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# Toward Credible Belief Base Revision

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**Abstract.** This paper deals with belief base revision, a form of belief change which consists in restoring consistency with the intention of incorporating a new piece of information from the environment, while minimally modifying the agent’s belief state represented by a finite set of propositional formulas. In an effort to guarantee more reliability and rationality for real applications while performing revision, we come up with the idea of credible belief base revision. We define two new formula-based revision operators using tools offered by evidence theory. These operators, uniformly presented in the same spirit as [93,13], stem from consistent sub-bases maximal with respect to credibility instead of set inclusion or cardinality. Logical properties and productivity of inference from these operators are established, along with complexity results.

**Keywords:** belief base revision, rationality, credibility, evidence theory, knowledge representation and reasoning.

## 1 Introduction

One important research topic in Artificial Intelligence is belief dynamics, or change, which is widely investigated in a large range of applications such as image processing, reliability expert opinions, robotics, radar detection and relational databases (see [8] for an overview of such applications). In this context, we are concerned with belief revision, where an intelligent agent faces incomplete, uncertain or inaccurate information, and thus often need a revision operation in order to manage belief changes in presence of a new and more reliable information. When the new information contradicts the agent’s current beliefs, the revision deals with keeping consistency in order to integrate the new information while modifying the initial set of beliefs as less as possible.

In this context, several operators have been proposed. Some of them are based on the construction of maximally consistent sub-bases according to different criteria [4,58,60]. Recently, Creignou and colleagues in [13] focused particularly on two operators, namely RSRG and RSRW, that are respectively similar to

Ginsberg’s one [42] and WIDTIO (When In Doubt Throw It Out) [91], using set cardinality instead of set inclusion as maximality criterion.

An important issue is to introduce *efficient* tools that fulfill the needs of others in their investigations. Indeed, in many real applications, belief bases are quite large and choosing, as a revision result, all maximal consistent sub-bases like Ginsberg’s approach [42] can be an expensive and exhaustive solution. On the one hand, keeping only beliefs that are not questioned (stemming from the intersection of all maximal-consistent sub-bases [91]) can be a plausible strategy, but is not always reliable in the sense that it can cause, in numerous cases, a huge loss of information, since there is no guarantee that this intersection does not result in the empty set. On the other hand, the selection of some maximal consistent sub-bases according to well-defined criteria can be shown also plausible, and the choice of set cardinality as in [93,13] sounds rather reasonable, because it respects the minimality change criterion of belief revision. Nonetheless, the selection of consistent sub-bases maximal with respect to set cardinality does not always guarantee the selection of the most relevant information (which can be in conflict with many other less relevant information), and can consequently neglect potential formulas from the agent’s initial beliefs (as shown and illustrated in Example 2).

This paper tries to address this problem by investigating a sub-base selection criterion which respects the minimal change principle of belief revision in a sense which seems more *credible*, while trying to perform the change by capturing the most valuable or potential information from initial beliefs. In order to achieve this goal, we investigate selection functions inspired by the basic mechanism of evidence theory [18,76]. As a matter of fact, in a similar vein of [13], we propose two formula-based operators similar to RSRG and RSRW, namely CSRG and CSRW, based on the selection of the most credible consistent sub-bases. The computation of credibility is performed using suitable tools offered by evidence theory. A concise definition of these operators is given. Moreover, we study the logical properties of our operators in terms of satisfaction of postulates that should hold for any rational belief base revision operator. This paper also investigates the productivity of the inference relation from the revision result. Finally, we study the complexity of model checking problem for our revision operators.

The rest of this paper is organized as follows. Section 2 fixes some notations in propositional logic, introduces briefly some mathematical formalism underlying evidence theory and gives a refresher on belief revision, as well as some notions about complexity classes. Section 3 is divided into four sub-sections: in the first one (sub-section 3.1), we formally define and develop our new belief operators stemming from consistent sub-bases maximal with respect to *credibility* degree, illustrating the difference between these operators and others by some examples. Sub-section 3.2 is dedicated to establish results about the productivity of the inference relation from the revision result. This is followed by the study of logical properties in terms of satisfaction of postulates in sub-section 3.3. More details are presented in sub-section 3.4 by looking into the computational complexity of model checking problem for our revision operators. In the last part of the

paper, we position this work compared to other previous ones in the literature in section 4. Finally, a conclusion is given in section 5.

## 2 Preliminaries

### 2.1 Propositional Logic

We assume that the reader has some background in propositional classical logic. We briefly present the notions and terminology used throughout this paper. Let  $\mathcal{L}$  be the language of propositional logic built from an infinite countable set of variables (atoms) denoted by  $\mathcal{V}$  and equipped with standard connectives  $\neg$ ,  $\wedge$ ,  $\vee$ ,  $\rightarrow$ ,  $\leftrightarrow$  and constants  $\top$ ,  $\perp$ . We recall that a literal is an atom (positive literal) or the negation of an atom  $\neg a$  (negative literal).

Let  $\mathcal{U}$  be a finite set of atoms. An interpretation  $v$  over  $\mathcal{U}$  is a function from  $\mathcal{U}$  to  $\{0, 1\}$  which associates to each atom a truth value, 0 denoting false and 1 denoting true. The interpretation of constants  $\top$  and  $\perp$  is respectively  $v(\top) = 1$  and  $v(\perp) = 0$ , and the interpretation of formulas is inductively defined as follows. Let  $\phi$  and  $\psi$  be any formulas of  $\mathcal{L}$ :  $v(\neg\phi) = 1 - v(\phi)$ ;  $v(\phi \vee \psi) = \max(v(\phi), v(\psi))$ ;  $v(\phi \wedge \psi) = \min(v(\phi), v(\psi))$ ;  $v(\phi \rightarrow \psi) = \max(1 - v(\phi), v(\psi))$ , and  $v(\phi \leftrightarrow \psi) = \min(\max(1 - v(\phi), v(\psi)), \max(v(\phi), 1 - v(\psi)))$ .

Generally, an interpretation  $v$  is represented by a subset of  $\mathcal{U}$  (corresponding to the variables set to true), the atoms being considered in lexicographical order. For instance if we consider  $\mathcal{U} = \{x_1, \dots, x_6\}$ , the interpretation  $x_1 = x_3 = x_6 = 1$  and  $x_2 = x_4 = x_5 = 0$  will be represented by  $\{x_1, x_3, x_6\}$ . When an interpretation  $v$  evaluates a formula  $\phi$  to 1, we write  $v \models \phi$ , and  $v$  is called a model of  $\phi$ .

If  $\phi$  is a formula, we denote by  $\text{Mod}(\phi)$  the set of models of  $\phi$ . A formula  $\psi$  is a logical consequence of  $\phi$ , denoted by  $\phi \models \psi$ , if  $\text{Mod}(\phi) \subseteq \text{Mod}(\psi)$ , and the two formulas are equivalent, denoted by  $\phi \equiv \psi$ , if  $\text{Mod}(\phi) = \text{Mod}(\psi)$ .

For a set  $K$  of formulas,  $Cn(K)$  is a consequence operation, that is, a function from  $\mathcal{P}(\mathcal{L})$  to  $\mathcal{P}(\mathcal{L})$  such that, for all subsets  $A$  and  $B$  of  $\mathcal{L}$ : a)  $A \subseteq Cn(A)$  (*inclusion*), b) if  $A \subseteq B$ , then  $Cn(A) \subseteq Cn(B)$  (*monotony*), c)  $Cn(A) = Cn(Cn(A))$  (*idempotence*). We assume that this operation includes classical consequences and verifies the standard properties of *supraclassicality* (if  $\alpha$  can be derived from  $A$ , then  $\alpha \in Cn(A)$ ), *deduction* ( $\beta \in Cn(A \cup \{\alpha\})$  if and only if  $(\alpha \rightarrow \beta) \in Cn(A)$ ) and *compactness* (if  $\alpha \in Cn(A)$  then  $\alpha \in Cn(A')$  for some finite subset  $A'$  of  $A$ ). A subset  $A$  of  $\mathcal{L}$  is *consistent* if and only if there is no formula  $\mu \in \mathcal{L}$  such that both  $\mu \in Cn(A)$  and  $\neg\mu \in Cn(A)$ .

A theory, or belief set in our context,  $K$  is a deductively closed set of formulas, that is  $K = Cn(K)$ . A belief base  $B$  is a finite set of propositional formulas, not deductively closed,  $B = \{\phi_1, \phi_2, \dots, \phi_p\}$  and it is understood as  $\bigwedge B$ , the conjunction of its formulas,  $\phi_1 \wedge \phi_2 \wedge \dots \wedge \phi_p$ .

When, for some set of sets of formulas  $S$  we want to express the disjunction of the conjunction of the elements of each  $B \in S$ , we write  $\bigvee S$  instead of  $\bigvee_{B \in S} B$ . Similarly, when we want to express the intersection of the elements of  $S$ , we write  $\bigcap S$  instead of  $\bigcap_{B \in S} B$ .

## 2.2 Belief base revision

Belief revision consists in incorporating a new belief into the existing beliefs of an agent, changing as few as possible the original beliefs while preserving consistency. Belief revision has been extensively studied within the framework of classical logic. Foundational work has concentrated on the definition of expected properties of the revision operations [40,41,1]. In particular, a set of rationality postulates for revision operations, known as AGM postulates, has been defined in [1].

In this framework, beliefs to be revised are represented by logically closed sets of propositional formulas, which are called belief sets (see section 2.1). A revision operation is represented by an operator, denoted by  $*$ , which is a function that takes as input a belief set  $K$  and a formula  $\mu$  representing new information and returns a new belief set  $K * \mu$  which incorporates the new belief  $\mu$ .

The AGM postulates are defined as follows. Let  $K$  be a belief set,  $\mu$  and  $\psi$  be propositional formulas, and  $*$  be a revision operation. The  $+$  operator is the *simple expansion operator*, which is a monotonic addition of a formula to a belief set defined as  $K + \mu = Cn(K \cup \{\mu\})$ :

- (\*1)  $K * \mu$  is a theory (closure);
- (\*2)  $\mu \in K * \mu$  (success);
- (\*3)  $K * \mu \subseteq K + \mu$  (inclusion);
- (\*4) if  $\neg\mu \notin K$ , then  $K + \mu \subseteq K * \mu$  (vacuity);
- (\*5) if  $\models \neg\mu$ , then  $K * \mu = Cn(\perp)$  (consistency preservation);
- (\*6) if  $\mu \equiv \psi$  then  $K * \mu = K * \psi$  (syntax independence);
- (\*7)  $K * (\mu \wedge \psi) \subseteq (K * \mu) + \psi$  (conjunctive inclusion);
- (\*8) if  $\neg\psi \notin K * \mu$ , then  $(K * \mu) + \psi \subseteq K * (\mu \wedge \psi)$  (conjunctive vacuity).

These postulates intend to capture an expected consistency of the results as well as the idea of minimal change. The first six postulates are called the basic postulates and are straightforward. Postulates (\*7) and (\*8) are called supplementary postulates and are less obvious. They express the idea that revising by the conjunction of two pieces of information amounts to a revision by the first one and an expansion by the second one whenever possible (whenever the second piece of information does not contradict any belief resulting from the first revision). This property is rather natural within different choice theories [59,75] (decision, social choice, etc.).

Two approaches can be considered for the definition of a revision operation. *Model based approaches* consider the models of the initial belief set and those of the revision formula [55]. *Syntax based approaches* (also called formula based approaches) focuses on the formulas to be revised [68]. When a formula based revision operation operates on a belief base and produces a belief base, it is called a *base revision operation*. A formula based revision operation can generally be extended to a revision operation by considering the deductive closure of the result. Namely, if  $K$  is a belief base and  $\mu$  a propositional formula, one can define a revision operation  $\otimes$  based on the formula based revision operation  $*$  as  $K \otimes \mu = Cn(K * \mu)$ . This is important in particular when the result of the

formula based revision operation is not itself a belief base, which will be the case for one of our operations.

Within this context, Hansson [48] proposed postulates that any base revision operation should satisfy. These postulates are as follows: let  $B$  and  $B'$  be consistent belief bases, and  $\mu$  and  $\phi$  be propositional formulas:

- (**Success**)  $\mu \in B * \mu$ .
- (**Inclusion**)  $B * \mu \subseteq B \cup \{\mu\}$ .
- (**Consistency**) If  $\mu$  is consistent then  $B * \mu$  is consistent.
- (**Vacuity**) if  $B \cup \{\mu\}$  is consistent then  $B * \mu = B \cup \{\mu\}$
- (**Core Retainment**) If  $\phi \in B$  and  $\phi \notin B * \mu$  then there exists  $B'$  such that  $B' \subseteq B \cup \{\mu\}$ ,  $B'$  is consistent but  $B' \cup \{\phi\}$  is inconsistent.
- (**Relevance**) If  $\phi \in B$  and  $\phi \notin B * \mu$  then there exists  $B'$  such that  $B * \mu \subseteq B' \subseteq B \cup \{\mu\}$ ,  $B'$  is consistent but  $B' \cup \{\phi\}$  is inconsistent
- (**Uniformity**) If for all subsets  $B' \subseteq B$ ,  $B' \cup \{\mu\}$  is inconsistent if and only if  $B' \cup \{\phi\}$  is inconsistent then  $B \setminus (B * \mu) = B \setminus (B * \phi)$

The intuitive meaning of these postulates is as follows. **Success** and **Consistency** express basic revision principles, namely that the new information should be accepted and that the result must be consistent. **Inclusion** states that the union of the initial belief base with the new information is the upper bound of any revision operation. **Vacuity** says that if new information is consistent with the initial belief base then the result of revision equals the non closing expansion. **Core Retainment** and **Relevance** express the intuition that nothing is removed from the original belief base unless its removal in some way contributes to make the result consistent. **Uniformity** states that if two formulas are consistent with the same subsets of the original belief base then the respective erased formulas should be identical.

In the literature, many formula-based operators stem from  $\mathcal{W}_{\subseteq}(B, \mu)$ , the set of maximal sub-bases of  $B$  consistent with  $\mu$  where maximality is considered in terms of set inclusion, and thus the following set is defined as follows:

$$\mathcal{W}_{\subseteq}(B, \mu) = \{B_i \subseteq B \mid B_i \not\models \neg\mu \text{ and for all } B_j, j \neq i \\ \text{s. t. } B_i \subset B_j \subseteq B, B_j \models \neg\mu\}.$$

This set is then used to define the revised belief base according to a given strategy. The maximality criterion as well as the strategy can vary. The selection of elements of  $\mathcal{W}_{\subseteq}(B, \mu)$  which will be retained for the construction of the outcome of a revision operation is frequently performed with a *selection function*. In the context of belief bases,  $\gamma$  is a *selection function* for a belief base  $B$  if and only if it is a function ( $\gamma : 2^{2^{\mathcal{L}}} \rightarrow 2^{2^{\mathcal{L}}}$ ) and for any formula  $\mu$  :

1. If  $\mathcal{W}_{\subseteq}(B, \mu) \neq \emptyset$  then  $\emptyset \neq \gamma(\mathcal{W}_{\subseteq}(B, \mu)) \subseteq \mathcal{W}_{\subseteq}(B, \mu)$ .
2. If  $\mathcal{W}_{\subseteq}(B, \mu) = \emptyset$  then  $\gamma(\mathcal{W}_{\subseteq}(B, \mu)) = \{B\}$ .

In [13], authors consider two different strategies. The first one can be viewed as “*permissive*”, as it considers that all maximal sub-bases are equally plausible.

The second one can be viewed as more “cautious”, as its result consists in the intersection of consistent maximal sub-bases, i.e. it only keeps beliefs that are not questioned. These two strategies provide two well-known operators, namely Ginsberg’s operator,  $*_G$ , [42] and WIDTIO operator,  $*_{wid}$ , [91] defined respectively as

$$B *_G \mu = \bigvee_{B' \in \mathcal{W}_{\subseteq}(B, \mu)} \bigwedge (B' \cup \{\mu\}),$$

$$B *_{wid} \mu = \bigwedge_{B' \in \mathcal{W}_{\subseteq}(B, \mu)} \bigcap (B' \cup \{\mu\}).$$

Authors in [13] then pay particular attention to maximality defined in terms of cardinality. This is a quite natural solution since in various applications the cardinality criterion is used because information acquisition is expensive. They consider the set of maximal sub-bases with respect to cardinality consistent with  $\mu$ ,  $\mathcal{W}_{card}(B, \mu)$ , instead of  $\mathcal{W}_{\subseteq}(B, \mu)$ . More formally:

$$\mathcal{W}_{card}(B, \mu) = \{B_i \subseteq B \mid B_i \not\models \neg\mu \text{ and } \forall B_j \subseteq B, j \neq i \\ \text{such that } |B_i| < |B_j|, B_j \models \neg\mu\}.$$

Analogously and respectively to Ginsberg’s and WIDTIO operators, the two strategies presented above provide two operators RSRG [3] and RSRW. Indeed, the notation RSR comes from the expression “*Removed Sets Revision*”, naming operators stemming from the removal of the smallest number of formulas from the initial belief base [3]. Formally, we have: let  $B$  be a belief base,  $\mu$  be a propositional formula, and let  $\gamma_{card}$  be a selection function such that  $\gamma_{card}(\mathcal{W}_{\subseteq}(B, \mu)) = \mathcal{W}_{card}(B, \mu)$ . Then

$$B *_{RSRG} \mu = \bigvee \gamma_{card}(\mathcal{W}_{\subseteq}(B, \mu)) \wedge \mu \quad \text{and}$$

$$B *_{RSRW} \mu = \bigcap \gamma_{card}(\mathcal{W}_{\subseteq}(B, \mu)) \cup \{\mu\}.$$

Note that  $*_{RSRW}$  is a belief base revision operator (the result is a set of formulas), while  $*_{RSRG}$  is not (the result is a single formula). Let us also mention that these formula-based operators are sensitive to the syntactic form of the knowledge representation. The following example illustrates this idea.

*Example 1.* Consider  $B_1 = \{a, b\}$ ,  $B_2 = \{a, a \rightarrow b\}$  two belief bases and a formula  $\mu = \neg b$  representing the new information.  $B_1$  and  $B_2$  are logically equivalent. The unique subset of  $B_1$  which is consistent with  $\mu$  is  $\{a\}$ , while there are two maximal subsets (with respect to set inclusion) of  $B_2$  which are consistent with  $\mu$ , namely,  $\{a\}$  and  $\{a \rightarrow b\}$ . Consequently,  $B_1 *_{wid} \mu = a \wedge \neg b$  and  $B_2 *_{wid} \mu = \neg b$ .

Therefore, in order to get rid of syntax dependency, Hansson [47] has shown that it seems natural to revise explicitly defined belief bases and to then extend these operations to belief sets, considering the deductive closure of the result of

revision. Thus it is possible to define, from a base revision operator  $*$ , a new one denoted by  $\otimes$  whose the result is a set of beliefs (or theory) such as

$$B \otimes \mu = Cn(B * \mu).$$

Adopting this point of view leads to the following four operators.

$$\begin{aligned} B \otimes_G \mu &= Cn\left(\bigvee_{B' \in \mathcal{W}_{\subseteq}(B, \mu)} \bigwedge B' \cup \{\mu\}\right) \\ B \otimes_{wid} \mu &= Cn\left(\bigwedge_{B' \in \mathcal{W}_{\subseteq}(B, \mu)} \bigcap \{B' \cup \{\mu\}\}\right) \\ B \otimes_{RSRG} \mu &= Cn\left(\bigvee \gamma_{card}(\mathcal{W}_{\subseteq}(B, \mu)) \wedge \mu\right) \\ B \otimes_{RSRW} \mu &= Cn\left(\bigcap \gamma_{card}(\mathcal{W}_{\subseteq}(B, \mu)) \cup \{\mu\}\right) \end{aligned}$$

While it is well-known that Ginsberg's and Widtjo operators satisfy respectively the first seven and the first six AGM postulates, it is proven in [3] that RSRG satisfies all of them.

### 2.3 Evidence theory

Uncertainty spreads in practice our daily lives, be it the result of sensing ambiguities or of incomplete knowledge. Generally, we do not even notice our skill at handling all these uncertainties. Away from this, when developing intelligent agents which are confronted to the real environment, it becomes obvious that modeling uncertainty is really a challenging problem. Nonetheless, in order to make these agents act in a robust way, they need to be able to cope with uncertainty. Hence, an agent with an explicit representation of uncertainty knows about the limitations of its knowledge and can consequently better predict its interactions.

There is a variety of mathematical frameworks for expressing uncertainty (see [57] for a review). The reason is that there are different types of uncertainty. Probability theory is the preferred one and is very widely applied when dealing with aleatory uncertainty related to randomness and chance. Uncertainty resulting from a lack of evidence is of a different nature though. This latter type of uncertainty is the result of ignorance rather than randomness<sup>6</sup>. The Bayesian view is that ignorance can be adequately represented using probability theory by applying the principle of indifference [56]. This principle states that, in the absence of evidence favoring any particular outcome, the probabilities representing the beliefs for each outcome should be the same for all outcomes. In contrast, evidence theory, which is also commonly known as Dempster-Shafer theory or

<sup>6</sup> There are other types of uncertainty (e.g., vagueness of natural language that can be described using fuzzy sets), nonetheless these are not taken into consideration in this paper.



belief functions theory [18,76], distinguishes between these types of uncertainty and thus makes ignorance explicit. That is why belief functions theory can be interpreted as a generalization of Bayesian probability theory [77]. Therefore, we can consider evidence theory as a convenient mathematical framework for modeling and reasoning with imperfect information and generally for describing *epistemic uncertainty* referring to quantified beliefs held by an agent.

Evidence theory was initially introduced by Arthur Dempster in 1967 [18] and then formalized by Glenn Shafer in his book titled "*A mathematical theory of evidence*" in 1976 [76]. It has been a tricky subject that has aroused the curiosity of numerous researchers from various disciplines such as artificial intelligence [89,87,88,5,97], clustering and classification [66,24,21,20,67,23], multi-criteria decision aid [94,44,7,6,10,11,12], etc. Moreover, it has been the starting point of several theoretical developments such as the transferable belief model [83,22]. Note that, while there are extensions to infinite domains [80,19], the belief functions considered here are usually assumed to have finite domains.

Let  $\Theta = \{S_1, \dots, S_n\}$  be a finite set of mutually exclusive and exhaustive statements called frame of discernment and  $2^\Theta$  be the power set of  $\Theta$ . A *Basic Belief Assignment* (BBA) [76] is the basic function used in evidence theory for modeling imperfect data. It is a mapping  $m$  from  $2^\Theta$  to  $[0, 1]$  such that  $m(\emptyset) = 0$  and  $\sum_{A \subseteq \Theta} m(A) = 1$ . The quantity  $m(A)$  represents the *belief mass* of subset  $A$ , i.e. the belief committed exactly to  $A$ . When  $m(A) \neq 0$ ,  $A$  is called a *focal element* or a *focal set*.

A BBA is said to be *Bayesian* if all its focal elements are singletons and *consonant* if all these elements are nested. It is called *vacuous* if the total belief is assigned only to  $\Theta$  ( $m(\Theta) = 1$ , total ignorance case) and *simple* if it has two focal elements and  $\Theta$  is one of these focal sets. In the latter case,  $m(\Theta)$  reflects an ignorance level since it is the belief mass which is not assigned to any subset  $A \neq \Theta$  and transferred to  $\Theta$ .

A BBA can be also represented by two functions called *credibility* (or belief) and *plausibility*, denoted in the literature respectively by *Bel* and *Pl* [76]. Formally, these two functions are defined from  $2^\Theta$  to  $[0, 1]$  as follows:

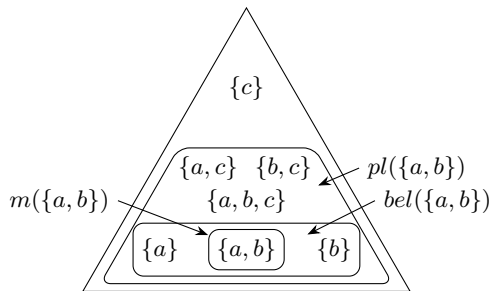
$$Bel(A) = \sum_{\substack{X \subseteq A \\ X \neq \emptyset}} m(X)$$

$$Pl(A) = \sum_{A \cap X \neq \emptyset} m(X)$$

$Bel(A)$  is the total belief of subsets  $X$  which are included in  $A$ , whereas  $Pl(A)$  is the total belief of subsets  $X$  having a non-empty intersection with  $A$ , i.e. the subsets that are included in  $A$  and those having a partial intersection with  $A$ .  $Bel(A)$  and  $Pl(A)$  are therefore the minimal and maximal total beliefs committed to  $A$ . They are also connected by the relation  $Pl(A) = 1 - Bel(\bar{A})$  where  $\bar{A}$  is the complement of  $A$  in  $\Theta$ .

An illustration of these different belief function representations can be shown in Figure 1. In this example, the frame of discernment  $\Theta$  consists of three ele-

ments  $a, b, c$  and the triangle contains all subsets of  $\Theta$  except for  $\emptyset$ . The indicated areas correspond to the mass of the different belief representations  $m$ ,  $Bel$  and  $Pl$ .



**Fig. 1.** Illustration of the different belief function representations [76]

When two pieces of information are coming from two distinct sources, it is essential to be able to combine them in order to obtain an overall result. Thus, combination is a fundamental notion in evidence theory, allowing the aggregation of imperfect information given by several sources and modeled by BBAs. Several combination rules have been proposed in the literature (see [82] for a review). Among them, Dempster’s rule [76] remains the most commonly used operator for the combination of independent BBAs. It is given by

$$m(A) = (1 - k)^{-1} \cdot \sum_{\substack{X \cap Y = A \\ A \neq \emptyset}} m_1(X) \cdot m_2(Y),$$

where  $m = m_1 \oplus m_2$  is the BBA deduced from the combination of  $m_1$  and  $m_2$  (called the orthogonal sum) and  $k = \sum_{X \cap Y = \emptyset} m_1(X) \cdot m_2(Y)$  is the belief mass that the combination assigns to the empty set. The ratio  $(1 - k)^{-1}$  is a normalization factor guaranteeing that no belief mass is given to the empty set and that the total belief is equal to one.

Dempster’s rule is a conjunctive operator, i.e. the resulting focal elements are intersections of those related to  $m_1$  and  $m_2$ . They are proved to be both commutative and associative. Thus, the combination result of several BBAs is independent of the order in which they are considered.

The decision-making is also an important step of evidence theory mechanism that aims to choose the “best” statement of  $\Theta$ . Among other rules, one can cite the maximum credibility rule which selects the most credible  $S_i$  [25], the maximum plausibility rule which chooses the most plausible  $S_i$  [25], and the maximum pignistic probability [81]. The latter operator is based on the idea of transforming a BBA into a function having properties similar to a probability distribution called *pignistic probability function*  $BetP$ . The decision is therefore to choose the statement having the maximal pignistic probability.

## 2.4 Complexity classes

The classes P and NP are the classes of decision problems solvable in deterministic resp. non deterministic polynomial time. The class coNP is the class of problems whose complementary is in NP [69]. An oracle for a complexity class  $\mathcal{C}$  is an entity able to solve any problem in the class  $\mathcal{C}$ , it is simply a “black box” which is able to decide for any instance of a given computational problem from  $\mathcal{C}$  whether it is a positive instance or not. We write  $P^{\mathcal{C}}$  (resp.  $NP^{\mathcal{C}}$ ) for the class of decision problems that can be decided by a deterministic (resp. non-deterministic) Turing machine in polynomial time using an oracle for the class  $\mathcal{C}$ . Stockmeyer inductively defined the polynomial hierarchy, where the first level consists of the P, NP and coNP classes [86]. Within this hierarchy we particularly use the classes of the second level, i.e.  $\Sigma_2P = NP^{NP}$ ,  $\Delta_2P = P^{NP}$  and  $\Pi_2P = coNP^{NP}$ . Also, the class  $\Theta_2P = P^{NP^{\lceil \log(n) \rceil}}$ , introduced by Wagner in [90], is a sub-class of  $\Delta_2P$ . The problems of  $\Theta_2P$  are those of  $\Delta_2P$  that can be solved in polynomial time with only a logarithmic number of calls to an NP-oracle.

## 3 Credible belief base revision

Combination, in the sense of evidence theory, is typically applied to pieces of information received from the “outside the environment”. However, an agent may have its proper prior opinion (from the inside), and then receive some input information coming from the environment. In such a case, the problem is no longer a combination. It is a revision problem which is intrinsically asymmetric as it adopts an insider point of view, so that either the new information or initial beliefs play specific roles, while combination is essentially a symmetric process, up to the proper handling of unequal reliability of sources. Remind also that if we assume that the new information is reliable, it should be retained, while the agent’s initial beliefs should comply with minimal change to that effect. In this context, we propose to use the credibility concept when performing revision by selecting prior beliefs which seem to be the most credible ones.

In this section, we investigate the idea considering maximality in terms of set credibility (instead of set inclusion or set cardinality) denoted throughout this paper by CSR (Credible Sets Revision). Our goal is to define belief base revision operators requiring rationality when revising in order to avoid losing valuable beliefs [66].

### 3.1 Credible belief operators

Our operators are derived from RSRG and RSRW operators. The key difference is that we initially evaluate each sub-base  $B_i$  not just by its cardinality but by its cardinality proportionally to the cardinality of the agent’s initial base. Indeed, it seems reasonable to consider that the importance or the relevance granted to a formula in an initial base of 10 formulas is not equal to the one granted to the same formula in a base of 100 formulas. Moreover, it is normal that, following

each revision iteration, the size of the agent's base changes. As a consequence, the relevance of a formula changes from one iteration to another.

We propose *credible belief base revision* as an attempt to define two new formula-based revision operators using suitable tools offered by evidence theory. To ensure uniformity with the RSRG and RSRW operators, we denote these operators by CSRG (referring to a *permissive* strategy) and CSRW (referring to a *drastic* strategy). We now elaborate on the set credibility notion which underlies our operators.

We start with  $\mathcal{W}_{\subseteq}(B, \mu)$ , i.e. the set of  $n$  maximal sub-bases  $B_i$  ( $1 \leq i \leq n$ ) with respect to set inclusion which are consistent with  $\mu$ . The first step consists in representing each  $B_i$  with a simple BBA denoted  $m_{i,\mu}$ . This function takes into account the cardinality of  $B_i$  relatively to the initial agent's belief base  $B$ .

**Definition 1 (individual basic belief assignment).** *Let  $B$  be a belief base and  $\mu$  a propositional formula. For each  $B_i \in \mathcal{W}_{\subseteq}(B, \mu)$ , we define the individual basic belief assignment  $m_{i,\mu}$  associated with  $B_i$  relatively to  $\mu$  as:*

$$\begin{cases} m_{i,\mu}(B_i) = \frac{|B_i|}{|B|}, \text{ and} \\ m_{i,\mu}(B) = 1 - \frac{|B_i|}{|B|} \end{cases}$$

where  $|B_i|$  is the cardinality of  $B_i$  and  $|B|$  is the cardinality of  $B$ .

The term  $m_{i,\mu}(B_i)$  represents the proportion of formulas belonging to  $B_i$  with regard to  $B$ . Also note that  $m_{i,\mu}(B)$  is interpreted as an ignorance level that reflects the belief mass which is not assigned to  $B_i$  and therefore transferred to  $B$ . Each  $m_{i,\mu}$  is a *simple* BBA.

In a second step, the BBAs describing the maximal consistent sub-bases are combined using Dempster's rule. The combined BBA is defined as the orthogonal sum of these BBAs.

**Definition 2 (global basic belief assignment).** *Let  $B$  be a belief base and  $\mu$  be a propositional formula. For each  $B_i \in \mathcal{W}_{\subseteq}(B, \mu)$ , let  $m_{i,\mu}$  be the individual basic belief assignment associated with  $B_i$ . We define  $m_{\mu}$  the global basic belief assignment on the belief base  $B$  with respect to  $\mu$  as the combined BBA defined by:*

$$m_{\mu} = m_{1,\mu} \oplus \dots \oplus m_{n,\mu}$$

Since Dempster's rule is a conjunctive operator and the focal elements of each  $m_{i,\mu}$  are  $B_i$  and  $B$ , therefore the focal sets of  $m_{\mu}$ , denoted by  $\mathcal{F}(B, \mu)$ , are all the sub-bases  $B_i \in \mathcal{W}_{\subseteq}(B, \mu)$ , all the sets derived from their possible non empty intersections, denoted by  $\mathcal{F}_{\cap}(B, \mu)$ , and the initial belief base  $B$ . This is due to the fact that  $B$  is a common focal element defined in each  $m_{i,\mu}$ .

**Definition 3 (focal sets of the global belief assignment).** *Let  $B$  be a belief base,  $\mu$  be a propositional formula, and  $m_{\mu}$  be the global belief assignment defined on  $B$  with respect to  $\mu$ , as defined in definition 2. The focal sets of  $m_{\mu}$  are defined as*

$$\mathcal{F}(B, \mu) = \mathcal{W}_{\subseteq}(B, \mu) \cup \mathcal{F}_{\cap}(B, \mu) \cup \{B\}$$

where

$$\mathcal{F}_\cap(B, \mu) = \left\{ \bigcap_{\substack{i,j \in \{1, \dots, n\} \\ i \neq j}} (B_i, B_j) \cup \bigcap_{\substack{i,j,k \in \{1, \dots, n\} \\ i \neq j \neq k}} (B_i, B_j, B_k) \right. \\ \left. \cup \dots \cup \bigcap (B_1, \dots, B_n) \right\} \setminus \{\emptyset\}$$

Thus,  $B$  plays a central role in the combination since it allows the appearance of all the  $B_i$  and their potential intersections.

At this point, let us note that the combination of the individual BBAs assigns belief masses to the intersection of the sub-bases. If an intersection (or several) supports completely a sub-base  $B_i$ , it is therefore a focal set affirming  $B_i$ . Thus, its belief mass can be added to  $m_\mu(B_i)$  which allows to define an overall degree of belief characterizing  $B_i$ . This measure is nothing else than its credibility degree defined in section 2.3. We denote it by  $Bel_\mu(B_i)$  in order to stress out the dependance on formula  $\mu$ .

**Definition 4 (credibility degree).** *Let  $B$  be a belief base and  $\mu$  a propositional formula. Let  $m_\mu$  be the global BBA defined on  $B$  with respect to  $\mu$ . Then, for each  $B_i \in \mathcal{W}_\subseteq(B, \mu)$  we define the credibility degree  $Bel_\mu(B_i)$  of  $B_i$  as :*

$$Bel_\mu(B_i) = m_\mu(B_i) + \sum_{\substack{X \subset B_i \\ X \in \mathcal{F}_\cap(B, \mu)}} m_\mu(X).$$

From these credibility degrees we define, for a given belief base  $B$  and a formula  $\mu$ , the set of sub-bases of  $B$  consistent with  $\mu$  and maximal with respect to credibility degree.

**Definition 5.** *Let  $B$  be a belief base,  $\mu$  be a propositional formula and  $Bel$  be a credibility degree function. We define:*

$$\mathcal{W}_{Bel}(B, \mu) = \{B_i \subseteq B \mid B_i \not\models \neg\mu \text{ and } \forall B_j \subseteq B, j \neq i \\ \text{such that } Bel(B_i) < Bel(B_j), B_j \models \neg\mu\}.$$

We are now able to define our credible revision operators. The  $*_{\text{CSRG}}$  revision operator takes into account all the sub-bases in  $\mathcal{W}_{Bel}(B, \mu)$ , considering them equally fair and favorable. This operator is defined as follows:

**Definition 6 (CSRG revision operator).** *Let  $B$  be a belief base,  $\mu$  a propositional formula and  $\gamma_{Bel}$  be a selection function such that  $\gamma_{Bel}(\mathcal{W}_\subseteq(B, \mu)) = \mathcal{W}_{Bel}(B, \mu)$ . The  $*_{\text{CSRG}}$  revision operator is defined as*

$$B *_{\text{CSRG}} \mu = \bigvee \gamma_{Bel}(\mathcal{W}_\subseteq(B, \mu)) \wedge \mu.$$

The CSRW revision operator  $*_{\text{CSRW}}$  considers the intersection of the most credible sub-bases consistent with  $\mu$ . It is defined as follows.

**Definition 7 (CSRW base revision operator).** Let  $B$  be a belief base and  $\mu$  be a propositional formula. Let  $\gamma_{Bel}$  be a selection function such that  $\gamma_{Bel}(\mathcal{W}_{\subseteq}(B, \mu)) = \mathcal{W}_{Bel}(B, \mu)$ . The CSRW base revision operator, denoted by  $*_{CSRW}$  is defined as

$$B *_{CSRW} \mu = \bigcap \gamma_{Bel}(\mathcal{W}_{\subseteq}(B, \mu)) \cup \{\mu\}.$$

Consequently, the associated operators  $\otimes_{CSRG}$   $\otimes_{CSRW}$  (returning a theory) respectively to  $*_{CSRG}$  and  $*_{CSRW}$  are the following:

**Definition 8 (CSRG and CSRW revision operators).** Let  $B$  be a belief base and  $\mu$  be a propositional formula. The CSRG and CSRW revision operators are defined as:

$$\begin{aligned} B \otimes_{CSRG} \mu &= Cn(\bigvee \gamma_{Bel}(\mathcal{W}_{\subseteq}(B, \mu)) \wedge \mu) \\ B \otimes_{CSRW} \mu &= Cn(\bigcap \gamma_{Bel}(\mathcal{W}_{\subseteq}(B, \mu)) \cup \{\mu\}) \end{aligned}$$

We can now define the notion of logical consequence for each of these operations. A formula  $\psi$  is a logical consequence of the revision result  $B * \mu$  if:

- In the case of CSRG operator, it is a logical consequence of any sub-base  $B_i$  in  $\mathcal{W}_{Bel}(B, \mu)$ , augmented with the new information  $\mu$ . Formally, we have

$$B *_{CSRG} \mu \models \psi \text{ if and only if } \forall B_i \in \mathcal{W}_{Bel}(B, \mu), B_i \cup \{\mu\} \models \psi.$$

- In the case of CSRW operator, it is a logical consequence of the intersection of all sub-bases in  $\mathcal{W}_{Bel}(B, \mu)$  augmented with the new information  $\mu$ . Formally, we have

$$B *_{CSRW} \mu \models \psi \text{ if and only if } \bigcap \gamma_{Bel}(\mathcal{W}_{\subseteq}(B, \mu)) \cup \{\mu\} \models \psi.$$

So far, we can conclude that an interpretation  $I$  is a model of the revised belief base ( $I \models B * \mu$ ) if and only if  $I$  satisfies  $\mu$  and satisfies at least one set of  $\mathcal{W}_{Bel}(B, \mu)$  in the case of CSRG operator, and every formula occurring in all selected maximal consistent sub-bases (i.e. in all sets of  $\mathcal{W}_{Bel}(B, \mu)$ ) in the case of CSRW operator.

Finally, it is interesting to note that the maximal credibility is generally used within evidence theory to select the most credible element of the frame of discernment. In this work, we have adapted the use of this rule according to the studied context since the objective is not to select one formula from the initial set  $B$ <sup>7</sup>. The decision should rather be taken on the different sub-bases (which are subsets of formulas) not on the formulas composing  $B$ . In addition, it is important to notice that we do not investigate the maximal plausibility in the context of CSRG and CSRW operators since the plausibility considers even the focal elements having a partial intersection with  $B_i$ . These focal elements are not the result of combining  $B_i$  with other sub-bases. They are rather induced by the

<sup>7</sup>  $B$  constitutes the frame of discernment and the formulas are the statements.

combination of other sets. Similarly, the maximum of pignistic probability (basically used to select the most likelihood element of the frame of discernment) is inappropriate in our context. Indeed, the pignistic transformation cannot be exploited correctly in this case since the objective is to select a maximal consistent sub-base not a unique formula of  $B$ .

We now provide a simple example illustrating the CSRG and the CSRW operators.

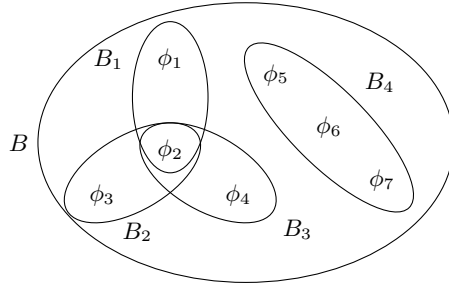
*Example 2.* Consider the following belief base<sup>8</sup>:

$$\begin{aligned}
B = \{ & \phi_1 : (\neg a \wedge b \wedge c) \vee (\neg c \wedge (a \leftrightarrow b)), \phi_2 : \neg a, \\
& \phi_3 : (\neg a \wedge c) \vee (a \wedge b \wedge \neg c), \phi_4 : (\neg a \wedge b) \vee (a \wedge \neg b \wedge \neg c), \\
& \phi_5 : c \wedge (\neg a \leftrightarrow b), \phi_6 : (c \wedge (\neg a \leftrightarrow b)) \vee (a \wedge \neg b \wedge \neg c), \\
& \phi_7 : (a \wedge (\neg b \leftrightarrow c)) \vee (\neg a \wedge b \wedge c) \}
\end{aligned}$$

and a formula  $\mu = (\neg a \wedge \neg b) \vee (a \wedge c) \vee (\neg a \wedge b \wedge \neg c)$  representing a new information.  $B$  is consistent but  $B \cup \{\mu\}$  is inconsistent. We have:

$$\begin{aligned}
\mathcal{W}_{\subseteq}(B, \mu) = \{ & B_1 = \{\phi_1, \phi_2\} & B_2 = \{\phi_2, \phi_3\} \\
& B_3 = \{\phi_2, \phi_4\} & B_4 = \{\phi_5, \phi_6, \phi_7\}
\end{aligned}$$

Figure 2 depicts  $\mathcal{W}_{\subseteq}(B, \mu)$ . Then we have



**Fig. 2.**  $\mathcal{W}_{\subseteq}(B, \mu)$  in example 2

$$\mathcal{W}_{card}(B, \mu) = \{B_4\}.$$

Hence,

$$\begin{aligned}
B *_{\text{RSRG}} \mu = & ((c \wedge (\neg a \leftrightarrow b)) \vee (a \wedge \neg b \wedge \neg c) \vee \\
& (a \wedge (\neg b \leftrightarrow c)) \vee (\neg a \wedge b \wedge c)) \wedge \\
& ((\neg a \wedge \neg b) \vee (a \wedge c) \vee (\neg a \wedge b \wedge \neg c))
\end{aligned}$$

<sup>8</sup> We name each formula in  $B$  in order to produce a more concise example.

and  $\bigcap_{B' \in \mathcal{W}_{card}(B, \mu)} B' = \{B_4\}$ , which leads to

$$B *_{\text{RSRW}} \mu = \{(c \wedge (\neg a \leftrightarrow b)), (c \wedge (\neg a \leftrightarrow b)) \vee (a \wedge \neg b \wedge \neg c), \\ (a \wedge (\neg b \leftrightarrow c)) \vee (\neg a \wedge b \wedge c), \\ (\neg a \wedge \neg b) \vee (a \wedge c) \vee (\neg a \wedge b \wedge \neg c)\}$$

and thus

$$B \circledast_{\text{RSRG}} \mu = \text{Cn}(((c \wedge (\neg a \leftrightarrow b)) \vee (a \wedge \neg b \wedge \neg c)) \vee \\ (a \wedge (\neg b \leftrightarrow c)) \vee (\neg a \wedge b \wedge c)) \wedge \\ ((\neg a \wedge \neg b) \vee (a \wedge c) \vee (\neg a \wedge b \wedge \neg c)))$$

and

$$B \circledast_{\text{RSRW}} \mu = \text{Cn}((c \wedge (\neg a \leftrightarrow b)) \wedge ((c \wedge (\neg a \leftrightarrow b)) \vee (a \wedge \neg b \wedge \neg c)) \wedge \\ ((a \wedge (\neg b \leftrightarrow c)) \vee (\neg a \wedge b \wedge c)) \wedge \\ ((\neg a \wedge \neg b) \vee (a \wedge c) \vee (\neg a \wedge b \wedge \neg c)))$$

Let us now consider the CSRG and CSRW operators. We first compute the individual BBAs related to each sub-base  $B_i$  ( $1 \leq i \leq 4$ ) according to definition 1:

$$\begin{array}{ll} m_{1, \mu}(B_1) = 2/7 & m_{2, \mu}(B_2) = 2/7 \\ m_{1, \mu}(B) = 5/7 & m_{2, \mu}(B) = 5/7 \\ m_{3, \mu}(B_3) = 2/7 & m_{4, \mu}(B_4) = 3/7 \\ m_{3, \mu}(B) = 5/7 & m_{4, \mu}(B) = 4/7 \end{array}$$

Dempster's rule is then applied to combine  $m_{1, \mu}$ ,  $m_{2, \mu}$ ,  $m_{3, \mu}$ , and  $m_{4, \mu}$ . This leads to  $m_\mu$ , the global BBA, according to definition 2:

$$\begin{array}{llll} m_\mu(B_1) = 200/1747 & m_\mu(B_1 \cap B_2) = 80/1747 & m_\mu(B_1 \cap B_2 \cap B_3) = 32/1747 & \\ m_\mu(B_2) = 200/1747 & m_\mu(B_1 \cap B_3) = 80/1747 & & \\ m_\mu(B_3) = 200/1747 & m_\mu(B_2 \cap B_3) = 80/1747 & & \\ m_\mu(B_4) = 375/1747 & & & \end{array}$$

Using this global BBA, we are able to compute the credibility degrees  $Bel_\mu(B_i)$ ,  $1 \leq i \leq 4$ :

$$\begin{array}{l} Bel_\mu(B_1) = m_\mu(B_1) + m_\mu(B_1 \cap B_2) + m_\mu(B_1 \cap B_3) + m_\mu(B_1 \cap B_2 \cap B_3) \\ \quad = 392/1747 \\ Bel_\mu(B_2) = m_\mu(B_2) + m_\mu(B_1 \cap B_2) + m_\mu(B_2 \cap B_3) + m_\mu(B_1 \cap B_2 \cap B_3) \\ \quad = 392/1747 \\ Bel_\mu(B_3) = m_\mu(B_3) + m_\mu(B_2 \cap B_3) + m_\mu(B_1 \cap B_3) + m_\mu(B_1 \cap B_2 \cap B_3) \\ \quad = 392/1747 \\ Bel_\mu(B_4) = m_\mu(B_4) = 375/1747 \end{array}$$



Thus  $\mathcal{W}_{Bel}(B, \mu) = \{B_1, B_2, B_3\}$ . Therefore

$$B \circledast_{\text{CSRSG}} \mu = Cn((B_1 \vee B_2 \vee B_3) \wedge ((\neg a \wedge \neg b) \vee (a \wedge c) \vee (\neg a \wedge b \wedge c))).$$

For  $\circledast_{\text{CSRW}}$ , we have  $B_1 \cap B_2 \cap B_3 = \{\phi_2\} = \{\neg a\}$ , hence

$$B \circledast_{\text{CSRW}} \mu = Cn(\neg a \wedge ((\neg a \wedge \neg b) \vee (a \wedge c) \vee (\neg a \wedge b \wedge c))).$$

Note that this example illustrates properly the fact that the selection of consistent sub-bases maximal with respect to set cardinality does not always guarantee the selection of the most relevant information since it can neglect potential formulas from the initial agent's beliefs and thus can induce a loss of rationality when revising. Additionally, the use of credibility in belief base revision seems reasonable regarding minimality as it captures relevant information playing a central role in the initial agent's beliefs, by giving extra importance to the intersection of the maximal sub-bases consistent with  $\mu$ .

The example presented above can be a proper illustration of the gap between cardinality and credibility as maximality criteria in the selection of consistent sub-bases. Nonetheless, this cannot deny the fact that the CSRSG and RSRG operators can lead to the same revision outcome if the intersection of each pair of sub-bases  $B_i \in \mathcal{W}_{\subseteq}$  is the empty set. Indeed, if it is the case, the intersection of any other group of sub-bases is also the empty set, i.e.  $\mathcal{F}_{\cap}(B, \mu) = \emptyset$ . As a result,  $\mathcal{F}(B, \mu)$ , the focal elements set of the combined BBA, consists only of  $B$  (the initial belief base) and the sub-bases  $B_i \in \mathcal{W}_{\subseteq}(B, \mu)$  (without intersections). This implies that there is no intersection which supports totally  $B_i$ , i.e.:

$$\sum_{\substack{X \subseteq B_i \\ X \in \mathcal{F}_{\cap}(B, \mu)}} m_{\mu}(X) = 0$$

Therefore,  $Bel_{\mu}(B_i) = m_{\mu}(B_i) = \frac{|B_i|}{|B|}$  and as a consequence, we obtain

$$\mathcal{W}_{card}(B, \mu) = \mathcal{W}_{Bel}(B, \mu).$$

### 3.2 Productivity of inference from credible belief revision operators

We now establish some results about the *productivity* of CSRSG and CSRW operators. For this purpose, we compare what can be *inferred* from the result of the revision operations by comparing the models of the result of the revision operations.

**Proposition 1.** *Let  $B$  a belief base and  $\mu$  a propositional formula. We have:*

- $\text{Mod}(B *_{\text{RSRG}} \mu) \subseteq \text{Mod}(B *_{\text{RSRW}} \mu)$  and
- $\text{Mod}(B *_{\text{CSRSG}} \mu) \subseteq \text{Mod}(B *_{\text{CSRW}} \mu)$ .

*Proof.* We begin by proving  $\text{Mod}(B *_{\text{RSRG}} \mu) \subseteq \text{Mod}(B *_{\text{RSRW}} \mu)$ . Let  $\alpha \in Cn(B *_{\text{RSRW}} \mu)$ . So, by definition, we have  $\alpha \in \{\phi \mid \bigcap_{B' \in \mathcal{W}_{card}(B, \mu)} B' \cup \{\mu\} \models \phi\}$ .

Consequently,  $\alpha \in \{\phi \mid \text{for each } B' \in \mathcal{W}_{card}(B, \mu), B' \cup \{\mu\} \models \phi\}$ . So,  $\alpha \in Cn(B *_{RSRG} \mu)$ . We obtain  $Cn(B *_{RSRW} \mu) \subseteq Cn(B *_{RSRG} \mu)$ . So,  $\text{Mod}(B *_{RSRW} \mu) \subseteq \text{Mod}(B *_{RSRG} \mu)$ .

We merely need to replace  $\mathcal{W}_{card}$  by  $\mathcal{W}_{Bel}$ ,  $*_{RSRG}$  by  $*_{CSRSG}$  and  $*_{RSRW}$  by  $*_{CSRW}$  to prove  $\text{Mod}(B *_{CSRSG} \mu) \subseteq \text{Mod}(B *_{CSRW} \mu)$ .

**Proposition 2.** *Let  $B$  a belief base and  $\mu$  a propositional formula. We have:*

- $\text{Mod}(B *_{RSRW} \mu) \not\subseteq \text{Mod}(B *_{RSRG} \mu)$  and
- $\text{Mod}(B *_{CSRW} \mu) \not\subseteq \text{Mod}(B *_{CSRSG} \mu)$ .

*Proof.* Let consider the example already cited in [13] where the belief base  $B = \{a \rightarrow \neg b, b, b \rightarrow c, c \rightarrow \neg a, b \rightarrow d, d \rightarrow \neg a, \neg d \rightarrow c\}$ , and the new information  $\mu = a$ . It is clear that

$$\begin{aligned} \mathcal{W}_{card}(B, \mu) &= \mathcal{W}_{Bel}(B, \mu) \\ &= \{\{a \rightarrow \neg b, b \rightarrow c, c \rightarrow \neg a, b \rightarrow d, d \rightarrow \neg a\}; \\ &\quad \{a \rightarrow \neg b, b \rightarrow c, c \rightarrow \neg a, b \rightarrow d, \neg d \rightarrow c\}; \\ &\quad \{a \rightarrow \neg b, b \rightarrow c, b \rightarrow d, d \rightarrow \neg a, \neg d \rightarrow c\}\} \end{aligned}$$

On the one hand we get

$$\begin{aligned} B *_{RSRG} \mu &= B *_{CSRSG} \mu \equiv (a \wedge \neg b \wedge \neg c \wedge \neg d) \vee (a \wedge \neg b \wedge \neg c \wedge d) \vee (a \wedge \neg b \wedge c \wedge \neg d) \\ &\equiv (a \wedge \neg b) \wedge \neg(c \wedge d). \end{aligned}$$

On the other hand, since  $\bigcap_{B' \in \mathcal{W}_{card}(B, \mu)} B' = \bigcap_{B' \in \mathcal{W}_{Bel}(B, \mu)} B' = \{a \rightarrow \neg b, b \rightarrow c, b \rightarrow d\}$ , we obtain

$$B *_{RSRW} \mu = B *_{CSRW} \mu \equiv \{a, (a \rightarrow \neg b), (b \rightarrow c), (b \rightarrow d)\}.$$

It is obvious that  $\text{Mod}(B *_{RSRG} \mu) = \text{Mod}(a \wedge \neg b \wedge \neg(c \wedge d)) = \{\{a\}, \{a, d\}, \{a, c\}\}$  and  $\text{Mod}(B *_{RSRW} \mu) = \{\{a\}, \{a, d\}, \{a, c\}, \{a, c, d\}\}$ . It is clear so that  $\text{Mod}(B *_{RSRW} \mu) \not\subseteq \text{Mod}(B *_{RSRG} \mu)$  and  $\text{Mod}(B *_{CSRW} \mu) \not\subseteq \text{Mod}(B *_{CSRSG} \mu)$ .

**Proposition 3.** *Let  $B$  a belief base and  $\mu$  a propositional formula.*

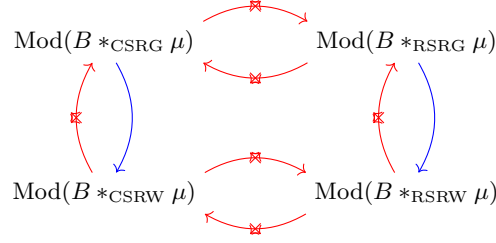
1. *On the one hand, we have*
  - $\text{Mod}(B *_{CSRSG} \mu) \not\subseteq \text{Mod}(B *_{RSRG} \mu)$  and
  - $\text{Mod}(B *_{CSRW} \mu) \not\subseteq \text{Mod}(B *_{RSRW} \mu)$ .
2. *On the other hand, we have*
  - $\text{Mod}(B *_{RSRG} \mu) \not\subseteq \text{Mod}(B *_{CSRSG} \mu)$  and
  - $\text{Mod}(B *_{RSRW} \mu) \not\subseteq \text{Mod}(B *_{CSRW} \mu)$ .

*Proof.* The proof is identical for the two cases.

Resume Example 2.

We have:  $\text{Mod}(B *_{CSRSG} \mu) = \{\{\}, \{c\}, \{b\}\}$ ,  $\text{Mod}(B *_{CSRW} \mu) = \{\{\}, \{c\}, \{b\}\}$ ,  $\text{Mod}(B *_{RSRG} \mu) = \{\{a, c\}\}$  et  $\text{Mod}(B *_{RSRW} \mu) = \{\{a, c\}\}$ . So  $\text{Mod}(B *_{CSRSG} \mu) \not\subseteq \text{Mod}(B *_{RSRG} \mu)$ ,  $\text{Mod}(B *_{CSRW} \mu) \not\subseteq \text{Mod}(B *_{RSRW} \mu)$ ,  $\text{Mod}(B *_{RSRG} \mu) \not\subseteq \text{Mod}(B *_{CSRSG} \mu)$  and  $\text{Mod}(B *_{RSRW} \mu) \not\subseteq \text{Mod}(B *_{CSRW} \mu)$ .

The results of propositions 1, 2 and 3 are summarized in figure 3.



**Fig. 3.** Comparing productivity of inference relations underlying the different revision operators. Red crossed arrows mean  $\not\subseteq$ , blue arrows mean  $\subseteq$ .

### 3.3 Logical properties of credible belief base operators

In this section we establish results about the logical properties of the  $*_{\text{CSRG}}$  and  $*_{\text{CSRW}}$  operators.

Unlike the case of  $*_{\text{CSRW}}$ , we cannot use the Hansson postulates in the case of  $*_{\text{CSRG}}$ , as the result of the operation is not a belief base. Thus, we study the  $\otimes_{\text{CSRG}}$  operator regarding to AGM postulates.

**Proposition 4.** *The  $\otimes_{\text{CSRG}}$  operator verifies (\*1), (\*2), (\*3), (\*4), (\*5), (\*6), (\*7). It does not verify (\*8).*

*Proof.* (\*1), (\*2) and (\*3) are obviously verified by definition 8.

Let  $\mathbf{K} = \text{Cn}(K)$ .

(\*4): If  $\neg\phi \notin \mathbf{K}$ , then  $\mathcal{W}_{\subseteq}(K, \phi) = \{K\}$  and  $\mathcal{W}_{\text{Bel}}(K, \phi) = \{K\}$ . Thus,  $K \otimes_{\text{CSRG}} \phi = \text{Cn}(K + \phi)$ .

(\*5): If  $\models \neg\phi$ , then  $\mathcal{W}_{\subseteq}(K, \phi) = \{\emptyset\}$  and  $\mathcal{W}_{\text{Bel}}(K, \phi) = \{\emptyset\}$ . Thus  $K \otimes_{\text{CSRG}} \phi = \text{Cn}(\phi) = \text{Cn}(\perp)$ .

(\*6): If  $\phi \equiv \psi$ , then  $\mathcal{W}_{\subseteq}(K, \phi) = \mathcal{W}_{\subseteq}(K, \psi)$ , and  $\mathcal{W}_{\text{Bel}}(K, \phi) = \mathcal{W}_{\text{Bel}}(K, \psi)$ . Thus  $K \otimes_{\text{CSRG}} \phi = K \otimes_{\text{CSRG}} \psi$ .

(\*7): if  $\text{Cn}(K \cup \{\psi\}) = \text{Cn}(\perp)$  the property is trivially verified.

Otherwise, ( $K$  is consistent with  $\psi$ ) we first show that  $\mathcal{W}_{\subseteq}(K, \mu) = \mathcal{W}_{\subseteq}(K, \mu \wedge \psi)$ .

( $\subseteq$ ) Let  $X \in \mathcal{W}_{\subseteq}(K, \mu)$ .  $X$  is consistent with  $\psi$  by hypothesis, and so  $X$  is consistent with  $\mu \wedge \psi$ . Moreover, by definition of  $\mathcal{W}_{\subseteq}(K, \mu)$  we have:  $\forall X'$  such that  $X \subset X' \subseteq K$ ,  $X'$  is inconsistent with  $\mu$ , so it is also inconsistent with  $\psi \wedge \mu$ . Thus,  $X \in \mathcal{W}_{\subseteq}(K, \mu \wedge \psi)$ .

( $\supseteq$ ) Let  $X \notin \mathcal{W}_{\subseteq}(K, \mu)$ . Two cases arise :

(i)  $X$  is consistent with  $\mu$  but there exists  $X'$ ,  $X \subset X' \subseteq K$  such that  $X'$  is consistent with  $\mu$ . But then  $X$  and  $X'$  are consistent with  $\mu \wedge \psi$  by hypothesis. Thus  $X \notin \mathcal{W}_{\subseteq}(K, \mu \wedge \psi)$ .

(ii)  $X$  is inconsistent with  $\mu$ . In this case,  $X$  is inconsistent with  $\mu \wedge \psi$ , so  $X \notin \mathcal{W}_{\subseteq}(K, \mu \wedge \psi)$ .

So  $\mathcal{W}_{Bel}(K, \mu) = \mathcal{W}_{Bel}(K, \mu \wedge \psi)$ , which means that  $K \otimes_{\text{CSRGM}} \mu = K \otimes_{\text{CSRGM}} (\mu \wedge \psi)$  and the postulate is verified.

(\*8): Not verified. Consider example 2, and  $\psi = \neg b \wedge (a \leftrightarrow c)$ . We have  $\mathcal{W}_{\subseteq}(B, \mu \wedge \psi) = \{B_1, B_4\} = \{\{\phi_1, \phi_2\}, \{\phi_5, \phi_6, \phi_7\}\}$ ,  $Bel_{\mu \wedge \psi}(B_1) = \frac{8}{43}$  and  $Bel_{\mu \wedge \psi}(B_4) = \frac{15}{43}$ , so  $\mathcal{W}_{Bel}(B, \mu \wedge \psi) = \{\{\phi_5, \phi_6, \phi_7\}\}$ . Hence  $B \otimes_{\text{CSRGM}} \mu \wedge \psi = Cn(\phi_5 \wedge \phi_6 \wedge \phi_7 \wedge \mu \wedge \psi)$ . we have :  $Mod(B \otimes_{\text{CSRGM}} \mu) = \{\{\}, \{c\}, \{b\}\}$  and  $Mod(\psi) = \{\{\}, \{a, c\}\}$ . So  $Mod(B \otimes_{\text{CSRGM}} \mu) \not\subseteq Mod(\psi)$ , so we can infer  $\neg \mu \notin B \otimes_{\text{CSRGM}} \mu$ . On the other hand,  $Mod(B \otimes_{\text{CSRGM}} \mu \wedge \psi) = \{\{a, c\}\}$  and  $Mod((B \otimes_{\text{CSRGM}} \mu) + \psi) = \{\{\}\}$ , so  $(B \otimes_{\text{CSRGM}} \mu) + \psi \not\subseteq B \otimes_{\text{CSRGM}} \mu \wedge \psi$ .

**Proposition 5.** *The  $*_{\text{CSRW}}$  operator verifies (**Success**), (**Inclusion**), (**Consistency**), (**Vacuity**), (**Relevance**), (**Uniformity**).*

For the sake of the proof of (**Uniformity**), we first establish a lemma.

**Lemma 1.** *If, for all  $B' \subseteq B$ ,  $B' \cup \{\mu\}$  is inconsistent if and only if  $B' \cup \{\phi\}$  is inconsistent, then  $\mathcal{W}_{\subseteq}(B, \mu) = \mathcal{W}_{\subseteq}(B, \phi)$ .*

*Proof (of lemma 1).* Suppose, without loss of generality, that there exist  $B' \subseteq B$  such that  $B' \in \mathcal{W}_{\subseteq}(B, \mu)$  but  $B' \notin \mathcal{W}_{\subseteq}(B, \phi)$ . Two cases arise:

1.  $B' \cup \{\phi\}$  is inconsistent, but  $B' \cup \{\mu\}$  is consistent, which contradicts the hypothesis.
2.  $B' \cup \{\phi\}$  is consistent, but there exists  $B''$ ,  $B' \subseteq B''$ , such that  $B'' \cup \{\phi\}$  is consistent. But then by lemma hypothesis  $B'' \cup \{\mu\}$  is consistent, thus  $B' \notin \mathcal{W}_{\subseteq}(B, \mu)$ , which contradicts the hypothesis.  $\square$

*Proof (of proposition 5).* (**Success**), (**Inclusion**) and (**Consistency**) are obviously verified by definition of the  $*_{\text{CSRW}}$  operator.

(**Vacuity**): if  $B \cup \{\mu\}$  is consistent, then  $\mathcal{W}_{Bel}(B, \mu) = \{B\}$ . Thus  $B *_{\text{CSRW}} \mu = B \cup \{\mu\}$ .

(**Relevance**): Let  $\phi \in B$  such that  $\phi \notin B *_{\text{CSRW}} \mu$ . Then, there exists  $X \in \mathcal{W}_{Bel}(B, \mu)$  such that  $\phi \notin X$ . Let  $B' = X \cup \{\mu\}$ . It follows immediately that  $B *_{\text{CSRW}} \mu \subseteq B' \subseteq B \cup \{\mu\}$ . On the other hand, since, by definition,  $X$  is maximally consistent with  $\mu$  with respect to set inclusion, it holds that  $B'$  is consistent, but  $B' \cup \{\phi\}$  is inconsistent.

(**Uniformity**): if  $\forall B' \subseteq B$   $B' \cup \{\mu\}$  is inconsistent if and only if  $B' \cup \{\phi\}$  is inconsistent, then, by lemma 1, we have  $\mathcal{W}_{\subseteq}(B, \mu) = \mathcal{W}_{\subseteq}(B, \phi)$ , but then, by construction, we have  $\mathcal{W}_{Bel}(B, \mu) = \mathcal{W}_{Bel}(B, \phi)$ , so  $B *_{\text{CSRW}} \mu = B *_{\text{CSRW}} \phi$ , and thus  $B \setminus (B *_{\text{CSRW}} \mu) = B \setminus (B *_{\text{CSRW}} \phi)$ .

Let us note that as the  $*_{\text{CSRW}}$  operator verifies (**Relevance**), it also verifies (**Core Retainment**).

### 3.4 Complexity of credible belief base operators

In this section we study the computational complexity of the following decision problem according to the considered belief revision operator  $*$ .

*Problem* : MODEL-CHECKING(\*)  
*Instance* :  $B$  a belief base,  $\mu$  a formula,  $I$  an interpretation  
*Question* :  $I \models B * \mu$  ?

In other words, given a set of beliefs  $B$  and new information  $\mu$ , the problem consists in deciding whether an interpretation  $I$  is a model of the set of revised beliefs. Liberatore and Schaerf [64] advocated that examining the complexity of model checking is worthwhile since this problem is the most fundamental computational task in the setting of model-based knowledge representation and reasoning and that the succinctness of a knowledge representation formalism is in reference to the complexity of model checking.

**Proposition 6.** MODEL-CHECKING(\* $_{CSRG}$ ) is in  $\Pi_2P$ .

*Proof.* The problem is to decide if an interpretation  $I$  satisfies  $B *_{CSRG} \mu$ , which amounts to searching a subset  $B' \subseteq B$  such that  $I \models B'$ , where  $B' \in \mathcal{W}_{Bel}(B, \mu)$ .

We first check that  $I \models \mu$ , otherwise  $I$  cannot be a model of the revised set. If  $I \models \mu$ , it remains to verify that  $I$  satisfies at least one of  $B' \in \mathcal{W}_{Bel}(B, \mu)$ . Considering  $I$ , there exists only one subset  $B' \subseteq B$  such that  $B' \in \mathcal{W}_{Bel}(B, \mu)$  and  $I \models B'$ : this is the set of formulas of  $B$  satisfied by  $I$ . More formally,  $B' = \{\alpha \in B : I \models \alpha\}$ .

Once we have established that  $I \models \mu$  and have built  $B'$ , it remains to be decided whether  $B'$  is indeed of maximum credibility. Answering negatively to this question amounts to showing a subset of formulas  $B_0 \subseteq B$  which does not contradict  $\mu$  and such that  $Bel(B_0) > Bel(B')$ . This amounts to exhibiting a pair  $(B_0, I_0)$  where  $B_0$  is a subset of  $B$  such that  $Bel(B_0) > Bel(B')$  and  $I_0$  is a model of  $B_0 \cup \{\mu\}$ . In other words, the following algorithm decides whether  $I \not\models B *_{CSRG} \mu$ .

1. Check if  $I \models \mu$ .
  - otherwise, we have  $I \not\models B *_{RSRG} \mu$ .
  - if so, continue.
2. Compute  $B' = \{\alpha \in B \mid I \models \alpha\}$ .
3. Check that  $B'$  is not an element of  $\mathcal{W}_{Bel}(B, \mu)$ .
  - Guess  $B_0 \subseteq B$  and check that  $Bel(B_0) > Bel(B')$ .
  - Guess  $m_0$  an interpretation and check that  $I_0$  is a model of  $B_0 \cup \{\mu\}$ , and so that  $B_0 \cup \{\mu\}$  is consistent.

The difficulty mainly lies in the third step, and particularly in the prediction of the pair  $(B_0, I_0)$  which forms a short and easily verifiable certificate of disqualification, where the verification that  $Bel(B_0) > Bel(B')$  actually requires the exhibition of all the focal sets of  $\mathcal{W}_{\subseteq}(B, \mu)$ , which is a non-deterministic step in polynomial time (see [69]). This can be performed using an NP oracle to find maximum credibility.

Finally we obtain an algorithm belonging to coNP which calls an NP oracle, which proves that the problem MODEL-CHECKING(\* $_{CSRG}$ ) lies in the coNP<sup>NP</sup> =  $\Pi_2P$  class of problems.

**Proposition 7.** MODEL-CHECKING( $*_{CSRW}$ ) is in  $\Sigma_2P$ .

*Proof.* Let  $B$  be a belief base and  $\mu$  a formula. We denote  $Bel_{max}$  the maximum credibility of subsets of  $B$  consistent with  $\mu$ . According to definition 7, In order to decide if  $I$  is a model of the revised base  $B *_{CSRW} \mu$ , we must be decide, for any formula  $\alpha \in B$  for which  $I$  is not a model, whether there exists a subset  $B'_\alpha$  of  $B \setminus \{\alpha\}$  such that  $B'_\alpha$  is consistent with  $\mu$  and its credibility is equal to  $Bel_{max}$ . The following algorithm summarize this process:

1. Check that  $I \models \mu$ .
  - otherwise we have  $I \not\models B *_{CSRW} \mu$ .
  - if so, continue.
2. Compute  $Bel_{max}$ .
3. For all  $\alpha \in B$  such that  $I \not\models \alpha$ , is there a subset  $B'_\alpha \subseteq B \setminus \{\alpha\}$  such that  $B'_\alpha \cup \{\mu\}$  is consistent (that is, there exists an interpretation  $I'_\alpha$  such that  $I'_\alpha \models B'_\alpha \cup \{\mu\}$ ) and  $Bel(B'_\alpha) = Bel_{max}$  ?
  - if yes,  $I \models B *_{CSRW} \mu$ .
  - otherwise,  $I \not\models B *_{CSRW} \mu$ .

The difficulty of the algorithm presented lies mainly in the second and the third step.

The second step, which consists in computing  $Bel_{max}$ , is performed by asking questions like “*Is there a subset of  $B$  consistent with  $\mu$  and of credibility greater than or equal to  $Bel$ ?*”. We proceed to exhibit all the focal sets until we obtain the value of  $Bel_{max}$ . We thus make calls to an oracle NP to find the maximum credibility of sub-bases of  $B$  consistent with  $\mu$ .

The third step is performed by setting, in a parallel way for all  $\alpha \in B$ , the question “*Is there a subset  $B'_\alpha$  of  $B$  that does not contain  $\alpha$  consistent with  $\mu$  and of credibility greater than or equal to  $Bel_{max}$ ?*”. This step requires in the algorithm above a single call to an oracle NP. Indeed, it suffices to guess pairs  $(B'_\alpha, I'_\alpha)$  where  $B'_\alpha \subseteq B \setminus \{\alpha\}$  with  $Bel(B'_\alpha) = Bel_{max}$  and  $I'_\alpha$  is an interpretation such as  $I'_\alpha \models \bigwedge (B'_\alpha \cup \{\mu\})$ .

Finally, we obtain a non-deterministic algorithm which uses calls to an NP oracle, which proves that the problem MODEL-CHECKING( $*_{RSRW}$ ) is in  $NP^{NP} = \Sigma_2P$ .

## 4 Related works

We remind that the belief revision process is based on the agent’s available assumptions about the world (background knowledge or experience) in order to construct new beliefs in agreement with a newly received information (observations, testimonies), while minimally modifying prior beliefs [32,50,33]. Indeed, the various revision operators can be split into two families, according to the formal framework used for representing information (or belief): Logical (syntactic, semantic) or quantitative (for example, numerical). The operators developed

in the present paper pertains to the family of logical approaches, while trying to exploit numerical apparatus stemming from the decision making field. To the best of our knowledge, this idea has not been studied yet.

Concerning the logical context, revision has been initially developed in the area of epistemology, under the name “theory change”, the starting point being the study of how scientific theories evolve [92]. First results on logical change operations go back to years 1975-1977, in the field of epistemology with Issaac Levi works [61,62,63] and in the field of history of sciences with William L. Harper works [52,51]. Shortly thereafter, the question of how to perform revision gave rise to numerous works according to the representation of beliefs [54,62,53,30]. Among the first attempts to formalize revision in artificial intelligence is the work of Alchourrón, Gärdenfors and Makinson [1,40], coming from philosophical logics. They proposed a set of rationality postulates, now widely known as the AGM postulates, characterizing desired properties of revision operations. Later on, they proposed first concrete revision operations for logical theories, named belief sets in this context, i.e. deductively closed sets of logical formulas.

Nonetheless, the AGM model has been subject to several criticisms. In particular the *impractical* use of belief sets to represent belief states and the *unrealistic* acceptance of any new information. This is why, shortly after the publication of the AGM model, alternative models started to appear in the literature (see [32] for an overview). In particular, in order to circumvent the computational complexity of handling deductively closed sets of formulas, the revision of belief bases (which are not deductively closed) has been considered [68,47,46,48]. In this context, a distinction is made between explicit beliefs, which are present in the base, and implicit beliefs, which can be inferred [42,91,70,47,3].

Furthermore, another class of operators called, *non-prioritized operators*, has been developed. This family of operators do not satisfy the success postulate. The motivation for the proposal of these operators is the fact that, as pointed out by Rott in [74], the *success* postulate is not a fully realistic requirement because an agent can have lots of (non-tautological) beliefs that it is not willing to give up for several reasons. The pioneering work on this topic is [34], where Fermé and Hansson introduced the concept of *shielded contraction*. Addressing further this issue, Marco Garapa, Eduardo Fermé and Maurício D. Luís Reis in [36] put forward various *shielded base contraction operators* which are defined on belief bases and constructed with the underlying idea that not all new information has to be accepted. Falappa et al. also covered this issue in [31], by proposing a prioritized (in which the input set is fully accepted) and a non-prioritized approach to multiple change applicable to belief bases. In a more recent work [37], Marco Garapa, Eduardo Fermé and Maurício D. Luís Reis criticized the belief revision process once again in the sense that the unconditional acceptance of the new information is not a realistic feature. The authors stressed the idea that, when facing new information, an intelligent agent should be able to reject part of it. This idea is underlying *credibility-limited revision*, which is a two step process. The first step consists in determining whether a given belief is credible or not. Then, in a second step the *credibility-limited revision* operator should:

- leave the set of beliefs unchanged if the belief by which it is revised is considered *non-credible*;
- behave as a standard revision operation when facing a credible belief.

More recently, Marco Garapa, in [35], continued addressing the problem of rationally when incorporating pieces of new information into an agent’s belief state. The adopted point of view is that, in real situations, one may want to reject the new information or only accept a part of it. The author proposes a constructive model called *Selective Revision*. In this model, an agent’s epistemic state is represented by a belief base. The model allows the acceptance of only part of the new information. Di Giusto and Governatori [43] presented approaches to revision of belief bases where sentences are split into two classes: *facts*, that can be deleted if new facts are necessary to be accommodated, and *rules*, which cannot be deleted, but can instead be changed. Obviously, Di Giusto and Governatori’s proposal (based on the modification of individual rules, instead of deletion) stipulate guidelines on the vulnerability of sentences of belief bases. Motivated by Hanssons’ remarks [49], Theofanis Aravanis introduced in [2] a new type of change operation, called *Deductive belief change (DBC)* (contraction and revision) that also imposes strong directives on the relation between the justificatory structure of a belief base and the vulnerability of its beliefs (which in turn reflects their resistance to change). This belief change operation illustrates the fact that the more explicit beliefs in  $B$  imply the information  $\mu$ , the more resistant to modify  $\mu$  is, with respect to  $B$ . Indeed, DBC was characterized constructively in terms of kernel belief change highlighting its simple and intuitive structure.

Let us emphasize now that our operators satisfy the *Success* postulate, so they don’t pertain to the class of *non-prioritized* operators. Moreover, in spite of our use of the word *credibility*, the comparison of our operators with all these latter is not feasible and thus each one cannot be reduced to the other. In our context, the word *credibility* applies to the beliefs which should be kept in the result of the revision operations, not to the information by which the revision is performed.

Now let us examine quantitative approaches to belief revision. Seminal works on this subject can be found in the literature on subjective probabilities, particularly the works of Richard Jeffrey in the 1960’s. In this framework of uncertainty theories (such as probability or possibility theories or the theory of ranking functions [96,95,71]), revision is taken from a different point of view, where agent’s beliefs are understood as a probability measure and belief revision is thus put into terms of what Jeffrey calls “probability kinematics” [54]. Belief revision in this context has been fully studied [29,27,28,84,85,72,73]. Indeed, both in probabilistic and possibilistic settings, a value in some totally ordered scale is associated to each possible world, which reflects the extent to which it can be considered as the real world. Such distributions shape epistemic states in a more refined attitude than in a pure boolean setting. Values may range in the unit interval  $[0; 1]$ , as in the probabilistic setting, but one may use the set of integers, or even just an ordinal scale, possibly finite. In such circumstances, the nature



of beliefs (sources, justification, distinction between explicit and derived beliefs) is not sufficiently taken into account; this approach is probably adequate in a quite simple representation framework where beliefs play the same role.

Without being exhaustive and under a very different framework, probability and numerical possibility theories are a special case of the theory of belief functions, and more generally of the theory of imprecise probabilities. In that framework, the generalization of Jeffrey's rule of revision from probability to belief functions has been initially studied in [78,79], then [45]. Later, Ma and colleagues [65] studied an adaptation of previous mass function based revision rules, insisting on the compliance of retaining the consistency of the set of beliefs. They proved that their revision operator generalizes Jeffrey's rule, Dempster's conditioning, and a form of AGM revision conjointly, and started investigating the issue of iterated revision. This unified view of the AGM revision has been extended to belief fusion by Dubois and colleagues in [26]. Interestingly, authors in [65] discussed the form of a revision operator (or rule) when the input is a general mass function rather than logical formulas as in our paper. At this point, we perceive that comparison of our paper with this work is not feasible and thus each one cannot be reduced to the other.

## 5 Conclusion

This paper contributes to the current line of research in belief change that has received considerable attention from the artificial intelligence, database and philosophy communities. In these contexts, revision is considered as the well-known belief change operation maintaining consistency when integrating a new piece of information, while modifying the initial beliefs as less as possible.

Let us remind that our paper deals with the use of the potential benefit offered by different tools stemming from evidence theory in the context of belief base revision, an idea that have seldom been addressed. We propose two revision operators, namely CSRG and CSRW. The notion of Basic Belief Assignment (BBA) is used to model the information related to each maximal consistent sub-base. Dempster's rule is applied to combine all these BBAs in order to yield a global BBA synthetizing the information given by the sub-bases set. Then maximal credibility is used as a selection criterion to choose the most credible sub-bases. The examples proposed throughout this paper illustrate and express to which extent the credibility criterion is interesting and allows to avoid the loss of valuable and relevant beliefs. Moreover, as an assessment of our operators, we establish results on the productivity of the inference relation from the revision result. We study their logical properties in terms of satisfaction of postulates. We are actually working on a representation theorem, which supposes the definition of new properties of the selection function. We also examine the membership to complexity classes of the model checking problem. It seems natural to think about carrying on the computational complexity study by looking for an upper bound. Also note that future work can include a thorough investigation of the computational complexity of these new revision operators in some particular

fragments (like Horn and Krom fragments) which has become a very active direction of research allowing for efficient reasoning methods [9,14,15,16,17].

Moreover, the work of Creignou et al. [13] defines PRSRG and PRSRW operators as the extension respectively of RSRG and RSRW operators to stratified belief bases. We remind that a stratified belief base  $B = (S_1, \dots, S_n)$  is provided by a partition of the belief base in strata  $S_i$  ( $1 \leq i \leq n$ ) representing priorities between formulas. So, the extension of our operators (CRSG and CSRW) to stratified belief bases is planned in a future work.

Let us mention also that the important property offered by the credibility degree  $Bel(B_i)$  of a consistent sub-base  $B_i$  in regard to the initial belief base  $B$  carries intuitively in some extent with a semantic aspect about this sub-base. Hence, it might be an interesting idea to define appropriate semantic counterpart operators for credible revision and this could perhaps be realised by examining the fact that credibility degree  $Bel(B_i)$  and plausibility degree  $Pl(B_i)$  are respectively the minimal and maximal total beliefs committed to  $B_i$ . This fact is indeed formalized by the duality relation  $Pl(B_i) = 1 - Bel(\overline{B_i})$  where  $\overline{B_i} = B \setminus B_i$ .

Furthermore, as explained previously, we have already proposed two extreme approaches for revising belief bases. The former is permissive and allows choosing all the maximal consistent sub-bases whereas the latter is drastic leading to keep only the beliefs that are not questioned. Between these two extremes, evidence theory let us shed some light on the idea of an intermediary or a *compromise* strategy. The underlying idea of this approach is to capture the maximal consistent sets stemming from all the possible intersections of maximal consistent sub-bases. This constitutes obviously an advantage with regard to the drastic strategy which has been intensively criticized in the literature since it is so prudent and can lead in many cases, and particularly with large belief bases, to lose all the initial belief's agents. In addition, it presents a benefit for the compromise strategy compared to the permissive approach which can lead to an exhaustive revision result especially in the domain of databases repair.

As regards belief revision and belief contraction, they are two sides of a same coin [61,62,53]. As a matter of fact, natural extension of this work is to study the equivalence between our belief base revision operators (CRSG and CSRW) and their corresponding in the context of belief contraction [39,38].

Finally, we consider an experimental study with two focal areas, namely evaluating the performance of such operators compared to other well known ones, and qualitatively comparing the nature of the results of our operators and those provided by others. For this purpose, we have already developed a prototype which is based on an ASP logic program for the enumeration of  $\mathcal{W}_{\subseteq}(B, \mu)$ , and a prolog program computing  $Bel_{\mu}(B_i)$  for each  $B_i \in \mathcal{W}_{\subseteq}(B, \mu)$ . We also designed a generator which produces bases  $B$  and formulas  $\mu$  based on structural properties of  $\mathcal{W}_{\subseteq}(B, \mu)$ .

## 6 Competing interests

The authors have no competing interests to declare that are relevant to the content of this article.

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