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# Generalization to differential-algebraic equations of Lyapunov-Schmidt type reduction at Hopf bifurcations

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## Abstract

The Lyapunov-Schmidt procedure, a well-known and powerful tool for the local reduction of nonlinear systems at bifurcation points or for ordinary differential equations (ODEs) at Hopf bifurcations, is extended to the context of strangeness-free differential-algebraic equations (DAEs), by generalizing the comprehensive presentation of the method for ODEs provided in the classical textbook by Golubitsky and Schaeffer [Applied mathematical sciences, **51**, Springer (1985)]. The appropriate setting in the context of DAEs at Hopf bifurcations is first detailed, introducing suitable operators and addressing the question of appropriate numerical algorithms for their construction as well. The different steps of the reduction procedure are carefully reinterpreted in the light of the DAE context and detailed formulas are provided for systematic and rational construction of the bifurcating local periodic solution, whose stability is shown, likely to the ODE context, to be predicted by the reduced equations. As an illustrative example, a classical DAE model for an electric power system is considered, exhibiting both supercritical and subcritical Hopf bifurcations, demonstrating the prediction capability of the reduced system with regard to the global dynamics.

*Keywords:* Differential-Algebraic Equations; Hopf bifurcation; Lyapunov-Schmidt Reduction

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## 1. Introduction

There are mainly two types of reduction procedures at bifurcation points of nonlinear dynamical systems, that is the celebrated center manifold reduction (for detailed presentations of this theory see for instance [1–3]) as well as the no less well known Lyapunov-Schmidt reduction. The mathematical framework for this latter method, which goes back to the early 20th

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century, has been reviewed in the near past by [4], while for instance in [5] the Lyapunov-Schmidt reduction and related formulas are developed for parameterized nonlinear systems as well as to Hopf bifurcations in autonomous ordinary differential equations (ODEs). This procedure appears to be particularly adapted when considering bifurcations of differential-algebraic equations (DAEs), given that it both applies to nonlinear algebraic equations and to dynamical systems.

Here DAEs of the form

$$\mathbf{M}(\mathbf{u}, \boldsymbol{\alpha}) \frac{d\mathbf{u}}{dt} + \mathbf{G}(\mathbf{u}, \boldsymbol{\alpha}) = 0$$

are considered with the solution  $\mathbf{u}(t) \in \mathbb{R}^n$ ,  $\mathbf{G}(\mathbf{u}, \boldsymbol{\alpha}) \in \mathbb{R}^n$  is a (nonlinear in general) function, the system is governed by a set of parameters  $\boldsymbol{\alpha}$  and  $\mathbf{M}(\mathbf{u}, \boldsymbol{\alpha}) \in \mathbb{R}^{n,n}$  is a  $n \times n$  matrix operator which does not admit an inverse (because if it were of rank  $n$ , one could recover a ODE-like system by inversion of this matrix). Such nonlinear systems naturally arise in engineering and science in general, for instance in classical mechanics where constrained motions of systems are ubiquitous, they occur for instance in chemical engineering due to process constraints, circuit theory models and power systems are classical examples for DAEs, they appear naturally in bioeconomic systems when for example economic profit is taken into account. The recent monograph [6] provides an exhaustive review on applied DAEs with numerous references therein.

The existence of Hopf bifurcation in such systems has found a lot of attention, for instance in [7] with application to circuit theory, in [8] considering a benchmark model of chemical engineering, for predator-prey models with nonlinear prey harvesting (e.g. [9] and references therein), and when DAE models for electric power systems are considered the question of local bifurcations has largely been addressed (see for instance [10–12]). In [7] a Hopf bifurcation theorem is assessed for DAEs such that the rank( $\mathbf{M}(\mathbf{u}, \boldsymbol{\alpha})$ ) =  $p < n$ , whereas in [13] the situation is considered where DAEs have singular solutions, that is the rank of the matrix varies with  $\mathbf{u}(t)$ .

The dynamic response to perturbations of ODEs and DAEs cannot always be reliably inferred from mere numerical integration of the global system and semi-analytic tools are still valuable, in particular to anticipate for instance the supercritical or subcritical behavior in the neighborhood of Hopf bifurcations. The center manifold theory as well as the Lyapunov-Schmidt reduction are such tools, providing a reduced equation from which the local behavior can be inferred and they have largely been applied to ODEs and in particular the center manifold theory (see for instance [14, 15], just to mention two recent publications).

The present work aims at providing a rigorously defined reduction tool for DAEs with a nonlinear algebraic part, by extending the well known Lyapunov-Schmidt reduction algorithm for ODEs at Hopf bifurcations. In Section 2 it is shown, which operators are conveniently to be introduced to assess a Hopf bifurcation by focusing on so-called strangeness-free DAEs. In Section 3 the theoretical framework for a Lyapunov-Schmidt reduction as exposed in [5] (chapter VIII) for ODEs is reinterpreted in the light of DAE systems and in Section 4 the formulas are derived to compute the expansion of the local bifurcating solution as well as the reduced equations. It is well known for ODEs that in the normal form associated to the reduced equation the sign of the coefficient of the cubic term in the amplitude indicates whether the local bifurcating solution is stable or unstable. The proof of this stability criterion will be readdressed in the DAE context. As case study, in Section 5 a power electric model is considered exhibiting both a supercritical and subcritical Hopf bifurcation. Some general considerations about the proposed local reduction approach are provided in the conclusion Section 6.

## 2. System setting and Hopf bifurcation

The DAE systems to be considered are

$$\mathbf{M}(\mathbf{u}, \boldsymbol{\alpha}) \frac{d\mathbf{u}}{dt} + \mathbf{G}(\mathbf{u}, \boldsymbol{\alpha}) = 0, \quad \mathbf{u}, \mathbf{G}(\mathbf{u}, \boldsymbol{\alpha}) \in \mathbb{R}^n, \quad \mathbf{M}(\mathbf{u}, \boldsymbol{\alpha}) \in \mathbb{R}^{n,n}, \quad (2.1)$$

with  $\boldsymbol{\alpha} \in \mathbb{R}^k$  a parameter vector, the matrix  $\mathbf{M}(\mathbf{u}, \boldsymbol{\alpha})$  being rank deficient. Such systems are called quasi-linear, given that the formal differential with respect to  $d\mathbf{u}/dt$  only depends on  $(\mathbf{u}, \boldsymbol{\alpha})$ . When addressing the existence and uniqueness of solutions of DAEs, the differentiation index (see for instance [16]) is often considered: the DAE is differentiated with respect to  $t$  and the system is inflated by the resulting equations. If  $d\mathbf{u}/dt$  may be uniquely determined by  $(t, \mathbf{u})$  and the augmented system, the DAE is said to have differentiation index one. If however one has to differentiate once again to still inflate the system, the index is two, so on and so forth. A more or less equivalent definition is the strangeness index and as discussed for instance in [17], strangeness index  $l$  would correspond to differentiation index  $l + 1$ . Here we suppose  $l = 0$ , that is the system is supposed to be strangeness-free. Index reduction methods for DAEs with differentiation index  $\geq 2$  are for instance discussed in [17], but it seems that those approaches are rather case-dependent, in particular in the presence of nonlinearities: therefore, whatever possible higher-index system is considered, it must be assured that it may be transformed to become strangeness-free.

As will be seen hereafter, to apply a Lyapunov-Schmidt reduction to the DAE

system and in contrast to the ODE case, rank and nullspace computations of square or rectangular linear operators are necessary. For the present work to be self-consistent, in Appendix A the classical  $LU$ -type decomposition, however with complete pivoting, is proposed as a reliable procedure for these computations. The starting point of the present analysis is a Hopf bifurcation occurring at a steady state solution  $\tilde{\mathbf{u}}$  for specific parameters  $\tilde{\boldsymbol{\alpha}}$ , that is

$$\mathbf{G}(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}}) = 0. \quad (2.2)$$

The matrix  $\mathbf{M}(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}})$  is supposed to have rank  $p = n - m$  (with  $m > 0$ ). Define  $\mathbf{Z}_2 \in \mathbb{R}^{n,m}$  such that its  $m$  columns form an orthonormal basis of the kernel of  $\mathbf{M}(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}})^T$ , the superscript  $T$  meaning transposition (see Appendix A for an algorithm to compute  $\mathbf{Z}_2$ ), that is

$$\mathbf{Z}_2^T \mathbf{M}(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}}) = 0. \quad (2.3)$$

Let  $D\mathbf{G}(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}})$  be the Jacobian matrix of  $\mathbf{G}$  (with respect to  $\mathbf{u}$ ). Some assumptions have to be made, the first one being

**Assumption 2.1.** *The rectangular operator*

$$\mathbf{Z}_2^T D\mathbf{G}(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}}) \in \mathbb{R}^{m,n}$$

*is supposed to be of maximum rank  $m$ .*

Its nullspace is represented by the matrix  $\mathbf{T} \in \mathbb{R}^{n,p}$  (recall that  $p = n - m$ ) and

$$\mathbf{Z}_2^T D\mathbf{G}(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}}) \mathbf{T} = 0. \quad (2.4)$$

The second assumption is

**Assumption 2.2.** *The operator*

$$\mathbf{M}(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}}) \mathbf{T} \in \mathbb{R}^{n,p}$$

*is of maximal rank  $r = p$ . Accordingly the nullspace of  $(\mathbf{M}(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}}) \mathbf{T})^T$  is of dimension  $m = n - p$  and hence represented by  $\mathbf{Z}_2 \in \mathbb{R}^{n,m}$  defined by (2.3).*

Now define  $\mathbf{Z}_1 \in \mathbb{R}^{n,p}$  such that its columns generate the subspace in  $\mathbb{R}^n$  orthogonal to the subspace generated by  $\mathbf{Z}_2$ . The column-vectors of  $\mathbf{Z}_1$  can be computed as the nullspace of  $\mathbf{Z}_2^T$  (followed by Gram-Schmidt orthonormalization) and accordingly the direct sum generates the entire vector space which is formally written as

$$\mathbf{Z}_1 \oplus \mathbf{Z}_2 = \mathbb{R}^n. \quad (2.5)$$

From Assumption 2.2 it follows that the square  $p \times p$  operator  $(\mathbf{M}(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}})\mathbf{T})^T \mathbf{Z}_1$  has maximal rank and hence its transposition too, that is the operator

$$\mathbf{B}_1 = \mathbf{Z}_1^T \mathbf{M}(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}})\mathbf{T} \in \mathbb{R}^{p,p} \quad (2.6)$$

admits an inverse. When the DAE is strangeness-free (and it is recalled that only those systems are considered in the present work) the two assumptions above are precisely fulfilled (see [17], chapter 4).

Now let  $\mathbf{x}(t)$  be the perturbation of the steady state, that is  $\mathbf{u}(t) = \tilde{\mathbf{u}} + \mathbf{x}(t)$ , for a deviation noted  $\lambda$  of one of the parameters, say  $\boldsymbol{\alpha} = \tilde{\boldsymbol{\alpha}} + \lambda \mathbf{e}_l$ . The general strangeness-free assumption implies that  $\mathbf{M}(\tilde{\mathbf{u}} + \mathbf{x}, \tilde{\boldsymbol{\alpha}} + \lambda \mathbf{e}_l)$  remains of rank  $p = n - m$ , but in the most general case  $\mathbf{Z}_2$  defined by (2.3) at the steady state could evolve with the perturbation. However, the local reduction procedure as exposed here is tractable only when this operator can be defined once for all and hence a further assumption has to be made.

**Assumption 2.3.** *It is supposed that  $\mathbf{Z}_2$  defined by (2.3) represents the kernel of  $\mathbf{M}(\tilde{\mathbf{u}} + \mathbf{x}, \tilde{\boldsymbol{\alpha}} + \lambda \mathbf{e}_l)^T$  in the neighborhood of the steady state, that is*

$$\mathbf{Z}_2^T \mathbf{M}(\tilde{\mathbf{u}} + \mathbf{x}, \tilde{\boldsymbol{\alpha}} + \lambda \mathbf{e}_l) = 0.$$

It is likely that this is fulfilled in many examples of DAE systems and of course always when the rank-deficient operator  $\mathbf{M}$  has constant coefficients. In the following the notations

$$\begin{aligned} \mathbf{M}_1(\mathbf{u}, \boldsymbol{\alpha}) &= \mathbf{Z}_1^T \mathbf{M}(\mathbf{u}, \boldsymbol{\alpha}), \quad \mathbf{G}_1(\mathbf{u}, \boldsymbol{\alpha}) = \mathbf{Z}_1^T \mathbf{G}(\mathbf{u}, \boldsymbol{\alpha}), \\ \mathbf{G}_2(\mathbf{u}, \boldsymbol{\alpha}) &= \mathbf{Z}_2^T \mathbf{G}(\mathbf{u}, \boldsymbol{\alpha}) \end{aligned} \quad (2.7)$$

are used. Also, for simplicity  $(0, 0)$  is written for the steady state  $(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}})$  and the perturbation  $(\mathbf{x}(t), \lambda)$  is solution of

$$\begin{aligned} \mathbf{M}_1(\mathbf{x}, \lambda) \frac{d\mathbf{x}}{dt} + \mathbf{G}_1(\mathbf{x}, \lambda) &= 0, \\ \mathbf{G}_2(\mathbf{x}, \lambda) &= 0. \end{aligned} \quad (2.8)$$

Writing  $D\mathbf{G}$  the Jacobian matrix of  $\mathbf{G}$ , the linearization of this system at the steady state is

$$\begin{aligned} \mathbf{M}_1(0, 0) \frac{d\mathbf{x}}{dt} + D\mathbf{G}_1(0, 0)\mathbf{x} &= 0, \\ D\mathbf{G}_2(0, 0)\mathbf{x} &= 0. \end{aligned} \quad (2.9)$$

According to the definition of  $\mathbf{T} \in \mathbb{R}^{n,p}$  (see (2.4)), the second equation implies that  $\mathbf{x}(t) = \mathbf{T}\mathbf{y}(t)$  with  $\mathbf{y}(t) \in \mathbb{R}^p$  and the linear stability of the steady state is governed by

$$\mathbf{B}_1 \frac{d\mathbf{y}}{dt} + D\mathbf{G}_1(0, 0)\mathbf{T}\mathbf{y} = 0$$

the matrix  $\mathbf{B}_1$  being defined by (2.6). Given that  $\mathbf{B}_1$  admits an inverse, one recovers a classical linear dynamical system

$$\frac{d\mathbf{y}}{dt} + \mathbf{A}\mathbf{y} = 0, \quad \mathbf{A} = \mathbf{B}_1^{-1}D\mathbf{G}_1(0,0)\mathbf{T}. \quad (2.10)$$

It should be remarked here that if the algebraic part of the system (2.8) is linear, it is solved once for all when writing  $\mathbf{x}(t) = \mathbf{T}\mathbf{y}(t)$  and with the help of the operator  $\mathbf{T}$  the DAE reduces to the dynamic part of (2.8). It is precisely for DAEs with a nonlinear algebraic part that the developments in [5] (chapter VIII) for ODEs do not extend straightforwardly. In that case and as will be seen in the next section, to generalize the Lyapunov-Schmidt reduction a supplementary operator  $\mathbf{S} \in \mathbb{R}^{n,m}$  is introduced such that its column vectors generate the vector space in  $\mathbb{R}^n$  orthogonal to that represented by  $\mathbf{T}$  and

$$\mathbf{T} \oplus \mathbf{S} = \mathbb{R}^n. \quad (2.11)$$

Again  $\mathbf{S}$  is obtained by computing the nullspace of  $\mathbf{T}^T$  together with orthonormalization. By construction the square operator

$$D\mathbf{G}_2(0,0)\mathbf{S} \in \mathbb{R}^{m,m} \quad (2.12)$$

admits an inverse. Given (2.11) the solution  $\mathbf{x}(t)$  of the nonlinear system (2.8) to be sought can be written

$$\mathbf{x}(t) = \mathbf{T}\mathbf{y}(t) + \mathbf{S}\mathbf{z}(t), \quad \mathbf{y}(t) \in \mathbb{R}^p, \mathbf{z}(t) \in \mathbb{R}^m. \quad (2.13)$$

Note that  $\mathbf{y}(t) = \mathbf{T}^T\mathbf{x}(t)$  and  $\mathbf{z}(t) = \mathbf{S}^T\mathbf{x}(t)$ , since  $\mathbf{T}^T\mathbf{S} = 0$  and  $\mathbf{T}^T\mathbf{T} = \mathbf{I}_p$ ,  $\mathbf{S}^T\mathbf{S} = \mathbf{I}_m$  (with the identity matrices  $p \times p$  and  $m \times m$  respectively).

It is now assumed that the steady state undergoes a Hopf bifurcation, that is the matrix  $\mathbf{A}$  defined by (2.10) admits a single pair of pure imaginary eigenvalues  $\pm i\omega$  and that the other eigenvalues have positive real parts. This is precisely the condition that at the steady state a Hopf bifurcation occurs. Note that for the present formulation of the system the eigenvalues with positive real parts correspond to the stable part of the eigenspectrum. The linear system (2.10) hence admits a  $2\pi/\omega$ -periodic solution  $\mathbf{y}(t)$  and  $\mathbf{x}(t) = \mathbf{T}\mathbf{y}(t)$  is the periodic solution of the linearized system (2.9).

In the next section the Lyapunov-Schmidt method as exposed in [5] for conventional dynamical systems will be generalized to the present DAE setting.

### 3. Lyapunov-Schmidt reduction for the DAE system

If a periodic solution exists for  $\lambda \neq 0$ , its period will change and as in [5] a parameter  $\tau$  and a scaled time  $s$  is introduced such that

$$s = \omega(1 + \tau)t, \quad (3.1)$$

that is the linear perturbation at the Hopf bifurcation is  $2\pi$ -periodic in the scaled time  $s$ . Given this time scaling,  $\mathbf{M}_1(\mathbf{u}, \boldsymbol{\alpha})$  in (2.7) is to be multiplied by  $\omega$  and we still write

$$\mathbf{M}_1 \quad \text{for} \quad \omega \mathbf{M}_1 \quad \text{and} \quad \mathbf{B}_1 \quad \text{for} \quad \omega \mathbf{B}_1.$$

Given the decomposition (2.13), the system

$$\mathbf{F}(\mathbf{y}, \mathbf{z}, \lambda, \tau) = (\mathbf{F}_1(\mathbf{y}, \mathbf{z}, \lambda, \tau), \mathbf{F}_2(\mathbf{y}, \mathbf{z}, \lambda)) = (0, 0) \quad (3.2)$$

is to be considered with  $\mathbf{F}_j$ ,  $j = 1, 2$  defined as

$$\begin{aligned} \mathbf{F}_1(\mathbf{y}, \mathbf{z}, \lambda, \tau) &= \mathbf{B}_1^{-1} \mathbf{M}_1(\mathbf{T}\mathbf{y} + \mathbf{S}\mathbf{z}, \lambda)(1 + \tau) \frac{d}{ds} (\mathbf{T}\mathbf{y} + \mathbf{S}\mathbf{z}) \\ &\quad + \mathbf{B}_1^{-1} \mathbf{G}_1(\mathbf{T}\mathbf{y} + \mathbf{S}\mathbf{z}, \lambda) \end{aligned} \quad (3.3)$$

$$\mathbf{F}_2(\mathbf{y}, \mathbf{z}, \lambda) = \mathbf{G}_2(\mathbf{T}\mathbf{y} + \mathbf{S}\mathbf{z}, \lambda). \quad (3.4)$$

The solution  $(\mathbf{y}(s), \mathbf{z}(s))$  of the system is sought as a continuous  $2\pi$ -periodic function from  $\mathbb{R}$  to  $\mathbb{R}^p \times \mathbb{R}^m$ . Let  $C_{2\pi}$  be the space of these functions and  $C_{2\pi}^1$  the space of those functions which are continuously differentiable and with these definitions

$$\mathbf{F} : C_{2\pi}^1 \times \mathbb{R}^2 \rightarrow C_{2\pi}$$

(the parameters  $(\lambda, \tau) \in \mathbb{R}^2$ ). The differentials of (3.3) and (3.4) with respect to  $\mathbf{y}$  and  $\mathbf{z}$  at  $(\mathbf{y}, \mathbf{z}, \lambda, \tau) = (0, 0, 0, 0)$  (in the following the argument  $(0, 0, 0, 0)$  standing for the steady state is omitted when writing the operators) are

$$D_{\mathbf{y}} \mathbf{F}_1 = \mathbf{L} = \frac{d}{ds} \mathbf{I} + \tilde{\mathbf{A}}, \quad (3.5)$$

where the  $p \times p$  identity matrix is written  $\mathbf{I}$  and  $\tilde{\mathbf{A}} = \frac{1}{\omega} \mathbf{A}$  with  $\mathbf{A}$  defined by (2.10),

$$\begin{aligned} D_{\mathbf{z}} \mathbf{F}_1 &= \mathbf{B}_1^{-1} \mathbf{M}_1 \mathbf{S} \frac{d}{ds} + \mathbf{B}_1^{-1} D\mathbf{G}_1 \mathbf{S}, \\ D_{\mathbf{y}} \mathbf{F}_2 &= D\mathbf{G}_2 \mathbf{T} = 0, \quad D_{\mathbf{z}} \mathbf{F}_2 = D\mathbf{G}_2 \mathbf{S} \end{aligned}$$

(it is recalled that  $\mathbf{T}$  represents the nullspace of  $D\mathbf{G}_2$ ). Note that  $\tilde{\mathbf{A}}$  in (3.5) by construction has  $\pm i$  as the pure imaginary eigenvalue pair, the other eigenvalues having a positive real part. Consequently, the differential operator of (3.2) at the origin has the following bloc structure

$$D\mathbf{F} = \left( \begin{array}{c|c} \mathbf{L} & D_{\mathbf{z}} \mathbf{F}_1 \\ \hline 0 & D\mathbf{G}_2 \mathbf{S} \end{array} \right). \quad (3.6)$$



Note that in the case of a ODE the differential operator would simply be  $\mathbf{L}$ . Let  $(\mathbf{y}, \mathbf{z})$  (with  $\mathbf{y} \in \mathbb{R}^p$  and  $\mathbf{z} \in \mathbb{R}^m$ ) be in the kernel (the nullspace) of  $D\mathbf{F}$  and according to the expression (3.6) one gets

$$\mathbf{L}\mathbf{y} = 0, \quad \mathbf{z} = 0$$

because  $D\mathbf{G}_2\mathbf{S}\mathbf{z} = 0$  implies  $\mathbf{z} = 0$ ,  $D\mathbf{G}_2\mathbf{S}$  being invertible. Writing  $\mathbf{0}$  the nullvector of  $\mathbb{R}^m$ ,  $\text{Ker } D\mathbf{F}$  is of dimension 2 and generated by

$$\mathbf{q}_1 = (\mathbf{v}_1, \mathbf{0}), \quad \mathbf{q}_2 = (\mathbf{v}_2, \mathbf{0}) \quad (3.7)$$

with  $\mathbf{v}_j \in \mathbb{R}^p$ ,  $j = 1, 2$  the vectors which generate  $\text{Ker } \mathbf{L}$ . Given the expression (3.5) of  $\mathbf{L}$  these vectors are (see [5], chapter VIII)

$$\mathbf{v}_1 = \mathcal{R}(e^{is}\mathbf{c}) = \frac{1}{2}(e^{is}\mathbf{c} + e^{-is}\bar{\mathbf{c}}), \quad \mathbf{v}_2 = \mathcal{I}(e^{is}\mathbf{c}) = \frac{1}{2}(-ie^{is}\mathbf{c} + ie^{-is}\bar{\mathbf{c}}), \quad (3.8)$$

$\mathbf{c}$  being eigenvector with eigenvalue  $-i$  of  $\tilde{\mathbf{A}}$ , (that is  $\tilde{\mathbf{A}}\mathbf{c} = -i\mathbf{c}$ ) (and  $\bar{\mathbf{c}}$  is its complex conjugate, that is eigenvector associated with  $i$ ). The real and imaginary parts are noted  $\mathcal{R}$  and  $\mathcal{I}$  respectively.

It is recalled that the solution space is  $C_{2\pi}$  and in the present DAE setting it is conveniently written  $C_{2\pi|_p} \times C_{2\pi|m}$ , that is as the Cartesian product of the spaces restricted to the functions in  $\mathbb{R}^p$  and  $\mathbb{R}^m$  respectively. The different steps are now outlined to generalize the theory in [5], chapter VIII, to the case of the DAE setting. Define the inner product for the space  $C_{2\pi|_p}$

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{u}_1(s)^T \mathbf{u}_2(s) ds, \quad \text{with } \mathbf{u}_j \in \mathbb{R}^p, j = 1, 2. \quad (3.9)$$

One may suppose that the operator  $\mathbf{L}$  of the DAE system is Fredholm with index zero, which is the assumption generally made for the linearized operators of nonlinear systems and ODEs (see [5] chapter VII), which allows to split  $C_{2\pi|_p}$  as

$$C_{2\pi|_p} = \text{Ker } \mathbf{L} \oplus \text{Range } \mathbf{L}, \quad \text{and} \quad \text{Range } \mathbf{L} = (\text{Ker } \mathbf{L}^*)^\perp. \quad (3.10)$$

To characterize the range of  $\mathbf{L}$  as the space orthogonal to the nullspace of  $\mathbf{L}^*$ , the adjoint for the inner product of  $\mathbf{L}$ , is known as the Fredholm alternative. This gives rise to the splitting of  $C_{2\pi}$

$$C_{2\pi} = \text{Ker } D\mathbf{F} \oplus R, \quad \text{with} \quad R = \text{Range } \mathbf{L} \times C_{2\pi|m} \quad (3.11)$$

Indeed, for  $(\mathbf{y}, \mathbf{z}) \in C_{2\pi} = C_{2\pi|_p} \times C_{2\pi|m}$  the splitting (3.10) allows to write  $\mathbf{y} = \mathbf{v} + \mathbf{w}$ , with  $\mathbf{v} \in \text{Ker } \mathbf{L}$  and  $\mathbf{w} \in \text{Range } \mathbf{L}$ . It follows that  $(\mathbf{y}, \mathbf{z}) =$

$(\mathbf{v}, \mathbf{0}) + (\mathbf{w}, \mathbf{z})$  ( $\mathbf{0}$  being the nullvector of  $\mathbb{R}^m$ ), that is the sum of a vector in the kernel of  $D\mathbf{F}$  (see (3.7) and a vector in  $\text{Range } \mathbf{L} \times C_{2\pi|m}^1$ . The space  $C_{2\pi}^1$  has to be splitted accordingly (the superscript 1 means that the functions are continuously differentiable) and

$$C_{2\pi}^1 = \text{Ker } D\mathbf{F} \oplus E, \quad E = N \times C_{2\pi|m}^1 \text{ with } N = C_{2\pi|p}^1 \cap \text{Range } \mathbf{L}. \quad (3.12)$$

For the Fredholm alternative, the kernel of the adjoint  $\mathbf{L}^*$  for the inner product (3.9), that is

$$\mathbf{L}^* = -\frac{d}{ds}\mathbf{I} + \tilde{\mathbf{A}}^T \mathbf{y}, \quad (3.13)$$

has to be computed. It is easy to verify that  $\text{Ker } \mathbf{L}^*$  is generated by the two vectors

$$\mathbf{v}_1^* = \frac{1}{2} (e^{is}\mathbf{d} + e^{-is}\bar{\mathbf{d}}), \quad \mathbf{v}_2^* = \frac{1}{2} (-ie^{is}\mathbf{d} + ie^{-is}\bar{\mathbf{d}}), \quad (3.14)$$

$\mathbf{d}$  being the eigenvector of  $\tilde{\mathbf{A}}^T$  associated with  $i$ , that is  $\tilde{\mathbf{A}}^T \mathbf{d} = i\mathbf{d}$ . The bi-orthogonal relation  $\mathbf{c}^T \mathbf{d} = 0$  holds and one can choose  $\mathbf{c}$  and  $\mathbf{d}$  in (3.8) and (3.14) such that

$$\bar{\mathbf{c}}^T \mathbf{c} = 2, \quad \mathbf{c}^T \bar{\mathbf{d}} = 2$$

and hence

$$\langle \mathbf{v}_j, \mathbf{v}_k \rangle = \delta_{jk}, \quad \langle \mathbf{v}_j^*, \mathbf{v}_k \rangle = \delta_{jk}, \quad j = 1, 2, \quad k = 1, 2 \quad (3.15)$$

with  $\delta_{jk}$  the Kronecker symbol. Note that these orthogonality conditions can easily be proven by considering that for any integers  $j$  and  $k$

$$\frac{1}{2\pi} \int_0^{2\pi} e^{is(j-k)} ds = \delta_{jk}. \quad (3.16)$$

For the reduction procedure, it is essential that the solution  $(\mathbf{y}, \mathbf{z})$  of (3.2) can be written as (given the splitting (3.12))

$$\begin{aligned} (\mathbf{y}, \mathbf{z}) &= (\mathbf{v} + \mathbf{w}, \mathbf{z}) \quad \text{with} \\ \mathbf{v} &= r_1 \mathbf{v}_1 + r_2 \mathbf{v}_2 \quad \text{and} \quad \langle \mathbf{v}_j^*, \mathbf{w} \rangle = 0, \quad j = 1, 2, \end{aligned} \quad (3.17)$$

$\mathbf{v}_j, j = 1, 2$ , given by (3.8) generating the kernel of  $\mathbf{L}$ . The inner product between  $\mathbf{w}$  and  $\mathbf{v}_j^*, j = 1, 2$ , (the generators of  $\text{Ker } \mathbf{L}^*$  given by (3.14)) being zero is equivalent to assert that  $\mathbf{w} \in \text{Range } \mathbf{L}$  by the Fredholm alternative. Before coming to the main result of the Lyapunov-Schmidt reduction, a projection operator  $P$  has to be defined for any function  $\mathbf{h} \in C_{2\pi|p}$ , that is

$$P\mathbf{h} = \mathbf{h} - \langle \mathbf{v}_1^*, \mathbf{h} \rangle \mathbf{v}_1 - \langle \mathbf{v}_2^*, \mathbf{h} \rangle \mathbf{v}_2. \quad (3.18)$$

Given the relations (3.15),  $\langle \mathbf{v}_j^*, P\mathbf{h} \rangle = 0, j = 1, 2$  and hence  $P\mathbf{h}$  is in the range of  $\mathbf{L}$ . The theorem hereafter is the main result of the Lyapunov-Schmidt reduction for DAEs at Hopf bifurcations, which generalizes the corresponding theorem for ODEs.

**Theorem 3.1.** Write  $(\mathbf{y}, \mathbf{z}) = (\mathbf{v} + \mathbf{w}, \mathbf{z})$  the solution of (3.2) according to (3.17) and consider the functional

$$\phi(\mathbf{v}, \mathbf{w}, \mathbf{z}, \lambda, \tau) = (P\mathbf{F}_1(\mathbf{v} + \mathbf{w}, \mathbf{z}, \lambda, \tau), \mathbf{F}_2(\mathbf{v} + \mathbf{w}, \mathbf{z}, \lambda)),$$

$P$  being the projection operator defined by (3.18). The differential with respect to  $\mathbf{w}$  and  $\mathbf{z}$  of the functional at the origin (the steady state) is an invertible operator from  $E$  defined by (3.12) onto  $R$  defined by (3.11) and by the implicit function theorem  $(\mathbf{w}, \mathbf{z})$  is locally solution of  $(\mathbf{v}, \lambda, \tau)$  such that

$$\begin{aligned} P\mathbf{F}_1(\mathbf{v} + \mathbf{w}(\mathbf{v}, \lambda, \tau), \mathbf{z}(\mathbf{v}, \lambda, \tau), \lambda, \tau) &= 0 \\ \mathbf{F}_2(\mathbf{v} + \mathbf{w}(\mathbf{v}, \lambda, \tau), \mathbf{z}(\mathbf{v}, \lambda, \tau), \lambda) &= 0, \end{aligned} \quad (3.19)$$

these functions vanishing at the origin, that is  $\mathbf{w}(0, 0, 0) = \mathbf{z}(0, 0, 0) = 0$ .

*Proof.* To prove this theorem, one observes first of all that according to (3.11) and (3.12)

$$\phi : \text{Ker } \mathbf{L} \times E \times \mathbb{R}^2 \rightarrow R$$

The differential with respect to  $\mathbf{w}$  and  $\mathbf{z}$  at the origin gives rise to the differential operator

$$\mathbf{J} = \left( \begin{array}{c|c} P\mathbf{L}|_N & PD_z\mathbf{F}_1 \\ \hline 0 & D\mathbf{G}_2\mathbf{S} \end{array} \right), \quad \mathbf{J} : E \rightarrow R$$

(with  $E = N \times C_{2\pi|m}^1$  and  $R = \text{Range } \mathbf{L} \times C_{2\pi|m}$ ). Indeed,  $\mathbf{J}$  is inferred from  $D\mathbf{F}$  (see (3.6)) replacing  $\mathbf{L}$  by  $\mathbf{L}|_N$  the operator restricted to  $N = C_{2\pi|p}^1 \cap \text{Range } \mathbf{L}$  and the projection operator  $P$  is applied. For any  $(\mathbf{b}_1, \mathbf{b}_2) \in R$ , it has to be shown, that there is a unique  $(\mathbf{a}_1, \mathbf{a}_2) \in E$  such that

$$\mathbf{J} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}.$$

Indeed,  $D\mathbf{G}_2\mathbf{S}$  being invertible,  $\mathbf{a}_2 = (D\mathbf{G}_2\mathbf{S})^{-1}\mathbf{b}_2$  and it remains to solve

$$P\mathbf{L}|_N \mathbf{a}_1 + PD_z\mathbf{F}_1 \mathbf{a}_2 = \mathbf{b}_1.$$

One observes that  $PD_z\mathbf{F}_1 \mathbf{a}_2 \in \text{Range } \mathbf{L}$  given the projection  $P$  defined by (3.18). The function  $\mathbf{b}_1$  is in  $\text{Range } \mathbf{L}$  (by definition of  $R$ ) and hence  $\mathbf{a}_1$  is uniquely determined because

$$P\mathbf{L}|_N : N \rightarrow \text{Range } \mathbf{L}$$

is invertible (the kernel of  $\mathbf{L}|_N$  being zero by construction). The differential operator  $\mathbf{J}$  being invertible, one can indeed apply the implicit function theorem, giving rise to the system (3.19).  $\square$

Finally, to solve the DAE, the equation

$$(I - P)\mathbf{F}_1(\mathbf{v} + \mathbf{w}(\mathbf{v}, \lambda, \tau), \mathbf{z}(\mathbf{v}, \lambda, \tau), \lambda, \tau) = 0 \quad (3.20)$$

is to be added to the system (3.19). Note that according to (3.18)

$$(I - P)\mathbf{F}_1 = \langle \mathbf{v}_1^*, \mathbf{F}_1 \rangle \mathbf{v}_1 + \langle \mathbf{v}_2^*, \mathbf{F}_1 \rangle \mathbf{v}_2.$$

Periodic functions are defined up to an arbitrary phase shift  $\theta$  and consider  $\mathbf{v}(s) = r_1 \mathbf{v}_1(s) + r_2 \mathbf{v}_2(s)$ . Given the expressions (3.8) it can easily be shown that

$$\mathbf{v}(s - \theta) = (r_1 \cos(\theta) - r_2 \sin(\theta)) \mathbf{v}_1(s) + (r_1 \sin(\theta) + r_2 \cos(\theta)) \mathbf{v}_2(s).$$

This means that one may only consider (see [5])

$$\mathbf{v} = r \mathbf{v}_1, \quad r \geq 0,$$

because any  $(r_1, r_2)$  may be written  $(r \cos(\theta), r \sin(\theta))$  and  $r \mathbf{v}_1(s - \theta) = r_1 \mathbf{v}_1(s) + r_2 \mathbf{v}_2(s)$ . One can therefore write  $\mathbf{v} = r \mathbf{v}_1$  in (3.19), (3.20) and from now on the dependence on  $r \mathbf{v}_1$  of  $\mathbf{w}, \mathbf{z}$  is simply written as dependence on  $r$ . Summarizing, (3.19) is to be solved as well as (3.20), the latter system being equivalent to the two so-called reduced equations

$$\begin{aligned} \varphi_1(r, \lambda, \tau) &= \langle \mathbf{v}_1^*, \mathbf{F}_1(r \mathbf{v}_1 + \mathbf{w}(r, \lambda, \tau), \mathbf{z}(r, \lambda, \tau), \lambda, \tau) \rangle = 0 \\ \varphi_2(r, \lambda, \tau) &= \langle \mathbf{v}_2^*, \mathbf{F}_1(r \mathbf{v}_1 + \mathbf{w}(r, \lambda, \tau), \mathbf{z}(r, \lambda, \tau), \lambda, \tau) \rangle = 0. \end{aligned} \quad (3.21)$$

In the next section details of the algorithm are provided to solve these equations by performing appropriate Taylor expansions.

#### 4. Lyapunov-Schmidt reduction algorithm and stability

In the following the procedure as exposed in [5] (chapter VIII) for ODEs will be closely followed, emphasizing however the differences giving rise to necessarily more complex formulas in the DAE context. The algorithm consists in computing the successive terms in  $r$  as well as the parameters  $\lambda$  and  $\tau$  in the Taylor expansions of  $\mathbf{w}(r, \lambda, \tau)$  and  $\mathbf{z}(r, \lambda, \tau)$  and these computations involve the classical multivariable derivative formula, which is recalled for any multivariable vector function  $\mathbf{h}(\mathbf{x}) \in \mathbb{R}^q$ ,  $\mathbf{x} \in \mathbb{R}^r$ , that is

$$D^j \mathbf{h}(0)[\mathbf{x}_1, \dots, \mathbf{x}_j] = \frac{\partial^j}{\partial t_1 \dots \partial t_j} \mathbf{h}(t_1 \mathbf{x}_1 + \dots + t_j \mathbf{x}_j) \Big|_{t_1 = \dots = t_j = 0}, \quad (4.1)$$

( $j$  a positive integer), this operator being multilinear and symmetric in the argument vectors  $\mathbf{x}_k, k = 1, \dots, j$ .

#### 4.1. Taylor expansion of the local solution

First of all, the expansions of  $\mathbf{F}_1$  and  $\mathbf{F}_2$  given by (3.3) and (3.4) have to be considered and only the terms necessary to obtain meaningful reduced equations are written out. The notation is that of the multivariable derivative formula, the argument  $(0, 0, 0, 0)$  being again omitted when writing the operators. Considering (3.3), the expansion for  $\mathbf{F}_1(\mathbf{y}, \mathbf{z}, 0, 0)$  up to the second order in the state variable is (it is recalled that  $\mathbf{y} = r\mathbf{v}_1 + \mathbf{w}$ )

$$\begin{aligned} & \mathbf{B}_1^{-1} \left( \mathbf{M}_1 \frac{d}{ds} (\mathbf{T}(r\mathbf{v}_1 + \mathbf{w}) + \mathbf{S}\mathbf{z}) + D\mathbf{G}_1[\mathbf{T}(r\mathbf{v}_1 + \mathbf{w}) + \mathbf{S}\mathbf{z}] \right. \\ & \quad \left. + D\mathbf{M}_1[\mathbf{T}(r\mathbf{v}_1 + \mathbf{w}) + \mathbf{S}\mathbf{z}] \frac{d}{ds} (\mathbf{T}(r\mathbf{v}_1 + \mathbf{w}) + \mathbf{S}\mathbf{z}) \right. \\ & \quad \left. + \frac{1}{2} D^2 \mathbf{G}_1[\mathbf{T}(r\mathbf{v}_1 + \mathbf{w}) + \mathbf{S}\mathbf{z}, \mathbf{T}(r\mathbf{v}_1 + \mathbf{w}) + \mathbf{S}\mathbf{z}] \right) + \dots \end{aligned} \quad (4.2)$$

(it is not necessary to write the third order term at this point, it is provided in Appendix B). Note that the expression  $D\mathbf{M}_1[\cdot]$  means that the multivariable derivative formula (4.1) is applied to each component of the matrix  $\mathbf{M}_1$ , if it depends explicitly on the variables. For the expansion in the parameter  $\lambda$ , the term  $\lambda \mathbf{F}_{1\lambda}(\mathbf{y}, \mathbf{z}, 0, 0)$  (the subscript  $\lambda$  meaning the derivative with respect to the parameter) has only to be expanded to the first order in the state variable which yields

$$\lambda \mathbf{B}_1^{-1} \left( \mathbf{G}_{1\lambda} + \mathbf{M}_{1\lambda} \frac{d}{ds} (\mathbf{T}(r\mathbf{v}_1 + \mathbf{w}) + \mathbf{S}\mathbf{z}) + D\mathbf{G}_{1\lambda}[\mathbf{T}(r\mathbf{v}_1 + \mathbf{w}) + \mathbf{S}\mathbf{z}] \right) \quad (4.3)$$

Concerning the dependence on the variable  $\tau$ , of course  $\mathbf{F}_1(0, 0, 0, \tau) = 0$  and the term combined with the first order term in the state variable is

$$\tau \mathbf{B}_1^{-1} \mathbf{M}_1 \frac{d}{ds} (\mathbf{T}(r\mathbf{v}_1 + \mathbf{w}) + \mathbf{S}\mathbf{z}) \quad (4.4)$$

Concerning the expansion of  $\mathbf{F}_2$ , according to (3.4) one gets for the expansion of  $\mathbf{F}_2(\mathbf{y}, \mathbf{z}, 0)$  up to the third order in the state variable

$$\begin{aligned} & D\mathbf{G}_2[\mathbf{S}\mathbf{z}] + \frac{1}{2} D^2 \mathbf{G}_2[\mathbf{T}(r\mathbf{v}_1 + \mathbf{w}) + \mathbf{S}\mathbf{z}, \mathbf{T}(r\mathbf{v}_1 + \mathbf{w}) + \mathbf{S}\mathbf{z}] \\ & \quad + \frac{1}{6} D^3 \mathbf{G}_2[\mathbf{T}(r\mathbf{v}_1 + \mathbf{w}) + \mathbf{S}\mathbf{z}, \mathbf{T}(r\mathbf{v}_1 + \mathbf{w}) + \mathbf{S}\mathbf{z}, \mathbf{T}(r\mathbf{v}_1 + \mathbf{w}) + \mathbf{S}\mathbf{z}] \dots \end{aligned} \quad (4.5)$$

(it is recalled that  $D\mathbf{G}_2\mathbf{T} = 0$ ). The expansion in the parameter  $\lambda$  to be retained for  $\mathbf{F}_2$  is

$$\lambda (\mathbf{G}_{2\lambda} + D\mathbf{G}_{2\lambda}[\mathbf{T}(r\mathbf{v}_1 + \mathbf{w}) + \mathbf{S}\mathbf{z}]) \quad (4.6)$$

For the Taylor expansions of  $\mathbf{w}(r, \lambda, \tau)$  and  $\mathbf{z}(r, \lambda, \tau)$ , in the following the subscripts  $r, \lambda, \tau$  mean the derivatives with respect to these variables and the argument  $(0, 0, 0)$  is omitted (that is for instance  $\mathbf{w}_{r,\tau}$  corresponds to the successive derivatives with respect to  $r$  and  $\tau$ ). The terms at the successive orders are computed by considering the system (3.19), taking into account the expansions above. First of all it is shown that a certain number of terms are zero.

**Proposition 4.1.** *The terms  $r\mathbf{w}_r, r\mathbf{z}_r, \tau\mathbf{w}_\tau, \tau\mathbf{z}_\tau, r\tau\mathbf{w}_{r,\tau}, r\tau\mathbf{z}_{r,\tau}$  in the Taylor expansion of  $\mathbf{w}, \mathbf{z}$  are equal to zero.*

*Proof.* In the expansion (4.5) the term in  $r$  is  $rD\mathbf{G}_2[S\mathbf{z}_r]$  and to put it to zero one gets  $\mathbf{z}_r = 0$  because  $D\mathbf{G}_2S$  admits an inverse. Retaining the term in  $r$  in the expansion (4.2) and putting it to zero by applying  $P$  (see system (3.19)) one gets (it is recalled that  $\mathbf{B}_1^{-1}\mathbf{M}_1\mathbf{T} = \mathbf{I}$  and  $\mathbf{L}$  is defined in (3.5))

$$P\mathbf{L}(\mathbf{v}_1 + \mathbf{w}_r) = P\mathbf{L}\mathbf{w}_r = 0.$$

By construction  $\mathbf{w}_r \in N$  (with  $N = C_{2\pi|p}^1 \cap \text{Range } \mathbf{L}$ ) and consequently  $\mathbf{w}_r = 0$ ,  $P\mathbf{L}|_N$  being invertible. The term  $\mathbf{z}_r = 0$ , the parameter  $\tau$  appearing explicitly only in  $\mathbf{F}_1$ . Considering  $\mathbf{w}_\tau$  in (4.2) one recovers  $\tau\mathbf{L}\mathbf{w}_\tau$  at order  $\tau$  and by (3.19) one gets  $P\mathbf{L}\mathbf{w}_\tau = 0$ , that is  $\mathbf{w}_\tau = 0$ . Consequently the term of order  $r\tau$  in (4.5) is  $r\tau D\mathbf{G}_2[\mathbf{S}\mathbf{z}_{r,\tau}]$  and putting it to zero, one gets  $\mathbf{z}_{r,\tau} = 0$ . Isolating the term in  $r\tau$  in the expansion (4.2) and adding the term in (4.4), by (3.19)

$$P\left(\mathbf{L}\mathbf{w}_{r,\tau} + \frac{d}{ds}\mathbf{v}_1\right) = 0.$$

It is easy to see that  $d\mathbf{v}_1/ds = -\mathbf{v}_2$  (see (3.8)) and  $P\mathbf{v}_2 = 0$ ,  $P$ , being the projection on the range of  $\mathbf{L}$  and one can conclude that  $\mathbf{w}_{r,\tau} = 0$ .  $\square$

It follows from the proposition, that the Taylor expansions of  $\mathbf{w}$  and  $\mathbf{z}$  are, where now the subscripts  $j = 1, 2, 3$  mean the first, second and third derivative with respect to  $r$

$$\begin{aligned}\mathbf{w} &= \lambda \mathbf{w}_\lambda + r\lambda \mathbf{w}_{1,\lambda} + \frac{1}{2} r^2 \mathbf{w}_2 + \frac{1}{6} r^3 \mathbf{w}_3 + \dots \\ \mathbf{z} &= \lambda \mathbf{z}_\lambda + r\lambda \mathbf{z}_{1,\lambda} + \frac{1}{2} r^2 \mathbf{z}_2 + \frac{1}{6} r^3 \mathbf{z}_3 + \dots\end{aligned}\tag{4.7}$$

It will be seen that it is sufficient to retain these terms in the expansions to ultimately recover the reduced equations at relevant order.

**Proposition 4.2.** *The term  $\mathbf{z}_\lambda$  in the expansion (4.7) is*

$$\mathbf{z}_\lambda = -(D\mathbf{G}_2\mathbf{S})^{-1}\mathbf{G}_{2\lambda} \quad (4.8)$$

and  $\mathbf{w}_\lambda$  is solution of

$$\tilde{\mathbf{A}}\mathbf{w}_\lambda = -\mathbf{B}_1^{-1}(\mathbf{G}_{1\lambda} + D\mathbf{G}_1[\mathbf{S}\mathbf{z}_\lambda]), \quad (4.9)$$

$\tilde{\mathbf{A}}$  being invertible (because 0 is not an eigenvalue).

*Proof.* Adding the terms at order  $\lambda$  of (4.5) and (4.6), one gets

$$\lambda(\mathbf{G}_{2\lambda} + D\mathbf{G}_2[\mathbf{S}\mathbf{z}_\lambda])$$

and putting it to zero (see (3.19)) one obtains indeed (4.8). Knowing  $\mathbf{z}_\lambda$ , the term in  $\lambda$  of (4.2) together with that of (4.3) writes

$$\lambda\tilde{\mathbf{A}}\mathbf{w}_\lambda + \lambda\mathbf{B}_1^{-1}(D\mathbf{G}_1[\mathbf{S}\mathbf{z}_\lambda] + \mathbf{G}_{1\lambda})$$

(where it has been taken into account that  $\mathbf{z}_\lambda$  and hence  $\mathbf{w}_\lambda$  do not depend on  $s$ , the DAE being autonomous). The projection  $P$  acts as identity on functions independent of  $s$  and to solve the first equation of (3.19) leads indeed to (4.9).  $\square$

Note that when the steady state is solution for all  $\lambda$  (that is  $\mathbf{G}(0, \lambda) = 0$ ), then of course  $\mathbf{z}_\lambda = \mathbf{w}_\lambda = 0$ . Now the terms  $\mathbf{z}_{1,\lambda}$  and  $\mathbf{w}_{1,\lambda}$  are addressed. Given that

$$\mathbf{v}_1 = \frac{1}{2}e^{is}\mathbf{c} + cc$$

(see (3.8), where  $cc$  means the complex conjugate expression), one observes that  $\mathbf{w}_{1,\lambda}$  and  $\mathbf{z}_{1,\lambda}$  are of the form

$$\mathbf{w}_{1,\lambda} = e^{is}\mathbf{w}_{1,\lambda}^{(1)} + cc, \quad \mathbf{z}_{1,\lambda} = e^{is}\mathbf{z}_{1,\lambda}^{(1)} + cc \quad (4.10)$$

Also, some remarks are necessary concerning the projection operator (3.18). Given (3.16) and the expressions (3.14) of  $\mathbf{v}_j^*$ ,  $j = 1, 2$ , it is clear that only terms containing  $e^{is}$  or  $e^{-is}$  have a nonzero contribution when performing the inner product of  $P$ . Let therefore  $\boldsymbol{\gamma}$  be any expression of the type

$$\boldsymbol{\gamma} = e^{is}\mathbf{b} + e^{-is}\bar{\mathbf{b}}$$

(with  $\mathbf{b} \in \mathbb{C}^p$ ) then it is easy to verify that

$$P\boldsymbol{\gamma} = e^{is}\tilde{P}\mathbf{b} + cc \quad \text{with} \quad \tilde{P}\mathbf{b} = \mathbf{b} - \frac{1}{2}(\mathbf{b}^T\bar{\mathbf{d}})\mathbf{c} \quad (4.11)$$

given the expressions (3.8) and (3.14).

**Proposition 4.3.** *The term  $\mathbf{z}_{1,\lambda}^{(1)}$  in (4.10) is solution of*

$$D\mathbf{G}_2\mathbf{S} \mathbf{z}_{1,\lambda}^{(1)} = -\frac{1}{2}D\mathbf{G}_{2\lambda}[\mathbf{T}\mathbf{c}] - \frac{1}{2}D^2\mathbf{G}_2[\mathbf{T}\mathbf{c}, \mathbf{T}\mathbf{w}_\lambda + \mathbf{S}\mathbf{z}_\lambda] \quad (4.12)$$

*$D\mathbf{G}_2\mathbf{S}$  being invertible, whereas the term  $\mathbf{w}_{1,\lambda}^{(1)}$  is solution of*

$$\begin{aligned} (i\mathbf{I} + \tilde{\mathbf{A}})\mathbf{w}_{1,\lambda}^{(1)} &= -\tilde{P}\mathbf{b} \quad \text{with} \\ \mathbf{b} &= \mathbf{B}_1^{-1} \left( i\mathbf{M}_1\mathbf{S}\mathbf{z}_{1,\lambda}^{(1)} + \frac{i}{2}\mathbf{M}_{1\lambda}\mathbf{T}\mathbf{c} + \frac{i}{2}D\mathbf{M}_1[\mathbf{T}\mathbf{w}_\lambda + \mathbf{S}\mathbf{z}_\lambda]\mathbf{T}\mathbf{c} \right. \\ &\quad \left. + D\mathbf{G}_1[\mathbf{S}\mathbf{z}_{1,\lambda}^{(1)}] + \frac{1}{2}D\mathbf{G}_{1\lambda}[\mathbf{T}\mathbf{c}] + \frac{1}{2}D^2\mathbf{G}_1[\mathbf{T}\mathbf{c}, \mathbf{T}\mathbf{w}_\lambda + \mathbf{S}\mathbf{z}_\lambda] \right) \end{aligned} \quad (4.13)$$

*with  $\tilde{P}$  defined by (4.11), which projects the right-hand side on the range of the matrix  $i\mathbf{I} + \tilde{\mathbf{A}}$  which is singular,  $-i$  being eigenvalue of  $\tilde{\mathbf{A}}$ ,  $\mathbf{w}_{1,\lambda}^{(1)}$  being the (unique) solution such that  $\tilde{P}\mathbf{w}_{1,\lambda}^{(1)} = \mathbf{w}_{1,\lambda}^{(1)}$ .*

*Proof.* First, one adds the expressions at order  $r\lambda$  in (4.5) and (4.6), retaining for  $\mathbf{z}_{1,\lambda}$  and  $\mathbf{v}_1$  the terms  $e^{is}\mathbf{z}_{1,\lambda}^{(1)}$  and  $\frac{1}{2}e^{is}\mathbf{c}$ . Putting to zero the resulting expression, one can convince oneself that  $\mathbf{z}_{1,\lambda}^{(1)}$  must be solution of (4.12). For the term  $\mathbf{w}_{1,\lambda}^{(1)}$ , one has to select in (4.2) the terms in  $r\lambda$  and to add those in (4.3). Again, only the terms in  $e^{is}$  are then retained and it has to be observed that in particular

$$\mathbf{L}\mathbf{w}_{1,\lambda} = e^{is} (i\mathbf{I} + \tilde{\mathbf{A}})\mathbf{w}_{1,\lambda}^{(1)} + cc$$

The projection  $P$  becomes  $\tilde{P}$  defined by (4.11), as the projection on the range of  $i\mathbf{I} + \tilde{\mathbf{A}}$  (the matrix being singular, having  $-i$  as eigenvalue). Putting to zero the total expression at order  $r\lambda$  after projection, one recovers indeed the solution (4.13), which has to verify  $\tilde{P}\mathbf{w}_{1,\lambda}^{(1)} = \mathbf{w}_{1,\lambda}^{(1)}$ .  $\square$

A remark should be made about how to solve the system (4.13) with the singular matrix  $i\mathbf{I} + \tilde{\mathbf{A}}$ . A  $LU$  decomposition with complete pivoting can be applied (see the end of Appendix A); the numerical solution vector will contain an arbitrary contribution belonging to the kernel of the matrix which is removed by the projection (4.11).

Now consider the term in  $r^2$  in the developments of  $\mathbf{w}$  and  $\mathbf{z}$  in (4.7). Given the dependence on  $r\mathbf{v}_1$  of  $\mathbf{w}_2$  and  $\mathbf{z}_2$ , one necessarily has

$$\mathbf{w}_2 = e^{i2s}\mathbf{w}_2^{(2)} + cc + \mathbf{w}_2^{(0)}, \quad \mathbf{z}_2 = e^{i2s}\mathbf{z}_2^{(2)} + cc + \mathbf{z}_2^{(0)} \quad (4.14)$$

( $\mathbf{w}_2^{(0)}$  and  $\mathbf{z}_2^{(0)}$  being real vectors). Writing out the terms in  $r^2$  in the expansion (4.5) as well as in the expansion (4.2) and taking into account that  $P$



acts as the identity operator on terms without  $e^{\pm is}$  as a factor, there is no difficulty to establish the following formulas, when considering the system (3.19) at order  $r^2$ .

**Proposition 4.4.** *The terms  $\mathbf{z}_2^{(2)}$  and  $\mathbf{z}_2^{(0)}$  of (4.14) are solution of the systems*

$$D\mathbf{G}_2\mathbf{S}\mathbf{z}_2^{(2)} = -\frac{1}{4}D^2\mathbf{G}_2[\mathbf{T}\mathbf{c}, \mathbf{T}\mathbf{c}], \quad D\mathbf{G}_2\mathbf{S}\mathbf{z}_2^{(0)} = -\frac{1}{2}D^2\mathbf{G}_2[\mathbf{T}\mathbf{c}, \mathbf{T}\bar{\mathbf{c}}] \quad (4.15)$$

( $D\mathbf{G}_2\mathbf{S}$  being invertible), whereas  $\mathbf{w}_2^{(2)}$  and  $\mathbf{w}_2^{(0)}$  are solution of

$$\begin{aligned} (2i\mathbf{I} + \tilde{\mathbf{A}})\mathbf{w}_2^{(2)} = & -\mathbf{B}_1^{-1} \left( 2i\mathbf{M}_1\mathbf{S}\mathbf{z}_2^{(2)} + \frac{i}{2}D\mathbf{M}_1[\mathbf{T}\mathbf{c}]\mathbf{T}\mathbf{c} \right. \\ & \left. + D\mathbf{G}_1[\mathbf{S}\mathbf{z}_2^{(2)}] + \frac{1}{4}D^2\mathbf{G}_1[\mathbf{T}\mathbf{c}, \mathbf{T}\mathbf{c}] \right) \end{aligned} \quad (4.16)$$

and

$$\begin{aligned} \tilde{\mathbf{A}}\mathbf{w}_2^{(0)} = & -\mathbf{B}_1^{-1} \left( D\mathbf{G}_1[\mathbf{S}\mathbf{z}_2^{(0)}] + \mathcal{R}(iD\mathbf{M}_1[\mathbf{T}\bar{\mathbf{c}}]\mathbf{T}\mathbf{c}) \right. \\ & \left. + \frac{1}{2}D^2\mathbf{G}_1[\mathbf{T}\mathbf{c}, \mathbf{T}\bar{\mathbf{c}}] \right) \end{aligned} \quad (4.17)$$

( $\mathcal{R}$  meaning the real part),  $2i\mathbf{I} + \tilde{\mathbf{A}}$  and  $\tilde{\mathbf{A}}$  admitting an inverse.

The third order terms in (4.7) are of the form

$$\mathbf{w}_3 = e^{is}\mathbf{w}_3^{(1)} + e^{i3s}\mathbf{w}_3^{(3)} + cc, \quad \mathbf{z}_3 = e^{is}\mathbf{z}_3^{(1)} + e^{i3s}\mathbf{z}_3^{(3)} + cc \quad (4.18)$$

Assembling the terms in  $r^3$  of the expansion (4.2), there is no major difficulty to establish the following formulas.

**Proposition 4.5.** *The terms  $\mathbf{z}_3^{(1)}$  and  $\mathbf{z}_3^{(3)}$  such that the second equation of (3.19) is solved at order  $r^3$  are solution of*

$$\begin{aligned} D\mathbf{G}_2\mathbf{S}\mathbf{z}_3^{(1)} = & -\frac{3}{2}D^2\mathbf{G}_2[\mathbf{T}\mathbf{c}, \mathbf{T}\mathbf{w}_2^{(0)} + \mathbf{S}\mathbf{z}_2^{(0)}] \\ & -\frac{3}{2}D^2\mathbf{G}_2[\mathbf{T}\bar{\mathbf{c}}, \mathbf{T}\mathbf{w}_2^{(2)} + \mathbf{S}\mathbf{z}_2^{(2)}] - \frac{3}{8}D^3\mathbf{G}_2[\mathbf{T}\mathbf{c}, \mathbf{T}\mathbf{c}, \mathbf{T}\bar{\mathbf{c}}] \end{aligned} \quad (4.19)$$

and

$$D\mathbf{G}_2\mathbf{S}\mathbf{z}_3^{(3)} = -\frac{3}{2}D^2\mathbf{G}_2[\mathbf{T}\mathbf{c}, \mathbf{T}\mathbf{w}_2^{(2)} + \mathbf{S}\mathbf{z}_2^{(2)}] - \frac{1}{8}D^3\mathbf{G}_2[\mathbf{T}\mathbf{c}, \mathbf{T}\mathbf{c}, \mathbf{T}\mathbf{c}]. \quad (4.20)$$

At this point of the analysis, all the necessary quantities to compute the expansion of the local equations (3.21) are available. For the expansions up to the third order in  $r$  of the local solution to be complete, the expression for  $\mathbf{w}_3$  is provided in Appendix B.

#### 4.2. Reduced bifurcation equation and existence of the periodic solution

To obtain the local equations (3.21), the inner products with  $\mathbf{v}_j^*$  has to be performed and only the terms in  $e^{\pm is}$  (see (3.16)) need to be taken into account. Therefore the terms  $r^{2k}$  with even powers of  $r$  vanish when performing the inner product, because they are either independent of  $s$ , or the associated exponential term is  $e^{i2qs}$  ( $q$  integer). Furthermore, in the expansion of  $\mathbf{F}_1$  there is no term in  $r$  and the only term in  $r\tau$  is  $r\tau \frac{d}{ds} \mathbf{v}_1 = -r\tau \mathbf{v}_2$  (see (4.4) and the results of proposition 4.1). By (3.15)  $\langle \mathbf{v}_1^*, \mathbf{v}_2 \rangle = 0$  and  $\langle \mathbf{v}_2^*, \mathbf{v}_2 \rangle = 1$ , it follows that in the first local equation there is no term in  $r\tau$  and in the second equation the term is  $-r\tau$ . Therefore, the reduced equations are (see [5])

$$\begin{aligned} \varphi_1(r, \lambda, \tau) &= rh_1(r^2, \lambda, \tau) \quad \text{with} \quad h_1 = a_1\lambda + b_1r^2 + \dots \\ \varphi_2(r, \lambda, \tau) &= rh_2(r^2, \lambda, \tau) \quad \text{with} \quad h_2 = -\tau + a_2\lambda + b_2r^2 + \dots \end{aligned} \quad (4.21)$$

Formulas are now derived for the coefficients  $a_j, j = 1, 2$  and  $b_j, j = 1, 2$ . Note first of all that the inner products with  $\mathbf{v}_j^*, j = 1, 2$  of the term  $\mathbf{L}\mathbf{w}$ , which is in the range of  $\mathbf{L}$ , cancel, the range being orthogonal to the kernel of  $\mathbf{L}^*$ . Now, only the terms

$$e^{is} \mathbf{F}_1^{(1)} + c\mathbf{c}$$

in the development of  $\mathbf{F}_1$  provide a nonzero contribution after projection. Writing

$$\mathbf{F}_1^{(1)} = r\lambda \mathbf{F}_{1;1,\lambda}^{(1)} + r^3 \mathbf{F}_{1;3}^{(1)}$$

some (a bit tedious) calculation provides the formulas (the terms associated with  $\mathbf{L}\mathbf{w}$  being discarded)

$$\begin{aligned} \mathbf{F}_{1;1,\lambda}^{(1)} &= \\ \mathbf{B}_1^{-1} &\left( i\mathbf{M}_1 \mathbf{S} \mathbf{z}_{1,\lambda}^{(1)} + \frac{i}{2} \mathbf{M}_{1,\lambda} \mathbf{T} \mathbf{c} + \frac{i}{2} D \mathbf{M}_1 [\mathbf{T} \mathbf{w}_\lambda + \mathbf{S} \mathbf{z}_\lambda] \mathbf{T} \mathbf{c} \right. \\ &\left. + D \mathbf{G}_1 [\mathbf{S} \mathbf{z}_{1,\lambda}^{(1)}] + \frac{1}{2} D \mathbf{G}_{1,\lambda} [\mathbf{T} \mathbf{c}] + \frac{1}{2} D^2 \mathbf{G}_1 [\mathbf{T} \mathbf{w}_\lambda + \mathbf{S} \mathbf{z}_\lambda, \mathbf{T} \mathbf{c}] \right) \end{aligned} \quad (4.22)$$

and

$$\begin{aligned} \mathbf{F}_{1;3}^{(1)} &= \mathbf{B}_1^{-1} \left( \frac{i}{6} \mathbf{M}_1 \mathbf{S} \mathbf{z}_3^{(1)} + \frac{i}{4} D \mathbf{M}_1 [\mathbf{T} \mathbf{w}_2^{(0)} + \mathbf{S} \mathbf{z}_2^{(0)}] \mathbf{T} \mathbf{c} \right. \\ &- \frac{i}{4} D \mathbf{M}_1 [\mathbf{T} \mathbf{w}_2^{(2)} + \mathbf{S} \mathbf{z}_2^{(2)}] \mathbf{T} \bar{\mathbf{c}} + \frac{i}{2} D \mathbf{M}_1 [\mathbf{T} \bar{\mathbf{c}}] (\mathbf{T} \mathbf{w}_2^{(2)} + \mathbf{S} \mathbf{z}_2^{(2)}) \\ &- \frac{i}{16} D^2 \mathbf{M}_1 [\mathbf{T} \mathbf{c}, \mathbf{T} \mathbf{c}] \mathbf{T} \bar{\mathbf{c}} + \frac{i}{8} D^2 \mathbf{M}_1 [\mathbf{T} \mathbf{c}, \mathbf{T} \bar{\mathbf{c}}] \mathbf{T} \mathbf{c} + \frac{1}{6} D \mathbf{G}_1 [\mathbf{S} \mathbf{z}_3^{(1)}] \\ &+ \frac{1}{4} D^2 \mathbf{G}_1 [\mathbf{T} \mathbf{c}, \mathbf{T} \mathbf{w}_2^{(0)} + \mathbf{S} \mathbf{z}_2^{(0)}] + \frac{1}{4} D^2 \mathbf{G}_1 [\mathbf{T} \bar{\mathbf{c}}, \mathbf{T} \mathbf{w}_2^{(2)} + \mathbf{S} \mathbf{z}_2^{(2)}] \\ &\left. + \frac{1}{16} D^3 \mathbf{G}_1 [\mathbf{T} \mathbf{c}, \mathbf{T} \mathbf{c}, \mathbf{T} \bar{\mathbf{c}}] \right), \end{aligned} \quad (4.23)$$

(the formulas for the different Taylor expansion terms of  $\mathbf{w}$  and  $\mathbf{z}$  being those derived in section 4.1). Given the expressions of  $\mathbf{v}_j^*$ ,  $j = 1, 2$  (see (3.14)) one gets

$$\langle \mathbf{v}_1^*, e^{is} \mathbf{F}_1^{(1)} + cc \rangle = \mathcal{R} \left( \bar{\mathbf{d}}^T \mathbf{F}_1^{(1)} \right), \quad \langle \mathbf{v}_2^*, e^{is} \mathbf{F}_1^{(1)} + cc \rangle = -\mathcal{I} \left( \bar{\mathbf{d}}^T \mathbf{F}_1^{(1)} \right),$$

and one recovers the coefficients in (4.21), that is

$$\begin{aligned} a_1 &= \mathcal{R} \left( \bar{\mathbf{d}}^T \mathbf{F}_{1;1,\lambda}^{(1)} \right), & b_1 &= \mathcal{R} \left( \bar{\mathbf{d}}^T \mathbf{F}_{1;3}^{(1)} \right) \\ a_2 &= -\mathcal{I} \left( \bar{\mathbf{d}}^T \mathbf{F}_{1;1,\lambda}^{(1)} \right), & b_2 &= -\mathcal{I} \left( \bar{\mathbf{d}}^T \mathbf{F}_{1;3}^{(1)} \right) \end{aligned} \quad (4.24)$$

Now set the local equations (4.21) to zero, the trivial solution (that is no perturbation) is given by  $r = 0$ . The small-amplitude periodic solution is provided by solving  $h_j(r^2, \lambda, \tau) = 0$ ,  $j = 1, 2$ . In the following it is supposed that  $a_1 \neq 0$  and  $b_1 \neq 0$ , which are the conditions that the Hopf bifurcation is non-degenerate (see the classifications in [5], chapter 8.5.). If those coefficients were zero, one would have to pursue the different expansions, which are already rather tedious in the present non-degenerate case, which can be considered as the generic situation. There are no terms in  $\tau^k$ ,  $k \geq 2$  in the expansions of  $h_j$ ,  $j = 1, 2$ , the function  $\mathbf{F}_1$  depending linearly on  $\tau$  and by Proposition 4.1. It follows that  $\frac{\partial}{\partial \tau} h_2(0, 0, \tau) = -1$ , which allows to solve  $\tau(r^2, \lambda)$ , that is

$$h_2(r^2, \lambda, \tau) \equiv 0 \quad \text{for} \quad \tau(r^2, \lambda). \quad (4.25)$$

It remains the equation

$$h(r^2, \lambda) = h_1(r^2, \lambda, \tau(r^2, \lambda)) = 0. \quad (4.26)$$

Supposing a small amplitude, say  $r = O(\epsilon)$ , and making the assumption that the parameter deviation  $\lambda = O(\epsilon^2)$ , one has  $h(r^2, \lambda) = a_1 \lambda + b_1 r^2 + O(\epsilon^4)$  and neglecting higher order terms one gets

$$r^2 = -\frac{a_1}{b_1} \lambda, \quad \text{the sign of } \lambda \text{ being such that } \frac{a_1}{b_1} \lambda < 0. \quad (4.27)$$

This provides the link between the amplitude  $r$  and the parameter  $\lambda$  for a periodic small-amplitude nonlinear solution to exist, the sign of  $a_1/b_1$  determining the sign of the parameter deviation. For  $\lambda$  and the corresponding  $r > 0$  one determines  $\tau$  by the second equation of (4.21) and

$$\tau = a_2 \lambda + b_2 r^2 \quad (4.28)$$

(where again the terms of order  $r^q$ ,  $q \geq 4$  are discarded). Note that to only keep these terms to qualitatively characterize the periodic solution in the

vicinity of the bifurcation point can be justified by what is called normal-form theory (see for instance [5]). Of course, the developments performed allow to write the periodic bifurcating solution (at the Hopf bifurcation of the DAE's steady state) up to the third order in amplitude  $r$ , that is by writing

$$\tilde{\omega} = (1 + \tau)\omega$$

(given the scaled time relation  $s = \omega(1 + \tau)t$ )

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{T}\mathbf{y}(t) + \mathbf{S}\mathbf{z}(t) \quad \text{with} \\ \mathbf{y}(t) &= \lambda\mathbf{w}_\lambda + \frac{1}{2}r^2\mathbf{w}_2^{(0)} + 2\mathcal{R}\left[e^{i\tilde{\omega}t}\left(r\frac{1}{2}\mathbf{c} + \lambda r\mathbf{w}_{1,\lambda}^{(1)} + \frac{1}{6}r^3\mathbf{w}_3^{(1)}\right)\right] \\ &\quad + 2\mathcal{R}\left[\frac{1}{2}r^2e^{i2\tilde{\omega}t}\mathbf{w}_2^{(2)} + \frac{1}{6}r^3e^{i3\tilde{\omega}t}\mathbf{w}_3^{(3)}\right], \\ \mathbf{z}(t) &= \lambda\mathbf{z}_\lambda + \frac{1}{2}r^2\mathbf{z}_2^{(0)} + 2\mathcal{R}\left[e^{i\tilde{\omega}t}\left(\lambda r\mathbf{z}_{1,\lambda}^{(1)} + \frac{1}{6}r^3\mathbf{z}_3^{(1)}\right)\right] \\ &\quad + 2\mathcal{R}\left[\frac{1}{2}r^2e^{i2\tilde{\omega}t}\mathbf{z}_2^{(2)} + \frac{1}{6}r^3e^{i3\tilde{\omega}t}\mathbf{z}_3^{(3)}\right] \end{aligned} \quad (4.29)$$

In the next subsection the question of stability of the small-amplitude periodic solution is addressed, again following the development in [5] emphasizing where the DAE setting differs from the classical dynamical system framework.

#### 4.3. Stability of the periodic solution

The local bifurcating perturbation  $\mathbf{x}(t)$  of the steady state is solution of (2.8) or equivalently (the scaled time  $s$  is considered) by solving (3.2), taking into account the expansion

$$\mathbf{x}(s) = \mathbf{T}\mathbf{y}(s) + \mathbf{S}\mathbf{z}(s), \quad \mathbf{y}(s) \in \mathbb{R}^p, \quad \mathbf{z}(s) \in \mathbb{R}^m, \quad (4.30)$$

the operators  $\mathbf{T}$ ,  $\mathbf{S}$  having been defined in section 2. Let

$$\tilde{\mathbf{x}}(s) = \mathbf{T}\tilde{\mathbf{y}}(s) + \mathbf{S}\tilde{\mathbf{z}}(s)$$

be a perturbation of the periodic solution, that is

$$D\mathbf{F}(\mathbf{y}(s), \mathbf{z}(s), \lambda, \tau) \begin{pmatrix} \tilde{\mathbf{y}}(s) \\ \tilde{\mathbf{z}}(s) \end{pmatrix} = 0, \quad \begin{pmatrix} \tilde{\mathbf{y}}(2\pi) \\ \tilde{\mathbf{z}}(2\pi) \end{pmatrix} = \gamma \begin{pmatrix} \tilde{\mathbf{y}}(0) \\ \tilde{\mathbf{z}}(0) \end{pmatrix}. \quad (4.31)$$

The number  $\gamma$  is called a Floquet multiplier. The periodic solution is asymptotically stable if all Floquet multipliers are such that  $|\gamma| < 1$  (if however one

Floquet multiplier is such that  $|\gamma| > 1$ , the solution is unstable). In the DAE setting  $D\mathbf{F} = (D\mathbf{F}_1, D\mathbf{F}_2)$  and given the expressions (3.3), (3.4)

$$D\mathbf{F}_1(\mathbf{y}(s), \mathbf{z}(s), \lambda, \tau) \begin{pmatrix} \tilde{\mathbf{y}}(s) \\ \tilde{\mathbf{z}}(s) \end{pmatrix} = \mathbf{B}_1^{-1} D\mathbf{G}_1(\mathbf{x}(s), \lambda) \tilde{\mathbf{x}}(s) \\ + (1 + \tau) \mathbf{B}_1^{-1} \left( \mathbf{M}_1(\mathbf{x}(s), \lambda) \frac{d}{ds} \tilde{\mathbf{x}}(s) + D\mathbf{M}_1(\mathbf{x}(s), \lambda) [\tilde{\mathbf{x}}(s)] \frac{d}{ds} \mathbf{x}(s) \right)$$

whereas

$$D\mathbf{F}_2(\mathbf{y}(s), \mathbf{z}(s), \lambda) \begin{pmatrix} \tilde{\mathbf{y}}(s) \\ \tilde{\mathbf{z}}(s) \end{pmatrix} = D\mathbf{G}_2(\mathbf{x}(s), \lambda) \tilde{\mathbf{x}}(s)$$

Now, a generalized eigenvalue relation has to be introduced for the DAE setting. Let

$$\mathbf{q} = \mathbf{T}\mathbf{q}_T + \mathbf{S}\mathbf{q}_S \quad (4.32)$$

(the reference to time  $s$  is omitted when writing the functions) be such that

$$D\mathbf{F}_1(\mathbf{y}, \mathbf{z}, \lambda, \tau) \begin{pmatrix} \mathbf{q}_T \\ \mathbf{q}_S \end{pmatrix} = \mu \mathbf{B}_1^{-1} \mathbf{M}_1(\mathbf{x}, \lambda) \mathbf{q}, \\ D\mathbf{F}_2(\mathbf{y}, \mathbf{z}, \lambda) \begin{pmatrix} \mathbf{q}_T \\ \mathbf{q}_S \end{pmatrix} = 0, \quad \text{with} \quad \mathbf{q}(2\pi) = \mathbf{q}(0), \quad (4.33)$$

then it is easy to see that

$$\begin{pmatrix} \tilde{\mathbf{y}} \\ \tilde{\mathbf{z}} \end{pmatrix} = e^{-\mu s/(1+\tau)} \begin{pmatrix} \mathbf{q}_T \\ \mathbf{q}_S \end{pmatrix} \quad \text{with} \quad \gamma = e^{-\mu 2\pi/(1+\tau)}$$

is solution of (4.31). It follows that

$$\mu = -\frac{1+\tau}{2\pi} \log(\gamma)$$

and accordingly if  $\mathcal{R}(\mu) > 1$  then  $|\gamma| < 1$  and if  $\mathcal{R}(\mu) < 1$  then  $|\gamma| > 1$  (for  $\tau > -1$ , by emphasizing that  $|\tau|$  has to be supposed small). As explained in [5], the eigenfunction of the small-amplitude periodic solution may be considered as to be a perturbation of the steady state's eigenfunction. It is recalled that the stability of the steady state is inferred from the eigenvalues  $\mu_j$  of  $\tilde{\mathbf{A}} = \frac{1}{\omega} \mathbf{A}$  (see 2.10) with  $\mu_1 = -i$ ,  $\mu_2 = i$  and according to the assumptions made  $\mu_j = (\mu_j)_r + i(\mu_j)_i$  with  $(\mu_j)_r > 0$ ,  $j \geq 3$ . The periodic linear perturbations of the steady state is such that

$$D\mathbf{F}(0, 0, 0, 0) \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{0} \end{pmatrix} = 0,$$

( $\mathbf{0}$  being the null vector of  $\mathbb{R}^m$ ) with  $\mathbf{v}_1(2\pi) = \mathbf{v}_1(0)$  and hence  $\gamma = 1$  (again phase-shift arguments justify that only  $\mathbf{v}_1$  of (3.8) is considered). Also, it is to be supposed that the modulus of the Floquet multipliers  $\gamma$  of possible extensions to the small-amplitude periodic state of the stable part of the steady state eigenspectrum (that is corresponding to  $\mu_j, j \geq 3$ ) remain smaller than one.

Adapting the developments in [5] (chapter VIII, §4) to the present DAE context, the relevant eigenfunction for the stability of the small-amplitude periodic state is written as

$$\tilde{\mathbf{q}} = \mathbf{T}(\mathbf{v}_1 + \tilde{\mathbf{w}}) + \mathbf{S}\tilde{\mathbf{z}} \quad (4.34)$$

with  $(\tilde{\mathbf{w}}, \tilde{\mathbf{z}})$   $2\pi$ -periodic functions to be sought in the space  $E$  defined by (3.12). Before pursuing, it has to be observed, by taking the derivative with respect to time  $s$  of (3.2), that

$$D\mathbf{F}(\mathbf{y}, \mathbf{z}, \lambda, \tau) \begin{pmatrix} d\mathbf{y}/ds \\ dz/ds \end{pmatrix} = 0 \quad (4.35)$$

that is  $(d\mathbf{y}/ds, dz/ds)$  is eigenfunction with eigenvalue  $\mu = 0$  (and hence Floquet multiplier  $\gamma = 1$ ) It will be seen a little later why this is important for the analysis. Also, it has to be taken into account that the periodic solution  $\mathbf{x}$  as it has been constructed can be written

$$\mathbf{x}(r, \lambda) = \mathbf{T}(r\mathbf{v}_1 + \mathbf{w}(r, \lambda, \tau(r^2, \lambda))) + \mathbf{S}\mathbf{z}(r, \lambda, \tau(r^2, \lambda)), \quad (4.36)$$

the dependence of  $\tau(r^2, \lambda)$  on  $r^2$  and  $\lambda$  being such that the second of the reduced equations (3.21), with the function  $h_2$  introduced in (4.21), is identically equal to zero (see (4.25)).

A variable  $\eta$  is to be introduced and a solution  $\eta, \mu, \tilde{\mathbf{w}}, \tilde{\mathbf{z}}$  is sought as function of  $r$  and  $\lambda$ , such that

$$\begin{aligned} D\mathbf{F}_1(\mathbf{y}, \mathbf{z}, \lambda, \tau) \begin{pmatrix} \mathbf{v}_1 + \tilde{\mathbf{w}} \\ \tilde{\mathbf{z}} \end{pmatrix} + \eta \mathbf{B}_1^{-1} \mathbf{M}_1(\mathbf{x}, \lambda) \frac{1}{r} \frac{d}{ds} \mathbf{x} \\ - \mu \mathbf{B}_1^{-1} \mathbf{M}_1(\mathbf{x}, \lambda) \tilde{\mathbf{q}} = 0, \\ D\mathbf{F}_2(\mathbf{y}, \mathbf{z}, \lambda) \begin{pmatrix} \mathbf{v}_1 + \tilde{\mathbf{w}} \\ \tilde{\mathbf{z}} \end{pmatrix} = 0 \end{aligned} \quad (4.37)$$

( $\tilde{\mathbf{q}}$  being defined by (4.34)). Observe that the expression

$$\frac{1}{r} \frac{d}{ds} \mathbf{x} \rightarrow \frac{d}{ds} \mathbf{T}\mathbf{v}_1 = -\mathbf{T}\mathbf{v}_2$$

when  $r$  and  $\lambda$  tend to zero. Indeed, it has been seen that  $\mathbf{w}$  and  $\mathbf{z}$  have terms of order  $r^3$ ,  $r\lambda$  and higher order terms, but also a term in  $\lambda$ , whenever the steady state varies with  $\lambda$ . Given the relation (4.27) between  $r$  and  $\lambda$ , one may however conclude that  $\lambda/r$  tends to zero too.

The system (4.37) can be considered as a functional  $\Psi(\eta, \mu, \tilde{\mathbf{w}}, \tilde{\mathbf{z}}, r, \lambda) = 0$ , by observing that  $(\tilde{\mathbf{w}}, \tilde{\mathbf{z}}) \in E$  ( $E$  being defined by (3.12), cf. section 3). Taking the derivative with respect to  $\eta$  of the functional, one obtains at the limit the vector  $(-\mathbf{v}_2, \mathbf{0})$ , given that at the origin  $\mathbf{B}_1^{-1}\mathbf{M}_1\mathbf{T} = \mathbf{I}$  (again  $\mathbf{0}$  is the null vector in  $\mathbb{R}^m$ ). For the derivative with respect to  $\mu$ , one gets at the limit  $(-\mathbf{v}_1, \mathbf{0})$ . For the derivatives with respect to  $\tilde{\mathbf{w}}$  and  $\tilde{\mathbf{z}}$ , one recovers for  $r = \lambda = 0$  the operator  $D\mathbf{F}$  given by (3.6) replacing  $\mathbf{L}$  by  $\mathbf{L}|_N$ , the range of the resulting operator being however equal to the range of  $D\mathbf{F}$  (which is Fredholm of index zero). Consequently the differential of the functional with respect to  $\eta, \mu, \tilde{\mathbf{w}}, \tilde{\mathbf{z}}$  at the origin generates

$$\mathbb{R}(\mathbf{v}_1, \mathbf{0}) \oplus \mathbb{R}(\mathbf{v}_2, \mathbf{0}) \oplus \text{Range } D\mathbf{F} = C_{2\pi}$$

(the first two terms of the sum being the kernel of  $D\mathbf{F}$ ) and is hence invertible. By the implicit function theorem one can assert that locally  $\eta, \mu, \tilde{\mathbf{w}}, \tilde{\mathbf{z}}$  are functions of  $(r, \lambda)$ , these functions vanishing at  $(0, 0)$  and in particular  $\mu(r, \lambda)$  exists such that  $\mu(0, 0) = 0$ . It follows by considering (4.35) and (4.37) that the eigenfunction (4.32) associated to  $\mu \neq 0$  in (4.33) is such that

$$\mathbf{q}_T = \mathbf{v}_1 + \tilde{\mathbf{w}}(r, \lambda) - \frac{\eta}{r\mu} \frac{d\mathbf{y}}{ds}, \quad \mathbf{q}_S = \tilde{\mathbf{z}}(r, \lambda) - \frac{\eta}{r\mu} \frac{d\mathbf{z}}{ds}.$$

Now suppose that  $\mu(r, \lambda) = 0$  and consider the periodic small amplitude solution in the form (4.36), then it can easily be verified that when taking the derivative of  $\mathbf{F}$  with respect to  $r$  one gets (to simplify the notations  $\mathbf{w}(r, \lambda, \tau(r^2, \lambda))$  is written  $\mathbf{w}(r, \lambda)$ )

$$\begin{aligned} & \frac{\partial}{\partial r} \mathbf{F}_1(r\mathbf{v}_1 + \mathbf{w}(r, \lambda), \mathbf{z}(r, \lambda), \lambda, \tau(r^2, \lambda)) \\ &= D\mathbf{F}_1(r\mathbf{v}_1 + \mathbf{w}(r, \lambda), \mathbf{z}(r, \lambda), \lambda, \tau(r^2, \lambda)) \begin{pmatrix} \mathbf{v}_1 + \mathbf{w}_r(r, \lambda) \\ \mathbf{z}_r(r, \lambda) \end{pmatrix} \\ &+ \frac{\partial \tau(r^2, \lambda)}{\partial r} \mathbf{B}_1^{-1} \mathbf{M}_1(\mathbf{x}, \lambda) \frac{d\mathbf{x}(r, \lambda)}{ds} \\ & \frac{\partial}{\partial r} \mathbf{F}_2(r\mathbf{v}_1 + \mathbf{w}(r, \lambda), \mathbf{z}(r, \lambda), \lambda, \tau(r^2, \lambda)) \\ &= D\mathbf{F}_2(r\mathbf{v}_1 + \mathbf{w}(r, \lambda), \mathbf{z}(r, \lambda), \lambda, \tau(r^2, \lambda)) \begin{pmatrix} \mathbf{v}_1 + \mathbf{w}_r(r, \lambda) \\ \mathbf{z}_r(r, \lambda) \end{pmatrix} \end{aligned}$$

Comparing with the system (4.37), it is seen that if indeed

$$\frac{\partial}{\partial r} \mathbf{F}_j(r\mathbf{v}_1 + \mathbf{w}(r, \lambda), \mathbf{z}(r, \lambda), \lambda, \tau(r^2, \lambda)) = 0, \quad j = 1, 2, \quad (4.38)$$

one obtains for  $\mu(r, \lambda) = 0$  the (unique) solution of (4.37) with

$$\eta(r, \lambda) = r \frac{\partial}{\partial r} \tau(r^2, \lambda), \quad \tilde{\mathbf{w}}(r, \lambda) = \mathbf{w}_r(r, \lambda), \quad \tilde{\mathbf{z}}(r, \lambda) = \mathbf{z}_r(r, \lambda)$$

Recall that the local bifurcating solution is obtained by solving (3.19) and (4.38) is verified for  $\mathbf{F}_2$  and  $P\mathbf{F}_1$ . As mentioned above,  $\tau(x^2, \lambda)$  is such that the second of the reduced equations (3.21) is identically zero (see (4.25)) and it hence remains zero when taking its derivative with respect to  $r$ . Taking the derivative with respect to  $r$  of the first local equation of (3.21), one gets (the subscript  $r$  meaning the derivative)

$$\varphi_r(r, \lambda) = \left\langle \mathbf{v}_1^*, \frac{\partial}{\partial r} \mathbf{F}_1(\mathbf{x}(r, \lambda), \lambda, \tau(r^2, \lambda)) \right\rangle. \quad (4.39)$$

with  $\varphi(r, \lambda) = rh(r^2, \lambda)$ ,  $h$  being defined in (4.26).

It can be concluded that

$$\varphi_r(r, \lambda) = 0 \quad \text{implies} \quad \mu(r, \lambda) = 0$$

because then (4.38) is indeed verified. As explained in [5], one would like to go a step further and to show that if  $\varphi_r(r, \lambda) \neq 0$  then  $\mu(r, \lambda) \neq 0$  and that in that case

$$\varphi_r(r, \lambda) / \mu(r, \lambda) > 0$$

(that is these quantities have the same sign). In [5] (see chapter I, §4) it is shown that this is true in some neighborhood of the origin, if when by taking the derivative with respect to some variable (say  $\beta$ ), the functions  $\varphi_{r,\beta}$  and  $\mu_\beta$  are nonzero at the origin and have the same sign. Note that this can only be shown here by introducing  $\beta$  as an extra-parameter and by adding an extra-term in the system (3.2). In the present DAE setting this can be conveniently done for the system (2.8) replacing  $\mathbf{G}_1$  by

$$\tilde{\mathbf{G}}_1(\mathbf{x}, \lambda, \beta) = \mathbf{G}_1(\mathbf{x}, \lambda) + \beta \mathbf{B}_1 \mathbf{T}^T \mathbf{x} \quad (4.40)$$

(note that  $\mathbf{T}^T \mathbf{x}$  corresponds in fact to  $\mathbf{y}$  in the decomposition (2.13)) and  $\mathbf{B}_1$  is defined by (2.6)). Of course, in the discussion above one has to replace  $\varphi_j$  by  $\tilde{\varphi}_j$  adding the parameter  $\beta$  and  $\mu$  is replaced by  $\tilde{\mu}$ . The extra terms in the reduced equations have to be computed. Referring to the reduced equation



algorithm of section 4.1, the function  $\tilde{\mathbf{F}}_{1;1,\beta}^{(1)}$  (see (4.22), replacing  $\lambda$  by  $\beta$ ) has to be found and it is easy to see that it is simply

$$\frac{1}{2}\mathbf{B}_1^{-1}\mathbf{B}_1\mathbf{T}^T\mathbf{T}\mathbf{c} = \frac{1}{2}\mathbf{c}.$$

Consequently  $\tilde{\mathbf{d}}^T\tilde{\mathbf{F}}_{1;1,\beta}^{(1)} = 1$  because  $\tilde{\mathbf{d}}^T\mathbf{c} = 2$ . This means that the additional term  $\tilde{a}_1 r \beta$  in  $\tilde{\varphi}_1$  is  $r\beta$ , that is  $\tilde{a}_1 = 1$ . Note that  $\tilde{a}_2 = 0$  (see (4.24)) and when solving  $\tilde{h}_2(r^2, \lambda, \tilde{\tau}, \beta) = 0$  in the second reduced equation (4.21) one gets  $\tilde{\tau}(r^2, \lambda, \beta)$  with  $\tilde{\tau}(0, 0, \beta) = 0$ . Furthermore,  $\tilde{a}_1 = 1$  implies that  $\tilde{\varphi}_r(0, 0, \beta) = \beta$ . In the local solution (4.29) the additional term is  $\beta r \mathbf{w}_{1,\beta}^{(1)}$  and it is easy to verify (see the reduced equation algorithm)

$$(i\mathbf{I} + \tilde{\mathbf{A}})\mathbf{w}_{1,\beta}^{(1)} = -\tilde{P}\frac{1}{2}\mathbf{c} = 0$$

by definition of  $\tilde{P}$  (see (4.11)). This implies that  $\mathbf{w}_{1,\beta}^{(1)} = 0$ , because it cannot belong to the kernel of  $i\mathbf{I} + \tilde{\mathbf{A}}$ ,  $\mathbf{w}$  being in range of  $\mathbf{L}$ . This means that  $\mathbf{x}(r, \lambda, \beta) = \mathbf{x}(r, \lambda)$  and hence  $\mathbf{x}(0, 0, \beta) = 0$  and one has also

$$\frac{1}{r}\frac{d}{ds}\tilde{\mathbf{x}}(r, \lambda, \beta) \rightarrow \frac{d}{ds}\mathbf{T}\mathbf{v}_1 = -\mathbf{T}\mathbf{v}_2$$

when  $(r, \lambda, \beta) \rightarrow (0, 0, \beta)$ .

Now the system (4.37) is to be considered with the modified function  $\tilde{\mathbf{F}}_1$ . One can convince oneself when putting  $r = \lambda = 0$  (and consequently  $\tilde{\tau} = 0$ ) that  $D\tilde{\mathbf{F}}(0, 0, 0, 0, \beta)$  has the structure (3.6), but  $\mathbf{L}$  has to be replaced by  $\mathbf{L} + \beta\mathbf{I}$  with  $\mathbf{I}$  the  $p \times p$  identity matrix. Taking  $\tilde{\mathbf{q}} = (\mathbf{v}_1, \mathbf{0})$ , the second equation is verified and the first equation becomes (putting  $r = \lambda = 0$ )

$$(\mathbf{L} + \beta\mathbf{I})\mathbf{v}_1 - \tilde{\eta}(0, 0, \beta)\mathbf{v}_2 - \tilde{\mu}(0, 0, \beta)\mathbf{v}_1 = 0.$$

and one can conclude that the solution is  $\tilde{\eta}(0, 0, \beta) = 0$ , because  $\mathbf{L}\mathbf{v}_1 = 0$  and  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly independent, and hence  $\tilde{\mu}(0, 0, \beta) = \beta$ . Consequently one gets  $\tilde{\mu}_\beta(0, 0, 0) = \tilde{\varphi}_{r,\beta}(0, 0, 0) = 1$ . As mentioned above, this is precisely the supplementary condition to conclude (going back to  $\varphi_r$  and  $\mu$  setting  $\beta = 0$ ), that  $\varphi_r(r, \lambda)$  and  $\mu(r, \lambda)$  are nonzero and of same sign in some neighborhood of the origin.

It is to be recalled that  $\mu(r, \lambda) > 0$  implies stability and  $\mu(r, \lambda) < 0$  instability. Now,

$$\varphi_r(r, \lambda) = h(r^2, \lambda) + 2r^2b_1 + O(r^4) = 2r^2b_1 + O(r^4),$$

because the nontrivial bifurcating solution exists for  $h(r^2, \lambda) = 0$  with  $h$  defined by (4.26). One may hence conclude for the DAE system (as for ODEs) that the periodic solution in the neighborhood of the Hopf bifurcation is locally

$$\text{stable if } b_1 > 0 \quad \text{and unstable if } b_1 < 0.$$

## 5. Case study: an electric power system model

Electric power system models are frequently considered as examples of DAEs (e.g. [10–12, 18]) which are likely to exhibit Hopf bifurcations, which may be subcritical or supercritical as for instance discussed in [19]. Such models are also discussed in chapter 4 of the monograph [6] devoted to an overview on applied differential-algebraic equations. A center-manifold reduction for a power system model, however structured as an ODE, has been performed in [20]. In [21] multimachine dynamic models are derived, that is the network between generators and loads, the so-called buses being elements (transmission lines or transformers) by which the components of the network are connected. For the present purpose it is out of the scope to go into details about the justification of the classical model (see for instance chapters 6 and 7 in [21]) which writes for  $n_g + 1$  generators and  $n_l$  loads

$$\begin{aligned} \frac{d\beta_i}{dt} - \omega_i &= 0, \quad i = 1, \dots, n_g + 1, \\ h_i \frac{\omega_i}{dt} - T_i + \sum_{k=1}^{n_g+n_l+1} b_{ik} v_i v_k \sin(\beta_i - \beta_k) &= 0, \quad i = 1, \dots, n_g + 1, \\ P_i - \sum_{k=1}^{n_g+n_l+1} b_{n_g+1+i,k} v_{n_g+1+i} v_k \sin(\beta_{n_g+1+i} - \beta_k) &= 0, \quad i = 1, \dots, n_l, \\ Q_i + \sum_{k=1}^{n_g+n_l+1} b_{n_g+1+i,k} v_{n_g+1+i} v_k \cos(\beta_{n_g+1+i} - \beta_k) &= 0, \quad i = 1, \dots, n_l, \end{aligned}$$

where  $\beta_j$  is the angle,  $\omega_j$  the frequency and  $v_j$  the voltage magnitude of the  $j$ th bus,  $h_i$  is the inertia and  $T_i$  the mechanical power injection of the  $i$ th generator,  $P_i$  and  $Q_i$  are the real and reactive power loads and  $b_{jk}$  is the transfer susceptance between buses  $j$  and  $k$ . The quantities  $h_i$ ,  $T_i$ ,  $P_i$ ,  $Q_i$  as well as  $b_{jk}$  (with  $b_{jk} = b_{kj}$ ) are parameters for the system as well as the voltage magnitudes  $v_i$ ,  $i = 1, \dots, n_g + 1$  of the generators,

$$\mathbf{u} = (\beta_1, \omega_1, \dots, \beta_{n_g+1}, \omega_{n_g+1}, \beta_{n_g+2}, \dots, \beta_{n_g+n_l+1}, v_{n_g+2}, \dots, v_{n_g+n_l+1})^T$$

being the vector of the unknowns. A difficulty arises with the above system when seeking for a steady state. Indeed, the Jacobian matrix of the nonlinear function at any vector  $\tilde{\mathbf{u}}$  is singular: indeed, for any constant  $c$ , the vector  $\mathbf{x}$  with  $x_{2i-1} = c$ ,  $i = 1, \dots, n_g + 1$  and  $x_{2(n_g+1)+i} = c$ ,  $i = 1, \dots, n_l$  (the other components of  $\mathbf{x}$  being zero) is in its nullspace. This is a consequence of what is mentioned in [21], that is that every rotational system must have a reference for angles. Therefore, the classical approach is to define the angles relative to generator 1 and the relative angles and frequencies

$$\theta_i = \beta_{i+1} - \beta_1, \quad \Omega_i = \omega_{i+1} - \omega_1, \quad i = 1, \dots, n_g, \quad \theta_{n_g+i} = \beta_{n_g+1+i} - \beta_1, \quad i = 1, \dots, n_l \quad (5.1)$$

are hence introduced. To recover the new system, the dynamic equations for  $\beta_1, \omega_1$  have to be subtracted from those for  $\beta_i, \omega_i, i = 2, \dots, n_g + 1$  and to recover equivalent equations for  $\theta_i, \Omega_i$  the assumption has to be made that  $h_i = h, i = 1, \dots, n_g + 1$ . The (known) voltage magnitude  $v_1$  is combined with  $b_{i1}$  writing

$$B_i = v_1 b_{i+1,1}, \quad i = 1, \dots, n_g + n_l. \quad (5.2)$$

It is convenient to introduce the renumbered voltage amplitudes and the transfer susceptance

$$V_i = v_{i+1}, \quad i = 1, \dots, n_g + n_l, \quad B_{ik} = b_{i+1,k+1}, \quad i = 1, \dots, n_g + n_l, \quad k = 1, \dots, n_g + n_l. \quad (5.3)$$

In the following  $T_i$  is written for  $T_{i+1} - T_1$  and the system becomes

$$\begin{aligned} \frac{d\theta_i}{dt} - \Omega_i &= 0, \quad i = 1, \dots, n_g, \\ h \frac{d\Omega_i}{dt} + \gamma_i \Omega_i - T_i + B_i V_i \sin(\theta_i) \\ &+ \sum_{k=1}^{n_g+n_l} (B_k V_k \sin(\theta_k) + B_{ik} V_i V_k \sin(\theta_i - \theta_k)) = 0, \quad i = 1, \dots, n_g, \end{aligned} \quad (5.4)$$

the algebraic part being

$$\begin{aligned} P_i - B_{n_g+i} V_{n_g+i} \sin(\theta_{n_g+i}) \\ - \sum_{k=1}^{n_g+n_l} B_{n_g+i,k} V_{n_g+i} V_k \sin(\theta_{n_g+i} - \theta_k) &= 0, \quad i = 1, \dots, n_l, \\ Q_i + B_{n_g+i} V_{n_g+i} \cos(\theta_{n_g+i}) \\ + \sum_{k=1}^{n_g+n_l} B_{n_g+i,k} V_{n_g+i} V_k \cos(\theta_{n_g+i} - \theta_k) &= 0, \quad i = 1, \dots, n_l. \end{aligned} \quad (5.5)$$

A similar system (with  $n_g = 2$  and  $n_l = 1$ ) has been considered in [22]. It will become clear later why in (5.4) damping coefficients  $\gamma_i$  have been introduced. The dimension of the system above is  $n = 2(n_g + n_l)$  and defining the components of the solution vector  $\mathbf{u} \in \mathbb{R}^n$  are as follows

$$\begin{aligned} u_{2i-1} &= \theta_i, \quad u_{2i} = \Omega_i, \quad i = 1, \dots, n_g, \\ u_{2n_g+i} &= \theta_{n_g+i}, \quad u_{2n_g+n_l+i} = V_{n_g+i}, \quad i = 1, \dots, n_l \end{aligned} \quad (5.6)$$

the system is of the form (2.1) with

$$\mathbf{M} \in \mathbb{R}^{n,n} \quad \text{with} \quad m_{2k-1,2k-1} = 1, \quad m_{2k,2k} = h, \quad k = 1, \dots, n_g$$

and all other coefficients of this matrix are zero.

The first task is to compute steady state solutions for specific parameter values, by first fixing the dimension of the system and for the example to be discussed hereafter  $n_g = 3$  and  $n_l = 4$ , that is  $n = 14$ . The parameters have been fixed (quite arbitrarily) as follows

$$\begin{aligned} B_{1j} &= 1, \quad j = 2, \dots, 7, & B_{2j} &= 2, \quad j = 3, \dots, 7, & B_{3j} &= 3, \quad j = 4, \dots, 7, \\ B_{44} &= -2, & B_{4j} &= 2, \quad j = 5, \dots, 7 & B_{55} &= -3, & B_{5j} &= 2, \quad j = 6, 7, \\ B_{66} &= -4, & B_{67} &= 1, & B_{77} &= -3, \\ B_i &= 1, \quad i = 1, 2, 3, & B_4 &= 5, & B_5 &= 1, & B_6 &= 5, & B_7 &= 1, \\ V_i &= 1, \quad i = 1, 2, 3, & h &= 1. \end{aligned}$$

(only  $B_{ij}, j \geq i$  are written given that  $B_{ji} = B_{ij}$  and note that  $B_{ii}, i = 1, \dots, n_g$  are not specified, because the corresponding terms in (5.4) are zero). It is not pretended here to provide a meaningful parameter set in the context of electric power engineering, but rather to illustrate for a hopefully meaningful example the reduction procedure. Note that for instance in [22] similar parameter value magnitudes for a power system model with  $n_g = 2$  and  $n_l = 1$  have been considered (see also the examples in [6], chapter 4).

The mechanical power injection parameters for the generators have been chosen as

$$T_1 = 1, \quad T_2 = 2, \quad T_3 = 0.5$$

and the key parameters are the real and reactive power loads which have been fixed in (5.5) as

$$P_2 = 1.8, \quad Q_2 = 0.9, \quad P_3 = 1, \quad Q_3 = 0.5, \quad P_4 = 0.8, \quad Q_4 = 0.4.$$

A bifurcation parameter  $\tilde{\alpha}$  is introduced such that the real and reactive power parameter of the first load are

$$P_1 = 2.2 (1 + \tilde{\alpha}), \quad Q_1 = 1.1 (1 + \tilde{\alpha}). \quad (5.7)$$

These parameter values proved appropriate to compute a family of steady states for a continuous set of values  $\tilde{\alpha}$ .

Writing the system (5.4), (5.5) in the form (2.1), the operators  $\mathbf{Z}_2$  and  $\mathbf{Z}_1$  of (2.3) and (2.5) are immediately evident and (with the notations of section 2 and 3)  $p = 2n_g = 6$ ,  $m = 2n_l = 8$ . The Matlab programming language has been used for the computations (see Appendix A for the algorithms to reliably compute the ranks and kernels of the operators). At the steady states  $(\tilde{\mathbf{u}}, \tilde{\alpha})$  the operator  $D\mathbf{G}_2(\tilde{\mathbf{u}}, \tilde{\alpha})$  has maximum rank  $m = 8$  and the nullspace operator  $\mathbf{T} \in \mathbb{R}^{14,6}$  according to (2.4) is to be computed, as well as  $\mathbf{S} \in \mathbb{R}^{14,8}$  in the

decomposition (2.11). The operator  $\mathbf{B}_1$  given by (2.6) is formed and the linear stability at the steady state when writing  $\mathbf{x}(t) = \mathbf{T}\mathbf{y}(t)$  is provided by the system (2.10), where  $\mathbf{A}$  for the present case is a  $6 \times 6$  matrix. In the framework of the Hopf bifurcation theory there has to be only one complex conjugate eigenvalue pair  $\sigma_1 \pm i\omega_1$  with the real part changing sign at specific points, the real parts of the other eigenvalues have to be positive, which stands for stability in the present formulation. The damping coefficients in (5.4) have precisely been introduced to achieve this conditions and the values

$$\gamma_1 = 0.1 \quad \gamma_2 = 0.01, \quad \gamma_3 = 0$$

have been chosen: one may indeed consider that more or less slight damping is also present in a real system. For  $\tilde{\alpha}$  in (5.7) within a certain range, steady states could indeed be computed and the linear stability analysis provided 3 complex conjugate eigenvalue pairs

$$\sigma_j \pm i\omega_j, \quad j = 1, 2, 3$$

where  $\sigma_2$  and  $\sigma_3$  are positive, whereas  $\sigma_1$  changes sign at specific points. The  $\sigma_1$  value as function of  $\tilde{\alpha} \in [-0.8, 1.1]$  is provided in figure 1 and two points are seen to exist, that is the Hopf bifurcation point marked H1 where the system transits from stability to instability and a second point H2 where for increasing  $\tilde{\alpha}$  the steady state becomes stable again. In this range of the parameter the steady state solution evolves continuously as well as the imaginary part of the eigenvalue pair  $\sigma_1 \pm i\omega_1$ , and its real part is such that  $\sigma_1 < \sigma_j$ ,  $j = 2, 3$ . The steady state solution, the eigenvalues and bifurcation parameter at point H1 are (see (5.6) for the link between the coefficients of  $\mathbf{u}$  and the power system state variables)

$$\begin{aligned} \tilde{\mathbf{u}}_1 = & (0.0283, 0, 0.4587, 0, 0.3006, 0, 2.2568, 2.2954, 2.2546, 2.3911, \\ & 0.1850, 0.2914, 0.1069, 0.1338), \\ \tilde{\alpha}_1 = & -0.21544 \quad \text{with eigenvalues:} \\ & \pm i 1.69766, \quad 0.0027324 \pm i 2.28574, \quad 0.0522676 \pm i 1.82009 \end{aligned}$$

whereas at H2 one gets

$$\begin{aligned} \tilde{\mathbf{u}}_2 = & (-0.2824, 0, 0.2688, 0, 0.1495, 0, 2.1085, 2.2030, 2.2004, 2.3054, \\ & 0.4500, 0.3000, 0.1081, 0.1378), \\ \tilde{\alpha}_2 = & 0.9536814 \quad \text{with eigenvalues:} \\ & \pm i 1.61938, \quad 0.0029354 \pm i 2.21595, \quad 0.052065 \pm i 1.66224. \end{aligned}$$

It is assured that at the points H1 and H2 the real part  $\sigma_1$  of the first eigenvalue pair is indeed computationally zero (of the order  $10^{-10}$ ). Note that

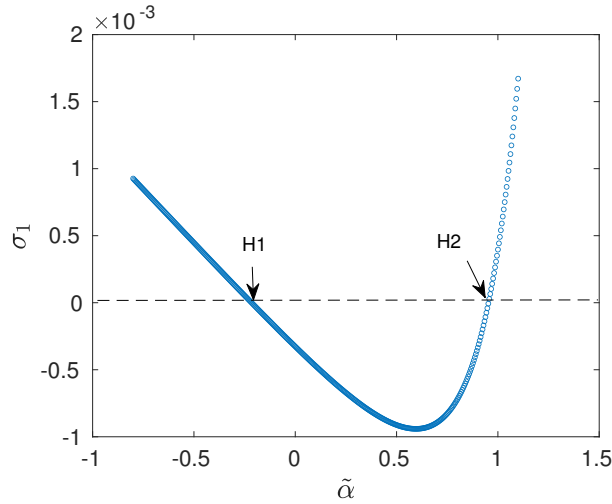


Figure 1: The real part  $\sigma_1$  of the least stable (and unstable between the points H1 and H2) complex eigenvalue pair as function of the bifurcation parameter  $\tilde{\alpha}$ , the points H1 and H2 where  $\sigma_1 = 0$  being the Hopf bifurcation points.

the angles  $\theta_j$  are defined as to be within  $]-\pi, \pi]$ . At both Hopf bifurcation points the Lyapunov-Schmidt reduction has been performed. The different formulas are provided in section 4.1 which can directly be applied; in the present case only  $\mathbf{G}_2$  depends on the bifurcation parameter and the matrix  $\mathbf{M}$  has constant coefficients, which means that the terms  $\mathbf{G}_{1,\lambda}$ ,  $\mathbf{M}_{1,\lambda}$ ,  $D\mathbf{M}_1[\cdot]$  and  $D^2\mathbf{M}_1[\cdot, \cdot]$  can be discarded in the formulas. Also, in Appendix C it is indicated how the multivariable expressions of the kind (4.1) can be computed in the present case. The coefficients  $a_j, b_j, j = 1, 2$  are computed according to (4.24) and the local solution (4.29) associated with the reduced equation is obtained as function of  $r$  with the frequency  $\tilde{\omega} = (1 + \tau)\omega$ ,  $\pm i\omega$  being the imaginary part of the eigenvalue pair with zero real part at the Hopf bifurcation. Let  $\lambda$  be the local deviation of the bifurcation parameter at the Hopf bifurcation point. It is recalled that for small amplitudes  $r = O(\epsilon)$  and neglecting terms of order  $\epsilon^q, q \geq 4$  in the reduced equations (4.21)

$$r^2 = -\frac{a_1}{b_1}\lambda, \quad \tau = a_2\lambda + b_2r^2,$$

and that the local periodic solution is stable if  $b_1 > 0$  and unstable when  $b_1 < 0$ .

The values at the first Hopf bifurcation H1 are

$$a_1 = -8.6858 \cdot 10^{-4}, \quad b_1 = 1.1521 \cdot 10^{-4}, \quad a_2 = -0.0443, \quad b_2 = -0.0194.$$

Given that  $-a_1/b_1 = 7.5389$  is positive,  $\lambda$  is to be taken positive and the

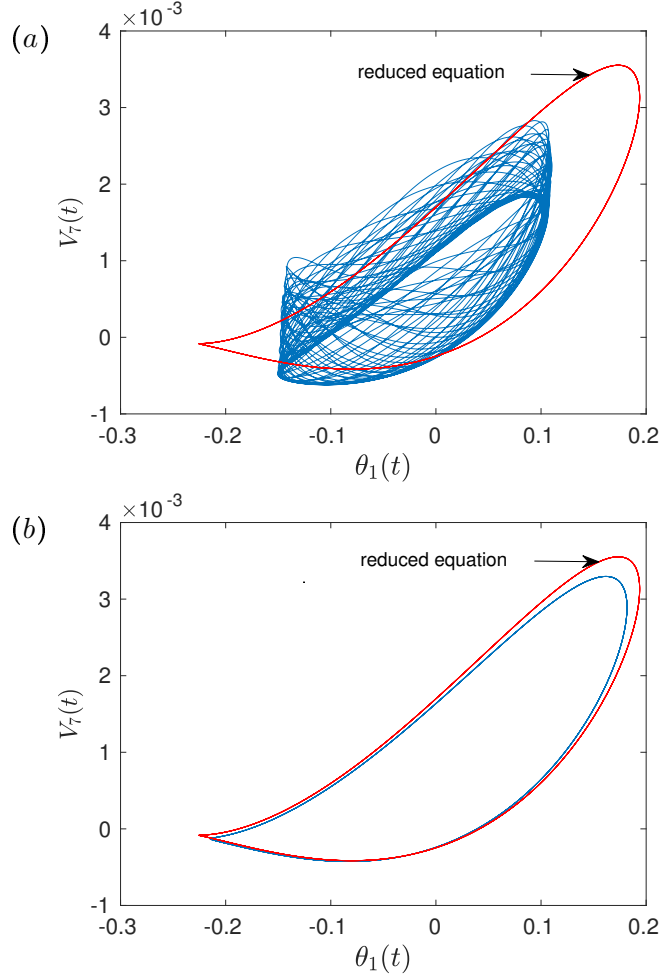


Figure 2: Phase-portrait in the  $(\theta_1(t), V_7(t))$ -plane for the global solution (blue, starting with a random initial condition with  $0 < x_j < 1, j = 1, \dots, 14$  and  $\|\mathbf{x}(0)\|_2 = 1$ ) near the Hopf bifurcation point H1 ( $\lambda = 0.1$ ) during  $[50T, 100T]$  (a) and the asymptotic regime (b), the red curve being the solution (4.29) of the theory.

value  $\lambda = 0.1$  has been chosen, even though the corresponding  $r = 0.8683$  is not particularly small, which yields  $\tau = -0.019$ . The small amplitude local solution is predicted as to be stable,  $b_1$  being positive, and to see whether this prediction is reliable, the global nonlinear system (2.8), where the steady state  $(\tilde{\mathbf{u}}_1, \tilde{\mathbf{a}}_1)$  is simply written  $(0, 0)$ , has been integrated in time, using a second-order implicit backward-Euler discretization in time together with Newton-Raphson iterations at each time step. The components of  $\mathbf{x}$  are noted as in (5.6). A random initial condition for the perturbation has been considered, each component being a random number between 0 and 1, the 2-norm of  $\mathbf{x}(0)$  being 1. The phase-portrait in the  $(x_1, x_{14}) = (\theta_1(t), V_7(t))$ -plane is shown in figure 2. Figure 2(a) shows the solution for the time interval  $[50T, 100T]$  ( $T = 2\pi/\tilde{\omega}$  being the period) and the integrated solution is seen to slowly evolve, the reduced equation solution as defined by (4.29) being the red contour. By stepping further on in time, the global solution ultimately almost superimposes with the theoretical solution given by the

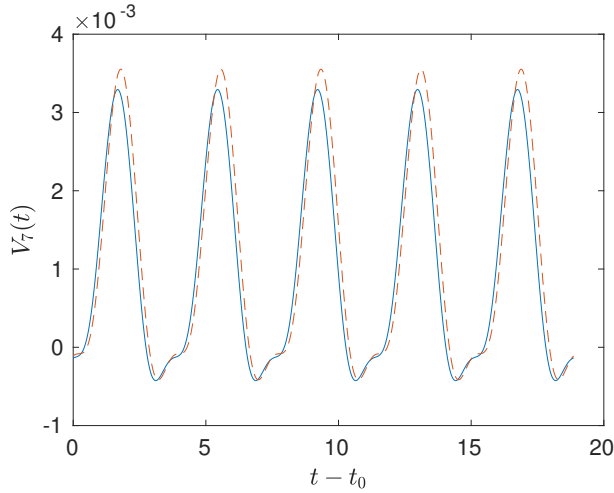


Figure 3: Time history of the asymptotic regime of figure 2(b), the  $V_7(t)$  component of the solution (blue) being shown, the red broken line being the theoretical prediction.

reduced equation as seen in figure 2(b), which means that the local solution up to the third order in  $r$  is already very accurate. The time evolution at the asymptotic state for  $t > t_0$  (with  $t_0$  sufficiently large) over 5 periods is shown in figure 3, the broken red curve being the local solution which coincides remarkably well (in particular the predicted and computed period is almost identical) with the computed global solution. This first Hopf bifurcation point corresponds consequently to a so-called supercritical bifurcation, with the existence of a local periodic nonlinear state.

The situation is different at the second Hopf bifurcation H2, where the values for the reduced equation are

$$a_1 = 4.56888 \cdot 10^{-3}, \quad b_1 = -4.6387 \cdot 10^{-4}, \quad a_2 = -0.0423, \quad b_2 = -0.0258.$$

Now  $-a_1/b_1 = 9.8495$  and  $\lambda$  has to be chosen as a positive value and  $b_1$  being negative, the theoretically predicted local periodic solution is unstable. This corresponds to what is known as a subcritical bifurcation. The gradient of  $\sigma_1$  is much stiffer at H2 than at H1 and a smaller value  $\lambda = 0.01$  has been chosen for to remain in the validity region of the local prediction. Note that for this value  $r = 0.3138$  and  $\tau = -0.003$ . A subcritical Hopf bifurcation is characterized by the absence of small-amplitude periodic solution with however the possible existence of large amplitude perturbations. Again, a random initial condition for the global system has been considered, first with a 2-norm of 0.7. The result is shown as a phase-portrait again in the  $(\theta_1(t), V_7(t))$  plane in figure 4(a) for the time interval  $[50T, 100T]$  and the global solution is seen to linger somehow erratically around the theoretical



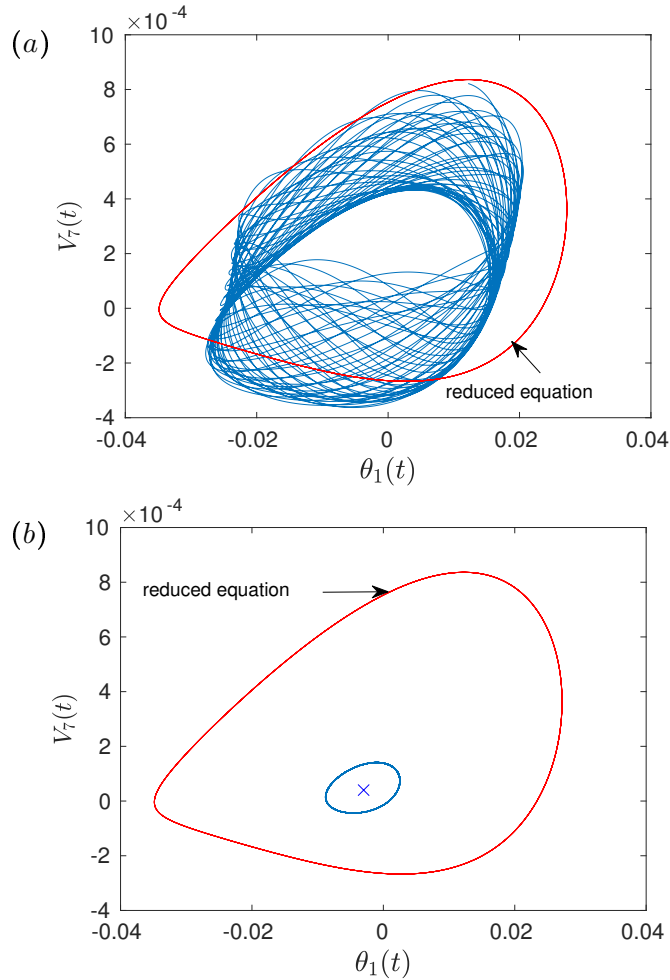


Figure 4: Phase-portrait in the  $(\theta_1(t), V_7(t))$ -plane for the global solution (blue, starting with a random initial condition with  $0 < x_j < 1, j = 1, \dots, 14$  and  $\|\mathbf{x}(0)\| = 0.7$ ) near the Hopf bifurcation point H2 ( $\lambda = 0.01$ ) during  $[50T, 100T]$  (a) and the asymptotic regime (b), the red curve being the solution (4.29) of the theory, the  $\times$  symbol in (b) being the steady state deviation towards which the solution ultimately shrinks.

periodic solution provided by the reduced equation. But ultimately the global solution shrinks as seen in figure 4(b) towards an asymptotic point, which corresponds to the small deviation from the steady state at  $\tilde{\alpha}_2$  to that at  $\tilde{\alpha}_2 + \lambda$ . When however considering a larger random initial condition with 2-norm equal to 1, the global solution is seen in figure 5 to again linger around the predicted but unstable solution for some time (the time-interval  $[50T, 100T]$  being shown), before it ultimately converges to a large amplitude asymptotic state far from the local state as predicted by the reduced equation. Note that the existence of large amplitude perturbations associated with subcritical bifurcations has often been mentioned in the literature in the context of power system models (see for instance [18], [19] among others). It is in such situation where a reduced equation analysis is particularly valuable as a detection tool, because those solutions are often difficult to find by merely integrating the global system by varying the parameters.

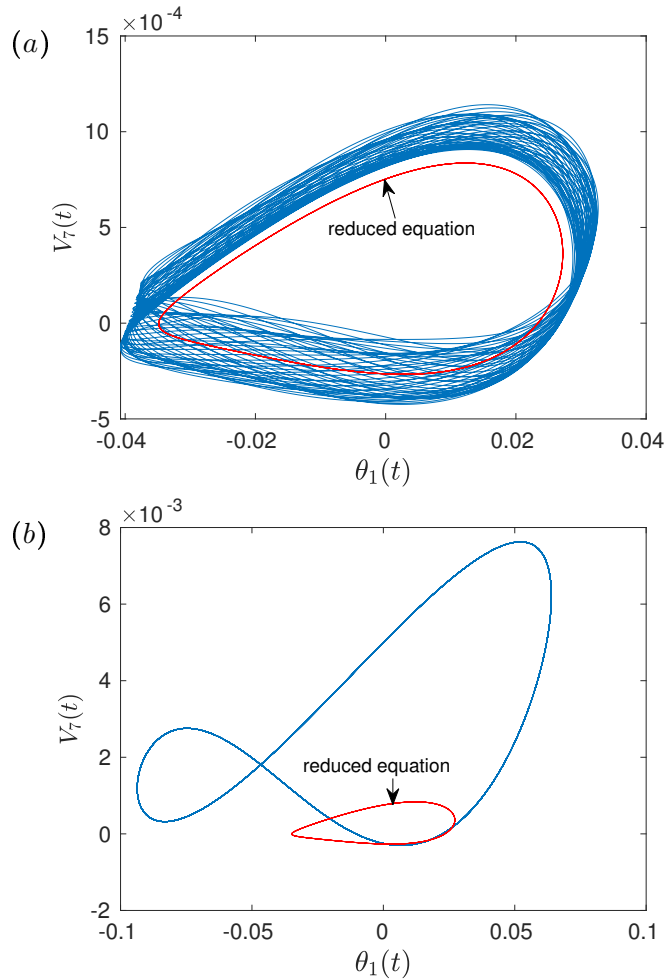


Figure 5: Phase-portrait in the  $(\theta_1(t), V_7(t))$ -plane for the global solution (blue, starting with a random initial condition with  $0 < x_j < 1, j = 1, \dots, 14$  and  $\|\mathbf{x}(0)\| = 1$ ) near the Hopf bifurcation point H2 ( $\lambda = 0.01$ ) during  $[50T, 100T]$  (a) and the asymptotic regime (b), the red curve being the solution (4.29) of the theory.

## 6. Concluding remarks

Center manifold theory continues to be employed for reduction of dynamical systems with for example very recently the application of deep learning in this context [15]. Differential-algebraic systems do however not directly enter in the framework of this theory, unless some manipulations allow to transform the system into a pure dynamical system. This may be possible when the algebraic constraint of the problem under investigation is linear or in the nonlinear case, when through implicit differentiating of algebraic constraints the system may be transformed by invoking the implicit function theorem.

Here the context of nonlinear DAE systems has been addressed which exhibit Hopf bifurcations. In this situation and given the dynamic as well as algebraic part of the equations, the idea was to see, whether the classical Lyapunov-Schmidt reduction could be directly applied without *a priori* transformations of the system. This seems not to have been attempted so far and it is rig-

ously shown, how the resolution of a hierarchy of systems, alternating the algebraic and the dynamic part, leads ultimately to the reduced equations and the local solution. The formulas are naturally more complex than in the classical theory but the reduction procedure is numerically treatable.

The aim of the reduction is of course not to substitute itself to a direct integration of the global system. It may however be a helpful tool, given that direct numerical integration results are not always easily interpretable, for instance when questioning large amplitude deviations from the steady state (or of course complex and chaotic behavior in time). To be capable of predicting the nature (supercritical or subcritical) of the Hopf bifurcation and to have a description of the small-amplitude bifurcating nonlinear periodic solution are valuable information and inputs when treating with nonlinear DAEs. The reliability of the method has been illustrated for an electric power system model, whose dimension has been chosen quite arbitrarily. Given that no specific assumption are made, besides the strangeness-free hypothesis, the procedure outlined here can in principle be applied to any more or less large DAE system.

## Appendix A. Rank and kernel computation

The so-called *LU* decomposition of a square matrix  $\mathbf{A} \in \mathbb{R}^{n,n}$  (all what follows applies to complex matrices too) is a classical numerical recipe, the matrix  $\mathbf{A}$  being transformed for to be written as the product of a lower triangular matrix  $\mathbf{L}$  and an upper triangular matrix  $\mathbf{U}$ . In [23] for instance it is shown how this algorithm by performing what is called complete pivoting (see algorithm 3.4.2 in this textbook) can be used to determine in particular the rank of the matrix. This algorithm can be generalized to a rectangular matrix  $\mathbf{A} \in \mathbb{R}^{n,m}$  which is for instance briefly mentioned in [23] and the procedure is summarized hereafter, for self-consistency.

Note  $\mathbf{A}^{(k-1)}$  the transformed matrix at step  $k-1$  of the algorithm ( $\mathbf{A}^{(0)} = \mathbf{A}$ ) and note  $\mathbf{L}^{(k-1)}$  such that  $\mathbf{L}^{(0)} = \mathbf{I}_n$  (the  $n \times n$  identity matrix). At the next step  $k$ , permutations of the rows and the columns are applied such that

$$|(\mathbf{P}_k \mathbf{A}^{(k-1)} \mathbf{Q}_k)_{kk}| = \max_{\substack{k \leq i \leq n \\ k \leq j \leq m}} |(\mathbf{A}^{(k-1)})_{ij}|$$

Gaussian elimination is performed such that

$$\mathbf{A}^{(k)} = (\mathbf{I}_n - \mathbf{l}_k \mathbf{e}_k^T) \mathbf{P}_k \mathbf{A}^{(k-1)} \mathbf{Q}_k,$$

$\mathbf{e}_k \in \mathbb{R}^n$  being the  $k$ th canonical basis vector and the elements of  $\mathbf{l}_k \in \mathbb{R}^n$  are

$$(\mathbf{l}_k)_i = 0, \quad 1 \leq i \leq k, \quad (\mathbf{l}_k)_i = \frac{(\mathbf{P}_k \mathbf{A}^{(k-1)} \mathbf{Q}_k)_{ik}}{v_k}, \quad k+1 \leq i \leq n,$$

where

$$v_k = (\mathbf{P}_k \mathbf{A}^{(k-1)} \mathbf{Q}_k)_{kk}$$

is the (non zero) pivot value, the new matrix  $\mathbf{L}^{(k)}$  being

$$\mathbf{L}^{(k)} = \mathbf{P}_k \mathbf{L}^{(k-1)} + \mathbf{l}_k \mathbf{e}_k^T.$$

The last possible step for the rectangular matrix is  $l = \min(n, m) - 1$ , if all the pivot values  $v_k \neq 0, k = 1, \dots, l$  and if the coefficient

$$v_{l+1} = (\mathbf{A}^{(l)})_{l+1, l+1} \neq 0$$

the matrix is of maximum rank. Suppose now that the pivot values are such that

$$v_k \neq 0, k = 1, \dots, r \quad \text{and} \quad v_{r+1} = 0, \quad r + 1 \leq \min(n, m) \quad (\text{A.1})$$

The Gaussian elimination procedure has to stop at step  $r$  and let  $\mathbf{P} = \mathbf{P}_r \dots \mathbf{P}_1$  and  $\mathbf{Q} = \mathbf{Q}_r \dots \mathbf{Q}_1$  be the products of the successive row and column permutation, one gets the decomposition

$$\mathbf{P} \mathbf{A} \mathbf{Q}^T = \mathbf{L}^{(r)} \mathbf{A}^{(r)}. \quad (\text{A.2})$$

To store the different permutation, it is not necessary to construct permutation matrices, but one merely permutes the elements of index vectors  $\mathbf{p} \in \mathbb{R}^n$  and  $\mathbf{q} \in \mathbb{R}^m$  (such as before starting the algorithm  $p_i = i, i = 1, \dots, n$  and  $q_i = i, i = 1, \dots, m$ ). The matrix  $\mathbf{A}^{(r)}$  has the following bloc structure

$$\mathbf{A}^{(r)} = \left( \begin{array}{c|c} \mathbf{U}^{(r)} & \mathbf{A}_{m-r}^{(r)} \\ \hline (0) & (0) \end{array} \right), \quad \mathbf{U}^{(r)} = \begin{pmatrix} v_1 & \cdots & \cdots \\ (0) & \ddots & (*) \\ & (0) & v_r \end{pmatrix} \quad (\text{A.3})$$

where  $\mathbf{U}^{(r)} \in \mathbb{R}^{r,r}$  is upper triangular (the symbol  $*$  stands for the coefficients resulting from the Gaussian algorithm),  $\mathbf{A}_{m-r}^{(r)} \in \mathbb{R}^{r, m-r}$  is the upper right part of  $\mathbf{A}^{(r)}$  after the algorithm is completed and the elements in the rows  $r + 1, \dots, n$  are zero. This transformed matrix has rank  $r$  (the rank of the submatrix  $\mathbf{U}^{(r)}$ ), which is, given that  $\mathbf{A}^{(r)}$  results from rows and columns permutations and Gaussian elimination, the rank of the matrix  $\mathbf{A}$ . The matrix  $\mathbf{L}^{(r)}$  has the following bloc structure

$$\mathbf{L}^{(r)} = \left( \begin{array}{c|c} \mathbf{L}_r^{(r)} & \frac{(0)}{\mathbf{I}_{n-r}} \\ \hline & \end{array} \right), \quad \mathbf{L}_r^{(r)} = \begin{pmatrix} 1 & (0) \\ (*) & \ddots & (0) \\ \cdots & (*) & 1 \\ \cdots & \cdots & \cdots \end{pmatrix} \quad (\text{A.4})$$

with the submatrix  $\mathbf{L}_r^{(r)} \in \mathbb{R}^{n,r}$  (again the symbol  $*$  stands for non zero or zero elements) and  $\mathbf{I}_{n-r}$  is the  $(n-r) \times (n-r)$  identity matrix. Note that if  $\mathbf{A}$  has maximal rank, that is in (A.1)  $v_k \neq 0$ ,  $k = 1, \dots, \min(n, m)$ , then if  $n > m$ , the Gaussian elimination steps have to be performed up to  $k = r = m$  (and in that case the bloc  $\mathbf{A}_{m-r}^{(r)}$  in the decomposition (A.3) does not exist), whereas if  $m \geq n$  the Gaussian elimination steps stop at  $n-1$  and one has only to assure that in (A.3) for  $r = n$  the last coefficient  $v_n \neq 0$ . Of course, if  $n = m$  and the rank is  $n$  one recovers conventional  $LU$  decomposition. The mathematical rank condition (A.1) must be interpreted in terms of the numerical zero. In practice an  $\epsilon$ -value has to be introduced according to machine precision in the way that  $v_{r+1} \approx 0$  is assured through a condition such as  $|v_{r+1}|/|v_1| < \epsilon$ , given that  $|v_1|$  is the first maximum pivot value. Of course, caution has always to be exerted for the numerical rank condition to be reliable, because round-off errors are always present, which however with complete pivoting are bounded (see [23]).

The rank-kernel theorem for a rectangular matrix  $n \times m$  states that

$$m = \dim \text{Ker}(\mathbf{A}) + \dim \text{Range}(\mathbf{A})$$

the rank being the dimension of the range of  $\mathbf{A}$ . The matrix  $\mathbf{A}$  has a kernel whenever its rank  $r < \min(n, m)$  but also if  $\mathbf{A}$  is of maximal rank and  $m > n$ . To compute the basis vectors of the kernel of  $\mathbf{A}$ , one has to solve  $\mathbf{A} \mathbf{x} = 0$ , that is by using the decomposition (A.2)

$$\mathbf{P} \mathbf{A} \mathbf{Q}^T \mathbf{y} = \mathbf{L}^{(r)} \mathbf{A}^{(r)} \mathbf{y} = 0, \quad \mathbf{x} = \mathbf{Q}^T \mathbf{y} \quad (\text{A.5})$$

and given that  $\mathbf{L}^{(r)}$  sketched in (A.4) admits an inverse, one has to solve  $\mathbf{A}^{(r)} \mathbf{y} = 0$  with  $\mathbf{A}^{(r)}$  given by (A.3) to recover the kernel. The linearly independent vectors  $\mathbf{y}_j, j = 1, \dots, m-r$ , can be sought as to be of the form

$$\mathbf{y}_j = \begin{pmatrix} \mathbf{z}_j \\ \mathbf{e}_j \end{pmatrix}$$

with  $\mathbf{z}_j \in \mathbb{R}^r$  and  $\mathbf{e}_j \in \mathbb{R}^{m-r}$  such that  $(\mathbf{e}_j)_k = \delta_{jk}$ ,  $k = 1, \dots, m-r$  ( $\delta_{jk}$  being the Kronecker symbol). Note that these  $m-r$  vectors are by construction linearly independent. According to the matrix structure (A.3), to solve  $\mathbf{A}^{(r)} \mathbf{y}_j = 0$  it is equivalent to find  $\mathbf{z}_j$  such that

$$\mathbf{U}^{(r)} \mathbf{z}_j = -\mathbf{A}_{m-r}^{(r)} \mathbf{e}_j$$

and this system can be solved easily,  $\mathbf{U}^{(r)}$  being upper triangular and it admits an inverse. The  $j$ th vector of the kernel is then recovered by  $\mathbf{Q} \mathbf{x}_j = \mathbf{y}_j$

(see (A.5)), given that  $\mathbf{Q}^T = \mathbf{Q}^{-1}$  (the permutation being the product of simple transpositions). As mentioned above, the vector of indices  $\mathbf{q}$  provides the permutations and accordingly

$$(\mathbf{x}_j)_{q_i} = (\mathbf{y}_j)_i, \quad i = 1, \dots, m.$$

The nullspace can be represented by  $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_{m-r}) \in \mathbb{R}^{m, m-r}$ , where the nullspace's basis vectors  $\mathbf{x}_j$  are the columns of  $\mathbf{X}$ . It may often be appropriate to perform a Gram-Schmidt orthogonalization for the standard inner-product of these vectors (dedicated algorithms being largely available) and a representation of the nullspace is then provided by

$$\mathbf{T} \in \mathbb{R}^{m, m-r}, \quad \text{with} \quad \mathbf{t}_k^T \mathbf{t}_l = \delta_{kl}, \quad 1 \leq k, l \leq m-r$$

( $\mathbf{t}_j$  being the column vectors of  $\mathbf{T}$ ). The nullspace of the transposed matrix  $\mathbf{A}^T \in \mathbb{R}^{m, n}$  can also be computed, by transposing the relation (A.2), that is

$$\mathbf{Q} \mathbf{A}^T \mathbf{P}^T = (\mathbf{A}^{(r)})^T (\mathbf{L}^{(r)})^T \tag{A.6}$$

and the dimension of the kernel is now  $n-r$  if the rank of  $\mathbf{A}$  is such that  $r < n$ . Now  $\mathbf{A}^T \mathbf{x} = 0$  if  $\mathbf{x} = \mathbf{P}^T \mathbf{y}$  with

$$(\mathbf{A}^{(r)})^T (\mathbf{L}^{(r)})^T \mathbf{y} = 0. \tag{A.7}$$

Given that  $\mathbf{L}^{(r)}$  is a lower triangular matrix,  $(\mathbf{L}^{(r)})^T$  is upper triangular and admits an inverse, while the structure of  $(\mathbf{A}^{(r)})^T$  is

$$(\mathbf{A}^{(r)})^T = \left( \begin{array}{c|c} (\mathbf{U}^{(r)})^T & (0) \\ \hline (\mathbf{A}_{m-r}^{(r)})^T & (0) \end{array} \right) \tag{A.8}$$

Take the vectors

$$\mathbf{z}_j = \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_j \end{pmatrix} \in \mathbb{R}^n, \quad j = 1, \dots, n-r,$$

$\mathbf{0}$  being the zero vector of length  $r$  and  $\mathbf{e}_j \in \mathbb{R}^{n-r}$  with coefficients  $(\mathbf{e}_j)_k = \delta_{jk}, k = 1, \dots, n-r$ , then according to (A.8)  $(\mathbf{A}^{(r)})^T \mathbf{z}_j = 0$ . Hence (A.7) is fulfilled if

$$(\mathbf{L}^{(r)})^T \mathbf{y}_j = \mathbf{z}_j, \quad j = 1, \dots, n-r$$

and this system can easily be solved. Note that the vectors  $\mathbf{z}_j, j = 1, \dots, n-r$  are linearly independent and so are the vectors  $\mathbf{y}_j$ . A vector basis for the kernel of the transposed matrix is now formed by the vectors  $\mathbf{x}_j \in \mathbb{R}^n, j = 1, \dots, n-r$  such that

$$(\mathbf{x}_j)_{p_i} = (\mathbf{y}_j)_i, \quad i = 1, \dots, n$$

Now, consider the case of a system

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

with  $\mathbf{A} \in \mathbb{R}^{n,n}$  a square matrix such that  $r < n$ . It is supposed that  $\mathbf{b}$  is in the range of  $\mathbf{A}$  and performing the decomposition (A.2) one gets

$$\mathbf{L}^{(r)} \mathbf{A}^{(r)} \mathbf{y} = \mathbf{P}\mathbf{b}, \quad \mathbf{x} = \mathbf{Q}^T \mathbf{y}.$$

One first solves  $\mathbf{L}^{(r)} \mathbf{z} = \mathbf{P}\mathbf{b}$  and then the singular system

$$\mathbf{A}^{(r)} \mathbf{y} = \mathbf{z}$$

is to be addressed. If  $\mathbf{b}$  is in the range of  $\mathbf{A}$ , then necessarily the coefficients  $z_j = 0$ ,  $j = r + 1, \dots, n$ , given the structure (A.3) of  $\mathbf{A}^{(r)}$ . The solution  $\mathbf{y}$  can be chosen such that  $y_j = 0$ ,  $j = r + 1, \dots, n$  and  $y_j$ ,  $j = 1, \dots, r$  are obtained by considering  $\mathbf{U}^{(r)}$ , the upper left  $r \times r$  bloc of  $\mathbf{A}^{(r)}$  (which is invertible). Finally one recovers a solution  $\mathbf{x}$  (which of course is not unique because defined up to a contribution belonging to the nullspace of the singular matrix).

### Appendix B. Computation of $\mathbf{w}_3$

To compute  $\mathbf{w}_3$ , the third-order term has to be added to the Taylor expansion (4.2) which is

$$\frac{1}{2} D^2 \mathbf{M}_1[\mathbf{T}\mathbf{y} + \mathbf{S}\mathbf{z}, \mathbf{T}\mathbf{y} + \mathbf{S}\mathbf{z}] \frac{d}{ds} (\mathbf{T}\mathbf{y} + \mathbf{S}\mathbf{z}) + \frac{1}{6} D^3 \mathbf{G}_1[\mathbf{T}\mathbf{y} + \mathbf{S}\mathbf{z}, \mathbf{T}\mathbf{y} + \mathbf{S}\mathbf{z}, \mathbf{T}\mathbf{y} + \mathbf{S}\mathbf{z}]$$

and  $\mathbf{w}_3$  is of the form

$$\mathbf{w}_3 = \mathbf{w}_3^{(1)} e^{is} + \mathbf{w}_3^{(3)} e^{i3s} + cc$$

Some tedious developments lead to the following expression, canceling the terms in  $r^3$ ,

$$\begin{aligned} (i\mathbf{I} + \tilde{\mathbf{A}}) \mathbf{w}_3^{(1)} &= -\tilde{\mathbf{P}}\mathbf{b} \quad \text{with} \\ \mathbf{b} &= \mathbf{B}_1^{-1} \left( i\mathbf{M}_1 \mathbf{S}\mathbf{z}_3^{(1)} - \frac{3i}{2} D\mathbf{M}_1[\mathbf{T}\mathbf{w}_2^{(2)} + \mathbf{S}\mathbf{z}_2^{(2)}] \mathbf{T}\bar{\mathbf{c}} \right. \\ &\quad + \frac{3i}{2} D\mathbf{M}_1[\mathbf{T}\mathbf{w}_2^{(0)} + \mathbf{S}\mathbf{z}_2^{(0)}] \mathbf{T}\mathbf{c} + 3i D\mathbf{M}_1[\mathbf{T}\bar{\mathbf{c}}](\mathbf{T}\mathbf{w}_2^{(2)} + \mathbf{S}\mathbf{z}_2^{(2)}) \\ &\quad + \frac{3i}{4} D^2 \mathbf{M}_1[\mathbf{T}\mathbf{c}, \mathbf{T}\bar{\mathbf{c}}] \mathbf{T}\mathbf{c} - \frac{3i}{8} D^2 \mathbf{M}_1[\mathbf{T}\mathbf{c}, \mathbf{T}\mathbf{c}] \mathbf{T}\bar{\mathbf{c}} + D\mathbf{G}_1[\mathbf{S}\mathbf{z}_3^{(1)}] \\ &\quad + \frac{3}{2} D^2 \mathbf{G}_1[\mathbf{T}\bar{\mathbf{c}}, \mathbf{T}\mathbf{w}_2^{(2)} + \mathbf{S}\mathbf{z}_2^{(2)}] + \frac{3}{2} D^2 \mathbf{G}_1[\mathbf{T}\mathbf{c}, \mathbf{T}\mathbf{w}_2^{(0)} + \mathbf{S}\mathbf{z}_2^{(0)}] \\ &\quad \left. + \frac{3}{8} D^3 \mathbf{G}_1[\mathbf{T}\mathbf{c}, \mathbf{T}\mathbf{c}, \mathbf{T}\bar{\mathbf{c}}] \right) \end{aligned} \tag{B.1}$$

with  $\tilde{P}$  the projection defined by (4.11). The above singular system can be solved by the method explained in Appendix A (and  $\tilde{P}\mathbf{w}_3^{(1)} = \mathbf{w}_3^{(1)}$ ). The function  $\mathbf{w}_3^{(3)}$  can be shown to be solution of

$$\begin{aligned} (3i\mathbf{I} + \tilde{\mathbf{A}})\mathbf{w}_3^{(3)} = & -\mathbf{B}_1^{-1} \left( 3i\mathbf{M}_1\mathbf{S}\mathbf{z}_3^{(3)} + \frac{3i}{2}D\mathbf{M}_1[\mathbf{T}\mathbf{w}_2^{(2)} + \mathbf{S}\mathbf{z}_2^{(2)}]\mathbf{T}\mathbf{c} \right. \\ & + 3iD\mathbf{M}_1[\mathbf{T}\mathbf{c}](\mathbf{T}\mathbf{w}_2^{(2)} + \mathbf{S}\mathbf{z}_2^{(2)}) + \frac{3i}{8}D^2\mathbf{M}_1[\mathbf{T}\mathbf{c}, \mathbf{T}\mathbf{c}]\mathbf{T}\mathbf{c} + D\mathbf{G}_1[\mathbf{S}\mathbf{z}_3^{(3)}] \\ & \left. + \frac{3}{2}D^2\mathbf{G}_1[\mathbf{T}\mathbf{c}, \mathbf{T}\mathbf{w}_2^{(2)} + \mathbf{S}\mathbf{z}_2^{(2)}] + \frac{1}{8}D^3\mathbf{G}_1[\mathbf{T}\mathbf{c}, \mathbf{T}\mathbf{c}, \mathbf{T}\mathbf{c}] \right). \end{aligned} \quad (\text{B.2})$$

### Appendix C. Multivariable derivative formula for the case study

In the reduction algorithm the multivariable derivative formula of the type (4.1) has to be computed. In the case study each component of  $\mathbf{G}$  given by (5.4), (5.5) is the sum of scalar function involving the product of components of the solution vector with sine and cosine functions. Consider for instance the highest order derivative in the reduction algorithm

$$D^3\mathbf{G}(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}})[\mathbf{q}, \mathbf{r}, \mathbf{s}].$$

where  $[\mathbf{q}, \mathbf{r}, \mathbf{s}]$  stands for the arguments in the different expressions. Selecting one of the scalar function for illustration, say

$$u_j u_l \sin(u_i - u_k),$$

the function

$$d(t_1, t_2, t_3) = (\tilde{u}_j + \phi_j)(\tilde{u}_l + \phi_l) \sin(\tilde{u}_i - \tilde{u}_k + \phi_i - \phi_k)$$

is to be formed with  $\phi_i = t_1 q_i + t_2 r_i + t_3 s_i$  and of course  $\phi_j$  etc are defined the same way by changing the subscript. This contribution to the derivative formula according to (4.1) is

$$\frac{\partial^3}{\partial t_1 \partial t_2 \partial t_3} d(t_1, t_2, t_3)|_{t_1=t_2=t_3=0}.$$

Expanding the sine function in a Taylor series at  $\tilde{u}_i - \tilde{u}_k$  (up to the third order for the example), the products between  $\phi_j$ ,  $\phi_l$ ,  $(\phi_i - \phi_k)^p$ ,  $p = 1, 2, 3$  are developed and in the resulting multivariable polynomial the terms in  $t_1 t_2 t_3$  are to be assembled and the resulting coefficient is the required expression. These



computations are more or less tedious but only simple algebra is involved. For the example one gets

$$\begin{aligned} & [(q_j r_l + q_l r_j)(s_i - s_k) + (q_j s_l + q_l s_j)(r_i - r_k) + (r_j s_l + r_l s_j)(q_i - q_k)] \cos(\tilde{u}_i - \tilde{u}_k) \\ & - [q_l(r_i - r_k)(s_i - s_k) + r_l(q_i - q_k)(s_i - s_k) + s_l(q_i - q_k)(r_i - r_k)] \tilde{u}_j \sin(\tilde{u}_i - \tilde{u}_k) \\ & - [q_j(r_i - r_k)(s_i - s_k) + r_j(q_i - q_k)(s_i - s_k) + s_j(q_i - q_k)(r_i - r_k)] \tilde{u}_l \sin(\tilde{u}_i - \tilde{u}_k) \\ & - (q_i - q_k)(r_i - r_k)(s_i - s_k) \tilde{u}_j \tilde{u}_l \cos(\tilde{u}_i - \tilde{u}_k) \end{aligned}$$

Of course, for lower derivative order the expressions are simpler and when computing  $D^2 \mathbf{G}(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}})[\mathbf{q}, \mathbf{r}]$  one proceeds in a similar manner by extracting the term in  $t_1 t_2$ .

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