{k}'_1)$ such that the change – of – basis matrix

${}^t\mathcal{B} = \mathcal{P}(\mathcal{B}'_1(\vec{c\tau}', \vec{i}'_p, \vec{j}'_p, \vec{k}'_1), \mathcal{B}'_1(\vec{c\tau}', \vec{i}'_p, \vec{j}'_p, \vec{k}'_1))$ is an orthogonal matrix of the form

$${}^t\mathcal{B} = \begin{bmatrix} 1 & 0 \\ 0 & R_2 \end{bmatrix} \text{ où } R_2, \text{ an orthogonal operator, transforms the spatial components of}$$

\vec{i}' to those of \vec{i}'_1 such that it is parallel and has the same direction as $\vec{\beta}$, and \vec{j}'_1 and \vec{k}'_1 are parallel and have the same direction as \vec{j}_1 , and \vec{k}_1 respectively, (rotation around the first axis).

Let's calculate the operator N that allows us to transform the basis

$\mathcal{B}(\vec{c\tau}, \vec{i}_p, \vec{j}_p, \vec{k}_1)$ to the basis $\mathcal{B}'_1(\vec{c\tau}', \vec{i}'_p, \vec{j}'_p, \vec{k}'_1)$:

$$N = \mathcal{P}(\mathcal{B}'_1(\vec{c\tau}', \vec{i}'_p, \vec{j}'_p, \vec{k}'_1), \mathcal{B}(\vec{c\tau}, \vec{i}_p, \vec{j}_p, \vec{k}_1)).$$

Since $\vec{\beta}$ is perpendicular to the spatial components of \vec{j}_1 and \vec{k}_1 , we have :

$$N = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega_{1,1} & \omega_{1,2} & \omega_{1,3} \\ 0 & \omega_{2,1} & \omega_{2,2} & \omega_{2,3} \\ 0 & \omega_{3,1} & \omega_{3,2} & \omega_{3,3} \end{bmatrix} = \begin{bmatrix} \gamma & \gamma\beta\omega_{1,1} & \gamma\beta\omega_{1,2} & \gamma\beta\omega_{1,3} \\ \gamma\beta & \gamma\omega_{1,1} & \gamma\omega_{1,2} & \gamma\omega_{1,3} \\ 0 & \omega_{2,1} & \omega_{2,2} & \omega_{2,3} \\ 0 & \omega_{3,1} & \omega_{3,2} & \omega_{3,3} \end{bmatrix}.$$

As the spatial components of the new spatial vectors are parallel and in the same direction as those of the old one, we necessarily have :

$\omega_{2,1} = \omega_{3,1} = \omega_{3,2} = \omega_{1,2} = \omega_{2,3} = \omega_{1,3} = 0$ et $\omega_{1,1} > 0$ then :

$$N = \begin{bmatrix} \gamma & \gamma\beta\omega_{1,1} & 0 & 0 \\ \gamma\beta & \gamma\omega_{1,1} & 0 & 0 \\ 0 & 0 & \omega_{2,2} & 0 \\ 0 & 0 & 0 & \omega_{3,3} \end{bmatrix},$$

as $\gamma^2 - \gamma^2\beta^2 = 1$ and ${}^tN \cdot G \cdot N = G$ imply that $\gamma^2\beta^2\omega_{1,1}^2 - \gamma^2\omega_{1,1} = -1$,

$$\text{and we have } \omega_{1,1} = \omega_{2,2} = \omega_{3,3} = 1 \text{ et } N = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now we have :

$$M = \mathcal{P}(\mathcal{B}(\vec{c\tau}, \vec{i}, \vec{j}, \vec{k}), \mathcal{B}'_1(\vec{c\tau}', \vec{i}'_p, \vec{j}'_p, \vec{k}'_1)) \\ = \mathcal{P}(\mathcal{B}(\vec{c\tau}, \vec{i}, \vec{j}, \vec{k}), \mathcal{B}(\vec{c\tau}, \vec{i}_p, \vec{j}_p, \vec{k}_1)) \mathcal{P}(\mathcal{B}(\vec{c\tau}, \vec{i}_p, \vec{j}_p, \vec{k}_1), \mathcal{B}'_1(\vec{c\tau}', \vec{i}'_p, \vec{j}'_p, \vec{k}'_1)) \mathcal{P}(\mathcal{B}'_1(\vec{c\tau}', \vec{i}'_p, \vec{j}'_p, \vec{k}'_1), \mathcal{B}'_1(\vec{c\tau}', \vec{i}'_p, \vec{j}'_p, \vec{k}'_1))$$

$$= A \cdot N \cdot B = (A \cdot N \cdot {}^t A) \cdot (A \cdot B) = \begin{bmatrix} \gamma & {}^t[\vec{\gamma\beta}] \\ [\vec{\gamma\beta}] & C \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}.$$

By the unicity of the decomposition, we have $\begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} = A \cdot B$

$$\text{and } \begin{bmatrix} \gamma & {}^t[\vec{\gamma\beta}] \\ [\vec{\gamma\beta}] & C \end{bmatrix} = A \cdot \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot {}^t A.$$

Ω is therefore independent of $\vec{\beta}$ since A and B are independent of $\vec{\beta}$

$$\text{As } \lim_{\beta \rightarrow 0} \left(\begin{bmatrix} \gamma & {}^t[\vec{\gamma\beta}] \\ [\vec{\gamma\beta}] & C \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix},$$

$\begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}$ is the Lorentz matrix M for $\vec{\beta} = \vec{0}$.

Transformation of Galileo :

This also allows us to show that the Galileo transformation is the limit as $c \rightarrow +\infty$ of the Lorentz transformation . We notice that if we write the Lorentz matrix in the form :

$$M = \begin{bmatrix} \gamma & {}^t[\vec{\gamma\beta}] \\ [\vec{\gamma\beta}] & Id_{\mathbb{R}^3} + \frac{\gamma^2}{(1+\gamma)} [\vec{\beta}] \cdot {}^t[\vec{\beta}] I \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}$$

$$= \begin{bmatrix} \gamma & {}^t[\vec{\gamma\beta}] \Omega \\ [\vec{\gamma\beta}] & \left(Id_{\mathbb{R}^3} + \frac{\gamma^2}{(1+\gamma)} [\vec{\beta}] \cdot {}^t[\vec{\beta}] \right) \Omega \end{bmatrix}.$$

On the other hand :

$$MY' = \begin{bmatrix} \gamma & {}^t[\vec{\gamma\beta}] \Omega \\ [\vec{\gamma\beta}] & \left(Id_{\mathbb{R}^3} + \frac{\gamma^2}{(1+\gamma)} [\vec{\beta}] \cdot {}^t[\vec{\beta}] \right) \Omega \end{bmatrix} \cdot \begin{bmatrix} ct' \\ X' \end{bmatrix}$$

$$= \begin{bmatrix} c\boldsymbol{\gamma} & {}^t[\boldsymbol{\gamma}\vec{\boldsymbol{\beta}}]\boldsymbol{\Omega} \\ c[\vec{\boldsymbol{\gamma}}] & \left(Id_{\mathbb{R}^3} + \frac{\boldsymbol{\gamma}^2}{(1+\boldsymbol{\gamma})} [\vec{\boldsymbol{\beta}}] \cdot {}^t[\vec{\boldsymbol{\beta}}] \right) \boldsymbol{\Omega} \end{bmatrix} \cdot \begin{bmatrix} t' \\ X' \end{bmatrix} = \begin{bmatrix} ct \\ X \end{bmatrix}$$

and then :

$$\begin{bmatrix} \boldsymbol{\gamma} & {}^t[\boldsymbol{\gamma}\vec{\boldsymbol{\beta}}]\frac{\boldsymbol{\Omega}}{c} \\ c[\vec{\boldsymbol{\gamma}}] & \left(Id_{\mathbb{R}^3} + \frac{\boldsymbol{\gamma}^2}{(1+\boldsymbol{\gamma})} [\vec{\boldsymbol{\beta}}] \cdot {}^t[\vec{\boldsymbol{\beta}}] \right) \boldsymbol{\Omega} \end{bmatrix} \cdot \begin{bmatrix} t' \\ X' \end{bmatrix} = \begin{bmatrix} t \\ X \end{bmatrix}$$

since $[\vec{\boldsymbol{\beta}}] = \left[\frac{\vec{V}}{c} \right]$ and $\boldsymbol{\gamma} = \frac{1}{1 - \left(\frac{\vec{V}}{c} \right)^2}$ hence :

$$\begin{bmatrix} \boldsymbol{\gamma} & {}^t \left[\boldsymbol{\gamma} \frac{\vec{V}}{c} \right] \frac{\boldsymbol{\Omega}}{c} \\ [\vec{\boldsymbol{\gamma}}] & \left(Id_{\mathbb{R}^3} + \frac{\boldsymbol{\gamma}^2}{(1+\boldsymbol{\gamma})} \left[\frac{\vec{V}}{c} \right] \cdot {}^t \left[\frac{\vec{V}}{c} \right] \right) \boldsymbol{\Omega} \end{bmatrix} \cdot \begin{bmatrix} t' \\ X' \end{bmatrix} = \begin{bmatrix} t \\ X \end{bmatrix} \text{ then}$$

For a given velocity \vec{V} , $\boldsymbol{\Omega}$ is independent of c ,

$$\text{if } c \rightarrow +\infty, \text{ we have } \begin{bmatrix} 1 & 0 \\ [\vec{V}] & \boldsymbol{\Omega} \end{bmatrix} \cdot \begin{bmatrix} t' \\ X' \end{bmatrix} = \begin{bmatrix} t \\ X \end{bmatrix},$$

that is the Galileo transformation .

Calculation of eigenvalues;

We can determine the eigenvalues directly :

$$\text{As } \mathbf{A} = \begin{bmatrix} 1 + \frac{\boldsymbol{\gamma}^2}{(1+\boldsymbol{\gamma})} \boldsymbol{\beta}_1^2 & \frac{\boldsymbol{\gamma}^2}{(1+\boldsymbol{\gamma})} \boldsymbol{\beta}_1 \boldsymbol{\beta}_2 & \frac{\boldsymbol{\gamma}^2}{(1+\boldsymbol{\gamma})} \boldsymbol{\beta}_1 \boldsymbol{\beta}_3 \\ \frac{\boldsymbol{\gamma}^2}{(1+\boldsymbol{\gamma})} \boldsymbol{\beta}_2 \boldsymbol{\beta}_1 & 1 + \frac{\boldsymbol{\gamma}^2}{(1+\boldsymbol{\gamma})} \boldsymbol{\beta}_2^2 & \frac{\boldsymbol{\gamma}^2}{(1+\boldsymbol{\gamma})} \boldsymbol{\beta}_2 \boldsymbol{\beta}_3 \\ \frac{\boldsymbol{\gamma}^2}{(1+\boldsymbol{\gamma})} \boldsymbol{\beta}_3 \boldsymbol{\beta}_1 & \frac{\boldsymbol{\gamma}^2}{(1+\boldsymbol{\gamma})} \boldsymbol{\beta}_3 \boldsymbol{\beta}_2 & 1 + \frac{\boldsymbol{\gamma}^2}{(1+\boldsymbol{\gamma})} \boldsymbol{\beta}_3^2 \end{bmatrix} = A \cdot \begin{bmatrix} \boldsymbol{\gamma} & \boldsymbol{\gamma}\boldsymbol{\beta} & 0 & 0 \\ \boldsymbol{\gamma}\boldsymbol{\beta} & \boldsymbol{\gamma} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

${}^T A,$

where A est orthogonal : $A \cdot {}^T A = Id$.

we note that $\det(\mathbf{\Lambda} - \lambda \mathbf{Id}) = \det \left(A \cdot \begin{bmatrix} \gamma & \gamma \cdot \boldsymbol{\beta} & 0 & 0 \\ \gamma \cdot \boldsymbol{\beta} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot {}^T A - \lambda \cdot A \cdot {}^T A \right)$

$$= \det(A) \det \left(\begin{bmatrix} \gamma & \gamma \cdot \boldsymbol{\beta} & 0 & 0 \\ \gamma \cdot \boldsymbol{\beta} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \lambda \cdot \mathbf{Id} \right) \det({}^T A).$$

and

$$\begin{bmatrix} \gamma & \gamma \cdot \boldsymbol{\beta} & 0 & 0 \\ \gamma \cdot \boldsymbol{\beta} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \\ +\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \gamma \cdot (1 + \boldsymbol{\beta}) & 0 & 0 & 0 \\ 0 & \gamma \cdot (1 - \boldsymbol{\beta}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} & +\frac{\sqrt{2}}{2} & 0 & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In the general case, it is shown that every Lorentz matrix admits at least one real eigenvalue :
(cf: **E • GOURGOULON** : Relativité restreinte EDP Sciences).

As it is of order 4, it has 2 eigenvalues. This is the maximum number for the general case :
consider the case where M is orthogonal to $\boldsymbol{\Omega}$ a rotation matrix around the \mathbf{Ox} axis.

Another expression of $\boldsymbol{\Omega}$:

We can directly calculate $\boldsymbol{\Omega}$ because

$$\text{if } M = \Lambda(\vec{\boldsymbol{\beta}}) \cdot \tilde{\boldsymbol{\Omega}} \Leftrightarrow \boldsymbol{\Omega} = \Lambda(-\vec{\boldsymbol{\beta}}) \cdot M.$$

It is possible to have a simpler calculation :

We have $\begin{pmatrix} \rightarrow \\ \boldsymbol{\beta} \end{pmatrix} \begin{pmatrix} \rightarrow \\ \boldsymbol{\beta} \end{pmatrix} \begin{pmatrix} \rightarrow \\ \boldsymbol{\beta} \end{pmatrix} \begin{pmatrix} \rightarrow \\ \boldsymbol{\beta} \end{pmatrix} = \vec{\boldsymbol{\beta}} \begin{pmatrix} \rightarrow \\ \boldsymbol{\beta} \end{pmatrix} \vec{\boldsymbol{\beta}} \begin{pmatrix} \rightarrow \\ \boldsymbol{\beta} \end{pmatrix} = \vec{\boldsymbol{\beta}} \begin{pmatrix} \rightarrow \\ \boldsymbol{\beta} \end{pmatrix} \begin{pmatrix} \rightarrow \\ \boldsymbol{\beta} \end{pmatrix} \begin{pmatrix} \rightarrow \\ \boldsymbol{\beta} \end{pmatrix} \begin{pmatrix} \rightarrow \\ \boldsymbol{\beta} \end{pmatrix}$ and $\gamma^2 \vec{\boldsymbol{\beta}}^2 = \gamma^2 - 1$,
that imply that :

$$\begin{aligned} & \left(Id_{\mathbf{R}^3} + \frac{\begin{bmatrix} \vec{\boldsymbol{\beta}} \\ \gamma \boldsymbol{\beta} \end{bmatrix} \begin{bmatrix} \vec{\boldsymbol{\beta}} \\ \gamma \boldsymbol{\beta} \end{bmatrix}}{(1 + \gamma)} \right) \cdot \left(Id_{\mathbf{R}^3} - \gamma \frac{\begin{pmatrix} \rightarrow \\ \boldsymbol{\beta} \end{pmatrix} \begin{pmatrix} \rightarrow \\ \boldsymbol{\beta} \end{pmatrix}}{(1 + \gamma)} \right) \\ &= Id_{\mathbf{R}^3} - \gamma \frac{\begin{pmatrix} \rightarrow \\ \boldsymbol{\beta} \end{pmatrix} \begin{pmatrix} \rightarrow \\ \boldsymbol{\beta} \end{pmatrix}}{(1 + \gamma)} + \gamma^2 \frac{\begin{pmatrix} \rightarrow \\ \boldsymbol{\beta} \end{pmatrix} \begin{pmatrix} \rightarrow \\ \boldsymbol{\beta} \end{pmatrix}}{(1 + \gamma)} - \gamma^3 \frac{\begin{pmatrix} \rightarrow \\ \boldsymbol{\beta} \end{pmatrix} \begin{pmatrix} \rightarrow \\ \boldsymbol{\beta} \end{pmatrix} \begin{pmatrix} \rightarrow \\ \boldsymbol{\beta} \end{pmatrix} \begin{pmatrix} \rightarrow \\ \boldsymbol{\beta} \end{pmatrix}}{(1 + \gamma)^2} \\ &= Id_{\mathbf{R}^3} + \frac{\begin{pmatrix} \rightarrow \\ \boldsymbol{\beta} \end{pmatrix} \begin{pmatrix} \rightarrow \\ \boldsymbol{\beta} \end{pmatrix}}{(1 + \gamma)^2} \left(-\gamma(1 + \gamma) + \gamma^2(1 + \gamma) - \gamma(\gamma^2 - 1) \right) = Id_{\mathbf{R}^3}. \end{aligned}$$

$$\text{Then } C^{-1} = \left(Id_{\mathbf{R}^3} + \frac{[\vec{\gamma\beta}]^t [\vec{\gamma\beta}]}{(1+\gamma)} \right)^{-1} = Id_{\mathbf{R}^3} - \gamma \frac{\vec{\beta} \vec{\beta}^t}{(1+\gamma)}$$

$$\text{As } M = \begin{bmatrix} \gamma & {}^t[\vec{\gamma\beta}] \\ [\vec{\gamma\beta}] & C \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} = \begin{bmatrix} \gamma & {}^t[\vec{\gamma\beta}]\Omega \\ [\vec{\gamma\beta}] & C\Omega \end{bmatrix},$$

$$\text{Now if we write } M = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{bmatrix}.$$

$$\text{We have } \Omega = \left(Id_{\mathbf{R}^3} - \gamma \frac{\vec{\beta} \vec{\beta}^t}{(1+\gamma)} \right) \begin{bmatrix} m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,2} & m_{4,3} & m_{4,4} \end{bmatrix}.$$

A final remark :

We can note :

M symmetric \Leftrightarrow the spatial components of the spatial vectors of $\mathcal{B}(\vec{c\tau}, \vec{i}, \vec{j}, \vec{k})$ and $\mathcal{B}'(\vec{c\tau'}, \vec{i}', \vec{j}', \vec{k}')$ are parallel.

Now we propose two other ways to calculate the terms of $\exp(\alpha N)$:

Second method :

The second method starts with a demonstration by J. Parizet in "La géométrie de la relativité restreinte (Ellipses) p.89."

Let's M a Lorentz matrix : $G = {}^t M G M$ that implies

$$G M^{-1} = {}^t M G \text{ and } M^{-1} = G M G.$$

We can write M as $\begin{bmatrix} \gamma & {}^t U \\ V & A \end{bmatrix}$ où $\gamma \in \mathbf{R}$, U et V 2 vecteurs de \mathbf{R}^3 et A une matrice 3×3 .

from $M^{-1} = G M G$ we have $M^{-1} = \begin{bmatrix} \gamma & -{}^t V \\ -U & {}^t A \end{bmatrix}$ and then :

$$M \cdot M^{-1} = \begin{bmatrix} \gamma^2 - {}^t V \cdot V & \gamma \cdot {}^t U - {}^t V \cdot A \\ -\gamma U + {}^t A \cdot V & {}^t A \cdot A - U \cdot {}^t U \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & Id_3 \end{bmatrix} \text{ that implies :}$$

$$\gamma^2 - {}^t V \cdot V = 1, -\gamma U + {}^t A \cdot V = \gamma \cdot {}^t U - {}^t V \cdot A = 0, {}^t A \cdot A - U \cdot {}^t U = Id_3.$$

We note $\gamma^2 \geq 1$ since $\gamma^2 - 1 = {}^tV \cdot V \geq 0$, ${}^tA \cdot V = \gamma U$ and ${}^tV \cdot A = \gamma \cdot {}^tU$.

We have $V=0 \Leftrightarrow \gamma^2=1$, $V=0 \Leftrightarrow U=0$, $V=0 \Leftrightarrow {}^tA \cdot A = Id_3$.

Now if $V=0$ then $M = \begin{bmatrix} \pm 1 & 0 \\ 0 & A \end{bmatrix}$ with ${}^tA \cdot A = Id_3$.

Consider the case $V \neq 0$ and then $\gamma^2 > 1$.

We then consider $\Omega = A - \frac{V \cdot {}^tU}{1 + \gamma}$.

$$\begin{aligned} \text{Let' calculate } {}^t\Omega \Omega &= \left({}^tA - \frac{U \cdot {}^tV}{1 + \gamma} \right) \left(A - \frac{V \cdot {}^tU}{1 + \gamma} \right) \\ &= {}^tA \cdot A - \frac{{}^tAV {}^tU}{1 + \gamma} - \frac{U {}^tVA}{1 + \gamma} + \frac{U {}^tV V {}^tU}{(1 + \gamma)^2}, \\ &= Id_3 + U \cdot {}^tU - \frac{2 \cdot \gamma U \cdot {}^tU}{1 + \gamma} + \frac{(\gamma^2 - 1) U \cdot {}^tU}{(\gamma^2 - 1)} \\ &= Id_3 + U \cdot {}^tU \left(1 - \frac{2 \cdot \gamma}{1 + \gamma} + \frac{(\gamma^2 - 1)}{(1 + \gamma)^2} \right) \\ &= Id_3 + U \cdot {}^tU (1 + \gamma - 2 \cdot \gamma + \gamma - 1) = Id_3. \end{aligned}$$

We conclude that Ω is an orthogonal matrix.

Now we can consider :

$${}^tV \cdot \Omega = {}^tV \cdot \left(A - \frac{V \cdot {}^tU}{1 + \gamma} \right) = {}^tV \cdot A - \frac{{}^tV \cdot V \cdot {}^tU}{1 + \gamma} = \gamma \cdot {}^tU - \frac{(\gamma^2 - 1) \cdot {}^tU}{1 + \gamma} = {}^tU.$$

Then $\Omega \cdot U = \Omega {}^t\Omega \cdot V = V$. Because $\Omega = A - \frac{V \cdot {}^tU}{1 + \gamma}$

$$M = \begin{bmatrix} \gamma & {}^tU \\ V & A \end{bmatrix} = \begin{bmatrix} \gamma & {}^tU \\ V & \Omega + \frac{V \cdot {}^tU}{1 + \gamma} \end{bmatrix} = \begin{bmatrix} \gamma & {}^tU \cdot {}^t\Omega \\ V & Id + \frac{V \cdot {}^tU \cdot {}^t\Omega}{1 + \gamma} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} \text{ and then :}$$

$$M = \begin{bmatrix} \gamma & {}^tV \\ V & Id + \frac{V {}^tV}{1 + \gamma} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}.$$

If we consider \mathcal{O} et \mathcal{O}' , \mathcal{O}' with a constant speed \vec{V} with respect to \mathcal{O} :

\mathcal{O}' has the coordinate $\begin{bmatrix} c \cdot t' \\ 0_{\mathbb{R}^3} \end{bmatrix}$ in the basis associated with \mathcal{O}' and

$\begin{bmatrix} c \cdot t \\ [\vec{V}] \cdot t \end{bmatrix}$ in the basis associated with \mathcal{O} .

$$\text{Then : } \begin{bmatrix} c \cdot t \\ \vec{V} \cdot t \end{bmatrix} = \begin{bmatrix} \gamma & {}^t V \\ V & Id + \frac{V {}^t V}{1 + \gamma} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} \begin{bmatrix} c \cdot t' \\ 0_{\mathbb{R}^3} \end{bmatrix}.$$

we deduce from it : $t = \gamma \cdot t'$ and $\vec{V} \cdot t = c \cdot t' \cdot V$ then $V = \frac{\gamma [\vec{V}]}{c} = \gamma [\vec{\beta}]$

Finally :

$$M = \begin{bmatrix} \gamma & \gamma \cdot {}^t [\vec{\beta}] \\ \gamma [\vec{\beta}] & Id + \frac{\gamma^2 [\vec{\beta}] [{}^t \vec{\beta}]}{1 + \gamma} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}.$$

$$\text{We can also write } M = \begin{bmatrix} -\gamma & -\gamma [{}^t \vec{\beta}] \\ -\gamma [\vec{\beta}] & - \left(Id + \frac{\gamma^2 [\vec{\beta}] [{}^t \vec{\beta}]}{1 + \gamma} \right) \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & -\Omega \end{bmatrix} \text{ and}$$

In a general way M can be written as :

$$M = \begin{bmatrix} \gamma & \gamma \cdot {}^t [\vec{\beta}] \\ \gamma [\vec{\beta}] & Id + \frac{\gamma^2 [\vec{\beta}] [{}^t \vec{\beta}]}{1 + \gamma} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix} \text{ with } \varepsilon = \pm 1, \gamma \geq 1, \gamma = \frac{1}{\sqrt{1 - \vec{\beta} \cdot \vec{\beta}}}$$

with Ω an orthonal matrix .

Third method :

We start from: Special relativity by Ulrich E. Schröder, page 92, World Scientific, modifying his demonstration.

(The book can be consulted at : <https://archive.org/details/specialrelativit0000schr/page/92/mode/2-up?view=theater>).

Following Schröder, let us consider a symmetric Lorentz matrix M of the form :

$$M = A \cdot \begin{bmatrix} \gamma & \gamma \cdot \beta & 0 & 0 \\ \gamma \cdot \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot {}^T A \text{ with } \beta = \|\vec{\beta}\|, A = \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}, \Omega \text{ an orthogonal matrix } 3 \times 3.$$

M is indeed a Lorentz matrix, since it can be expressed as the product of 3 Lorentz matrices.

Since Ω is a 3×3 rotation matrix,

it can be decomposed into a product of 3 elementary rotations (Euler angles) :

see, for example, https://en.wikipedia.org/wiki/Rotation_formalisms_in_three_dimensions .

If we define : $R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$, $R_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\varphi) & 0 & \sin(\varphi) \\ 0 & 0 & 1 & 0 \\ 0 & -\sin(\varphi) & 0 & \cos(\varphi) \end{bmatrix}$, $R_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\psi) & -\sin(\psi) & 0 \\ 0 & \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

We can write $A = R_z \cdot R_y \cdot R_x$.

This order of composition is chosen to simplify the calculations.

It is observed that due to rotational invariance around the the $x -$ axis.

$$R_x \cdot \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot {}^t R_x = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

After calculation (with Maple) :

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\psi) \cos(\varphi) & -\sin(\psi) \cos(\theta) + \cos(\psi) \sin(\varphi) \sin(\theta) & \sin(\psi) \sin(\theta) + \cos(\psi) \sin(\varphi) \cos(\theta) \\ 0 & \sin(\psi) \cos(\varphi) & \cos(\psi) \cos(\theta) + \sin(\psi) \sin(\varphi) \sin(\theta) & -\cos(\psi) \sin(\theta) + \sin(\psi) \sin(\varphi) \cos(\theta) \\ 0 & -\sin(\varphi) & \cos(\varphi) \sin(\theta) & \cos(\varphi) \cos(\theta) \end{bmatrix}$$

We have since the θ variable is eliminated :

$$M = \begin{bmatrix} \gamma & \cos(\psi) \cos(\varphi) \gamma\beta & \sin(\psi) \cos(\varphi) \gamma\beta & -\sin(\varphi) \gamma\beta \\ \cos(\psi) \cos(\varphi) \gamma\beta & 1 + \cos(\psi)^2 (-1 + \gamma) \cos(\varphi)^2 & \sin(\psi) \cos(\psi) \cos(\varphi)^2 (-1 + \gamma) & -\cos(\varphi) \cos(\psi) \sin(\varphi) (-1 + \gamma) \\ \sin(\psi) \cos(\varphi) \gamma\beta & \sin(\psi) \cos(\psi) \cos(\varphi)^2 (-1 + \gamma) & 1 + ((1 - \gamma) \cos(\psi)^2 - 1 + \gamma) \cos(\varphi)^2 & -\cos(\varphi) \sin(\psi) \sin(\varphi) (-1 + \gamma) \\ -\sin(\varphi) \gamma\beta & -\cos(\varphi) \cos(\psi) \sin(\varphi) (-1 + \gamma) & -\cos(\varphi) \sin(\psi) \sin(\varphi) (-1 + \gamma) & (1 - \gamma) \cos(\varphi)^2 + \gamma \end{bmatrix}$$

One can still directly evaluate the first column of a general Lorentz matrix with $\vec{\beta}$ by considering the coordinates of the observer O' in the two bases associated with O and O' :

If we write $M = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{bmatrix}$ then $M \cdot \begin{bmatrix} ct' \\ 0 \\ 0 \\ 0 \end{bmatrix}_{O'}$ = $\begin{bmatrix} ct \\ c \cdot \beta_1 \cdot t \\ c \cdot \beta_2 \cdot t \\ c \cdot \beta_3 \cdot t \end{bmatrix}_O$ hence :

if we write $\gamma = m_{1,1}$ then $\gamma \cdot t' = t$ moreover $\gamma > 0$. (Principle of causality) .

From this, we deduce that the first column of M is written as $\begin{bmatrix} \gamma \\ \gamma \beta_1 \\ \gamma \beta_2 \\ \gamma \beta_3 \end{bmatrix}$.

By identification : $\cos(\Psi) \cos(\varphi) = \frac{\beta_1}{\beta}$, $\sin(\Psi) \cos(\varphi) = \frac{\beta_2}{\beta}$, $\sin(\varphi) = -\frac{\beta_3}{\beta}$.

We recall that $\gamma^2 \cdot \beta^2 = \gamma^2 - 1$ since $1 = \gamma^2(1 - \beta^2)$ since ${}^tMGM = G$.
Calculation of the $m_{i,j}$:

$$m_{2,2} = 1 + \cos(\Psi)^2 (-1 + \gamma) \cos(\varphi)^2 = 1 + \cos(\Psi)^2 \cos(\varphi)^2 \frac{\gamma^2 \cdot \beta^2}{1 + \gamma}$$

$$= 1 + \left(\frac{\beta_1}{\beta} \right)^2 \frac{\gamma^2 \cdot \beta^2}{1 + \gamma} = 1 + \frac{\gamma^2}{1 + \gamma} \beta_1^2 .$$

$$m_{3,3} = 1 + ((1 - \gamma) \cos(\Psi)^2 - 1 + \gamma) \cos(\varphi)^2 = 1 + (1 - \gamma) (\cos(\Psi)^2 \cos(\varphi)^2 - \cos(\varphi)^2)$$

$$= 1 + (1 - \gamma) (\cos(\Psi)^2 \cos(\varphi)^2 - 1 + \sin(\varphi)^2) = 1 + (1 - \gamma) \left(\frac{\beta_1^2 - \beta^2 + \beta_3^2}{\beta^2} \right)$$

$$= 1 + \frac{\gamma^2}{1 + \gamma} \cdot \beta_2^2 .$$

$$m_{4,4} = (1 - \gamma) \cos(\varphi)^2 + \gamma = 1 + (\gamma - 1) (1 - \cos(\varphi)^2) = 1 + \frac{\gamma^2 \cdot \beta^2}{1 + \gamma} \sin(\varphi)^2 = 1 + \frac{\gamma^2}{1 + \gamma} \beta_3^2 .$$

$$m_{3,2} = \sin(\Psi) \cos(\Psi) \cos(\varphi)^2 (-1 + \gamma) = \sin(\Psi) \cos(\varphi) \cos(\Psi) \cos(\varphi) (-1 + \gamma) = \frac{\gamma^2 \cdot \beta^2}{1 + \gamma} \frac{\beta_2}{\beta} \frac{\beta_1}{\beta}$$

$$= \frac{\gamma^2}{1 + \gamma} \beta_1 \beta_2 .$$

$$m_{4,3} = -\cos(\varphi) \sin(\Psi) \sin(\varphi) (-1 + \gamma) = \frac{\gamma^2}{1 + \gamma} \beta_2 \beta_3 .$$

$$m_{4,2} = -\cos(\varphi) \cos(\Psi) \sin(\varphi) (-1 + \gamma) = \frac{\gamma^2}{1 + \gamma} \beta_1 \beta_3 .$$

We complete by symmetry .

Other methods :

See also

E • GOURGOULON : *Relativité restreinte EDP Sciences p.203* •
a geometric method based on the existence of an eigenvalue ,
and for a heuristic approach :

C • Semay B • Silvestre — Brac *Relativité Restreinte Dunod p.120* .

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