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QUANTITATIVE RECURRENCE FOR T, T^{-1} TRANSFORMATIONS

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Abstract. We are interested in the study of the asymptotic behaviour of return times in small balls for the T, T^{-1}-transformation. We exhibit different asymptotic behaviour (different scaling, different limit point process) depending on the respective dimensions of the measures of the two underlying dynamical systems. It behaves either as for the direct product of the underlying systems, or as for the $\mathbb{Z}$-extension of the driving system (also studied in this article), or as a more sophisticated process.

Contents

1. Introduction

Within the context of dynamical systems, quantitative recurrence forms a specific family of limit theorems where one either wants to precise the distribution of entrance times to certain regions of the phase space, or also to compute the time needed for an orbit to come back close to its starting point. It has been studied mainly for finite measure preserving transformations with good mixing properties. The typical situation is to obtain an exponential law in the first case and a recurrence rate equal to the dimension of the measure in the second case. We refer to the book [8] and references therein for a survey of these results and the relation with extreme value theory.

In this work we consider a map which preserves a finite measure, but its mixing is not strong enough to be treated by classical methods. Indeed its behavior has a lot to do with an underlying deterministic random walk, an infinite measure preserving dynamical system. The quantitative recurrence in the infinite measure case has been studied, among the few works, by Bressaud, Zweimüller and the authors in [2, 9, 11, 12].

More precisely, we study the particular case of the generalized T, T^{-1}-transformation, which is known, since [19, p. 682, Problem 2] and [7], to be Kolmogorov but not loosely Bernoulli. We will see that, depending on the measure dimensions of both dynamical

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systems defining the T, T^{-1}-transformation, the quantitative recurrence properties are either analogous to those of mixing subshift of finite type, or to those of an infinite measure preserving dynamical systems ($\mathbb{Z}$-extension of a subshift of finite type), or some more elaborate compound process.

Let us recall the definition of the generalized T, T^{-1}-transformation. Let (X, f, μ) and (Y, g, ν) be two ergodic probability measure preserving dynamical systems, where f (resp. g) is a transformation acting on a relatively compact metric spaces X (resp. Y), with g invertible. For simplicity, we focus in this work on the case where (X, f, μ) is a mixing subshift of finite type endowed with an equilibrium state of a Hölder potential. We endow the product space $Z := X \times Y$ with the product (sup) metric and denote all these metrics by d. Let $h\colon X \to \mathbb{Z}$ be a measurable centered function $\int_Y h \, d\mu = 0$. We define the generalized T, T^{-1} transformation by

$$F(x, y) = \left(f(x), g^{h(x)}(y) \right).$$

This map preserves the product probability measure $\rho := \mu \otimes \nu$. We denote by $h_n(x) = h(x) + \cdots + h(f^{n-1}x)$. Note that

$$F^n(x, y) = \left(f^n(x), g^{h_n(x)}(y) \right). \tag{1}$$

Our goal is to study fine (quantitative) recurrence properties of F, and more precisely to study the return times of the orbit $(F^n(x, y))_n$ in a ball $B_r^Z(x, y) = B_r^X(x) \times B_r^Y(y)$ around the initial point. To describe our results we suppose in this introduction that the system (Y, g, ν) is also a mixing subshift of finite type endowed with an equilibrium measure of a Hölder potential.

We will prove in Section 2 that the recurrence rate is given by $\min(2d_\mu, d_\rho)$, where d_m stands for the dimension of a measure m. We establish results of convergence in distribution in Section 3. In the particular case where $d_\mu < d_\nu$, we show in Section 3.1 how the first return time for F coincide with the first return time for the $\mathbb{Z}$-extension of (X, f, μ) by the cocycle h (dynamical system preserving an infinite measure), that has been studied by Yassine in [20, 21]. This $\mathbb{Z}$-extension consists in the dynamical system $(X \times \mathbb{Z}, \widetilde{F}, \mu \otimes \mathfrak{m})$ with

$$\widetilde{F}\colon X \times \mathbb{Z} \circlearrowleft, \quad \widetilde{F}(x, q) = (f(x), q + h(x)), \tag{2}$$

so that

$$\forall n \in \mathbb{N}^*, \quad \widetilde{F}^n(x, q) = (f^n(x), q + h_n(x)),$$

and where $\mathfrak{m}$ denotes the counting measure on $\mathbb{Z}$. Then, in Section 3.2, we study the return time point process of F with the time normalization $\max((\mu(B_r^X(x)))^2, \rho(B_r^Z(x, y)))$ and establish the convergence of this point process to:

- a standard Poisson process if $d_\mu > d_\nu$, as if F was the direct product $f \otimes g : (x, y) \mapsto (f(x), g(y))$;
- a standard Poisson process taken at the local time at 0 of the limit Brownian motion B of $(h_{\lfloor nt \rfloor}/\sqrt{n})_n$ if $d_\mu < d_\nu$, as for the $\mathbb{Z}$-extension $\widetilde{F}$ (the result for $\widetilde{F}$ will be proved in Section 5);
- a sum of Poisson processes of parameter a taken at the local time of B at some random points given by an independent Poisson process of parameter b, if $d_\mu = d_\nu$ (the couple of parameters (a, b) may be random, its distribution can then be explicitely computed).

In Section 4 we prove Theorem 3.3, the core approximation of the moments of our hitting point process. In Section 5 we prove the convergence, stated in Theorem 3.10, of the point process of the $\mathbb{Z}$-extension $\widetilde{F}$.

Finally the appendix contains results about the moments of the limiting processes discussed above.

2. Recurrence rate

We define, for $z = (x, y) \in Z$ and $r > 0$, the first return time $\tau_r = \tau_r^F$ in the r-th neighbourhood of the initial point, i.e.

$$\tau_r(z) = \tau_r^F(z) := \inf\{n \geq 1 \colon d(F^n(z), z) < r\}.$$

The first quantity we want to consider is the pointwise recurrence rate defined by

$$R(z) = R^F(z) = \lim_{r\to 0} \frac{\log \tau_r(z)}{-\log r}\,,$$

when it exists, otherwise we define $\overline{R}$ with a limsup and $\overline{R}$ with a liminf. We define the pointwise dimension $d_\mu(x)$ (resp. $d_\nu(y)$) of μ at x (resp. ν at y) as

$$d_\mu(x) := \lim_{r\to 0} \frac{\log(\mu(B_r^X(x)))}{-\log r} \quad \text{and} \quad d_\nu(y) := \lim_{r\to 0} \frac{\log(\nu(B_r^Y(y)))}{-\log r}\,,$$

setting $B_r^X(x)$ and $B_r^Y(y)$ for the balls of radius r respectively around x in X and around y in Y. In this paper we assume that the pointwise dimensions exist a.e. and are constant, in particular they are equal then to the Hausdorff dimension of the measures d_μ and d_ν.

It was proven in [17] that the upper recurrence rate is bounded from above by the pointwise dimension. Namely, for ρ-a.e. $z = (x, y)$ one has

$$\overline{R}(z) = \overline{R}^F(z) = \overline{\lim_{r\to 0}} \frac{\log \tau_r(z)}{-\log r} \leq d_\rho = d_\mu + d_\nu\,, \tag{3}$$

We write τ_r^f and τ_r^g for the respective first return times of f and g in the ball of radius r around the original point. With these notations, the ball $B_r^Z(x, y)$ of radius r around (x, y) in Z is $B_r^X(x) \times B_r^Y(y)$ and

$$\tau_r(x, y) = \inf\left\{n \geq 1 \,:\, f^n(x) \in B_r^X(x),\ g^{h_n(x)}(y) \in B_y^Y(y)\right\}\,.$$

Whenever the random walk h_n returns to the origin it produces an exact return for the second coordinate since $g^0 = id$. Therefore the study of the recurrence of the whole system may be estimated via the one of the $\mathbb{Z}$-extension $(X \times \mathbb{Z}, \widetilde{F}, \mu \otimes \mathfrak{m})$. Indeed,[1] one has for any $(x, y) \in Z$ and any $r < 1$,

$$\tau_r(x, y) \leq \tau_r^{\widetilde{F}}(x) = \inf\left\{n \geq 1 \,:\, f^n(x) \in B_r^X(x),\ h_n(x) = 0\right\}\,,$$

which immediately gives the

Proposition 2.1. *The upper recurrence rate for F is bounded from above by the one of the $\mathbb{Z}$-extension $\widetilde{F}$*

$$\overline{R}^F(x, y) := \overline{\lim_{r\to 0}} \frac{\log \tau_r(z)}{-\log r} \leq \overline{R}^{\widetilde{F}}(x, 0) := \overline{\lim_{r\to 0}} \frac{\log \tau_r^{\widetilde{F}}(x, 0)}{-\log r} \quad \textit{for } \mu\textit{-a.e. } x \textit{ and any } y.$$

[1] noticing that $\tau_r^{\widetilde{F}}(x, q)$ does not depend on $q \in \mathbb{Z}$

The latter was studied by Yassine [20]. She proved that when X is a mixing subshift of finite type endowed with an equilibrium measure associated to an Hölder potential and when h is continuous[2], then for μ-a.e. $x \in X$

$$\lim_{r\to 0} \frac{\log \tau_r^{\widetilde{F}}(x,0)}{-\log r} = 2d_\mu. \tag{4}$$

Combining Proposition 2.1 and (4) we obtain in this setting that for ρ-a.e. z

$$\overline{R}^F(z) \leq 2d_\mu. \tag{5}$$

When $d_\mu < d_\nu$, and under some hypotheses that we will describe later, this bound is optimal. However, the returns to the origin of the random walk h_n are quite sparse, and the first return of the whole system may happen much before. This is what happens when $d_\mu > d_\nu$ where (3) becomes optimal, again under the same hypotheses that we describe now.

Throughout the rest of this section we will make the following assumption.

Hypothesis 2.2. (A) *The system (X, f, μ) is a one-sided mixing subshift of finite type with finite alphabet $\mathcal{A}$, endowed with and equilibrium measure μ with respect to some Hölder potential.*

(B) *The system (Y, g, ν) is a two-sided mixing subshift of finite type with finite alphabet $\mathcal{A}'$, endowed with an equilibrium measure μ with respect to some Hölder potential, or more generally it has super-polynomial decay of correlations on Lipschitz functions[3] and Y has finite covering dimension[4] as in* [17].

(C) *The step function h is Lipschitz and μ-centered.*

Let $\lambda_X > 0$. We use the notations $C_m(x)$ for the cylinder[5] of generation m (also called m-cylinder) containing x and $\mathcal{C}_m$ for the set of the m-cylinders of X. We endow X with the ultrametric $d(x, x') = e^{-\lambda_X n}$ where n is the largest integer such that $x_i = x'_i$ for all $i < n$. We call it the metric with Lyapunov exponent λ_X. The cylinder $C_m(x)$ and the open ball $B^X(x, r)$ are equal when $m = \lfloor \frac{-1}{\lambda_X} \log r \rfloor$. Recall that the Hausdorff dimension of μ is the ratio of the entropy h_μ and the Lyapunov exponent λ_X: $d_\mu = h_\mu/\lambda_X > 0$, and $d_\rho = d_\mu + d_\nu$.

Theorem 2.3. *Assume Hypothesis 2.2. Then the lower recurrence rate is equal to the dimensions :*

$$R^F(z) = \min(2d_\mu, d_\rho) \quad \textit{for } \rho\textit{-a.e. } z.$$

Before proving this result, we will state some notations and useful intermediate results. It follows from (1) that, for a point $z = (x, y)$ we have the obvious equivalence

$$d(F^n(z), z) < r \text{ iff } \begin{cases} d(f^n(x), x) < r \\ h_n(x) \in G_r(y) := \left\{ k \in \mathbb{Z} \,:\, d(g^k y, y) < r \right\} \end{cases}. \tag{6}$$

[2]Since h has integer values, this implies that h is locally constant and takes a finite number of values.

[3]Actually, we just use the fact that $\underline{R}^g(y) = \underline{R}^{(g^{-1})}(y) = d_\nu$ for ν-almost every $y \in Y$ and that there exists $K > 0$ and $r_1 > 0$ such that, for any $r \in]0; r_1[$, any $y \in Y$ and any $k \geq r^{-d_\nu+\varepsilon}$, $\nu\left(B^Y_{2r}(y) \cap g^{-k}(B^Y_{2r}(y))\right) \leq K\nu(B^Y_r(y))^2$.

[4]meaning that there exists M such that for each $r > 0$ there exists a cover of Y by r-balls with multiplicity at most M, e.g. Y is a subset of euclidean space.

[5]$C_m(x)$ is here the set of $x = (x'_k)_{k\in\mathbb{Z}}$ such that $x'_k = x_k$ for all $k = 0, ..., m$.

We will apply a version of the local limit theorem [9] (see [21] for the precised error term); quantifying the so-called mixing local limit theorem (See e.g. [5]). Let us write $\mathcal{F}_m$ for the σ-algebra of sets made of union of cylinders of generation n.

Proposition 2.4. *Assume Hypothesis 2.2. There exists $C > 0$ such that, for any positive integers n, m satisfying $m \leq 3n$, and for any $A \in \mathcal{F}_m$ and any $B \in \sigma\left(\bigcup_{k\geq 0} f^{-k}(\mathcal{F}_{k+m})\right)$,*

$$\left|\mu\left(\{x \in A, h_n(x) = k, f^n(x) \in B\}\right) - \frac{\mu(A)\mu(B)}{\sigma\sqrt{2n\pi}} \exp\left(-\frac{k^2}{2\sigma^2 n}\right)\right| \leq \mu(A)\mu(B)C\frac{m}{n}.$$

A measurable set B is in $\sigma\left(\bigcup_{k\geq 0} f^{-k}(\mathcal{F}_{k+m})\right)$ if there exists a measurable function $f_0 : \mathcal{A}^{\mathbb{N}} \to \{0, 1\}$ (where $\mathcal{A}$ is the alphabet of the subshift X) such that $1_B(\mathbf{x}) = f_0(x_m, ...)$.

Lemma 2.5. *Assume Hypothesis 2.2 and $d_\nu > 0$. For any $\varepsilon > 0$, any decreasing family $(N_r)_{r>0}$ of positive integers, for ν-a.e. $y \in Y$, for any $r > 0$ sufficiently small*

$$\#(G_r(y) \cap [-N_r, N_r]) \leq 1 + 2N_{r/2}r^{d_\nu - 2\varepsilon},$$

where $G_r(y)$ is the set defined in (6).

Proof. Note that $k = 0 \in G_r(y)$. For non zero k, it suffices to estimate the number of positive k in the set, and then to apply the same estimate to g^{-1}, which still satisfies our assumption, to get the result for negative k. By assumption (B) on g and [17], the lower recurrence rate satisfies $\underline{R}^g(y) = d_\nu$ for ν-a.e. $y \in Y$. Let $\varepsilon > 0$ and set

$$Y_\varepsilon^{r_0} = \left\{y \in Y : \forall r < r_0, d(g^k(y), y) \geq r \text{ for } 1 \leq k < r^{-d_\nu+\varepsilon} \text{ and } \nu(B_{2r}^Y(y)) \leq r^{d_\nu-\varepsilon}\right\}$$

Since $\nu(Y_\varepsilon^{r_0}) \to 1$ as $r_0 \to 0$, it suffices to prove the results for $y \in Y_\varepsilon^{r_0}$. Let $y_0 \in Y$ and $r < r_0$. Set $B = B_{2r}^Y(y_0)$. When $y \in B_r^Y(y_0)$ and $d(g^k y, y) < r$ we have $g^k y \in B$. Moreover, if $y \in Y_\varepsilon^{r_0}$ this does not happen if $k < r^{-d_\nu+\varepsilon}$. Therefore, by Markov inequality

$$\nu\left(y \in B_r^Y(y_0) \cap Y_\varepsilon^{r_0} : \#\{k = 1..N_r : d(g^k(y), y) < r\} > L\right) \leq \frac{1}{L} \sum_{r^{-d_\nu+\varepsilon} \leq k \leq N_r} \nu(B \cap g^{-k}B). \tag{7}$$

By assumption[6], for such k we have

$$\nu(B \cap g^{-k}B) \leq 2\nu(B)^2.$$

Taking a finite cover of $Y_\varepsilon^{r_0}$ by balls of radius r of multiplicity at most M shows that

$$\nu(y \in Y_\varepsilon^{r_0} : \#G_r(y) \cap [1, N_r] > L) \leq 2M\frac{N_r}{L}r^{d_\nu-\varepsilon} = O(r^\varepsilon),$$

choosing $L = N_r r^{d_\nu-2\varepsilon}$. The result follows from the Borel Cantelli lemma, summing up over $r_m = 2^{-m}$ and then using the monotonicity of N_r. □

We follow the proof in [9], using the extra information about the growth rate of h_n given by the law of iterated logarithm.

Proof of Theorem 2.3. By (3) and (5) we only need to prove a lower bound.

We assume that $d_\nu > 0$. Let $\varepsilon > 0$ and $K^\varepsilon = K_\varepsilon^{m_0,n_0}$ be the set of points $x \in X$ such that $\forall m \geq n_0$, $\mu(C_m(x)) \leq e^{-m(h_\mu-\varepsilon)}$ and $\forall n \geq n_0$, $|h_n(x)| \leq (1+\varepsilon)\sigma\sqrt{n \log\log n}$. The Shannon-McMillan-Breiman theorem and the law of iterated logarithm ensures the

[6]This follows from ψ-mixing when (Y, g, ν) is a SFT with an equilibrium state; otherwise it follows by approximation of indicator function of balls by Lipschitz functions as in [17].

existence of m_0 and n_0 such that $\mu(K_\varepsilon^{m_0,n_0}) \geq 1-\varepsilon$. Let $N = N_n = (1+\varepsilon)\sigma\sqrt{n\log\log n}$. We fix now m such that $m := \lfloor -\frac{1}{\lambda_X}\log r \rfloor$.

First note that for any $y \in Y$

$$\mu\left(\{x \in K_\varepsilon, d(F^n(x,y),(x,y)) < r\}\right) \leq \sum_{C_m\,:\,C_m\cap K_\varepsilon \neq \emptyset} \mu\left(\{x \in C_m \cap f^{-n}C_m, h_n(x) \in G_r(y)\}\right). \tag{8}$$

By Proposition 2.4, for any $k \in \mathbb{Z}$, the quantity

$$\mu\left(\{x \in C_m \cap f^{-n}C_m, h_n(x) = k\}\right) \tag{9}$$

is the sum of a main term

$$\mu(C_m)^2 \frac{1}{\sigma\sqrt{2n\pi}} \exp\left(-\frac{-k^2}{\sigma^2 n}\right) \leq c\mu(C_m)^2 n^{-1/2} \tag{10}$$

and of an error term bounded in absolute value by

$$\mu(C_m)^2 \frac{m}{n}. \tag{11}$$

Summing the main contribution (10) among all m-cylinders intersecting K^ε and all integer $k \in [-N_n, N_n]$ such that $d(g^k y, y) < r$ gives, using Lemma 2.5 (r and n will be linked later), that for ν-a.e. y, provided r is small enough

$$\begin{aligned} E_n^\varepsilon(y,r) &:= \sum_{k\in[-N_n,N_n]} \sum_{C_m\,:\,C_m\cap K_\varepsilon\neq\emptyset} c\mu(C_m)^2 n^{-1/2} \\ &\leq ce^{h_\mu-\varepsilon} r^{d_\mu - \varepsilon/\lambda_X} n^{-1/2} \left(1 + \frac{m}{\sqrt{n}}\right) \#\{k \in [-N_n, N_n] \colon d(g^k y, y) < r\} \\ &\leq ce^{h_\mu-\varepsilon} r^{d_\mu - \varepsilon/\lambda_X} \left(1 + \frac{|\log r|}{\lambda_X\sqrt{n}}\right) \frac{1 + r^{d_\nu - 2\varepsilon}\sqrt{n\log\log n}}{\sqrt{n}}. \end{aligned}$$

If $d_\mu \leq d_\nu$, we take $r_n = n^{-\frac{1}{2d_\mu - \kappa\varepsilon}}$ with $\kappa = \max(6, 3/\lambda_X)$. The term in the numerator goes to one as $n \to \infty$ therefore the whole term is bounded with

$$E_n^\varepsilon(y, r_n) = O(r_n^{d_\mu - \varepsilon/\lambda_X} n^{-1/2}) = O(n^{-\frac{d_\mu - \varepsilon/\lambda_X}{2d_\mu - \kappa\varepsilon} - \frac{1}{2}})$$

which is summable in n.

If $d_\mu \geq d_\nu$ we take $r_n = n^{-\frac{1}{d_\mu + d_\nu - \kappa\varepsilon}}$ with $\kappa = 3 + 1/\lambda_X$. We get a bound

$$E_n^\varepsilon(y, r_n) = O\left(r_n^{d_\mu + d_\nu - (1/\lambda_X + 2)\varepsilon}\sqrt{\log\log n}\right),$$

which is again summable in n.

In both cases the error term is negligible with respect to the main term, therefore by Borel Cantelli lemma we conclude that for ρ-a.e. $z \in Z$, $d(F^n z, z) \geq r_n$ eventually.

Letting $\varepsilon \to 0$ ends the proof of Theorem 2.3 when $d_\nu > 0$.

In the case where $d_\nu = 0$, by [17] we have for ρ-a.e. (x,y)

$$d_\mu = R^f(x) \leq \underline{R}^F(x,y) \leq \overline{R^F}(x,y) \leq d_\mu$$

by (3), which proves the equality. □

3. Convergence in distribution

We still consider the case where (X, f, μ) is a mixing subshift of finite type. We will first state in Section 3.1 a result of convergence in distribution (Proposition 3.1) for the first return time in the particular case where[7] $\nu(B_r^Y) \ll \mu(B_r^X)$. This first convergence result will appear as a consequence of Yassine's convergence result for $\tau_r^{\widetilde{F}}$ established in [20]. In a second time, in Section 3.2, we will study the asymptotic behavior of the point process of visits to small balls. When[8] $\nu(B_r^Y) \ll \mu(B_r^X)$ we will retrieve a result analogous to Proposition 3.1. But we will also highlight other behaviors when $\mu(B_r^X) \ll \nu(B_r^Y)$ or $\mu(B_r^X) \approx \nu(B_r^Y)$. The normalization will be given by

$$n_r(x,y) := 1/\max\left(\mu(B_r^X(x))^2, \rho(B_r^Z(x,y))\right) .$$

3.1. **Study of the first return time when $\nu(B_r^Y) \ll \mu(B_r^X)$.** Then we set $n_r(x,y) = (\mu(B_r^X(x)))^{-2}$.

Proposition 3.1. *Assume (X, f, μ) is a mixing two-sided subshift of finite type and that h is bounded Hölder continuous, that $\nu(B_r^Y) \ll \mu(B_r^X)$ in ρ-probability and that*[9]

$$\lim_{\varepsilon\to 0} \overline{\lim_{r\to 0}}\, \nu(\nu(B_r^Y)\tau_r^g < \varepsilon) = \lim_{\varepsilon\to 0} \overline{\lim_{r\to 0}}\, \nu(\nu(B_r^Y)\tau_r^{g^{-1}} < \varepsilon) = 0\,.$$

Then $(\mu(B_r^X(\cdot))^2\tau_r^F)_r$ converges in distribution, as $r \to 0$ to $\sigma^2\mathcal{E}^2/\mathcal{N}^2$, where $\mathcal{E}$ and $\mathcal{N}$ are standard exponential and Gaussian random variable mutually independent, and where σ^2 is the asymptotic variance of $(h_n/\sqrt{n})_n$.

Proof. Let $\varepsilon > 0$ and $\eta > 0$. Set $D_{r,\eta} := \{\nu(B_r^Y(\cdot)) \le \eta\mu(B_r^X(\cdot))\}$. Furthermore,

$$\begin{aligned}\rho\left(\sup_{k\le n_r} |h_k| > \varepsilon/\nu(B_r^Y(\cdot)),\ D_{r,\eta}\right) &\le \rho\left(\sup_{k\le \eta^2(\nu(B_r^Y(\cdot)))^{-2}} |h_k| > \varepsilon/\nu(B_r^Y(\cdot))\right)\\ &\le \int_Y \mu\left(\sup_{k\le\eta^2(\nu(B_r^Y(y)))^{-2}} |h_k| > \varepsilon/\nu(B_r^Y(y))\right) d\nu(y)\,, \qquad (12)\end{aligned}$$

which converges to $\mathbb{P}(\eta W > \varepsilon)$ as $r \to 0$ since $\sup_{k\le\eta^2 n} \frac{|h_k|}{\sqrt{n}}$ converges to ηW (where $W = \sigma \sup_{[0;1]} |B|$, B being a standard Brownian motion). Let

$$\Omega_{r,\varepsilon} := \left\{ \min(\tau_r^g, \tau_r^{g^{-1}}) > \varepsilon/\nu(B_r^Y(\cdot)),\ \sup_{k\le n_r} |h_k| \le \varepsilon/\nu(B_r^Y(\cdot)) \right\} .$$

On $\Omega_{r,\varepsilon}$, for all $n = 1, ..., n_r$,

$$[d(f^n(x), x) < r,\ d(g^{h_n(x)}(y), y) < r] \quad \Leftrightarrow \quad [h_n(x) = 0,\ d(f^n(x), x) < r]\,.$$

Thus, on $\Omega_{r,\varepsilon}$,

$$\tau_r^F(x,y) = \tau_r^{\widetilde{F}}(x) := \inf\{n \ge 1 :\ h_n(x) = 0,\ d(f^n(x), x) < r\} .$$

[7] Throughout this article, the notation $a_r \ll b_r$ means that $a_r = o(b_r)$, i.e. that a_r is negligible with respect to b_r as $r \to 0$.

[8] The notation $a_r \ll b_r$ means that $a_r = o(b_r)$ as $r \to 0$.

[9] This happens, e.g. if $(\nu(B_r^Y)\tau_r^g)_r$ and $(\nu(B_r^Y)\tau_r^{g^{-1}})_r$ both converge, as $r \to 0$, to some random variable with no atom at 0.

But Yassine proved in [20] that, when (X, f, μ) is a subshift of finite type, then $(\mu(B_r^X(\cdot))^2\widetilde{\tau}_r)_r$ converges in distribution, with respect to μ (and so to ρ) as $r \to 0$, to $\mathcal{E}^2/\mathcal{N}^2$. We conclude as follows. For all $t > 0$,

$$\begin{aligned}
\overline{\lim_{r\to 0}} \left|\rho((\mu(B_r^X(\cdot))^2\tau_r^F > t) - \mathbb{P}(\mathcal{E}^2/\mathcal{N}^2 > t)\right| & \\
&\leq \overline{\lim_{r\to 0}} \left[\rho(\Omega_{r,\varepsilon}^c) + \left|\rho(\Omega_{r,\varepsilon},\ (\mu(B_r^X(\cdot))^2\tau_r^F > t) - \mathbb{P}(\mathcal{E}^2/\mathcal{N}^2 > t)\right|\right] | \\
&\leq \overline{\lim_{r\to 0}} \left[2\rho(\Omega_{r,\varepsilon}^c) + \left|\rho((\mu(B_r^X(\cdot))^2\tau_r^{\widetilde{F}} > t) - \mathbb{P}(\mathcal{E}^2/\mathcal{N}^2 > t)\right|\right] \\
&\leq 2\,\overline{\lim_{r\to 0}}\, \rho(\Omega_{r,\varepsilon}^c)\,.
\end{aligned}$$

Moreover, it follows from the convergence of (12) that, for all ε, η,

$$\begin{aligned}
\overline{\lim_{r\to 0}}\, \rho(\Omega_{r,\varepsilon}^c) &\leq \overline{\lim_{r\to 0}}[\nu(\tau_r^g \leq \varepsilon/\nu(B_r^Y)) + \nu(\tau_r^{g^{-1}} \leq \varepsilon/\nu(B_r^Y)) + \rho(D_{r,\eta}^c)] + \mathbb{P}(W > \varepsilon/\eta) \\
&\leq \overline{\lim_{r\to 0}}[\nu(\tau_r^g \leq \varepsilon/\nu(B_r^Y)) + \nu(\tau_r^{g^{-1}} \leq \varepsilon/\nu(B_r^Y))] + \mathbb{P}(W > \varepsilon/\eta)
\end{aligned}$$

We end the proof of Proposition 3.1 by taking $\overline{\lim}_{\varepsilon\to 0}\,\overline{\lim}_{\eta\to 0}$. □

3.2. **Study of the point process in the general case.** We are interested in the study of the asymptotic behavior of the point process generated by visits to the ball $B_r^Z(x, y) = B_r^X(x) \times B_r^Y(y)$, i.e.

$$\mathcal{N}_r(z) = \sum_{n\in\mathbb{N}\,:\,F^n(z)\in B_r^Z(z)} \delta_{n/n_r}\,. \tag{13}$$

To this end, we will consider moments of the multivariate variable $(\mathcal{N}_r([t_{v-1}; t_v]))_v$. To simplify the exposure of our proofs, we have chosen to restrict our study to the following case.

Hypothesis 3.2. *We assume that*

(I) *The system (X, f, μ) is a one-sided mixing subshift of finite type and μ is an equilibrium state of a (normalized) Hölder potential*[10]

(II) *The system (Y, g, ν) is a two-sided mixing subshift of finite type and the measure ν is an equilibrium state of a Hölder potential, or, more generally, it satisfies the following condition: For all integers J, K such that $2 \leq J \leq K$, there exists $\alpha \in (0; 1)$ and $c_0 \geq 1$ such that, for all integers $\ell_1 < ... < \ell_K$ and all $y \in Y$, the following holds true*[11]

$$\nu\left(\bigcap_{j=1}^{K} g^{-\ell_j}(B_r^Y(y))\right) = \left(1 + \mathcal{O}(\alpha^{\ell_J - \ell_{J-1} - c_0 \log r})\right) \nu\left(\bigcap_{j=1}^{J-1} g^{-\ell_j}(B_r^Y(y))\right) \nu\left(\bigcap_{j=J}^{K} g^{-\ell_j}(B_r^Y(y))\right)\,,$$

uniformly in $(r, y, \ell_1, ..., \ell_K)$.

(III) *The μ-centered function h is constant on 0-cylinders with asymptotic variance $\sigma^2 := \lim_{n\to+\infty} \frac{\mathbb{E}_\mu[h_n^2]}{n}$,*

[10] In particular the ball $B_r^X(x)$ corresponds to the $|\log r|$-cylinder containing x, i.e. to the set of points $(y_k)_{k\in\mathbb{Z}}$ such that $y_k = x_k$ for all non-negative integer $k \leq |\frac{1}{\lambda_X} \log r|$.

[11] Note that for mixing subshifts of finite type this assumption holds also true if we replace the 2-sided cylinders $B_r^Y(y) = \{z : z_k = y_k,\ \forall |k| \leq |\frac{1}{\lambda_Y} \log r|\}$ by the one-sided cylinders $\{z : z_k = y_k,\ \forall k = 0, ..., \lfloor|\frac{1}{\lambda_Y} \log r|\rfloor\}$.

(IV) *The function h is non-arithmetic, i.e. h is not cohomologous in $L^2(\mu)$ to a sub-lattice valued function.*

Under these assumptions, we know that $(h_{\lfloor nt \rfloor}/\sqrt{n})_{n\geq 1}$ converges in distribution, as $n \to +\infty$, to a centered Brownian process B of variance σ^2. Let $(L_t(x))_{t\geq 0, x\in\mathbb{R}}$ be a continuous compactly supported version of the local time of B, i.e. $(L_t(x))_{t,x}$ satisfies

$$\int_{\mathbb{R}} f(x) L_t(x)\, dx = \int_0^t f(B_s)\, ds\,,$$

i.e. L_t is the image measure of the Lebesgue measure on $[0;t]$ by the Brownian motion B. Recall that

$$n_r = n_r(x,y) = \min\left((\mu(B_r^X(x)))^{-2}, (\mu(B_r^X(x))\nu(B_r^Y(y)))^{-1}\right)\,.$$

We define $\alpha_r(x,y) := n_r(x,y)\mu(B_r^X(x))^2$ and $\beta_r(x,y) := n_r(x,y)\rho(B_r^Z(x,y))$. Note that

$$(\alpha_r(x,y), \beta_r(x,y)) = \begin{cases} \left(1, \frac{\nu(B_r^Y(y))}{\mu(B_r^X(x))}\right) & \text{if } \mu(B_r^X(x)) > \nu(B_r^Y(y)) \\ \left(\frac{\mu(B_r^X(x))}{\nu(B_r^Y(y))}, 1\right) & \text{otherwise} \end{cases}\,. \tag{14}$$

Let us now state our key result that will be proved in Section 4.

Theorem 3.3. *Assume Hypothesis 3.2. Let K be a positive integer and $\overline{\mathbf{m}} = (\overline{m}_1, ..., \overline{m}_K)$ be a K-uple of positive integers and let $(t_0 = 0, t_1, ..., t_K)$ be an increasing collection of nonnegative real numbers. There exist $C > 0$ and $u > 0$ such that, for every $(x,y) \in X \times Y$,*

$$\left| \mathbb{E}_\rho \left[\prod_{v=1}^K \mathcal{N}_r(]t_{v-1}, t_v])^{\overline{m}_v} \middle| B_r^Z(x,y) \right] - \mathbb{E}\left[\prod_{v=1}^K \left(\mathcal{Z}_r^{(x,y)}(t_v) - \mathcal{Z}_r^{(x,y)}(t_{v-1}) \right)^{\overline{m}_v} \right] \right|$$
$$\leq C \left(|\log r|^{3^m+m} \left(\mu\left(\tau^f_{B_r^X(x)} \leq |\log r|^{3^m} \middle| B_r^X(x) \right) + \nu\left(\tau^g_{B_r^Y(y)} \leq |\log r|^{3^m} \middle| B_r^Y(y) \right) \right.\right. \tag{15}$$

$$\left.\left. + e^{-u\sqrt{-\log r}} + r^{\frac{m^2}{2}} + \frac{\log n_r}{\sqrt{n_r}} \right) + |\log r|^{-\frac{1}{2}} + \varepsilon_0(|\log r|^2/n_r) \right)\,, \tag{16}$$

with $m = |\overline{\mathbf{m}}| = \overline{m}_1 + \cdots + \overline{m}_K$, ε_0 bounded, continuous, vanishing at 0, and with

$$\mathcal{Z}_r^{(x,y)} = \mathcal{Z}_{\alpha_r(x,y), \beta_r(x,y)}\,,$$

where $\mathcal{Z}_{0,1}$ is a standard Poisson process and where, for all $\alpha \in (0,1]$ and all $\beta \in [0;1]$,

$$\mathcal{Z}_{\alpha,\beta}(t) = \int_{\mathbb{R}} \mathcal{P}'_s(L_t(s))\, d(\delta_0 + \mathcal{P})(s)\,, \tag{17}$$

where $\mathcal{P}$, $\mathcal{B}$ and $(\mathcal{P}'_s)$ are mutually independent, $\mathcal{B}$ being a Brownian motion of variance σ^2 and of local time L, $(\mathcal{P}'_s)_{s\in\mathbb{R}}$ being a family of independent homogeneous Poisson processes with intensity $\sqrt{\alpha}$ and $\mathcal{P}$ being a two-sided Poisson process with intensity $\beta/\sqrt{\alpha}$.

Remark 3.4. *Observe that $\mathcal{Z}_{1,0}(t) = \mathcal{P}'_0(L_t(0))$. Furthermore, we will see in Appendix A that the moments $\mathbb{E}\left[\prod_{v=1}^K \left(\mathcal{Z}_{\alpha,\beta}(t_v) - \mathcal{Z}_{\alpha,\beta}(t_{v-1})\right)^{m'_v}\right]$ are continuous in (α,β).*

Corollary 3.5 (Conditional convergence in distribution)**.** *Assume the assumptions and keep the notations of Theorem 3.3. Suppose that $(x,y) \in X \times Y$ is such that the limit*

$$\lim_{r\to 0} (\alpha_r(x,y), \beta_r(x,y)) =: (\alpha, \beta)$$

exist and all the error terms of Theorem 3.3 satisfy $(15)+(16)=o(1)$ *as* $r\to 0$. *Then* $\mathcal{N}_r$ *converges in distribution for the vague convergence*[12] *as* $r\to 0$, *with respect to* $\rho(\cdot|B_r^X(x)\times B_r^Y(y))$, *to* $\mathcal{Z}_{\alpha,\beta}$ *defined in* (17).

Proof. It follows from Theorem 3.3 that the moments of any linear combination of the coordinates of multivariate variables

$$X_r := (\mathcal{N}_r(]t_{v-1},t_v]))_{v=1..K} \tag{18}$$

converges to those of $X_0 := (\mathcal{Z}_{\alpha,\beta}]t_{v-1},t_v])_{v=1..K}$.

Note that $\|X_0\|_\infty \le \mathcal{Z}_{\alpha,\beta}([t_0,t_K]) =: Y$, which is a random variable with Poisson distribution with random parameter bounded by $c(1+|N|)$ where N is a standard normal random variable. The convergence in distribution of X_r (an thus of the process $\mathcal{N}_r$ itself by convergence of its finite dimensional distributions) will follow from the multivariate Carleman's criterion (See Lemma A.4) provided

$$\sum_{m\ge 1}\mathbb{E}[Y^m]^{-\frac{1}{m}}=\infty\,.$$

It follows from Lemma A.1 (with the notations therein) that, for $m\ge 1$,

$$\begin{aligned}\mathbb{E}[Y^m] &= \sum_{q=1}^m S(m,q)c^q\sum_{k=0}^q\binom{n}{k}\frac{\Gamma(1+\frac12)^k}{\Gamma(1+\frac{k}{2})}\\ &\le (\sqrt{\pi}/2)^m\sum_{q=1}^m S(m,q)(2c)^q\\ &\le (2(1+c)\sqrt{\pi}/2)^m m^m\,,\end{aligned}$$

since $\Gamma(1+\frac12)=\sqrt{\pi}/2$, $\Gamma(1+\frac{k}{2})\ge 1$ and $\sum_{k=0}^q\binom{q}{k}=2^q$, and finally since the number of partitions of $\{1,...,m\}$ in non-empty sets is dominated by the number of its self maps m^m. Thus

$$\sum_{m\ge1}\mathbb{E}[Y^m]^{-\frac{1}{m}}\ge\sum_{m\ge1}(2m(1+c)\sqrt{\pi}/2))^{-1}=+\infty\,.$$

This implies the convergence of the finite distributions, which, combined with the convergence of their moments, implies the convergence in distribution in the space of positive measure endowed with the vague convergence (due to [16, Proposition 3.22]). □

Theorem 3.6. *Assume Hypothesis 3.2, that both systems are SFT (either with 2-sided or 1-sided cylinders), and that the error terms* $(15)+(16)$ *of Theorem 3.3 converge to* 0 *in* ρ*-probability.*

Assume furthermore that (α_r,β_r) *converges in distribution, under* ρ, *to some random variable with law* η.

Then $\mathcal{N}_r$ *converges in distribution, with respect to* ρ *for the vague convergence, as* $r\to 0$, *to the point process* $\mathcal{Z}_\pi$ *where* π *is a random variable independent of the* $\mathcal{Z}_{\alpha,\beta}$*'s and with distribution* η. *Namely*

$$\mathbb{E}[\phi(\mathcal{Z}_\pi)]=\int_{[0,1]^2}\mathbb{E}\left[\phi(\mathcal{Z}_{\alpha,\beta})\right]\,d\eta(\alpha,\beta)\,,$$

for all bounded continuous function ϕ *defined on the space of measures on* $(0,+\infty)$.

[12]See e.g. [16] for this convergence. Recall that this convergence implies also the convergence in distribution of $(\mathcal{N}_r([0;t]))_{t\in[0;T]}$ (seen as a càdlàg process) for the $J1$-metric (see [6]).

In particular, if π is a.s. constant equal to (α, β) then $\mathcal{N}_r$ converges in distribution, as $r \to 0$ and with respect to ρ, to $\mathcal{Z}_{\alpha,\beta}$ defined in (17).

Proof. Fix some K and (t_v)'s as in Theorem 3.3. We will apply the multivariate moments Lemma A.4 to prove the convergence in distribution

$$X_r = (\mathcal{N}_r([t_{v-1}, t_v]))_{v=1..K} \underset{r\to 0}{\longrightarrow} X_0 := (\mathcal{Z}_\pi([t_{v-1}, t_v]))_{v=1..K}.$$

The Carleman's criterion holds as in the proof of Corollary 3.5.

For any multi-index $m' = (m'_1, \dots m'_K)$ and $r > 0$ denote the corresponding error term in Theorem 3.3 by $\epsilon_r^{(m')}(z) := (15) + (16)$. By assumption $\epsilon_r^{(m')} \to 0$ in probability as $r \to 0$. Set $\Delta_r^{(d)} := \max_{|m'|\le d} \epsilon_r^{(m')}$. We still have $\Delta_r^{(d)} \to 0$ in probability as $r \to 0$. Therefore, for all $d \ge 1$, there exists $r_d \in (0; 1]$ (take the largest one) such that

$$\forall r \in]0; r_d[, \quad \rho(\Delta_r^{(d)} > d^{-1}) < d^{-1}.$$

The sequence $(r_d)_d$ is decreasing.
If it converges to some value $r_\infty > 0$, this means that, for all $r \in]0; r_\infty[$ and all d, $\rho(\Delta_r^{(d)} > d^{-1}) < d^{-1}$, and so that $\rho(\Delta_r^{(\infty)} > d^{-1}) < d^{-1}$, so that $\Delta_r^{(\infty)} = 0$ a.s. and we can take $\Omega_r = \{\Delta_r^{(\infty)} = 0\}$. If $(r_d)_d$ converges to 0, then, for any $d \ge 1$ and any $r \in]r_{d+1}; r_d]$, we set

$$\Omega_r := \left\{ \Delta_r^{(d)} \le \frac{1}{d} \right\}.$$

We notice that $\rho(\Omega_r) \to 1$ as $r \to 0$.

Let $\mathbf{m}'$ and set $m = |\mathbf{m}'|$ as in Theorem 3.3. For $(\alpha, \beta) \in [0, 1]^2$, set $G(\alpha, \beta) := \mathbb{E}[\prod_{v=1}^K \mathcal{Z}_{\alpha,\beta}(]t_{v-1}, t_v])^{m'_v}]$.

We partition the space $X \times Y$ by balls B_r^Z of radius r, noticing that Ω_r is a finite union of such balls and that α_r and β_r are constants on these balls and get

$$\begin{aligned} E_\rho(\prod_{v=1}^K (\mathcal{N}_r(]t_{v-1}; t_v]))^{m'_v} 1_{\Omega_r^d}) &= \sum_{B_r^Z :\, B_r^Z \subset \Omega_r} \rho(B_r^Z) E_\rho \left[\prod_{v=1}^K (\mathcal{N}_r(]t_{v-1}; t_v]))^{m'_v} \middle| B_r^Z \right] \\ &= \int_{\Omega_r} G(\alpha_r(B_r^Z), \beta_r(B_r^Z)) d\rho + O(\sup_{\Omega_r} \Delta_r^{(m)}) \\ &\underset{r\to 0}{\longrightarrow} \int_{[0,1]^2} G(\alpha, \beta) d\eta(\alpha, \beta) = \mathbb{E}\left[\prod_{v=1}^K (Z_\pi(]t_{v-1}; t_v]))^{m'_v} \right], \end{aligned}$$

by the convergence in distribution of (α_r, β_r) and the continuity and boundedness of F (coming from Lemmas A.1 and A.3). □

Corollary 3.7. *Assume Hypothesis 3.2, that both systems are SFT (f one-sided, g 2-sided) with equilibrium states of Hölder potentials. Let $\lambda_Y > 0$ and endow Y with the metric (resp. pseudo metric) with Lyapunov exponent λ_Y, so that B_r^Y are two-sided (resp. one-sided) cylinders; Set $d = 2$ (resp. $d = 1$). Then in the following cases, $\mathcal{N}_r$ converges in distribution to the random process $\mathcal{Z}_\pi$, where the random parameter π is equal to*

(a) $\pi = (1, 0)$ *a.s. if* $0 < d_\mu < d_\nu$;
(b) $\pi = (0, 1)$ *a.s. if* $d_\mu > d_\nu > 0$;
(c) $\pi = (1, 0)$ *or* $(0, 1)$ *with probability* $1/2$ *if* $d_\mu = d_\nu > 0$ *and if at least one of the measures is not of maximal entropy;*

(d) *π is a discrete random variable supported on $(0,1]^2$ if $d_\mu = d_\nu$, $\lambda_X = \lambda_Y$ and both measures are of maximal entropy*[13]*. In particular, if f and g are two full shifts with uniform distribution (on sets of respectively L^d and L elements), then $\pi = (L^{1-d}, 1)$.*

The remaining case (d) with $\lambda_X \neq \lambda_Y$ will be considered in Remark 3.8 below.

Proof. To apply Theorem 3.6 we first show that the error terms (15) and (16) go to zero. Let $0 < \gamma < \dim_H \mu = h_\mu$. Given $r > 0$, a summation over balls B_r^X allows to get

$$\sum_{B_r^X} \mu(B_r^X)\mu\left(\tau_{B_r^X}^f \leq |\log r|^{3m} \,\middle|\, B_r^X\right) \leq \mu(\tau_r^f \leq r^{-\gamma}) = o(r^\alpha),$$

by the large deviation estimates for return time proven in [3], for some $\alpha > 0$ and all r sufficiently small. The same arguments apply to the error term involving τ_r^g. Therefore the error terms go to zero in L^1_ρ, hence in probability.

By the existence of the pointwise dimension we have $\rho-$ a.e. (x,y)

$$\frac{1}{\log r} \log \frac{\mu(B_r(x))}{\nu(B_r(y))} \underset{r\to 0}{\longrightarrow} d_\mu - d_\nu.$$

Using (14) we get in the first case (a) that $(\alpha_r, \beta_r) \underset{r\to 0}{\longrightarrow} (1,0)$ a.s., and in the second case (b), $(\alpha_r, \beta_r) \underset{r\to 0}{\longrightarrow} (0,1)$ a.s.

Suppose that $d_\mu = d_\nu$ and that at least one of the measures is not of maximal entropy. Let $k_r^X = \lfloor \frac{-1}{\lambda_X} \log r \rfloor$ and $k_r^Y = \lfloor \frac{-1}{\lambda_Y} \log r \rfloor$. Then for ϕ_μ and ϕ_ν the respective normalized potentials we have

$$\frac{1}{\sqrt{|\log r|}} \log \frac{\mu(B_r^X(x))}{\nu(B_r^Y(y))} = \frac{1}{\sqrt{k_r^X}} \sum_{j=0}^{k_r^X} [\phi_\mu(f^j x) + h_\mu] - \frac{1}{\sqrt{k_r^Y}} \sum_{j=(1-d)k_r^Y}^{k_r^Y - 1} [\phi_\nu(g^k y) + h_\nu] + o(1),$$

since $k_r^X h_\mu - d k_r^Y h_\nu = O(1)$. By the central limit theorem (for f and g) these normalized Birkhoff sums converge in distribution under the product measure ρ to a sum S of two centered (since $\int \phi_\mu d\mu = -h_\mu$ and $\int \phi_\nu d\nu = -h_\nu$) independent gaussian random variables, with at least one of them of nonzero variance (otherwise both potentials are cohomologous to a constant and each measure has maximal entropy). Hence the variance of S is positive. Therefore, removing the normalization, the ratio of the measures converges in distribution to the uniform law on $\{-\infty, +\infty\}$, proving the result in the third case (c).

In the last case (d), the two measures are the Parry measure. Let A^X and A^Y denote the transition matrices of the subshifts. Denote by u^X, v^X and u^Y, v^Y their left and right positive eigenvectors, associated to the maximal eigenvalues e^{h_μ}, e^{h_ν}. We fix the normalization $u^X \cdot v^X = 1$ and $u^Y \cdot v^Y = 1$. Dropping the dependence on r we denote by k the common value of $k_r^X = k_r^Y$ (since $\lambda_X = \lambda_Y$). The measure of a cylinder is known to be equal to $\mu([a_0 a_1 \dots a_k]) = u^X_{a_0} v^X_{a_k} e^{-k h_\mu}$ and similarly for ν. Thus the distribution of the ratios

$$\frac{\mu([a_0 \dots a_k])}{\nu([b_{-k} \dots b_k])} = \frac{u^X_{a_0} v^X_{a_k}}{u^Y_{b_{-k}} v^Y_{b_k}} \text{ if } d = 2 \quad \text{and} \quad \frac{\mu([a_0 \dots a_k])}{\nu([b_0 \dots b_k])} = \frac{u^X_{a_0} v^X_{a_k}}{u^Y_{b_0} v^Y_{b_k}} \text{ if } d = 1 \qquad (19)$$

[13] The two measures are thus Markov. The limit distribution π is explicitly computed in terms of the stationary vector at the end of the proof of Corollary 3.7.

is given by the discrete measure

$$\sum_{a,a',b,b'} \mu([a]\cap f^{-k}[a'])\nu([b]\cap g^{-dk}[b'])\delta_{\frac{u_a^X v_{a'}^X}{u_b^Y v_{b'}^Y}} \underset{k\to\infty}{\longrightarrow} \sum_{a,a',b,b'} u_a^X v_a^X u_{a'}^X v_{a'}^X u_b^Y v_b^Y u_{b'}^Y v_{b'}^Y \delta_{\frac{u_a^X v_{a'}^X}{u_b^Y v_{b'}^Y}} =: \eta^*,$$

by mixing. The limiting distribution η is obtained by the continuous mapping theorem, applying the map $\Phi\colon \lambda \mapsto (1,\lambda^{-1})1_{\lambda>1} + (\lambda,1)1_{\lambda\le 1}$ to the distribution η^*.

In the particular case of two full shifts, $\frac{\mu([a_0\dots a_k])}{\nu([b_{-k}\dots b_k])} = \frac{L^{-2(k+1)}}{L^{-(2k+1)}} = L^{-1}$ if $d=2$ and $\frac{\mu([a_0\dots a_k])}{\nu([b_0\dots b_k])} = \frac{L^{-k-1}}{L^{-k-1}} = 1$ if $d=1$. □

Remark 3.8. *In the case when $d_\mu = d_\nu$ and both measures are of maximal entropy but $\lambda_X \neq \lambda_Y$, the random parameter does not converge. Indeed, the computations in the proof of (d) can be rewritten with k_X and k_Y defined as in the proof of (b). However, the entropic part in the expression of the measure of the cylinders do not cancel completely, so that in front of the ratio of the measures* (19) *a deterministic prefactor*

$$\zeta_r := \exp\left[-\left(\lfloor\frac{-\log r}{\lambda_X}\rfloor\lambda_X - \lfloor\frac{-\log r}{\lambda_Y}\rfloor\lambda_Y\right)\frac{h_\mu}{h_\nu}\right]$$

subsists. Note that if λ_X and λ_Y are rationally free, the accumulation points of ζ_r as $r\to 0$ is the whole interval $[e^{-dh_\nu}, e^{h_\mu}]$. Proceeding as in the proof of case (d), we conclude that $\mathcal{N}_r$ is asymptotic to $\mathcal{Z}_{\pi_r}$, where the parameter π_r is distributed as the image of η^* by the continuous map $\Phi(\zeta_r\cdot)$.

Remark 3.9. *Under the assumptions of the previous corollary, if $\nu(B_r^Y) \ll \mu(B_r^X)$ in ρ-probability, we retrieve the conclusion of Proposition 3.1. Indeed*

$$\tau_r^F = \inf\{t>0 : \mathcal{P}(L_t(0)) \ge 1\} = \inf\{t>0 : L_t(0) \ge \mathcal{E}\} = T_{\mathcal{E}}^{(0)}$$

where $\mathcal{E} := \inf\{u>0 : \mathcal{P}(u)\ge 1\}$ has exponential distribution of parameter 1 and $T_u^{(0)} := \inf\{t>0 : L_t(0)\ge u\}$ which has the same distribution as $\sigma^2u^2\mathcal{N}^{-2}$ where $\mathcal{N}$ is a standard gaussian random variable (see e.g. [15]*) combined with the fact that $L_t(0) = L'_t(0)/\sigma$ where L' is the local time of the standard Brownian motion B/σ.*

We end this section by stating a result that ensures that the limit process of $\mathcal{N}_r$ when $\nu(B_r^Y) \ll \mu(B_r^X)$ coincide with the limit of the analogous time process $\widetilde{\mathcal{N}}_r$ of return times of the $\mathbb{Z}$-extension $\widetilde{F}$ to the origin. This point process is given by

$$\forall x\in X,\quad \widetilde{\mathcal{N}}_r(x) = \sum_{n\in\mathbb{N}\,:\,\widetilde{F}^n(x,0)\in B_r^X(x)\times\{0\}} \delta_{n\mu(B_r^X(x))^2}\,, \tag{20}$$

where $\widetilde{F}$ has been defined in (2). We will prove in Section 5 the next result about the asymptotic behaviour of $\widetilde{\mathcal{N}}_r$ as $r\to 0$.

Theorem 3.10. *The family of point processes $(\widetilde{\mathcal{N}}_r)_{r>0}$ converges in distribution to $\mathcal{Z}_{1,0}$, as $r\to 0$, for the vague convergence, with respect to both $\mu(\cdot|B_r^X(x))$ and μ.*

4. Approximation of moments of the hitting process

We prove here Theorem 3.3. Let P be the transfer operator of (X,f,μ), i.e.

$$\forall G,H\in L^2(\mu),\quad \int_X P(G).H\,d\mu = \int_X G.H\circ f\,d\mu\,.$$

The following results come from Fourier perturbations $u\in\mathbb{C}, w\mapsto P_u(w) := P(e^{iuh}w)$ of the transfer operator P acting on the Banach space $\mathcal{B}_\theta$ of θ-Hölder continuous functions

endowed with the norm $\|w\| := |w|_\theta + \|w\|_1$ where $|w|_\theta := \inf\{K > 0 \colon \forall n,\, \mathrm{var}_n(w) \le K\theta^n\}$, where $\mathrm{var}_n(w)$ is the maximal variation of w on a n-cylinder, that is

$$\mathrm{var}_n(w) := \sup_{x,y\in X : x_0=y_0,\dots,x_n=y_n} |w(x) - w(y)|\,.$$

We recall that $P_u^n(w) = P^n(e^{iuh_n}w)$, using again the notation $h_n := \sum_{k=0}^{n-1} h\circ f^k$.

Proposition 4.1. *Assume Hypothesis 3.2. There exist three positive numbers δ, c' and $\alpha < 1$ and three continuous functions $u \mapsto \lambda_u \in \mathbb{C}$, $u \mapsto \Pi_u \in \mathcal{L}(\mathcal{B})$ and $u \mapsto N_u \in \mathcal{L}(\mathcal{B})$ defined on $\{z \in \mathbb{C} : |z| \le \delta\}$ such that*

(i) *for $|u| < \delta$, $P_u^k w = \lambda_u^k \Pi_u(w) + N_u^k(w)$ with $\|N_u^k(w)\| \le c'\alpha^k\|w\|$, $\lambda_u = 1 - \frac{\sigma^2}{2}u^2 + O(u^3) = e^{-\frac{\sigma^2}{2}u^2 + O(u^3)}$, $\left|\log(\lambda_u) + \frac{\sigma^2}{2}u^2\right| \le \frac{\sigma^2}{4}|u|^2$ and $\|\Pi_u(w) - \mu(w)\| \le c'|u|\|w\|$,*

(ii) *for $u \in [-\pi,\pi] \setminus [-\delta,\delta]$, $\|P_u^k w\| \le c'\alpha^k\|w\|$.*

Lemma 4.2. *For all constant $c > 0$ there exists a constant $C > 0$ such that for any $M \ge 1$ and any function H such that $\log|H|$ is uniformly θ-Hölder continuous on each M-cylinder with Hölder constant bounded by $c\theta^{-M}$,*

$$\textit{i.e.} \quad \forall D \in \mathcal{C}_M, \quad \forall y,z \in D, \quad |H(y)| \le |H(z)| e^{c\theta^{-M} d_\theta(y,z)}\,,$$

then for all $u \in [-\pi,\pi]$, $\|P_u^M(H)\| \le C\|H\|_1$.

Proof. Let us write φ for the (normalized) potential. Write $H = \sum_D H1_D$ where the sum runs over all M-cylinders. One has $\mathrm{var}_n(P_u^M(H1_D)) \le \left(|\varphi + iuh|_\theta \frac{\theta^n}{1-\theta} + e^c c\theta^n\right) \|P^M(|H|1_D)\|_\infty$. Note that $|\varphi + iuh|_\theta \le |\varphi|_\theta + \pi|h|_\theta$ is uniformly bounded, and

$$\left|P^M(|H|1_D)(x)\right| = |H(x_D)| \exp(S_M\varphi(x_D)) \le |H(x_D)| \kappa\, \mu(D)$$

by the Gibbs property, where $x_D \in D$ is the unique preimage in D of $x \in X$ by σ^M, for some constant κ. Furthermore

$$\int_D |H(x_D)|\, d\mu(y) \le \int_D |H(y)|\, |H(x_D)/H(y)|\, d\mu(y) \le e^c \int_D |H(y)|\, d\mu(y)\,.$$

Hence $|P_u^M(H1_D)|_\theta \le C'\|H1_D\|_1$ for the constant $C' = \left(\frac{|\varphi|_\theta + \pi|h|_\theta}{1-\theta} + ce^c\right)\kappa e^c$. Therefore, summing over D gives $|P_u^M(H)|_\theta \le C'\|H\|_1$. □

We will use the next lemma which is the operator estimate that is behind Proposition 2.4.

Lemma 4.3. *For all $c > 0$, there exists a constant $C > 0$ such that, for any positive integer M, any function H as in Lemma 4.2 and any $k > M^2$ we have*

$$\sup_{\ell\in\mathbb{Z}} \left\| P^k\left(1_{\{h_k=\ell\}} H\right) - \frac{1}{\sqrt{k}} \Phi\left(\frac{\ell}{\sqrt{k}}\right) \mu(H) \right\| \le \frac{C}{k}\|H\|_1\,, \tag{21}$$

where Φ is the density function of the centered Gaussian distribution with variance σ^2.

Proof. We start with the identity

$$P^k\left(1_{\{h_k=\ell\}} H\right) = \frac{1}{2\pi} \int_{[-\pi,\pi]} e^{-iu\ell} P_u^k(H) du\,. \tag{22}$$

Let $w = P_u^M(H)$. Let $d_u(k,w) = \left| P_u^{k-M} w - e^{-\frac{\sigma^2}{2}u^2(k-M)}\mu(w)\right|$. We apply Proposition 4.1. For $|u| \geq \delta$ we have

$$d_u(k,w) \leq \left(c'\alpha^{k-M} + e^{-\frac{\sigma^2}{2}\delta^2(k-M)} \right) \|w\|.$$

For $|u| < \delta$ we have

$$\begin{aligned} d_u(k,w) &\leq |P_u^{k-M}w - \lambda_u^{k-M}\Pi_u(w)| + |\lambda_u^{k-M}||\Pi_u(w) - \mu(w)| + |\lambda_u^{k-M} - e^{-\frac{\sigma^2}{2}u^2(k-M)}|\,|\mu(w))| \\ &\leq c'\alpha^{k-M}\|w\| + c'|\lambda_u|^{k-M}|u|\|w\| + e^{-\frac{\sigma^2}{4}u^2(k-M)}(K-M)|u|^3|\mu(w)|. \end{aligned}$$

The second term is handled by the change of variable $v = u\sqrt{k}$

$$\int_{-\delta}^{\delta} |\lambda_u^{k-M}|\ |u|du \leq \int_{-\delta}^{\delta} e^{-\frac{\sigma^2}{4}u^2(k-M)}|u|du \leq \frac{1}{k-M}\int_{\mathbb{R}} e^{-\frac{\sigma^2}{8}v^2}|v|dv \leq \frac{C'}{k},$$

for some constant C'. Next, the same change of variable and the dominated convergence theorem shows that the integral of the third term is $O(k^{-1})|\mu(w)|$.

Finally, the same change of variable yields

$$\begin{aligned} \frac{\sqrt{k}}{2\pi}\int_{-\delta}^{\delta} e^{-iu\ell}e^{-\frac{\sigma^2}{2}u^2(k-M)}du - \Phi\left(\frac{\ell}{\sqrt{k}}\right) &= \frac{1}{2\pi}\int_{-\delta\sqrt{k}}^{\delta\sqrt{k}} e^{-\frac{iv\ell}{\sqrt{k}}-\frac{\sigma^2}{2}v^2\frac{k-M}{k}}dv - \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\frac{iv\ell}{\sqrt{k}}-\frac{\sigma^2 v^2}{2}}dv \\ &= O\left(\frac{M}{k}\right) = O(k^{-\frac{1}{2}})\,. \end{aligned}$$

Therefore

$$\sup_{\ell\in\mathbb{Z}}\left|\frac{\sqrt{k}}{2\pi}\int_{[-\pi,\pi]} e^{-iu\ell}P_u^{k-M}(w)du - \Phi\left(\frac{\ell}{\sqrt{k}}\right)\mu(w)\right| \leq \eta_k'\|w\|,$$

where $\eta_k' = O(k^{-\frac{1}{2}})$. To conclude remark that $P_u^k(H) = P_u^{k-M}P_u^M(H)$, use (22) and Lemma 4.2. □

Lemma 4.4. *Let $c > 0$. There exists $K > 1$ such that for all $u > 0$ small enough,*

$$P^k(1_{\{h_k \geq L\}}H) \leq Ke^{-\frac{uL}{\sqrt{k}}}\|H\|_{L^1},$$

uniformly in L, M, in $k \geq M^2$, in H as in Lemma 4.2.

Proof. For all $u > 0$, we have

$$P^k(1_{\{h_k\geq L\}}H) \leq P^k\left(e^{\frac{u(h_k-L)}{\sqrt{k}}}|H|\right) \leq e^{-\frac{Lu}{\sqrt{k}}}P^k_{-iu/\sqrt{k}}(|H|)\ .$$

Noticing that $P_{-iu}(\cdot) = P(e^{uh}\cdot)$. Set $w = P^M_{-iu/\sqrt{k}}(|H|) = P^M\left(e^{\frac{u}{\sqrt{k}}h_M}|H|\right)$. Noticing that $\frac{M}{\sqrt{k}} \leq 1$ and that the θ-Hölder constant of h_M on each M-cylinder D satisfies $|(h_M)_{|D}|_\theta \leq \frac{|h|_\theta\theta^{-M}}{\theta^{-1}-1}$, it follows from Lemma 4.2 that $\|w\| \leq C\|H\|_1$.

We assume from now on that $|u| < M\delta$. By real perturbations in Proposition 4.1 we get

$$\begin{aligned} P^k_{-\frac{iu}{\sqrt{k}}}(|H|) = P^{k-M}_{-iu/\sqrt{k}}(w) &\leq \left|\lambda^{k-M}_{-iu/\sqrt{k}}\Pi_{-iu/\sqrt{k}}(w)\right| + c'\alpha^{k-M}\|w\| \\ &\leq (2e^{\frac{3u^2(k-M)}{4k}} + c'\alpha^{k-M})\|w\| \leq K\|H\|_1\,, \end{aligned}$$

since $|u/\sqrt{k}| \leq |u|/M < \delta$. □

For $m, k \in \mathbb{N}$ we denote the number of partitions with k atoms of a set of m elements by $S(m,k)$, the Stirling number of the second kind.

We now proceed with the proof of the theorem.

Proof of Theorem 3.3. It follows from the definition (13) of $\mathcal{N}_r$ combined with (6) and from the fact that the balls are cylinders that

$$\forall (x',y') \in B_r^X(x) \times B_r^Y(y), \quad \mathcal{N}_r(x',y') = \sum_{(n,k)\in\mathbb{N}\times\mathbb{Z}\,:\, f^n(x')\in B_r^X(x), h_n(x')=k, g^k(y)\in B_r^Y(y)} \delta_{n/n_r}\,.$$

We set $m_r := 3c_0|\log r|$, where $c_0 \geq 1$ is the constant appearing in (II) of Hypothesis 3.2 (noticing that, since $c_0 \geq 1$, this assumption holds also true if we replace (Y,g,ν) by (X,f,μ)). Changing c_0 is necessary, we assume that $c_0 \geq \frac{1}{\lambda_X}$. For any sequence $(k_j)_{j\geq 1}$, we denote the derived sequence $(k'_j = k_j - k_{j-1})_{j\geq 1}$ where we put $k_0 = 0$.

Step 1: Moments expressed thanks to the Fourier-perturbed transfer operator

In the following product, we first expand the $\overline{m}_v$ powers of the sums defining $\mathcal{N}_r([t_{v-1},t_v])$ as $\overline{m}_v$ sums of a product. Regrouping the indices which are equal we are left with $q_v = 1,...,\overline{m}_v$ distinct indices, which gives after reordering

$$\begin{aligned}\mathcal{M}_r(\mathbf{t},\overline{\mathbf{m}}) &:= \mathbb{E}_\rho\left[\prod_{v=1}^K \mathcal{N}_r(]t_{v-1},t_v])^{\overline{m}_v} 1_{B_r^X(x)\times B_r^Y(y)}\right] \\ &= \sum_{\substack{\mathbf{q}=(q_1,...,q_K)\\ q_v=1,...,\overline{m}_v}} \left(\prod_{v=1}^K S(\overline{m}_v,q_v) q_v!\right) A_{n_r;\mathbf{q}}(x,y)\,, \end{aligned} \tag{23}$$

with

$$A_{n_r;\mathbf{q}}(x,y) := \sum_{\mathbf{k}=(k_1,...,k_q)} \sum_{\boldsymbol{\ell}\in\mathbb{Z}^q} \mathbb{E}_\rho\left[1_{B_r^X(x)\times B_r^Y(y)} \prod_{j=1}^q \left(1_{B_r^Y(y)}\circ g^{\ell_j}\, 1_{\{h_{k_j}=\ell_j\}} 1_{B_r^X(x)}\circ f^{k_j}\right)\right] \tag{24}$$

where we set from now on

$$q := q_1 + ... + q_K$$

and where the first sum holds over the $\mathbf{k} = (k_1,...,k_q)$ corresponding to concatenation of $(k_1^1,...,k_{q_1}^1)$,...,$(k_1^v,...,k_{q_v}^v)$,..., $(k_1^K,...,k_{q_K}^K)$ such that

$$t_{v-1}n_r < k_1^v < \ldots < k_{q_v}^v \leq t_v n_r\,.$$

Recalling that

$$k'_i := k_i - k_{i-1} \quad \text{and} \quad \ell'_i := \ell_i - \ell_{i-1}, \quad k_0 = \ell_0 = 0\,,$$

we observe that $A_{n_r;\mathbf{q}}(x,y)$ can be rewritten

$$\sum_{\mathbf{k}} \sum_{\boldsymbol{\ell}\in\mathbb{Z}^q} \mathbb{E}_\rho\left[1_{B_r^X(x)\times B_r^Y(y)} \prod_{j=1}^q \left(1_{B_r^Y(y)}\circ g^{\ell_j}\, 1_{\{h_{k'_j}=\ell'_j\}}\circ f^{k_{j-1}} 1_{B_r^X(x)}\circ f^{k_j}\right)\right]$$

and so

$$A_{n_r;\mathbf{q}}(x,y) = \sum_{\mathbf{k}} \sum_{\boldsymbol{\ell}\in\mathbb{Z}^q} \nu\left(\bigcap_{j=0}^q g^{-\ell_j}(B_r^Y(y))\right) \mathbb{E}_\mu\left[Q^{(x)}_{k'_q,\ell'_q}\cdots Q^{(x)}_{k'_1,\ell'_1}(1_{B_r^X(x)})\right]\,, \tag{25}$$

with

$$Q^{(x)}_{k,\ell}(H) := 1_{B_r^X(x)} P^k\left(1_{\{h_k=\ell\}}\, H\right).$$

The strategy of the proof is then to apply inductively Condition (II) of Hypothesis 3.2 and Lemma 4.3 to say roughly

- that $\nu\left(\bigcap_{j=0}^{q} g^{-\ell_j}(B_r^Y(y))\right)$ behaves as $\left(\nu(B_r^Y(y))\right)^{q+1}$;
- and that $\mathbb{E}_\mu\left[Q^{(x)}_{k'_q,\ell'_q}\cdots Q^{(x)}_{k'_1,\ell'_1}(1_{B_r^X(x)})\right]$ behaves as $\left(\mu(B_r^X(x))\right)^{q+1}\prod_{j=1}^{q}\frac{\Phi\left(\frac{\ell'_j}{\sqrt{k'_j}}\right)}{\sqrt{k'_j}}$.

Unfortunately this requires some care since the Hölder norm of $1_{B_r^X(x)}$ (resp. $1_{B_r^Y(y)}$) explodes as r goes to 0. Nevertheless, this will be possible when there are gaps between the indices. In Step 2, we treat the bad situations where there are clusters of indices k_j's (and so lack of gaps). A second difficulty will come from the uniform error (in ℓ'_j) in the approximation by $\Phi\left(\frac{\ell'_j}{\sqrt{k'_j}}\right)/\sqrt{k'_j}$. To avoid this difficulty we will use Lemma 4.4 to control in Step 3 the contribution of the big values of ℓ'_j (and so to restrict the sums over the $\ell'_j s$). We will then be able to conclude with the use of Riemann sums (Step 4) and by sum-integral approximations and moment identifications (Step 5).

We say that $k \in \mathbb{Z}^q$ has clustering if $k'_j < m_r$ for some $j = 1, ..., q$.

Step 2: Neglectability of clusters of k_i's. We will prove that the contribution to $A_{n_r;\mathbf{q}}(x,y)$ of those $k \in \mathbb{Z}^q$ for which there is clustering gives rise to the error term (15).

For $k_1 < \cdots < k_q$ with clustering we denote by c_1 the minimal $j \geq 0$ such that $k'_{j+1} < m_r$. The length of the first cluster is $p_1 + 1$ where p_1 is the maximal integer such that $k'_{c_1}, \ldots, k'_{c_1+p_1} < m_r$. We then define inductively the s-th cluster (if any) and its length $p_s + 1$ by $c_s := \min\{j \geq c_{s-1} + p_{s-1} \colon k'_{j+1} < m_r^{3^{s-1}}\}$ and p_s is the maximal integer such that $k'_{c_s}, \ldots, k'_{c_s+p_s} < m_r^{3^{s-1}}$. Note that the s-th cluster starts at the index c_s and ends at the index $c_s + p_s$. Let $\mathcal{J}_{\mathbf{k}} = \{1, \ldots, q\} \setminus \bigcup_s \{c_s + 1, \ldots, c_s + p_s\}$. $\{0\} \cup \mathcal{J}_{\mathbf{k}}$ is the set of indices j which are isolated or where a cluster starts. It determines uniquely the sequences c_s and p_s. Note that the existence of a cluster means that $\#\mathcal{J}_{\mathbf{k}} < q$.

The sum over $\mathbf{k} = (k_1, ..., k_q)$ in the definition of $A_{n_r;\mathbf{q}}(x,y)$ detailed between (24) and (25) can be rewritten as a sum over $\mathcal{J} \subset \{1, \ldots, q\}$ of the sum over the $\mathbf{k}$'s such that $\mathcal{J}_{\mathbf{k}} = \mathcal{J}$.

Fix $\mathcal{J} \subset \{1, ..., q\}$ with $w := \#\mathcal{J} < q$. Consider $\mathbf{k}$ such that $\mathcal{J}_{\mathbf{k}} = \mathcal{J}$ for which there is a cluster. First we have

$$\mathbb{E}_\nu\left[\prod_{j=0}^{q} 1_{B_r^Y(y)} \circ g^{\ell_j}\right] \leq \nu\left(\bigcap_{j\in\mathcal{J}\cup\{0\}} g^{-\ell_j}(B_r^Y(y))\right).$$

Let us consider $j \in \mathcal{J} \cup \{0\}$ a beginning of say the s-th cluster. Inside this cluster, $(k'_{j+i})_{i=1,\ldots,p_j}$ can take at most $(m_r^{3^{s-1}})^{p_j}$ values and we know that at time k_j not only we are in the set $B_r^X(x)$ but also we return to it before time $m_r^{3^{s-1}}$. Hence

$$\sum_{k_{j+1}..k_{j+p_j}} \sum_{\ell_{j+1}..\ell_{j+p_j}} 1_{\{h_{k'_{j+i}} = \ell'_{j+i}\}} \circ f^{k_{j+i-1}} 1_{B_r^X(x)} \circ f^{k_{j+i}} \leq m_r^{3^s p_j} 1_{\{\tau^f_{B_r^X(x)} \leq m_r^{3^s}\}} \circ f^{k_j}.$$

This means that we can remove all the sums over the indices inside the clusters, provided we insert the above factor in the corresponding place. This finally gives a contribution to

$A_{n_r;\mathbf{q}}(x,y)$ of those k such that $\mathcal{J}_{\mathbf{k}}=\mathcal{J}$ bounded from above by

$$m_r^{3^q}\sum_{\overline{\mathbf{k}}}\sum_{\boldsymbol{\ell}\in\mathbb{Z}^w}\nu\left(\bigcap_{j=0}^{w}g^{-\ell_j}(B_r^Y(y))\right)\mathbb{E}_\mu\left[\widetilde{Q}^{(x)}_{w,\bar{k}'_w,\ell'_w}\circ\cdots\circ\widetilde{Q}^{(x)}_{1,\bar{k}'_1,\ell'_1}(1_{G_{r,0}(x)})\right]. \tag{26}$$

where the sum is taken over $\overline{\mathbf{k}}\in\mathbb{Z}^w$ which are images by the projection $\mathbf{k}\mapsto(k_j)_{j\in\mathcal{J}}$, and where

$$\widetilde{Q}^{(x)}_{i,k'_i,\ell'_i}(H):=1_{G_{r,i}(x)}P^{k'_i}\left(1_{\{h_{k'_i}=\ell'_i\}}\,H\right).$$

and $G_{r,i}(x)=B_r^X(x)\cap\{\tau^f_{B_r^X(x)}\le m_r^{3^s}\}$ if the $(i+1)$-th element of $\{0\}\cup\mathcal{J}$ is the beginning of the s-th cluster, and $G_{r,i}(x)=B_r^X(x)$ otherwise. Note that, since $c_0\ge 1$, $G_{r,i}(x)$ is a union of m_r-cylinders (if no cluster) or a union of $m_r+m_r^{3^s}$-cylinders, in any case $\bar{k}'_{i+1}\ge m_r^{3^{s+1}}\ge(m_r+m_r^{3^s})^2$, so that the lemmas apply. We observe that

$$\mathbb{E}_\mu\left[\widetilde{Q}^{(x)}_{w,\bar{k}'_w,\ell'_w}\circ\cdots\circ\widetilde{Q}^{(x)}_{1,\bar{k}'_1,\ell'_1}(1_{G_{r,0}(x)})\right]=\mathbb{E}_\mu[1_{G_{r,w}(x)}\rho_w^{\bar{\mathbf{k}}',\boldsymbol{\ell}'}], \tag{27}$$

with $\rho_0=1$ and defining inductively

$$\forall i=1,...,w,\quad \rho_i^{\bar{k}_{1..i},\ell_{1..i}}:=P^{\bar{k}'_i}\left(1_{\{h_{\bar{k}'_i}=\ell'_i\}}1_{G_{r,i-1}}\rho_{i-1}^{\bar{k}_{1..i-1},\ell_{1..i-1}}\right). \tag{28}$$

A computation by induction shows that the norms $\|\log\rho_i^{\bar{k}_{1..i},\ell_{1..i}}\|$ are bounded by a constant C_w independent of $\overline{\mathbf{k}},\boldsymbol{\ell}$.

We again need to decompose the sum over $\boldsymbol{\ell}\in\mathbb{Z}^w$ subject to clustering or not. Unfortunately clusters may now appear from non consecutive indices and we need to adapt the definition. Given $\boldsymbol{\ell}\in\mathbb{Z}^w$, for each $i=0,...,w$ we denote by $C_i^{\boldsymbol{\ell}}$ the set of indices i' such that there exists a chain of ℓ_j's, pairwise m_r-close, joining ℓ_i to $\ell_{i'}$. Next we denote by $\mathcal{I}_{\boldsymbol{\ell}}=\{C_i^{\boldsymbol{\ell}},i=0,...,w\}$ the set of such clusters.

Fix $\mathcal{I}$ a partition of $\{0,...,w\}$ and consider $\boldsymbol{\ell}$ such that $\mathcal{I}_{\boldsymbol{\ell}}=\mathcal{I}$. Denote by $\mathcal{I}^*=\{\min C,C\in\mathcal{I}\}\setminus\{0\}$ the set of minimal index of each cluster, zero excluded. Let $p=\#\mathcal{I}^*=\#\mathcal{I}-1$. It follows from (II) of Hypothesis 3.2 on g that

$$\begin{aligned}\nu\left(\bigcap_{j=0}^{w}\left(g^{-\ell_j}(B_r^Y(y))\right)\right)&=(1+O(\alpha^{m_r}))\prod_{C\in\mathcal{I}}\nu\left(\bigcap_{i\in C}g^{-\ell_i}(B_r^Y(y))\right)\\&=(1+O(\alpha^{m_r}))\nu(B_r^Y(y))^{\alpha_\ell}\nu\left(B_r^Y(y)\cap\{\tau^g_{B_r^Y(y)}<m_r^{3^q}\}\right)^{\beta_\ell} \quad (29)\\&\le C\nu(B_r^Y(y))^{p+1}\nu\left(\tau^g_{B_r^Y(y)}<m_r^{3^q}\,\middle|\,B_r^Y(y)\right)^{\beta_\ell}, \quad (30)\end{aligned}$$

where $\alpha_\ell=\#\{C\in\mathcal{I}:\#\{\ell_i,i\in C\}=1\}$ and $\beta_\ell=\#\{C\in\mathcal{I}:\#\{\ell_i,i\in C\}>1\}=p+1-\alpha_\ell$. It follows from Lemma 4.3 that

$$\mathbb{E}_\mu\left[1_{G_{r,i}}\rho_i^{\bar{k}_{1..i},\ell_{1..i}})\right]=\mu(G_{r,i})A_i(\bar{\mathbf{k}},\boldsymbol{\ell})\mathbb{E}_\mu\left[1_{G_{r,i-1}}\rho_{i-1}^{\bar{k}_{1..i-1},\ell_{1..i-1}}\right]$$

where, setting $\gamma=\|h\|_\infty$,

$$\left|\frac{b_i(\mathbf{k},\boldsymbol{\ell})}{k'_i}:=A_i(\mathbf{k},\boldsymbol{\ell})-\frac{1}{\sqrt{k'_i}}\Phi\left(\frac{\ell'_i}{\sqrt{k'_i}}\right)\right|\le\frac{C}{k'_i}\quad\text{if}\quad|\ell'_i|\le\gamma k'_i \tag{31}$$

and $A_i(\mathbf{k}, \boldsymbol{\ell}) = 0$ if $|\ell_i'| > \gamma k_i'$. Hence an immediate induction gives

$$\mathbb{E}_\mu \left[1_{G_{r,w}} \rho_w^{\bar{\mathbf{k}}, \boldsymbol{\ell}} \right] \leq \left(\prod_{i=0}^{w} \mu(G_{r,i}) \right) \prod_{i=1}^{w} A_i(\bar{\mathbf{k}}, \boldsymbol{\ell}). \tag{32}$$

Now we fix $\overline{\mathbf{k}}$ and make the summation over $\boldsymbol{\ell}$ such that $\mathcal{I}_{\boldsymbol{\ell}} = \mathcal{I}$:

$$\begin{aligned} S_{\mathcal{I}}(\overline{\mathbf{k}}) &:= \sum_{\boldsymbol{\ell} \colon \mathcal{I}_{\boldsymbol{\ell}} = \mathcal{I}} \mathbb{E}_\mu [1_{G_{r,w}(x)} \rho_w^{\bar{\mathbf{k}}', \boldsymbol{\ell}'}] \\ &= \sum_{\boldsymbol{\ell} \colon \mathcal{I}_{\boldsymbol{\ell}} = \mathcal{I}} \left(\prod_{j=0}^{w} \mu(G_{r,j}) \right) \prod_{i=1}^{w} \frac{1}{\sqrt{\bar{k}_i'}} \left(\Phi \left(\frac{\ell_i'}{\sqrt{\bar{k}_i'}} \right) + \frac{b_i(\bar{\mathbf{k}}, \boldsymbol{\ell})}{\sqrt{\bar{k}_i'}} \right) 1_{\{|\ell_i'| \leq \gamma k_i'\}} \\ &= \sum_{\boldsymbol{\ell}} \left(\prod_{j=0}^{w} \mu(G_{r,j}) \right) \prod_{i=1}^{w} \left(\frac{1}{\sqrt{\bar{k}_i'}} \left(\Phi \left(\frac{\ell_i'}{\sqrt{\bar{k}_i'}} \right) + \frac{b_i(\bar{\mathbf{k}}, \boldsymbol{\ell})}{\sqrt{\bar{k}_i'}} \right) \right), \end{aligned}$$

where the sums are restricted to $|\ell_i'| \leq \gamma \bar{k}_i'$ for $i = 1, ..., w$ and $\mathcal{I}_{\boldsymbol{\ell}} = \mathcal{I}$. During this summation, when $i \in \mathcal{I}^*$ we bound the sum over ℓ_i by

$$K_0 := \sup_{r<1} \sup_{m_r < k < n_r} \sum_{|\ell| \leq \gamma k} \frac{1}{\sqrt{k}} \left(\Phi \left(\frac{\ell}{\sqrt{k}} \right) + \frac{C}{\sqrt{k}} \right) < \infty. \tag{33}$$

Otherwise when $i \notin \mathcal{I}^*$, there are at most $2pm_r + 1$ choices for ℓ_i, since ℓ_i is close to ℓ_j where $j = \min C_i < i$ (hence ℓ_j is already fixed). Moreover $\Phi \leq 1$ therefore the sum over ℓ_i is bounded by

$$((p+1)m_r + 1) \frac{1 + C}{\sqrt{\bar{k}_i'}}. \tag{34}$$

Therefore

$$S_{\mathcal{I}}(\overline{\mathbf{k}}) = O \left(\left(\prod_{j=0}^{w} \mu(G_{r,j}) \right) \prod_{i \notin \mathcal{I}^*} \frac{m_r}{\sqrt{\bar{k}_i'}} \right).$$

Putting this last estimate together with (27) and (29) and summing up over $\bar{\mathbf{k}}$ gives

$$\begin{aligned} (26) &\leq C m_r^{3^w} \nu(B_r^Y(y))^{p+1} \left(\prod_{i=0}^{w} \mu(G_{r,i}) \right) (m_r \sqrt{n_r})^{w-p} n_r^p \\ &\leq C m_r^{3^w} m_r^q \mu(B_r^X(x)) \nu(B_r^Y(y)) \left(\mu(B_r^X(x)) \nu(B_r^Y(y)) n_r \right)^p \left(\sqrt{n_r} \mu(B_r^X(x)) \right)^{w-p} \\ &\quad \mu \left(\tau^f_{B_r^X(x)} \leq m_r^{3^q} | B_r^X(x) \right) \\ &\leq C \rho(B_r^Z(x,y)) m_r^{3^q+q} \mu \left(\tau^f_{B_r^X(x)} \leq m_r^{3^q+q} | B_r^X(x) \right) \end{aligned} \tag{35}$$

since there were at least one cluster for k ($w < q$) and due to the definition of $n_r = n_r(x, y)$.

We henceforth suppose that there are no cluster of k_j's in the definition of $A_{n_r;\mathbf{q}}(x, y)$, that is $k_j' > m_r$ for $j = 1..q$, hence $w = q$, and $G_{r,i} = B_r^X(x)$ for all i.

Step 3: Neglectability of big values of ℓ_i'. Let $c_r := q\sqrt{-\log r}$. The contribution of those ℓ such that $\ell_i' > L_i := c_r \sqrt{k_i'}$ for some i contributes to the error term (16).

Set $\mathcal{I}_\ell' := \{i = 1, ..., q\colon \ell_i' > L_i\}$. Fix $\emptyset \neq \mathcal{I}' \subset \mathbb{Z}^q$. We follow the idea in the previous discussion: fix a partition $\mathcal{I} \subset \{0, \dots, q\}$, define as previously $\mathcal{I}^*$ as the set of starting indices of clusters and $p = \#\mathcal{I}^*$, consider the sum over ℓ such that $\mathcal{I}_\ell = \mathcal{I}$ and $\mathcal{I}_\ell' = \mathcal{I}'$.

Recall that $\rho_{i-1}^{k_{1..i-1},\ell_{1..i-1}}$ has been defined inductively in (28). Set, for the indices $i \in \mathcal{I}' \cap \mathcal{I}^*$ we obtain via Lemma 4.4

$$\begin{aligned}\sum_{\ell_i : |\ell_i'| > L_i} \rho_i^{k_{1..i},\ell_{1..i}} &= P^{k_i'}(1_{\{|h_{k_i'}| > L_i\}} 1_{G_{r,i-1}} \rho_{i-1}^{k_{1..i-1},\ell_{1..i-1}}) \\ &\leq K e^{-\frac{L_i u}{\sqrt{k_i'}}} \mathbb{E}_\mu \left[1_{G_{r,i-1}} \rho_{i-1}^{k_{1..i-1},\ell_{1..i-1}} \right] .\end{aligned}$$

Therefore

$$\sum_{|\ell_i'| > L_i} \mathbb{E}_\mu \left[1_{G_{r,i}} \rho_i^{k_{1..i},\ell_{1..i}} \right] \leq K \mu(G_{r,i}) e^{-uc_r} \mathbb{E}_\mu \left[1_{G_{r,i-1}} \rho_{i-1}^{k_{1..i-1},\ell_{1..i-1}} \right], \tag{36}$$

that is instead of (33) we get the better bound Ke^{-uc_r}.

For the indices $i \in \mathcal{I}' \setminus \mathcal{I}^*$, we notice that $\Phi\left(\frac{\ell_i'}{\sqrt{k_i'}}\right) \leq \exp(-\frac{\sigma^2 c_r^2}{2})$, therefore the sum over ℓ_i is bounded by

$$m_r \left(\frac{e^{-\frac{\sigma^2 c_r^2}{2}}}{\sqrt{k_i'}} + \frac{C}{k_i'} \right),$$

instead of (34).

For the other indices $i \notin \mathcal{I}'$, we keep the estimates of the previous discussion, using using (33) or (34), wether $i \in \mathcal{I}^*$ or not. We finally end up with a contribution in

$$\begin{aligned}&\leq C\mu(B_r^X(x))^{q+1} \nu(B_r^Y(y))^{\#\mathcal{I}} (n_r e^{-uc_r})^{\#\mathcal{I}' \cap \mathcal{I}^*} (m_r \sqrt{n_r} [e^{-\frac{\sigma^2 c_r^2}{2}} + \frac{\log n_r}{\sqrt{n_r}}])^{\#\mathcal{I}' \setminus \mathcal{I}^*} \times \\ &\qquad \times n_r^{\#(\mathcal{I}^* \setminus \mathcal{I}')} (m_r \sqrt{n_r})^{q - \#(\mathcal{I}^* \cup \mathcal{I}')} \\ &\leq C\mu(B_r^X(x)) \nu(B_r^Y(y)) \left(n_r \mu(B_r^X(x)) \nu(B_r^Y(y)) \right)^p \left(\sqrt{n_r} \mu(B_r^X(x)) \right)^{q-p} m_r^q \epsilon_r \\ &\leq C m_r^q \epsilon_r \rho(B_r^Z(x,y)) ,\end{aligned}$$

with $\epsilon_r := \max(e^{-uc_r}, e^{-\frac{1}{2}c_r^2} + \frac{\log n_r}{\sqrt{n_r}})$.

Step 4: Reduction to a Riemann sum. Recall that $c_r = q\sqrt{-\log r}$. Thus, up to error terms in (15) and (16), in view of (23), $A_{n_r,\mathbf{q}}(x,y)/\rho(B_r^Z(x,y))$ behaves as

$$\sum_{\mathbf{k} : k_j' > m_r} \mu(B_r^X(x))^q \sum_{\ell_j : |\ell_j'| \leq c_r \sqrt{k_j'}} \nu\left(\bigcap_{i=0}^q g^{-\ell_i}(B_r^Y(y)) \,\middle|\, B_r^Y(y) \right) \prod_{j=1}^q A_j(k,\ell) ,$$

with $|A_j(k,\ell) - a_j(k,\ell)| \leq C/k_j'$ by Lemma 4.3, where $a_j(k,\ell) = \frac{1}{\sqrt{k_j'}} \Phi\left(\frac{\ell_j'}{\sqrt{k_j'}}\right)$.

We aim to replace each $A_j(k,\ell)$ by $a_j(k,\ell)$. Note that they are both bounded by $C/\sqrt{k_j'}$ and their difference is bounded by C/k_j'. Fix a partition $\mathcal{I}$ of $\{1, \dots, q\}$ as above. Using

a telescopic sum we get

$$\left|\prod_{j=1}^{q} A_j(k,\ell) - \prod_{j=1}^{q} a_j(k,\ell)\right| \leq \sum_{i'=1}^{q} \prod_{j=1}^{i'-1} A_j(k,\ell)\,|A_{i'}(k,\ell) - a_{i'}(k,\ell)| \prod_{j=i'+1}^{q} a_j(k,\ell)$$
$$\leq \sum_{i'=1}^{q} \frac{C}{k'_{i'}} \prod_{j\neq i'} \left(a_j(k,\ell) + \frac{C}{k'_j}\right).$$

We now fix some i' and sum over ℓ such that $\mathcal{I}_\ell = \mathcal{I}$ as in the proof of the Step 2, with the additional condition $|\ell'_i| \leq c_r\sqrt{k'_i}$ for all i. For the indices $i \neq i'$ we use the same estimates, and for $i = i'$ two cases can happen:

If $i = i' \in \mathcal{I}^*$ we replace the estimate (33) by

$$\sum_{k_i}\sum_{\ell_i} \frac{C}{k'_i} \leq \sum_{k_i} \frac{1 + 2c_r\sqrt{k'_i}}{k'_i} \leq Cc_r\sqrt{n_r}.$$

Otherwise, $i = i' \notin \mathcal{I}^*$ and we replace the estimate (34) by

$$\sum_{k_i}\sum_{\ell_i} \frac{C}{k'_i} \leq \sum_{k_i} \frac{1 + 2m_r}{k'_i} \leq Cm_r \log n_r.$$

In both cases we gain a factor $\max(m_r \log n_r/n_r, c_r/\sqrt{n_r})$, so that the total contribution is dominated by (16). Thus, writing $e_r := (15) + (16)$, we have proved that

$$\mathbb{E}_\rho\left[(\mathcal{N}_r)^m | B_r^Z(x,y)\right] = O\left(e_r\right) + (1 + O(e_r))\mu(B_r^X(x))^q$$
$$\sum_{\mathbf{k}:k'_j>m_r} \sum_{\ell_j:|\ell'_j|\leq c_r\sqrt{k'_j}} \nu\left(\bigcap_{i=0}^{q} g^{-\ell_i}(B_r^Y(y))\,\middle|\, B_r^Y(y)\right) \prod_{j=1}^{q} \frac{1}{\sqrt{k'_j}} \Phi\left(\frac{\ell'_j}{\sqrt{k'_j}}\right).$$

Step 5 : Final step. It remains to estimate the following quantity

$$\mu(B_r^X(x))^q \sum_{\mathbf{k}:k'_j>m_r} \sum_{\ell_j:|\ell'_j|\leq c_r\sqrt{k'_j}} \nu\left(\bigcap_{i=0}^{q} g^{-\ell_i}(B_r^Y(y))\,\middle|\, B_r^Y(y)\right) \prod_{j=1}^{q} \frac{1}{\sqrt{k'_j}} \Phi\left(\frac{\ell'_j}{\sqrt{k'_j}}\right), \quad (37)$$

with $c_r = q\sqrt{-\log r}$. Recall from (29) that

$$\nu\left(\bigcap_{j=0}^{q} g^{-\ell_j} B_r^Y(y)\,\middle|\, B_r^Y(y)\right) = (1+O(\alpha^{m_r}))\nu(B_r^Y(y))^{\#\mathcal{I}_\ell - 1}\nu\left(\{\tau^g_{B_r^Y(y)} < m_r^{3^q}\}\,\middle|\, B_r^Y(y)\right)^{\beta_\ell},$$

with $\mathcal{I}_{\boldsymbol{\ell}}$ the partition of $\{0,...,q\}$ consisting in gathering the indices i corresponding to clusters of ℓ_i's (as defined in step 2) and with $\beta_{\boldsymbol{\ell}} = \#\{C \in \mathcal{I}_{\boldsymbol{\ell}} : \#\{\ell_i, i \in C\} > 1\}$. Let $q_0 \in \{0,\ldots,q\}$. Since $\sup_{k>m_r}\sum_\ell \frac{\Phi(\ell/\sqrt{k})}{\sqrt{k}} < \infty$, the contribution of terms with $\#\mathcal{I}_{\boldsymbol{\ell}} = q_0 + 1$ and $\beta_\ell = \beta$ is bounded from above by

$$C\mu(B_r^X(x))^q \nu(B_r^Y(y))^{q_0} \nu\left(\{\tau_r^g < m_r^{3^q}\}\middle|\, B_r^Y(y)\right)^\beta \left(\Phi(0)\sum_{k=m_r}^{n_r} k^{-\frac{1}{2}}\right)^{q-q_0} n_r^{q_0} m_r^{3^m 1_{\beta\neq 0}}$$
$$\leq C'(\mu(B_r^X(x))^2)^{\frac{q-q_0}{2}} (\mu(B_r^X(x))(\nu(B_r^Y(y)))^{q_0} \nu\left(\{\tau_r^g < m_r^{3^q}\}\middle|\, B_r^Y(y)\right)^\beta n_r^{\frac{q-q_0}{2}+q_0} m_r^{3^m 1_{\beta\neq 0}}$$
$$\leq C' m_r^{3^m 1_{\beta\neq 0}} \nu\left(\{\tau_r^g < m_r^{3^q}\}\middle|\, B_r^Y(y)\right)^\beta,$$

due to the definition of $n_r = n_r(x,y)$. Thus the terms with $\beta_{\boldsymbol{\ell}} \geq 1$ contribute to the error term (15).

We now consider the terms corresponding to $\beta_{\boldsymbol{\ell}} = 0$, that is we are led to the study of

$$\mathcal{M}_{\mathbf{t}}^{\overline{\mathbf{m}}}(x,y) := \sum_{\substack{\mathbf{q}=(q_1,\dots,q_K)\\ q_v=1,\dots,m'_v}} \left(\prod_{v=1}^{K} S(m'_v,q_v) q_v! \right) A'_{n_r;\mathbf{q}}(x,y)$$

where

$$A'_{n_r;\mathbf{q}}(x,y) := \sum_{\boldsymbol{\ell}\,:\,\beta_{\boldsymbol{\ell}}=0} \mu(B_r^X(x))^q (\nu(B_r^Y(y)))^{\#\mathcal{I}_\ell - 1} D_{\boldsymbol{\ell}}(\mathbf{q})\,, \tag{38}$$

as $n_r/m_r^2 \to +\infty$, i.e. as $\mu(B_r^X(x))/m_r \to +\infty$ and $\mu(B_r^X(x))\nu(B_r^Y(y))/(m_r)^2 \to +\infty$, with

$$D_\ell(\mathbf{q}) := \sum_{\mathbf{k}:k'_j>m_r} \prod_{j=1}^{q} \frac{1}{\sqrt{k'_j}} (\Phi\, 1_{[-c_r,c_r]}) \left(\frac{\ell'_j}{\sqrt{k'_j}} \right),$$

the sum over $\mathbf{k}$ being still constrained by the q_v's as in (24).

At this step, we can point out the two exteme cases:

(A) if $\mu(B_r^X(x)) = o(\nu(B_r^Y(y)))$, then the terms with $q_0 < q$ (i.e. $\#\mathcal{I}_{\boldsymbol{\ell}} < q+1$) are negligeable.
(B) if $\nu(B_r^X(x)) = o(\mu(B_r^Y(y)))$, then the terms with $q_0 > 0$ are negligeable, hence the remaining term is $q_0 = 0$ and $\beta_{\boldsymbol{\ell}} = 0$ thus $\ell_j = 0$ for all j.

Let us start with the study of the case $q_0 = q$ and $\beta_{\boldsymbol{\ell}} = 0$, i.e. $|\ell_j - \ell_{j'}| > m_r$ (the dominating term in Case (A) above). The contribution of these terms is

$$\begin{aligned}
&(\rho(B_r^Z(x,y)))^q \sum_{\mathbf{k}} \sum_{\boldsymbol{\ell}} \prod_{j=1}^{q} \frac{\Phi\left(\frac{\ell'_j}{\sqrt{k'_j}}\right)}{\sqrt{k'_j}} 1_{\{k'_j>m_r\}} 1_{\{|\ell'_j|\leq c_r\sqrt{k'_j}\}} \\
&\sim (\rho(B_r^Z(x,y)))^q \sum_{\mathbf{k}} \prod_{j=1}^{q} \frac{\sum_{\ell'_j} \Phi\left(\frac{\ell'_j}{\sqrt{k'_j}}\right)}{\sqrt{k'_j}} \\
&\sim (\rho(B_r^Z(x,y)))^q \sum_{\mathbf{k}} \prod_{j=1}^{q} \int_{\mathbb{R}} \Phi(s)\, ds \\
&\sim (\rho(B_r^Z(x,y)))^q \#\{\mathbf{k}\} \\
&\sim (\rho(B_r^Z(x,y)) n_r)^q \prod_{v=1}^{K} \frac{(t_v - t_{v-1})^{q_v}}{q_v!}\,.
\end{aligned}$$

A careful analysis would have shown that this equivalence is an equality up to a multiplicative factor $1 + O(e^{-\frac{c_r^2}{2}}) + O({m_r}^{-\frac{1}{2}}) = 1 + O(|\log r|^{-\frac{1}{2}})$. Furthermore, we recognize the distribution of a standard Poisson process.

Second, we study the contribution of terms of (38) such that $q_0 = 0$ and $\beta_{\boldsymbol{\ell}} = 0$, i.e. such that $\ell_j = 0$ for all j. This contribution is

$$\mu(B_r^X(x))^q \sum_{\mathbf{k}} \frac{(\Phi(0))^q}{\prod_{j=1}^q \sqrt{k'_j}} \sim (\sqrt{n_r}\mu(B_r^X(x))\Phi(0))^q \int_{\mathcal{E}} \prod_{j=1}^q (s'_j)^{-\frac{1}{2}}\, ds$$

$$\sim (\sqrt{n_r}\mu(B_r^X(x)))^q \mathbb{E}_\mu \left[\prod_{v=1}^K \frac{(L_{t_v}(0) - L_{t_{v-1}}(0))^{q_v}}{q_v!} \right], \qquad (39)$$

where $\mathcal{E}$ is the set of $x \in \mathbb{R}^q$ obtained by concatenation of $(x_1^1, ..., x_{q_1}^1)$,..., $(x_1^v, ..., x_{q_v}^v)$,..., $(x_1^K, ..., x_{q_K}^K)$ such that $t_{v-1} < x_1^v < ... < x_{q_v}^v < t_v$, and where the last identification follows e.g. from [14, Proposition B.1.2].

We return to the general case. Given $q_0 \in \{0, ..., q\}$, we study of the asymptotic as $n_r/m_r^2 \to +\infty$ of the terms of (38) with $\#\mathcal{I}_\ell = q_0 + 1$ and $\beta_\ell = 0$ (i.e. clusters in ℓ corresponds to repetitions of a same value).

Setting $J_{q\to q_0}$ for the set of surjections $\psi : \{0, ..., q\} \to \{0, ..., q_0\}$ such that $\psi(0) = 0$, we observe that

$$\sum_{\ell\,:\,\#\mathcal{I}_\ell = q_0+1,\, \beta_\ell = 0} D_\ell(\mathbf{q}) = \frac{1}{q_0!} \sum_{\psi \in J_{q\to q_0}} \sum_{\mathbf{k}: k'_i > m_r} \sum_{(\ell_v)_{v=1,...,q_0} : |\ell_v - \ell_{v'}| > m_r} \prod_{j=1}^q \frac{(\Phi\, 1_{[-c_r, c_r]}) \left(\frac{\ell_{\psi(j)} - \ell_{\psi(j-1)}}{\sqrt{k'_j}} \right)}{\sqrt{k'_j}}$$

$$\sim \frac{1}{q_0!} \int_{n_r\mathcal{E}} \left(\int_{\mathbb{R}^{q_0}} \prod_{j=1}^q \frac{\Phi\left(\frac{w_{\psi(j)} - w_{\psi(j-1)}}{\sqrt{s'_j}} \right)}{\sqrt{s'_j}}\, dw \right) ds$$

where we set $\mathcal{E}$ the set of $s \in \mathbb{R}^q$ obtained by concatenation of $(s_1^1, ..., s_{q_1}^1)$,..., $(s_1^v, ..., s_{q_v}^v)$,..., $(s_1^K, ..., s_{q_K}^K)$ such that $t_{v-1} < s_1^v < ... < s_{q_v}^v < t_v$, and using again the notation $s'_j := s_j - s_{j-1}$ with the conventions $s_0 := 0$ and $w_0 = 0$. It follows that, if $n_r/m_r^2 \to +\infty$,

$$\sum_{\ell\,:\,\#\mathcal{I}_\ell = q_0+1 \beta_\ell = 0} D_\ell(\mathbf{q}) \sim \sum_{\psi \in J_{q\to q_0}} \frac{n_r^{\frac{q}{2}}}{q_0!} \int_{\mathcal{E}} \left(\int_{\mathbb{R}^{q_0}} \prod_{j=1}^q \frac{\Phi\left(\frac{w_{\psi(j)} - w_{\psi(j-1)}}{\sqrt{n_r s'_j}} \right)}{\sqrt{s'_j}}\, dw \right) ds$$

$$\frac{n_r^{\frac{q+q_0}{2}}}{q_0!} \sum_{\psi \in J_{q\to q_0}} \int_{\mathcal{E}} \left(\int_{\mathbb{R}^{q_0}} \prod_{j=1}^q \frac{\Phi\left(\frac{w_{\psi(j)} - w_{\psi(j-1)}}{\sqrt{s'_j}} \right)}{\sqrt{s'_j}}\, dw \right) ds$$

$$\sim \frac{n_r^{\frac{q+q_0}{2}}}{q_0!} \sum_{\psi \in J_{q\to q_0}} \int_{\mathcal{E}} \left(\int_{\mathbb{R}^{q_0}} \phi_{q,s_1,...,s_q}\left((w_{\psi(j)})_j \right)\, dw_1 ... dw_{q_0} \right) ds_1 ... ds_q\,,$$

where we set $\phi_{q,s_1,...,s_q}$ for the density function of $(B_{s_1}, ..., B_{s_q})$.

On $\mathcal{E}$ the s_j's are in increasing order. For a.e. $s \in \prod_{v=1}^K (t_{v-1}, t_v]^{q_v} =: \overline{\mathcal{E}}$ there exists a unique permutation π, preserving the K blocks, such that $(s_{\pi(j)})_j \in \mathcal{E}$. Applying this change of variables is balanced by substituting ψ by $\psi' = \psi \circ (0 \mapsto 0, \pi)$. Thus the above

quantity is

$$= \frac{n_r^{\frac{q+q_0}{2}}}{q_0!} \sum_{\psi\in J_{q\to q_0}} \left(\prod_{v=1}^{K}\frac{1}{q_v!}\right) \int_{\mathbb{R}^{q_0}} \left(\int_{\overline{\mathcal{E}}} \phi_{q,s_1,\dots,s_q}\left((w_{\psi(j)})_j\right)\, ds_1...ds_q\right) dw_1...dw_{q_0}$$

$$\sim \frac{n_r^{\frac{q+q_0}{2}}}{q_0!} \sum_{\psi\in J_{q\to q_0}} \left(\prod_{v=1}^{K}\frac{1}{q_v!}\right) \int_{\mathbb{R}^{q_0}} \mathbb{E}\left[\prod_{j=1}^{q}(L_{t_{v_q(j)}} - L_{t_{v_q(j)-1}})(w_{\psi(j)})\right] dw_1...dw_{q_0}\,,$$

where $v_q(j)$ is the smallest integer v such that $j \le q_1 + \cdots + q_v$. So

$$\sum_{\ell\,:\,\#\mathcal{I}_\ell = q_0+1, \beta_\ell = 0} D_\ell(\mathbf{q}) \sim n_r^{\frac{q+q_0}{2}} J_r(\mathbf{q}, q_0)\,,$$

with

$$J_r(\mathbf{q}, q_0) := \sum_{\psi\in J_{q\to q_0}} \left(\prod_{v=0}^{K}\frac{1}{q_v!}\right) \mathbb{E}\left[\int_{\mathbb{R}^{q_0}} \prod_{j=1}^{q}(L_{t_{v_q(j)}} - L_{t_{v_q(j)-1}})(w_{\psi(j)})\, dw_1...dw_{q_0}\right].$$

In view of (23) and (38), we study

$$\mathcal{M}_{\mathbf{t}}^{\overline{\mathbf{m}}}(x,y) \sim \sum_{\substack{\mathbf{q}=(q_1,\dots,q_K)\\ q_v=1,\dots,m'_v}} \left(\prod_{v=1}^{K} S(m'_v, q_v) q_v!\right) A'_{n_r;\mathbf{q}}(x,y)\,.$$

As $n_r/m_r^2 \to +\infty$, this quantity is equivalent to

$$\sum_{\substack{\mathbf{q}=(q_1,\dots,q_K)\\ q_v=1,\dots,m'_v}} \left(\prod_{v=1}^{K} S(m'_v, q_v)\right) \sum_{q_0=0}^{q} \left((n_r\mu(B_r^X(x))^2\right)^{\frac{q}{2}} \left(n_r\nu(B_r^Y(y)^2)\right)^{\frac{q_0}{2}} \times$$

$$\times \sum_{\psi\in J_{q\to q_0}} \frac{1}{q_0!}\mathbb{E}\left[\int_{\mathbb{R}^{q_0}} \prod_{j=1}^{q}(L_{t_{v_q(j)}} - L_{t_{v_q(j)-1}})(w_{\psi(j)})\, dw_1...dw_{q_0}\right].$$

We then conclude by Lemmas A.1 and A.3.

□

5. Study for the $\mathbb{Z}$-extension

This section is devoted to the proof of Theorem 3.10 about the convergence in distribution, as $r \to 0$, of the return time point process $\widetilde{\mathcal{N}}_r$ defined in (20) of the $\mathbb{Z}$-extension $\widetilde{F}$. The proof of the next result appears as an easy consequence of parts of the proof of Theorem 3.3.

Theorem 5.1. *Assume Hypothesis 3.2 except (II). Let K be a positive integer and $\overline{\mathbf{m}} = (\overline{m}_1, ..., \overline{m}_K)$ be a K-uple of positive integers and let $(t_0 = 0, t_1, ..., t_K)$ be an increasing collection of nonegative real numbers. There exist a constant $C > 0$ and a continuous function ε_1 vanishing at 0 such that, for all $x \in X$,*

$$\left|\mathbb{E}_\mu\left[\prod_{v=1}^{K}\widetilde{\mathcal{N}}_r(]t_{v-1}, t_v])^{\overline{m}_v}\middle| B_r^X(x)\right] - \mathbb{E}\left[\prod_{v=1}^{K}(\mathcal{Z}_{1,0}(t_v) - \mathcal{Z}_{1,0}(t_{v-1}))^{\overline{m}_v}\right]\right|$$

$$\le \varepsilon_1(\mu(B_r^X(x)))+,+C|\log r|^{3^m+m}\mu\left(\tau^f_{B_r^X(x)} \le |\log r|^{3^m}\middle| B_r^X(x)\right), \qquad (40)$$

with $m = |\overline{\mathbf{m}}| = \overline{m}_1 + \cdots + \overline{m}_K$, and with $\mathcal{Z}_{1,0}$ as in Remark 3.4.

Proof. Setting this time $n_r := (\mu(B_r^X(x)))^{-2}$, we follow the scheme of the proof of Theorem 3.3 with some simplifications coming from the fact that first $\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d}$ therein is replaced by $\boldsymbol{\ell} = \mathbf{0}$ and second that $B_r^Y(y)$ disappears. As in Step 1 of the proof of Theorem 3.3, we observe that

$$\mathbb{E}_\mu\left[\prod_{v=1}^K \widetilde{\mathcal{N}}_r(]t_{v-1},t_v])^{\overline{m}_v} 1_{B_r^X(x)}\right] = \sum_{\substack{\mathbf{q}=(q_1,\ldots,q_K)\\ q_v=1,\ldots,\overline{m}_v}} \left(\prod_{v=1}^K S(\overline{m}_v,q_v)q_v!\right) A_{n_r;\mathbf{q}}(x,y)\,,$$

where we denote again $S(m,k)$ for the Stirling number of the second kind (i.e. for the number of partitions with k atoms of a set of m elements), and where we set this time

$$A_{n_r;\mathbf{q}}(x) := \sum_{\mathbf{k}=(k_1,\ldots,k_q)} \mathbb{E}_\mu\left[1_{B_r^X(x)}\prod_{j=1}^q \left(1_{\{h_{k_j}=0\}}\, 1_{B_r^X(x)}\circ f^{k_j}\right)\right] = \sum_{\mathbf{k}} \mathbb{E}_\mu\left[Q^{(x)}_{k'_q,\ell'_q}\cdots Q^{(x)}_{k'_1,\ell'_1}(1_{B_r^X(x)})\right]\,, \tag{41}$$

with the notations

$$q := q_1 + \ldots + q_K, \quad k'_i := k_i - k_{i-1}, \quad k_0 = 0\,, \quad Q^{(x)}_{k,\ell}(H) := 1_{B_r^X(x)} P^k\left(1_{\{h_k=\ell\}}\, H\right)\,,$$

and where the sum over the $\mathbf{k} = (k_1,\ldots,k_q)$ corresponds to concatenation of $(k_1^1,\ldots,k_{q_1}^1)$,...,$(k_1^v,\ldots,k_{q_v}^v)$,..., $(k_1^K,\ldots,k_{q_K}^K)$ such that

$$t_{v-1}n_r < k_1^v < \ldots < k_{q_v}^v \le t_v n_r\,.$$

In Step 2 of the proof of Theorem 3.3, since $\boldsymbol{\ell} = \mathbf{0}$, $\mathcal{I}^* = \emptyset$ and $p = 0$, (26) and (35) ensure that the contribution of clusters of the m_r-clusters of k_j's with $\mathcal{J}_{\mathbf{k}} = \mathcal{J}$ is

$$\begin{aligned}
&m_r^{3^q}\sum_{\overline{\mathbf{k}}}\sum_{\boldsymbol{\ell}\in\mathbb{Z}^w} \mathbb{E}_\mu\left[\widetilde{Q}^{(x)}_{w,\bar{k}'_w,\ell'_w}\circ\cdots\circ\widetilde{Q}^{(x)}_{1,\bar{k}'_1,\ell'_1}(1_{G_{r,0}(x)})\right]\\
&\le C m_r^{3^w}\left(\prod_{i=0}^w \mu(G_{r,i})\right)(m_r\sqrt{n_r})^w\\
&\le C m_r^{3^w} m_r^q \mu(B_r^X(x))\left(\sqrt{n_r}\mu(B_r^X(x))\right)^w \mu\left(\tau^f_{B_r^X(x)} \le m_r^{3^q}|B_r^X(x)\right)\\
&\le C\mu(B_r^X(x)) m_r^{3^q+q}\mu\left(\tau^f_{B_r^X(x)} \le m_r^{3^q+q}|B_r^X(x)\right)\,.
\end{aligned}$$

Step 3 of the proof of Theorem 3.3 disappears (as well as the first part of (16)) since $\ell'_j = 0$ for all j.

As in Step 4 of the proof of Theorem 3.3, it follows from Lemma 4.3 that

$$\begin{aligned}
\frac{A_{n_r,\mathbf{q}}(x)}{\mu(B_r^X(x))} &= \mu(B_r^X(x))^q \sum_{\mathbf{k}}\left(\left(\prod_{j=1}^q \frac{\Phi(0)}{\sqrt{k'_j}}\right) + \mathcal{O}\left(\sum_{i'=1}^q \frac{1}{k'_{i'}}\prod_{j\neq i'}\frac{1}{\sqrt{k'_j}}\right)\right)\\
&= \frac{1}{\sqrt{n_r}^q}\sum_{\mathbf{k}}\prod_{j=1}^q \frac{\Phi(0)}{\sqrt{k'_j}} + \mathcal{O}\left(n_r^{-\frac{1}{2}}\log(n_r)\right)
\end{aligned}$$

uniformly in $x\in X$, and we conclude by using (39) (note that we do not need anymore the control of the probability that τ_r^g is small used in Step 5 of the proof of Theorem 3.3, so that the second part of (15) does not appear here). □

Proof of Theorem 3.10. This result follows from Theorem 5.1 as Corollary (3.5) and Theorem 3.6 follow from Theorem 3.3. □

Appendix A. Moments of integrals with respect to a Poisson process

The main goal of this appendix is to prove that for any $\alpha, \beta \in [0;1]$ such that $\max(\alpha,\beta) = 1$, for all K, $m'_1, ..., m'_K$ with $m = m'_1 + ... + m'_K$, $t_1 < ... < t_K$,

$$\begin{aligned} &\mathbb{E}\left[\prod_{v=1}^{K} \left(\mathcal{Z}_{\alpha,\beta}(t_v) - \mathcal{Z}_{\alpha,\beta}(t_{v-1})\right)^{m'_v}\right] \qquad (42)\\ &= \sum_{q_0=0}^{m} \frac{1}{q_0!} \sum_{\substack{\mathbf{q}=(q_1,...,q_K)\\ q_v=1,...,m'_v}} \left(\prod_{v=1}^{q_1+...+q_K} S(m'_v, q_v)\right) \sum_{\psi \in J_{q_1+...+q_K \to q_0}} \alpha^{\frac{q-q_0}{2}} \beta^{q_0}\\ &\quad \mathbb{E}\left[\int_{\mathbb{R}^{q_0}} \prod_{j=1}^{q} (L_{t_{v_q(j)}}(s_{\psi(j)}) - L_{t_{v_q(j)-1}}(s_{\psi(j)}))\, ds_1 ... ds_{q_0}\right], \end{aligned}$$

keeping the notations $S(m,q)$, $v_q(j)$ and $J_{q\to q_0}$ introduced above in the proof of Theorem 3.3. The case $\alpha = 0$ will follow from Lemma A.1, whereas the case $\alpha > 0$ will be studied in Lemma A.3 (applied with $a := \sqrt{\alpha}$ and $b := \beta/\sqrt{\alpha}$).

Lemma A.1. *For any nonnegative integer m, the moment of order m of a Poisson random variable $\mathcal{P}_\lambda$ of intensity $\lambda > 0$ is*

$$\mathbb{E}[\mathcal{P}_\lambda^m] = \sum_{q=0}^{m} S(m,q)\lambda^q\,. \qquad (43)$$

Proof. Recall that $S(m,q)$ is the number of partitions of a set of m elements in q nonempty subsets. The proof of Lemma A.1 is standard and follows e.g. from the following computation

$$\begin{aligned} \mathbb{E}[e^{t\mathcal{P}_\lambda}] = e^{\lambda(e^t-1)} &= \sum_{q\geq 0} \frac{(\lambda(e^t-1))^q}{q!} = e^{\lambda(e^t-1)} = \sum_{q\geq 0} \frac{\left(\lambda \sum_{m\geq 1} \frac{t^m}{m!}\right)^q}{q!}\\ &= \sum_{m\geq 0} \left(\sum_{q=0}^{m} S(m,q)\lambda^q\right) \frac{t^m}{m!}\,. \end{aligned}$$

□

Lemma A.2. *Let $\mathcal{P}$ be a Poisson process with intensity η on $\mathbb{R}$ and g_j, $j = 1..m$, be bounded integrable functions from $\mathbb{R}$ to $\mathbb{R}$. Then*

$$\mathbb{E}\left[\prod_{j=1}^{m} \int_{\mathbb{R}} g_j(s)\, d\mathcal{P}(s)\right] = \sum_{q=1}^{m} \frac{1}{q!} \sum_{p_i\geq 1\,:\, p_1+...+p_q=m} \sum_{\chi} \int_{\mathbb{R}^q} \prod_{j=1}^{m} g_j(s_{\chi(j)})\, d\eta(s_1) ... d\eta(s_q)\,,$$

where the last sum is taken over the set of maps $\chi : \{1,...,m\} \to \{1,...,q\}$ such that $\#\chi^{-1}(\{j\}) = p_j$.

Proof. We first claim that

$$\mathbb{E}\left[\left(\int_{\mathbb{R}} g(s)\, d\mathcal{P}(s)\right)^m\right] = \sum_{q=1}^{m} \frac{1}{q!} \sum_{p_i\geq 1\,:\, p_1+\ldots+p_q=m} \frac{m!}{\prod_{j=1}^{q}(p_j!)} \int_{\mathbb{R}^q} \prod_{j=1}^{q} (g(s_j))^{p_j}\, d\eta(s_1)...d\eta(s_q)\,.$$

Indeed, using the functional Fourier transform of a Poisson measure [4]

$$\begin{aligned}
\mathbb{E}\left[e^{i\theta\int_{\mathbb{R}} g(s)\, d\mathcal{P}(s)}\right] &= \exp\left(\int_{\mathbb{R}} (e^{i\theta g(s)}-1)\, d\eta(s)\right)\\
&= 1+\sum_{q\geq 1} \frac{\left(\int_{\mathbb{R}}(e^{i\theta g(s)}-1)\, d\eta(s)\right)^q}{q!}\\
&= 1+\sum_{q\geq 1} \frac{\int_{\mathbb{R}^q}(\prod_{i=1}^{q}(e^{i\theta g(s_i)}-1)\, d\eta(s_1)...d\eta(s_q)}{q!}\\
&= 1+\sum_{q\geq 1} \int_{\mathbb{R}^q} \sum_{p_1\geq 1} ... \sum_{p_q\geq 1} \prod_{j=1}^{q} \frac{(i\theta g(s_j))^{p_j}}{p_j!}\, d\eta(s_1)...d\eta(s_q)\\
&= 1+\sum_{m\geq 1} \frac{(i\theta)^m}{m!} \sum_{q=1}^{m} \sum_{p_i\geq 1\,:\, p_1+\ldots+p_q=m} \frac{m!}{\prod_{j=1}^{q}(p_j!)} \int_{\mathbb{R}^q} \prod_{j=1}^{q} (g(s_j))^{p_j}\, d\eta(s_1)...d\eta(s_q).
\end{aligned}$$

This proves the claim by expanding the exponential in $\mathbb{E}\left[e^{i\theta\int_{\mathbb{R}} g(s)\, d\mathcal{P}(s)}\right]$ and identifying the m-th coefficient.

The lemma follows by identification of the coefficients of $t_1\cdots t_m$ in the following identity, obtained with the claim applied to $\sum_i t_i g_i$ and by direct computation:

$$\begin{aligned}
&\sum_{j_1,\ldots,j_m=1}^{m} \left(\prod_{u=1}^{m} t_{j_u}\right) \mathbb{E}\left[\prod_{u=1}^{m}\int_{\mathbb{R}} g_{j_u}(s)\, d\mathcal{P}(s)\right] = \mathbb{E}\left[\left(\int_{\mathbb{R}} \sum_{i=1}^{m} t_i g_i(s)\, d\mathcal{P}(s)\right)^m\right]\\
&= \sum_{q\geq 1} \sum_{p_i\geq 1\,:\, p_1+\ldots+p_q=m} \frac{m!}{q!\prod_{j=1}^{q}(p_j!)} \int_{\mathbb{R}^q} \prod_{j=1}^{q} \left(\sum_{i=1}^{m} t_i g_i(s_j)\right)^{p_j} d\eta(s_1)...d\eta(s_q)\,.
\end{aligned}$$

□

Lemma A.3. *Let $\mathcal{B}$ be a Brownian motion of variance σ^2 and $(L_t(\cdot))_t$ its local time. Let $\mathcal{P}$ be a two-sided Poisson process with intensity $b>0$ and let $(\mathcal{P}'_s)_{s\in\mathbb{R}}$ be a family of independent homogeneous Poisson processes with intensity $a>0$. We assume that $\mathcal{P}$, $\mathcal{B}$ and $(\mathcal{P}'_s)$ are mutually independent. Let $\overline{\mathbf{m}} = (\overline{m}_1,\ldots,\overline{m}_K)$ be a K-uple of positive integers and $\mathbf{t} = (t_1,\ldots,t_K)$ be a K-uple of positive real numbers as in Theorem 3.3. Then*

$$\begin{aligned}
\overline{\mathcal{M}}(\mathbf{t},\overline{\mathbf{m}}) :=& \mathbb{E}\left[\prod_{v=1}^{K}\left(\int_{\mathbb{R}} \mathcal{P}'_s(L_{t_v}(s)) - \mathcal{P}'_s(L_{t_{v-1}}(s))\, d(\delta_0+\mathcal{P})(s)\right)^{\overline{m}_v}\right]\\
&= \sum_{q_0=0}^{m} \frac{1}{q_0!} \sum_{\substack{\mathbf{q}=(q_1,\ldots,q_K)\\ q_v=1,\ldots,\overline{m}_v}} \left(\prod_{v=1}^{q} S(\overline{m}_v,q_v)\right) \sum_{\psi\in J_{q\to q_0}} a^q b^{q_0}\\
&\qquad \mathbb{E}\left[\int_{\mathbb{R}^{q_0}} \prod_{j=1}^{q} (L_{t_{v_q(j)}}(s_{\psi(j)}) - L_{t_{v_q(j)-1}}(s_{\psi(j)}))\, ds_1...ds_{q_0}\right],
\end{aligned}$$

with the convention $s_0 = 0$, $q = q_1 + \cdots + q_K$, $J_{q\to q_0}$ denotes the set of surjections from $\{1,\dots,q\}$ to $\{0,\dots,q_0\}$, $v_q(j)$ is the smallest integer v such that $j \le q_1 + \cdots + q_v$ and $m = \overline{m}_1 + \cdots + \overline{m}_K$.

Proof. Take $g_j(s) = \mathcal{P}'_s(L_{t_{v(j)}}(s)) - \mathcal{P}'_s(L_{t_{v(j)-1}}(s))$. Expanding the product of the sum then using Lemma A.2 applied to $(g_j)_{j\notin I_0}$ and $E(\cdot|\mathcal{B},(\mathcal{P}'_s))$ give

$$\begin{aligned}
\overline{\mathcal{M}}(\mathbf{t},\overline{\mathbf{m}}) =& \mathbb{E}\left[\prod_{j=1}^{m}\int_{\mathbb{R}} g_j(s)\, d(\delta_0+\mathcal{P})(s)\right]\\
=& \sum_{I_0\subset\{1,\dots,m\}} \mathbb{E}\left[\prod_{j_0\in I_0} g_{j_0}(0) \prod_{j\in\{1,\dots,m\}\setminus I_0}\int_{\mathbb{R}} g_j(s)\, d\mathcal{P}(s)\right]\\
=& \sum_{p_0=0}^{m}\sum_{q_0=0}^{m-p_0}\sum_{p_i\ge 1\,:\,p_1\dots+p_{q_0}=m-p_0} \frac{1}{q_0!}\sum_{\chi}\int_{\mathbb{R}^{q_0}} \mathbb{E}\left[\prod_{j=1}^{m} g_j(s_{\chi(j)})\right] b^{q_0} ds_1\dots ds_{q_0}\,,
\end{aligned}$$

with the convention $s_0 = 0$ and where the last sum is taken over the set of maps $\chi : \{1,\dots,m\} \to \{0,\dots,q_0\}$ such that $\#\chi^{-1}(\{u\}) = p_u$ for $u = 0,\dots,q_0$. Since the $\mathcal{P}'_{s_u}$ are a.e independent conditionally to B, it follows that

$$\mathbb{E}\left[\prod_{j=1}^{m} g_j(s_{\chi(j)})\,\middle|\, B\right] = \prod_{u=0}^{q_0} K_u(s)$$

with

$$\begin{aligned}
K_u(s) :=& \mathbb{E}\left[\prod_{j\in\chi^{-1}(u)} (\mathcal{P}'_{s_u}(L_{t_{v(j)}(s_u)}) - \mathcal{P}'_{s_u}(L_{t_{v(j-1)}}(s_u))\,\middle|\, B\right]\\
=& \prod_{w=1}^{K} \mathbb{E}\left[(\mathcal{P}'_{s_u}(L_{t_v}(s_u) - L_{t_{v-1}}(s_u)))^{\#\{j\in\chi^{-1}(u):v(j)=w\}}\,\middle|\, B\right]\\
=& \prod_{w=1}^{K}\sum_{z_{u,w}=0}^{m^{\chi}_{u,w}} S(m^{\chi}_{u,w}, z_{u,w})(b.(L_{t_w}(s_u) - L_{t_{w-1}}(s_u)))^{z_{u,w}}\,.
\end{aligned}$$

with $m^{\chi}_{u,w} := \#\{j = 1,\dots,m\colon \chi(j) = u, v(j) = w\}$. So

$$\begin{aligned}
&\int_{\mathbb{R}^{q_0}} \mathbb{E}\left[\prod_{j=1}^{m} g_j(s_{\chi(j)})\right] ds_1\dots ds_{q_0}\,.\\
&= \sum_{D}\left(\prod_{u,w} S(m^{\chi}_{u,w}, z_{u,w})\right) a^{|D|} H(D)\,,
\end{aligned}$$

where the sum is over all the matrices $Z' = (z'_{u,w})_{u=0,\dots,q_0,w=1,\dots,K}$ such that $\forall u,w, 0 \le z_{u,w} \le m^{\chi}_{u,w}$ with $|Z'| = \sum_{u,w} z_{u,w}$ and

$$H(Z') := \mathbb{E}\left[\int_{\mathbb{R}^{q_0}}\prod_{u=0}^{q_0}\left(\prod_{w=1}^{K}(L_{t_w}(s_u) - L_{t_{w-1}}(s_u))^{z_{u,w}}\right) ds_1\dots ds_{q_0}\right]. \tag{44}$$

Hence, we have proved that

$$\begin{aligned}
&\overline{\mathcal{M}}(\mathbf{t}, \overline{\mathbf{m}}) = \\
&= \sum_{q_0=0}^{m} \frac{1}{q_0!} b^{q_0} \sum_{p_0 \geq 0, p_i \geq 1 \,:\, p_0+\ldots+p_{q_0}=m} \sum_{\chi} \sum_{Z'} \left(\prod_{u,w} S(m^{\chi}_{u,w}, z_{u,w}) \right) a^{|Z'|} H(Z') \\
&= \sum_{q_0=0}^{m} \frac{1}{q_0!} \sum_{\substack{\mathbf{q}=(q_1,\ldots,q_K) \\ q_v=1,\ldots,\overline{m}_v}} a^{q} b^{q_0} \sum_{Z'} H(Z') \left[\sum_{\chi} \left(\prod_{u,w} S(m^{\chi}_{u,w}, z_{u,w}) \right) \right],
\end{aligned}$$

where in the last line the matrices Z' are such that $\sum_{u=0}^{q_0} z_{u,w} = q_w$ for all $w = 1..K$, and the surjection χ is such that $m^{\chi}_{u,w} \geq z_{u,w}$.

Let Z' be such that $\prod_{u,w} S(m^{\chi}_{u,w}, z_{u,w}) \neq 0$. Then, for every u, w such that $m^{\chi}_{u,w} \geq 1$, we also have $z_{u,w} \geq 1$. This ensures that both $q_w := \sum_{u=0}^{q_0} z_{u,w}$ and $\sum_{w=1}^{K} z_{u,w}$ are non null.

Denote by $C(Z')$ the term into brackets above. We claim that $C(Z')$ is the number of colored partitions in the following sense: a partition of $\{1, \ldots, m\}$ which refines the partition in successive temporal blocks of size m'_v and to each element of the partition is assigned a value in $0..q_0$, with the constraint that for each u, w there are $z_{u,w}$ elements of the partition in the wth temporal block giving the value u.

Indeed, choosing χ assigns to each integer in $\{1, ..., m\}$ a unique value in $\{0, ..., q_0\}$. For each temporal block w and each value u, the set of integers j in the wth temporal block such that $\chi(j) = u$ has cardinality $m^{\chi}_{u,w}$. We partition it into $z_{u,w}$ atoms. There are $S(m^{\chi}_{u,w}, z_{u,w})$ possibilities. This defines uniquely a partition with the prescribed property, and there are $C(Z')$ possibilities.

Another method to construct such a colored partition is to first partition each temporal v block in q_v atoms. There are $S(m'_v, q_v)$ possibilities. This refined partition has q elements. Any surjection $\psi \in J_{q,q_0}$ assigns to the jth atom (ordered by their minimal element) of the partition a value $\psi(j)$. Let Z^{ψ} be the matrix with entries $z^{\psi}_{u,w}$ equal to the number of atoms in the wth temporal block with value u, that is the number of j such that $v_q(j) = w$ and $\psi(j) = u$. We restrict to those ψ such that $Z^{\psi} = Z'$. Note that there are $\prod_{v=1}^{K} S(\overline{m}_v, q_v) \#\{\psi \in J_{q,q_0} \colon Z^{\psi} = Z'\}$ possibilities. Therefore

$$\overline{\mathcal{M}}(\mathbf{t}, \overline{\mathbf{m}}) = \sum_{q_0=0}^{m} \frac{1}{q_0!} \sum_{\substack{\mathbf{q}=(q_1,\ldots,q_K) \\ q_v=1,\ldots,m'_v}} a^{q} b^{q_0} \left(\prod_{v=1}^{q} S(\overline{m}_v, q_v) \right) \sum_{Z'} \sum_{\psi \in J_{q \to q_0} \,:\, Z^{\psi}=Z'} H(Z').$$

□

Finally, we give for completeness a convergence result with the moments method under assumptions slightly weaker than usual. The subtlety is due to the fact that the moments converge in restriction to a good set that may depend on the exponent. Given $x \in \mathbb{R}^K$ and $m \in \mathbb{N}^K$ we let $x^m = \prod_{v=1}^{K} x_v^{m_v}$.

Lemma A.4. *Let W, X_n's be $\mathbb{R}^K$ valued random variables, such that*

- *There exists a sequence $(\Omega_n)_n$ of measurable sets such that $P(\Omega_n) \to 1$ as $n \to \infty$*
- *For every $m \in \mathbb{N}^K$, $E(X_n^m 1_{\Omega_n}) \to E(W^m)$ as $n \to \infty$*
- *W satisfies the Carleman's criterion $\sum_{m=0}^{\infty} E(\|W\|^m)^{-1/m} = \infty$.*

Then X_n converges in distribution to W.

Proof. It follows from the classical Carleman's criterion (see e.g. [18]) that the linear combinations of $(X_n 1_{\Omega_n})_n$ converge in distribution to those of W, which implies the convergence in distribution of $(X_n 1_{\Omega_n})_n$ to W. Furthermore $X_n - X_n 1_{\Omega_n}$ converges in distribution to 0. We conclude (using e.g. the Slutzky lemma) that $(X_n)_n$ converges in distribution to W. □

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References

[1] P. Billingsley, Convergence of probability measures, 2nd ed. (English) Wiley Series in Probability and Statistics. Chichester: Wiley. x, 277 p. (1999).
[2] X. Bressaud and R. Zweimüller, Non exponential law of entrance times in asymptotically rare events for intermittent maps with infinite invariant measure, Ann. I.H.P. Phys. Th. **2** (2001) 1–12
[3] A. Coutinho, J. Rousseau, B. Saussol, Large deviation for return times, Nonlinearity 31 , no. 11 (2018) 5162-5179.
[4] D.J. Daley, D. Vere-Jones, An Introduction to the Theory of Point Processes: Volume II: General Theory and Structure (Probability and Its Applications), 2007.
[5] S. Gouëzel, Variations around Eagleson's Theorem on mixing limit theorems for dynamical systems, Ergodic Theory and Dynamical Systems 40 (2020) 3368–3374.
[6] P. Jagers, Aspects of random measures and point processes. (English) Zbl 0333.60059 Advances Probab. related Topics 3, 179-239 (1974).
[7] S. A. Kalikow, T, T-1 Transformation is Not Loosely Bernoulli. Ann. Math. 115, (1982), no 2, 393–409.
[8] Valerio Lucarini, Davide Faranda, Ana Cristina Moreira Freitas, Jorge Milhazes Freitas, Mark Holland, Tobias Kuna, Matthew Nicol, Mike Todd, Sandro Vaienti, Extremes and Recurrence in Dynamical Systems, Wiley Interscience, 2016, Pure and Applied Mathematics : A Wiley Series of Texts, Monographs and Tracts.
[9] F. Pène, B. Saussol, Quantitative recurrence in two-dimensional extended processes, Ann. Instit. H. Poincaré - Proba. stat. **45**-4 (2009) 1065-1084.
[10] F. Pène, B. Saussol, Spatio-temporal Poisson Point Processes, in preparation.
[11] F. Pène, B. Saussol, R. Zweimüller, Recurrence rates and hitting-time distributions for random walks on the line, Annals of Probability 41 (2), 619–635 (2013).
[12] F. Pène, B. Saussol, R. Zweimüller, Return- and Hitting-time limits for rare events of null-recurrent Markov, Ergodic Theory and Dynamical Systems 37 (1), 244–276 (2017).
[13] F. Pène , D. Thomine, Potential kernel, hitting probabilities and distributional asymptotics, Ergodic Theroy and Dynamical Systems, 40 (2020), no 7 , 1894-1967.
[14] M. Phalempin, Théorèmes Limites en mesure infinie : auto-intersections et flots perturbés moyennés. PhD-Thesis (french), Université de Bretagne occidentale - Brest (2022), NNT : 2022BRES0057, tel-03881987.
[15] J. Pitman, M. Yor, Hitting, occupation and inverse local times of one-dimensional diffusions: martingale and excursion approaches, Bernoulli 9, no. 1 (2003) 1-24.
[16] S. I. Resnick, Extreme values, regular variation and point processes. Reprint of the 1987 original. New York, NY: Springer (2008).
[17] J. Rousseau, B. Saussol, Recurrence rate for observations, TAMS.
[18] K. Schmüdgen, The moment problem, Graduate Texts in Mathematics 277, Springer, xii, 535 p. (2017).
[19] B. Weiss, The isomorphism problem in ergodic theory. Bull. AMS 78, (1972), 668–684.
[20] N. Yassine, Quantitative recurrence of some dynamical systems with an infinite measure in dimension one, Discrete and Continuous Dynamical Systems-A 38 (2018) 343—361.
[21] N. Yassine, Quantitative properties of recurrence of some dynamical systems with an infinite measure, PhD Thesis (2018)

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