



HAL
open science

An Analysis of Symmetry in Quantitative Semantics

Pierre Clairambault, Simon Forest

► **To cite this version:**

Pierre Clairambault, Simon Forest. An Analysis of Symmetry in Quantitative Semantics. 2024. hal-04427678

HAL Id: hal-04427678

<https://amu.hal.science/hal-04427678>

Preprint submitted on 30 Jan 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

An Analysis of Symmetry in Quantitative Semantics

Pierre Clairambault

Aix Marseille Univ, Université de Toulon, CNRS, LIS
Marseille, France

Pierre.Clairambault@cnrs.fr

Simon Forest

Aix Marseille Univ, CNRS, LIS
Marseille, France

Simon.Forest@univ-amu.fr

ABSTRACT

Drawing inspiration from linear logic, *quantitative semantics* aim at representing quantitative information about programs and their executions: they include the relational model and its numerous extensions, game semantics, and syntactic approaches such as non-idempotent intersection types and the Taylor expansion of λ -terms. The crucial feature of these models is that programs are interpreted as witnesses which consume “bags” of resources.

“Bags” are often taken to be finite multisets, *i.e.* quotiented structures. Another approach typically seen in categorifications of the relational model is to work with unquotiented structures (*e.g.* sequences) related with explicit morphisms referred to here as *symmetries*, which express the exchange of resources. Symmetries are obviously at the core of these categorified models, but we argue their interest reaches beyond those – notably, symmetry *leaks* in some non-categorified quantitative models (such as the weighted relational model, or Taylor expansion) under the form of numbers whose combinatorial interpretation is not always clear.

In this paper, we build on a recent bicategorical model called *thin spans of groupoids*, introduced by Clairambault and Forest. Notably, thin spans feature a decomposition of symmetry into two sub-groupoids of *polarized* – *positive* and *negative* – symmetries. We first construct a variation of the original exponential of thin spans, based on sequences rather than families. Then we give a syntactic characterisation of the interpretation of simply-typed λ -terms in thin spans, in terms of rigid intersection types and rigid resource terms. Finally, we formally relate thin spans with the weighted relational model and generalized species of structure. This allows us to show how some quantities in those models reflect polarized symmetries: in particular we show that the weighted relational model counts witnesses from generalized species of structure, *divided* by the cardinal of a group of positive symmetries.

1 INTRODUCTION

Denotational semantics is an approach to the semantics of programming languages that consists in associating to every program a denotation in an adequate mathematical universe; crucially this is done compositionally, by induction on syntax. Most denotational models are *qualitative*: a term $\vdash M : A \rightarrow B$ is typically represented by a function from the denotation of A to the denotation of B , giving us the input/output behaviour of M , but omitting *quantitative* information, such as resources, time, probabilities...

Within denotational semantics, *quantitative semantics* is a family of models whose distinguishing feature is to record quantitative information – first and foremost, displaying *how many times* a function $\vdash M : A \rightarrow B$ must evaluate its argument in order to produce a given result. Originally prompted by Girard’s linear logic [15], quantitative semantics has developed into a wide research

topic with numerous models and approaches, including the relational model [15] and its weighted [8, 19, 20] or categorical [3, 12] extensions, resource terms and the Taylor expansion of λ -terms [10], non-idempotent intersection types [5, 14], game semantics [1, 17], and others. This is not merely a subjective methodological difference: quantitative models are well-suited to model quantitative features such as probabilistic [9] or quantum [26] primitives, reflecting quantitative property such as execution time [7], or the number of non-deterministic branches [19], and many others.

To keep track of quantitative information, quantitative models must represent all individual resource accesses, but this is trickier than it might seem. Linear logic decomposes the intuitionistic arrow $A \rightarrow B$ as $!A \multimap B$ where \multimap is the *linear arrow* (for functions calling their argument exactly once), and $!$ is the *exponential modality*, allowing arbitrary duplications of resources. Typically, the difficulty in designing a quantitative model arises with handling the exponential: how to keep track of all individual resource accesses while ensuring the laws required for a $!$ in models of linear logic?

Quotients. If resource accesses in $!A$ are ordered in a sequence

$$\langle \alpha_1, \dots, \alpha_n \rangle,$$

then this will generally fail the commutations laws for the exponential, which require a *commutative comonoid* [23]¹. So sequences are often quotiented out by commutativity, as in the relational model [15] (and in general the so-called web-based models of linear logic), where $!A = \mathcal{M}(A)$ the set of finite multisets. This quotient is also found in quantitative notions of program approximation: for instance, the *Taylor expansion* of λ -terms [10] approximates λ -terms via the *resource calculus*, a strongly finitary calculus where an application MN from the λ -calculus is approximated with

$$m [n_1, \dots, n_k]$$

the application of a resource term m , approximating M , to a *finite multiset* of resource terms n_1, \dots, n_k , all approximating N . This expresses one of the possible behaviours of MN , where M will call its argument *exactly* k times, each call associated to one of the n_i ’s.

This quotient, at the heart of quantitative semantics, is by no means innocent: in situations when quantitative semantics manipulate numerical coefficients, the underlying symmetries on multisets *leak*, yielding scalars which are not clearly related to the computational situation, but instead reflect some aspect of its underlying symmetries. For instance, the relational model weighted by (completed) natural numbers [19], which in this paper we refer to as $\mathbf{WRel}_!$, counts distinct execution branches for non-deterministic programs when applied at ground type. But at higher-order type it yields non-trivial coefficients, even for plain simply-typed λ -terms:

¹Though some games models, notably simple games with the Hyland exponential [18], get away with that exploiting that copy accesses are totally chronologically ordered.

what do these numbers mean? Are those numbers related to the coefficients appearing in the Taylor expansion of λ -terms?

Rigid structures. It is tempting to avoid these quotients: in the quantitative semantics literature, the corresponding structures are often called *rigid*. Developing rigid models is subtle; for instance naively replacing finite multisets with sequences in the resource calculus yields a non-confluent reduction [25]; while naive rigid non-idempotent intersection types fail subject reduction.

Proper treatments of rigid structures may be found in *categorifications* of the relational model, the prime example being the cartesian closed bicategory \mathbf{Esp} of generalized species of structure [12]. There, types are interpreted as *categories* (or *groupoids*) and the exponential $!A$ is the *free strict symmetric monoidal category* $\mathbf{Sym}(A)$ on A , where objects are sequences $\langle a_1, \dots, a_n \rangle$ of objects of A , and where a morphism from $\langle a_1, \dots, a_n \rangle$ to $\langle a'_1, \dots, a'_m \rangle$ is a bijection $\sigma : n \simeq m$ along with $f_i : a_i \rightarrow a'_{\sigma(i)}$ in A for all $1 \leq i \leq n$. A term $\Gamma \vdash M : A$ is interpreted as a *distributor* from $\mathbf{Sym}(\llbracket \Gamma \rrbracket)$ to $\llbracket A \rrbracket$, i.e.

$$\llbracket M \rrbracket_{\mathbf{Esp}} : \mathbf{Sym}(\llbracket \Gamma \rrbracket)^{\text{op}} \times \llbracket A \rrbracket \rightarrow \mathbf{Set},$$

a functor which to $\vec{y} \in \mathbf{Ob}(\mathbf{Sym}(\llbracket \Gamma \rrbracket))$ and $a \in \mathbf{Ob}(\llbracket A \rrbracket)$ associates a set $\llbracket M \rrbracket_{\mathbf{Esp}}(y, a)$ of *witnesses* – crucially, $\llbracket M \rrbracket_{\mathbf{Esp}}$ also has a functorial action, making the symmetries (morphisms) of $\mathbf{Sym}(\llbracket \Gamma \rrbracket)$ and $\llbracket A \rrbracket$ act on witnesses. Tsukada *et al.* [27] and Olimpieri [24] have studied the nature of these witnesses, showing that they can be regarded as terms of a rigid resource calculus. Their calculi are not the naive rigid resource calculus mentioned above: they refine it by letting resource terms carry *morphisms/symmetries* from the types – but the precise location of these symmetries in the term is irrelevant, and it must be forgotten by yet another quotient!

Nevertheless, as \mathbf{Esp} is a generalization of \mathbf{Rel} properly accounting for symmetries, it looks a natural candidate to illuminate the scalars arising from the weighted relational model: we may expect

$$(\llbracket M \rrbracket_{\mathbf{WRel}})_{y,a} = \#(\llbracket M \rrbracket_{\mathbf{Esp}})(y, a) \quad (1)$$

(conflating for now objects and symmetry classes). But this fails, and we shall see that the link between the two involves data that is missing from the theory of \mathbf{Esp} : *polarized symmetries*.

Contributions. Recently, Clairambault and Forest have introduced a new bicategorical model \mathbf{Thin} , called *thin spans of groupoids* [3], also a categorification of the relational model, inspired concurrent game semantics [2] – our first contribution is to show that it supports an exponential based on sequences rather than families.

We then delve deeper into the interpretation of the simply-typed λ -calculus in the Kleisli bicategory $\mathbf{Thin}_!$. Just like for \mathbf{Esp} [24, 27], we show that an intersection type system (and matching resource terms) is implicit in thin spans. Perhaps surprisingly, it turns out to be the naive rigid intersection type system discussed above, obtained by merely replacing finite multisets with sequences (or the similarly naive rigid resource calculus), not carrying any symmetries, and without any quotient. Though subject reduction fails on the nose, our results entail that it does hold in a relaxed sense, *up to symmetry*. Beyond just characterising the witnesses as in [24, 27], we go further and also give a syntactic description of *symmetries* between derivations, obtaining a syntactic description of the full groupoid obtained as the interpretation of a term.

A central feature of \mathbf{Thin} is that objects are certain groupoids A admitting two sub-groupoids A_- and A_+ , respectively of *negative* and *positive* symmetries. Those are reminiscent from ideas in game semantics: *negative* symmetries exchange resources controlled by the environment, while *positive* symmetries exchange resources controlled by the program. Not every symmetry is negative or positive, but every symmetry factors uniquely as a negative composed with a positive. Far from being a technicality of the model construction, we argue that these polarized sub-symmetries are *fundamental*. In particular, they are the key to illuminate some of the questions mentioned earlier: in this paper, we characterise the coefficients obtained by $\mathbf{WRel}_!$ as counting witnesses in $\mathbf{Thin}_!$ – i.e. rigid resource terms – *up to positive symmetry*, or symmetry classes of witnesses – i.e. standard resource terms – with a correcting coefficient involving *negative symmetries*. Drawing inspiration from recent work linking thin concurrent games with generalized species of structure [4], we also construct an interpretation-preserving pseudofunctor from $\mathbf{Thin}_!$ to \mathbf{Esp} , allowing us overall to express the coefficients obtained through $\mathbf{WRel}_!$ directly in terms of \mathbf{Esp} , correcting (1) – again, the correct equation involves polarized symmetries.

Related work. *Polarized symmetries* are central to the construction of thin spans of groupoids (and before that, thin concurrent games [2]), but they predate those models: to our knowledge, they first appear in Melliès’ approach to uniformity by bi-invariance, in the setting of asynchronous games [22]. They also make an appearance in Tsukada *et al.*’s study of *weighted generalized species* [28], though they are not part of the general theory but computed *a posteriori* for groupoids arising from simple types.

This work is part of an ongoing effort from the community to refine our understanding of resources in quantitative models, replacing quotients with rigid structures related with explicit morphisms and explore the corresponding categorical structures. Aside from work on generalized species of structure, a work complementary to ours is Melliès’ *homotopy template games* [21], also based on categorical spans, focusing on links with homotopy theory.

Outline. In Section 2 we recall the definition of \mathbf{Thin} from [3], replacing their exponential with a new one based on \mathbf{Sym} . In Section 3, we give our syntactic characterisation of the interpretation of the simply-typed λ -terms in $\mathbf{Thin}_!$. Finally, in Section 4 we explore the link between $\mathbf{Thin}_!$ and relational models: first the plain relational model \mathbf{Rel} , then the weighted (by completed natural numbers) relational model \mathbf{WRel} , and finally generalized species \mathbf{Esp} .

2 THIN SPANS ON SEQUENCES

We start with a brief reminder on \mathbf{Thin} [3], along with the definition of the new exponential based on sequences. In the following, we write \mathbf{Gpd} for the 2-category of groupoids, functors between groupoids and natural transformations between such functors. We will also often call *symmetries* the morphisms of a groupoid.

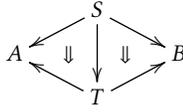
2.1 Reminder on Thin Spans of Groupoids

A *span* from A to B in a category C is simply a diagram like

$$A \xleftarrow{\partial_l^S} S \xrightarrow{\partial_r^S} B$$

which in **Set** (or **Cat**, of **Gpd**) is regarded as a generalized relation: a pair (a, b) may be related via a number of distinct **witnesses**, *i.e.* elements $s \in S$ s.t. $\partial_l^S(s) = a$ and $\partial_r^S(s) = b$ – in this paper, we often write $\partial_l^S(s) = s_A$ and $\partial_r^S(s) = s_B$, keeping ∂_l^S and ∂_r^S implicit. Here we focus on spans over groupoids: those form a bicategory **Span** where objects are groupoids, and a morphism from A to B is a span $A \leftarrow S \rightarrow B$. The identity span Id_A is $A \leftarrow A \rightarrow A$ with two identity functors, and spans are composed by pullback.

In **Span**, the 2-cells from a span $A \leftarrow S \rightarrow B$ to $A \leftarrow T \rightarrow B$ are functors $S \rightarrow T$ making the two triangles commute, and their horizontal composition is given by the universal property of pullbacks. Unfortunately, these 2-cells are too strict for many purposes; in particular they are incompatible with the laws required for the exponential modality of linear logic. Alternative 2-cells relax the hypothesis that the two triangles commute, asking instead for



two natural isomorphisms. This allows us to relate more spans and indeed supports the laws for the exponential modality. However, the universal property of pullbacks then fails to provide a definition of horizontal composition for those. This mismatch has different solutions, either replacing the pullbacks with adequate notions of homotopy pullbacks, or requiring additional fibrational conditions on spans – in almost all cases this concretely means importing the morphisms of groupoids inside witnesses, as in generalized species of structure or in template games [21].

In [3], an alternative idea was introduced. In **Span**, some pullbacks happen to behave well *w.r.t.* homotopy (they are *bipullbacks*, see below). The key observation is that as it turns out, the pullbacks arising from the denotational interpretation of programs actually *always are* bipullbacks! The bicategory **Thin** of *thin spans* captures this via a biorthogonality construction, morally cutting **Span** down and keeping only certain spans – those deemed “uniform” – ensuring that their composition pullbacks are always bipullbacks.

2.1.1 Uniformity. Given a groupoid A , a **prestrategy**² on A is a pair (S, ∂^S) of a groupoid S and a functor $\partial^S: S \rightarrow A$, the **display map**. We write $\text{PreStrat}(A)$ for the class of prestrategies on A .

Given two prestrategies (S, ∂^S) and (T, ∂^T) , we write $(S, \partial^S) \perp (T, \partial^T)$ (or, more simply, $S \perp T$), when the following pullback

$$\begin{array}{ccc} & P & \\ \swarrow l & \downarrow & \searrow r \\ S & & T \\ \searrow \partial^S & & \swarrow \partial^T \\ & A & \end{array} \quad (2)$$

is a **bipullback**. In **Gpd**, this means that for every $s \in S, t \in T$ and $\theta: s_A \rightarrow t_B$, there is $u: s \rightarrow s' \in S$ and $v: t' \rightarrow t \in T$ such that $\theta = v_A \circ u_A$ in A : when two states can synchronize up to symmetry, we can find symmetric states that can synchronize on the nose, coherently. Given a set, or even a class \mathcal{S} of prestrategies on A , we write S^\perp for the class $\{T \in \text{PreStrat}(A) \mid \forall S \in \mathcal{S}, S \perp T\}$.

A **uniform groupoid** is a pair $A = (A, \mathbf{U}_A)$ where A is a groupoid and $\mathbf{U}_A \subseteq \text{PreStrat}(A)$ is a class of prestrategies such that $S^{\perp\perp} = S$. One can define several constructions on uniform groupoids [3]. The

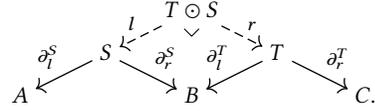
²Some terminology in [3] is game-theoretic, reflecting the game semantics inspirations.

dual A^\perp of the uniform groupoid A has (A, \mathbf{U}_A^\perp) . Given another uniform groupoid $B = (B, \mathbf{U}_B)$, one can define binary constructions like the **tensor** $A \otimes B$ and its de Morgan dual the **par** $A \wp B$, both having underlying groupoid $A \times B$. From these two constructions, one then defines the **linear arrow** $A \multimap B$ as $A^\perp \wp B$. Finally, the **with** $A \& B$ has underlying groupoid $A + B$.

2.1.2 Spans. The underlying groupoid of $A \multimap B$ is $A \times B$ so that $S \in \mathbf{U}_{A \multimap B}$ is a prestrategy on $A \times B$, equivalently seen as a span

$$A \leftarrow S \rightarrow B$$

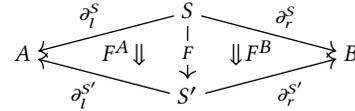
in **Gpd**. In the following, we call such S a **uniform span** to emphasize that it is a prestrategy of $\mathbf{U}_{A \multimap B}$. Notably, the identity span on a uniform groupoid A , is uniform. Given uniform groupoids A, B, C , $S \in \mathbf{U}_{A \multimap B}$ and $T \in \mathbf{U}_{B \multimap C}$, the composition via the pullback



is uniform (*i.e.* in $\mathbf{U}_{A \multimap C}$) by [3, Lem. 2] – and the composition pullback is a bipullback, as stated in our motivation for **Thin**.

2.1.3 Morphisms of spans. As introduced above, uniform spans must be related via adequate notions of morphisms between spans:

Definition 2.1 ([3, Def. 1]). A **weak morphism** from $A \leftarrow S \rightarrow B$ to $A \leftarrow S' \rightarrow B$ is (F, F^A, F^B) , with F^A and F^B natural isos, and



We call this a **strong morphism** if F^A and F^B are identities.

The bipullback property, for the composition pullback, ensures the existence of candidates for the horizontal composition of weak morphisms. However, it is not uniquely defined, and the bipullback property is insufficient to guarantee a canonical choice satisfying the laws of a bicategory (see [3, Par. III-B4]). We thus need additional structure in order to ensure the existence of a canonical choice.

2.1.4 Thinness. For this we must capture a more subtle property observed in the denotational interpretation of programs: non-trivial symmetries between states always originate from the environment – in a closed world interaction, no non-trivial symmetry is left. This is called *thinness*, and again is captured by orthogonality.

Given a uniform groupoid A , $S \in \mathbf{U}_A$ and $T \in \mathbf{U}_A^\perp$, we write $S \perp\!\!\!\perp T$ when the pullback vertex of (2) is a discrete groupoid. Given a class $\mathcal{S} \subseteq \mathbf{U}_A$, we write $S^{\perp\!\!\!\perp}$ for the class $\{T \in \mathbf{U}_A^\perp \mid \forall S \in \mathcal{S}, S \perp\!\!\!\perp T\}$.

Definition 2.2 ([3, Def. 10]). A **thin groupoid** is a tuple $A = (A, A_-, A_+, \mathbf{U}_A, \mathbf{T}_A)$ where (A, \mathbf{U}_A) is a uniform groupoid, and

- A_- and A_+ are subgroupoids of A with the same objects, with embedding functors $\text{id}_A^-: A_- \rightarrow A$ and $\text{id}_A^+: A_+ \rightarrow A$;
- $\mathbf{T}_A \subseteq \mathbf{U}_A$ is a class of prestrategies such that $\mathbf{T}_A^{\perp\!\!\!\perp} = \mathbf{T}_A$, satisfying that $(A_-, \text{id}_A^-) \in \mathbf{T}_A$ and $(A_+, \text{id}_A^+) \in \mathbf{T}_A$.

In a groupoid G with $x, y \in G$, we often write $\theta: x \cong_G y$ to mean that $\theta \in G[x, y]$. For A a thin groupoid, $\theta: a \cong_A^+ a'$

indicates that $\theta \in A_+[a, a']$ – we say that θ is a **positive** symmetry – likewise, $\theta : a \cong_{A^-}^+ a'$ indicates that $\theta \in A_-[a, a']$, and we say that θ is **negative**. Intuitively, this polarity tells us who, among the program or the environment, is responsible for a permutation. If it is a permutation among resources called upon by the environment (e.g., coming from an occurrence of $!$ in covariant position), then the symmetry is *negative*. If it permutes resources controlled by the program (e.g. with a $!$ in contravariant position), then the symmetry is *positive*. In general a symmetry may mix the two and can be neither negative nor positive, but from Definition 2.2 we get:

LEMMA 2.3. *For any $\theta : a \cong_A a'$ in a thin groupoid A , there are unique $a'' \in A$ and $\theta^+ : a \cong_{A^+}^+ a''$, $\theta^- : a'' \cong_{A^-}^- a'$ s.t. $\theta = \theta^- \circ \theta^+$.*

See [3, Lem. 3]. The constructions introduced before on uniform groupoids $((-)^{\perp}, \otimes, \wp, \&)$ extend to thin groupoids [3].

2.1.5 *Thin spans.* Given thin groupoids A and B , a **thin span** is a prestrategy $S \in \mathbf{T}_{A \rightarrow B}$. As above the underlying groupoid of $A \rightarrow B$ is $A \times B$, so S can be seen as a span between A and B . Given a thin groupoid A , we have $\text{Id}_A \in \mathbf{T}_{A \rightarrow A}$; and for thin spans $A \leftarrow S \rightarrow B$ and $B \leftarrow T \rightarrow C$, we have $T \odot S \in \mathbf{T}_{A \rightarrow C}$ (see [3, Prop. 2]).

Together, uniformity and thinness guarantee strong properties for the composition of thin spans. For thin spans $A \leftarrow S \rightarrow B$ and $B \leftarrow T \rightarrow C$, recall that (following the obvious pullback construction in \mathbf{Gpd}) elements of $T \odot S$ are simply pairs (s, t) such that $s_B = t_B$. However, it is central in the construction of **Thin** (in particular for the horizontal composition of 2-cells that we shall not detail here) that thin spans may synchronize *up to symmetry*:

LEMMA 2.4. *Consider $A \leftarrow S \rightarrow B$ and $B \leftarrow T \rightarrow C$ thin spans, $s \in S, t \in T$, linked with a symmetry $\theta : s_B \cong_B t_B$.*

Then there are unique $s' \in S, t' \in T$ and $\varphi : s \cong_S s', \psi : t' \cong_T t$ such that φ_A negative, ψ_C positive, and $\theta = \psi_B \circ \varphi_B$.

See [3, Lem. 2]. Another important consequence of the definition of thin spans is that symmetries *act* on thin spans:

LEMMA 2.5. *Consider $A \leftarrow S \rightarrow B$ a thin span, $s \in S$, with $\theta_A : a \cong_A s_A$ and $\theta_B : s_B \cong_B b$. Then, there are unique $s' \in S$, $\varphi : s \cong_S s'$, ϑ_A^- and ϑ_B^+ such that the two triangles commute:*

$$\begin{array}{ccc} & \theta_A & s_A \\ & \nearrow & \downarrow \varphi_A \\ a & & s'_A \\ & \searrow & \downarrow \vartheta_A^- \\ & \vartheta_A^- & s'_A \end{array} \quad \begin{array}{ccc} & s_B & \theta_B \\ & \nearrow & \downarrow \vartheta_B^+ \\ & \vartheta_B^+ & b \\ & \searrow & \downarrow \vartheta_B^+ \\ & s'_B & \vartheta_B^+ \end{array}$$

See Appendix A.1. So $s \in S$ may be reindexed by symmetries θ_A and θ_B , though we will not exactly hit the targets a and b : only up to positive (or negative, depending on the variance) symmetry.

2.1.6 *Positive weak morphisms.* This additional structure may be leveraged to get the canonicity of horizontal composition of 2-cells – modulo a final fine-tuning of their definition:

Definition 2.6. Given two thin groupoids A and B , a weak morphism (F, F^A, F^B) between A and B as in Definition 2.1 is **positive** when, for every $s \in S$, $F_s^B : s_B \cong_B^+ F(s)_B$ and $F_s^A : s_A \cong_A^- F(s)_A$.

We call it *positive* since it is positive on $A \rightarrow B$. Positivity lets us use the uniqueness property of Lemma 2.4 to give a *unique* choice for horizontal composition of positive weak morphisms, and:

THEOREM 2.7 ([3, THM 2]). *There is a bicategory **Thin** of thin groupoids, thin spans, and positive weak morphisms. The identity on A is Id_A , and the composition of thin spans is given by plain pullbacks.*

2.2 The Sym Exponential on Thin

Thin was originally developed using the **Fam** functor as exponential, mapping a groupoid A to $\mathbf{Fam}(A)$ with objects families $(a_i)_{i \in I}$ indexed by finite sets of integers I . Instead, we consider here the **Sym** functor (used as exponential modality on distributors to construct generalized species of structure), which extends to groupoids the list functor of **Set**. This seems a minor difference since **Fam** and **Sym** are equivalent as endofunctors of **Gpd**, but it is actually a non-trivial shift since thin spans do not respect the principle of equivalence, by relying on strict pullbacks in a 2-categorical setting.

2.2.1 *The Sym monad on Gpd.* We start by considering the functor

$$\mathbf{Sym} : \mathbf{Gpd} \rightarrow \mathbf{Gpd}$$

mapping A to the free strict symmetric monoidal groupoid $\mathbf{Sym}(A)$. Concretely, the objects of $\mathbf{Sym}(A)$ are sequences $\langle a_i \rangle_{i \in \{1, \dots, n\}} = \langle a_1, \dots, a_n \rangle$ of objects of A , and its morphisms from $\langle a_1, \dots, a_n \rangle$ to $\langle b_1, \dots, b_m \rangle$ are pairs $(\pi, \langle f_i \rangle_{i \in \{1, \dots, n\}})$ where π is a bijection between $\{1, \dots, n\}$ and $\{1, \dots, m\}$, and $\langle f_i \rangle_i$ is a sequence of morphisms $f_i : a_i \rightarrow b_{\pi(i)}$ for $i \in \{1, \dots, n\}$. **Sym** can be extended to a monad $(\mathbf{Sym}, \eta, \mu)$ on **Gpd**: on objects, the unit $\eta_A : A \rightarrow \mathbf{Sym}(A)$ maps $a \in A$ to $\langle a \rangle$, and $\mu_A : \mathbf{Sym}(\mathbf{Sym}(A)) \rightarrow \mathbf{Sym}(A)$ concatenates sequences – this extends to symmetries as expected.

2.2.2 *The pseudocomonad.* The definition of a pseudocomonad $!$ for **Thin** based on **Sym** is done as in [3, Sec. IV-A], we recall the salient elements here. Given $A = (A, A_-, A_+, \mathbf{U}_A, \mathbf{T}_A)$, we set

$$!A = (\mathbf{Sym}(A), \mathbf{Sym}(A_-), \mathbf{Sym}^+(A_+), (\mathbf{Sym} \mathbf{U}_A)^{\perp\perp}, (\mathbf{Sym} \mathbf{T}_A)^{\perp\perp})$$

where $\mathbf{Sym}^+(A_+)$ is a subgroupoid of $\mathbf{Sym}(A_+)$ with the same objects but morphisms only the $(\text{id}, \langle f_i \rangle_i)$; where $\mathbf{Sym} \mathbf{U}_A$ has all $(\mathbf{Sym}(S), \mathbf{Sym}(\partial^S))$ for all $(S, \partial^S) \in \mathbf{U}_A$, and likewise for $\mathbf{Sym} \mathbf{T}_A$.

Sym lifts to a pseudofunctor $!$ on **Thin** via the functorial action

$$! \left(\begin{array}{ccc} \partial_i^S & S & \partial_i^S \\ \swarrow & & \searrow \\ A & & B \end{array} \right) = \begin{array}{ccc} \mathbf{Sym}(\partial_i^S) & \mathbf{Sym}(S) & \mathbf{Sym}(\partial_i^S) \\ \swarrow & & \searrow \\ \mathbf{Sym}(A) & & \mathbf{Sym}(B) \end{array}$$

on thin spans, defining similarly the image of 2-cells as the image by **Sym** of their underlying components.

When instantiated on the underlying groupoid of a thin groupoid A , the natural transformations η_A and μ_A are not only functors, but **renamings** in the sense of [3]. Recall from there the pseudofunctor $\checkmark : \mathbf{Ren}^{\text{op}} \rightarrow \mathbf{Thin}$ from the (dualized) 2-category of renamings to the bicategory of thin spans, mapping a renaming $f : A \rightarrow B$ to

$$B \xleftarrow{f} A \xrightarrow{\text{id}_A} A$$

a thin span, yielding a counit $\checkmark_A \in \mathbf{Thin}[!A, A]$ and a comultiplication $\checkmark_A \in \mathbf{Thin}[!A, !A]$ for $!$. We have (see Appendix B):

THEOREM 2.8. *We have a pseudocomonad $!$ on **Thin** based on **Sym**.*

$$\begin{array}{c}
\frac{(\alpha \triangleleft A_i)}{\dots, x_i : \langle \alpha \rangle \triangleleft A_i, \dots \triangleleft x_1 : A_1, \dots, x_n : A_n \vdash x_i^\alpha \triangleleft x_i : \alpha \triangleleft A_i} \\
\frac{\Theta \triangleleft \Gamma \vdash m \triangleleft M : \vec{\alpha} \multimap \beta \triangleleft A \rightarrow B \quad \Theta' \triangleleft \Gamma \vdash \vec{n} \triangleleft N : \vec{\alpha} \triangleleft A}{\Theta \bullet \Theta' \triangleleft \Gamma \vdash m \vec{n} \triangleleft MN : \beta \triangleleft B} \\
\frac{\forall i \in \{1, \dots, k\}, \quad \Theta_i \triangleleft \Gamma \vdash m_i \triangleleft M : \alpha_i \triangleleft A}{\Theta_1 \bullet \dots \bullet \Theta_n \triangleleft \Gamma \vdash \langle m_1, \dots, m_k \rangle \triangleleft M : \langle \alpha_1, \dots, \alpha_k \rangle \triangleleft A} \\
\frac{(\Theta, x : \vec{\alpha} \triangleleft A) \triangleleft \Gamma, x : A \vdash m \triangleleft M : \beta \triangleleft B}{\Theta \triangleleft \Gamma \vdash \lambda x. m \triangleleft \lambda x. M : \vec{\alpha} \multimap \beta \triangleleft A \rightarrow B}
\end{array}$$

Figure 1: Intersection types and approximation

3.2.1 *Rigid intersection types.* The **rigid intersection types** are:

$$\begin{array}{l}
\alpha, \beta, \dots \quad ::= \quad \star \quad | \quad \vec{\alpha} \multimap \beta \\
\vec{\alpha}, \vec{\beta}, \dots \quad ::= \quad \langle \alpha_1, \dots, \alpha_n \rangle \quad (n \in \mathbb{N}).
\end{array}$$

As we study the simply-typed λ -calculus, we shall not consider these intersection types as standalone objects but only as refinements of simple types – we now move to the refinement relation.

3.2.2 *Refinement.* The refinement relation is defined with

$$\frac{}{\star \triangleleft o} \quad \frac{\vec{\alpha} \triangleleft A \quad \beta \triangleleft B}{\vec{\alpha} \multimap \beta \triangleleft A \rightarrow B} \quad \frac{\forall i \in \{1, \dots, n\} \quad \alpha_i \triangleleft A_i}{\langle \alpha_1, \dots, \alpha_n \rangle \triangleleft A},$$

noting that both intersection and sequence types may refine simple types. This refinement judgement correctly captures the objects in the groupoid interpreting a type A , as expressed by the following:

PROPOSITION 3.2. *For every simple type A , there are bijections*

$$K_A : \mathbf{Ob}(\llbracket A \rrbracket) \simeq \{\alpha \mid \alpha \triangleleft A\}, \quad K_A^! : \mathbf{Ob}(!\llbracket A \rrbracket) \simeq \{\vec{\alpha} \mid \vec{\alpha} \triangleleft A\}.$$

3.2.3 *Resource contexts.* To extend this to contexts, it is convenient to introduce *resource contexts*. A **resource context** for $\Gamma = x_1 : A_1, \dots, x_n : A_n$ is a sequence of bindings $\Theta = (x_1 : \vec{\alpha}_1 \triangleleft A_1, \dots, x_n : \vec{\alpha}_n \triangleleft A_n)$ – we then write $\Theta \triangleleft \Gamma$. Clearly, the bijections above extend to $K_\Gamma : \mathbf{Ob}(\llbracket \Gamma \rrbracket) \simeq \{\Theta \mid \Theta \triangleleft \Gamma\}$. Given resource contexts for Γ

$$\Sigma = (x_i : \vec{\alpha}_i \triangleleft A_i)_{1 \leq i \leq n} \quad \text{and} \quad \Theta = (x_i : \vec{\beta}_i \triangleleft A_i)_{1 \leq i \leq n},$$

their **concatenation** $\Sigma \bullet \Theta$ is the resource context $(x_i : (\vec{\alpha}_i \bullet \vec{\beta}_i) \triangleleft A_i)_i$, where $\vec{\alpha}_i \bullet \vec{\beta}_i$ is the **concatenation** of sequence types.

3.2.4 *Intersection type judgements.* We now introduce typing judgements for rigid intersection types. There are two kinds of judgements, respectively for single intersection types and for sequences:

$$\Theta \triangleleft \Gamma \vdash M : \alpha \triangleleft A \quad \text{and} \quad \Theta \triangleleft \Gamma \vdash M : \vec{\alpha} \triangleleft A.$$

The rules appear in Figure 1 ignoring, for the moment, the $\dots u/\vec{v} \triangleleft \dots$ parts in the middle. In the variable rule, we only display variables with non-empty sequences. The rules may appear heavy due to the multiple components of judgments as required for the simple type refinement. But ignoring simple type refinements, what remains is the standard ruleset for non-idempotent intersection types as appears e.g. in [6], just without commutativity.

Given a derivation $\Gamma \vdash M : A$, $\gamma \in \mathbf{Ob}(\llbracket \Gamma \rrbracket)$ and $a \in \mathbf{Ob}(\llbracket A \rrbracket)$, we write $\llbracket M \rrbracket_{\gamma, a}$ for the **witnesses** of γ, a , i.e. the objects of $\llbracket M \rrbracket$ that project on γ and a through $\partial_l^{\llbracket M \rrbracket}$ and $\partial_r^{\llbracket M \rrbracket}$. As the definition of $\llbracket M \rrbracket$ directly follows the syntax, it is relatively direct that:

PROPOSITION 3.3. *Given a simply-typed $\Gamma \vdash M : A$, for every $\gamma \in \mathbf{Ob}(\llbracket \Gamma \rrbracket)$, for every $a \in \mathbf{Ob}(\llbracket A \rrbracket)$, we have a bijection*

$$\llbracket M \rrbracket_{\gamma, a} \simeq \{\pi \mid \pi \text{ is a derivation of } K_\Gamma(\gamma) \triangleleft \Gamma \vdash M : K_A(a) \triangleleft A\}.$$

Combined with Theorem 3.1, this shows that for any simply-typed λ -term $\Gamma \vdash M : A$, for any $\gamma \in \llbracket \Gamma \rrbracket$ and $a \in \llbracket A \rrbracket$, the set of $m \in \mathbf{Ob}(\llbracket M \rrbracket)$ mapping to γ, a may be regarded as the set of derivations of $K_\Gamma(\gamma) \triangleleft \Gamma \vdash M : K_A(a) \triangleleft A$ in our rigid intersection type system. This result is to be compared with existing works providing similar characterisations in generalized species of structure [24, 27], where the rigid intersection type systems considered are much more complex, in particular importing symmetries in derivations – and derivations must be quotiented by relations forgetting the exact position of symmetries in the derivations. In contrast, our derivations are the simple inductive structures they appear to be, no quotient is required to obtain our characterisation.

3.3 Extension to symmetries

Proposition 3.3 is analogous to earlier results of Tsukada *et al.* [27] and Olimpieri [24] set in generalized species of structures, but here we go further and characterise the full *groupoid* by also giving an inductive, syntax-directed presentation of the *symmetries*.

3.3.1 *Intersection type morphisms.* The linear, sequence and multilinear intersection type morphisms are defined by the grammar

$$\begin{array}{l}
\phi, \psi, \dots \quad ::= \quad \text{id}_\star \quad | \quad \vec{\phi} \multimap \psi \\
\vec{\phi}, \vec{\psi}, \dots \quad ::= \quad \langle \phi_1, \dots, \phi_n \rangle \quad (n \in \mathbb{N}) \\
\tilde{\phi}, \tilde{\psi}, \dots \quad ::= \quad (\sigma, \vec{\phi}) \quad (\sigma \in \mathcal{S}_n, |\vec{\phi}| = n)
\end{array}$$

where \mathcal{S}_n is the symmetric group on n elements. Given two multilinear morphisms $\vec{\phi}_1$ and $\vec{\phi}_2$ where $\vec{\phi}_i = (\sigma_i, \langle \phi_{i,1}, \dots, \phi_{i,n_i} \rangle)$, we define their **concatenation** $\vec{\phi}_1 \bullet \vec{\phi}_2$ as $(\sigma_1 \oplus \sigma_2, \langle \phi_{1,i} \rangle_i \bullet \langle \phi_{2,i'} \rangle_{i'})$.

3.3.2 *Groupoids of refinements for types.* We extend our refinement relations to morphisms and introduce the **linear and multilinear morphism refinement judgements**, of the form $\phi :: \alpha \Rightarrow \alpha' \triangleleft A$ and $\vec{\phi} :: \vec{\alpha} \Rightarrow \vec{\alpha}' \triangleleft A$. The former states that ϕ is a linear morphism from α to α' within refinements of simple type A , and likewise for the latter. Those are defined inductively through:

$$\frac{}{\text{id}_\star :: \star \Rightarrow \star \triangleleft o} \quad \frac{\vec{\phi} :: \vec{\alpha} \Rightarrow \vec{\alpha}' \triangleleft A \quad \psi :: \beta \Rightarrow \beta' \triangleleft B}{(\vec{\phi} \multimap \psi) :: (\vec{\alpha} \multimap \beta) \Rightarrow (\vec{\alpha}' \multimap \beta') \triangleleft A \rightarrow B} \\
\frac{n \in \mathbb{N} \quad \sigma \in \mathcal{S}_n \quad \forall i \in \{1, \dots, n\} \quad \phi_i :: \alpha_i \Rightarrow \alpha'_{\sigma(i)} \triangleleft A}{(\sigma, \langle \phi_1, \dots, \phi_n \rangle) :: \langle \alpha_1, \dots, \alpha_n \rangle \Rightarrow \langle \alpha'_{\sigma(1)}, \dots, \alpha'_{\sigma(n)} \rangle \triangleleft A}$$

It is immediate that if $\phi :: \alpha \Rightarrow \alpha' \triangleleft A$, then $\alpha \triangleleft A$ and $\alpha' \triangleleft A$, and that likewise, if $\vec{\phi} :: \vec{\alpha} \Rightarrow \vec{\alpha}' \triangleleft A$, then $\vec{\alpha} \triangleleft A$ and $\vec{\alpha}' \triangleleft A$.

As suggested by the syntax, the linear (*resp.* multilinear) intersection types and the associated morphisms that refine a common simple type A organize into a groupoid $\mathbf{IT}(A)$ (*resp.* $\mathbf{IT}_!(A)$). The

composition operation is defined by induction on derivations, with:

$$\begin{aligned} \text{id}_\star \circ \text{id}_\star &= \text{id}_\star \\ (\tilde{\phi}' \circ \psi') \circ (\tilde{\phi} \circ \psi) &= (\tilde{\phi}' \circ \tilde{\phi}) \circ (\psi' \circ \psi) \\ (\sigma', \langle \phi'_i \rangle_{1 \leq i \leq n}) \circ (\sigma, \langle \phi_i \rangle_{1 \leq i \leq n}) &= (\sigma' \circ \sigma, \langle \phi'_{\sigma(i)} \circ \phi_i \rangle_{1 \leq i \leq n}) \end{aligned}$$

The inverse of a morphism is defined by induction similarly. This allows us to extend the correspondence of Proposition 3.2:

PROPOSITION 3.4. *For A a simple type, there are groupoid isos:*

$$K_A : \llbracket A \rrbracket \cong \text{IT}(A) \quad \text{and} \quad K_A^! : \llbracket !A \rrbracket \cong \text{IT}_!(A).$$

As $\llbracket A \rrbracket$ is a thin groupoid, it comes equipped with its two polarized sub-groupoids $\llbracket A \rrbracket_-$ and $\llbracket A \rrbracket_+$ – via the proposition above, they transport to two sub-groupoids $\text{IT}_-(A)$ and $\text{IT}_+(A)$ of $\text{IT}(A)$.

3.3.3 *Groupoids of refinements for contexts.* Consider Γ a context and $\Theta, \Theta' \triangleleft \Gamma$. A **context morphism** from Θ to Θ' is a sequence

$$\Xi = (x_1 : \tilde{\phi}_1 :: \tilde{\alpha}_1 \Rightarrow \tilde{\alpha}'_1, \dots, x_n : \tilde{\phi}_n :: \tilde{\alpha}_n \Rightarrow \tilde{\alpha}'_n)$$

where $\Theta = (x_i : \tilde{\alpha}_i \triangleleft A_i)_{1 \leq i \leq n}$ and $\Theta' = (x_i : \tilde{\alpha}'_i \triangleleft A_i)_{1 \leq i \leq n}$ – we also write $\Xi :: \Theta \Rightarrow \Theta' \triangleleft \Gamma$ to mean that Ξ is a morphism of refinements of Γ from Θ to Θ' ; in that case we write $\Theta = \text{dom}(\Xi)$ and $\Theta' = \text{cod}(\Xi)$. Given two such morphisms $\Xi_1 :: \Theta_1 \Rightarrow \Theta'_1 \triangleleft \Gamma$ and $\Xi_2 :: \Theta_2 \Rightarrow \Theta'_2 \triangleleft \Gamma$ for a common context Γ , their **concatenation**

$$\Xi_1 \bullet \Xi_2 :: \Theta_1 \bullet \Theta_2 \Rightarrow \Theta'_1 \bullet \Theta'_2 \triangleleft \Gamma$$

is defined by componentwise concatenation. The resource contexts and resource context morphisms form a groupoid $\text{IT}(\Gamma)$ which can be seen as the product of the $\text{IT}_!(A_i)$, so we have a groupoid iso

$$K_\Gamma : \llbracket \Gamma \rrbracket \cong \text{IT}(\Gamma).$$

3.3.4 *Morphisms between derivations.* We finally set to construct a *groupoid* of derivations in our rigid intersection type system. The morphisms will be given by two kinds of judgements, of the form

$$\Xi \triangleleft \Gamma \vdash M : \phi :: \alpha \Rightarrow \alpha' \triangleleft A \quad \text{and} \quad \Xi \triangleleft \Gamma \vdash M : \tilde{\phi} :: \tilde{\alpha} \Rightarrow \tilde{\alpha}' \triangleleft A$$

read as stating that ϕ is a morphism from $\text{dom}(\Xi) \triangleleft \Gamma \vdash M : \alpha \triangleleft A$ to $\text{cod}(\Xi) \triangleleft \Gamma \vdash M : \alpha' \triangleleft A$, and likewise for multilinear refinements.

The rules appear in Figure 2. The most subtle case is the last, corresponding to *promotion* and introducing new symmetries following an arbitrary permutation σ . In particular, swapping derivations for M by σ requires swapping accordingly the resource accesses in the context. This uses an operation that to a family $(\tilde{\phi}_i :: \tilde{\alpha}_i \Rightarrow \tilde{\alpha}'_i \triangleleft A)_{1 \leq i \leq n}$ of morphisms of refinements of A associates

$$\sigma \circ (\tilde{\phi}_i)_{1 \leq i \leq n} :: \tilde{\alpha}_1 \bullet \dots \bullet \tilde{\alpha}_n \Rightarrow \tilde{\alpha}'_{\sigma^{-1}(1)} \bullet \dots \bullet \tilde{\alpha}'_{\sigma^{-1}(n)} \triangleleft A$$

a single morphism defined in the obvious way. This generalizes to context refinement morphisms transparently, variable by variable.

Now, given a derivation $\Gamma \vdash M : A$, its associated intersection type derivations $\Theta \triangleleft \Gamma \vdash M : \alpha \triangleleft A$ and intersection type morphism derivations $\Xi \triangleleft \Gamma \vdash M : \phi :: \alpha \Rightarrow \alpha' \triangleleft A$ organize into a groupoid $\text{IT}(M)$, whose composition is directly derived from the ones of refinement types and resource contexts. By considering the two projection functors defined in the obvious way, we get a span

$$\text{IT}(\Gamma) \xleftarrow{\partial_l^M} \text{IT}(M) \xrightarrow{\partial_r^M} \text{IT}(A)$$

which can be seen as a syntactic description of $\llbracket M \rrbracket$ by the result:

THEOREM 3.5. *For any simply-typed λ -term $\Gamma \vdash M : A$, there is an iso of groupoids $K_M : \llbracket M \rrbracket \rightarrow \text{IT}(M)$ making the diagram commute:*

$$\begin{array}{ccccc} \llbracket \Gamma \rrbracket & \xleftarrow{\partial_l^{\llbracket M \rrbracket}} & \llbracket M \rrbracket & \xrightarrow{\partial_r^{\llbracket M \rrbracket}} & \llbracket A \rrbracket \\ K_\Gamma \downarrow & & \downarrow K_M & & \downarrow K_A \\ \text{IT}(\Gamma) & \xleftarrow{\partial_l^M} & \text{IT}(M) & \xrightarrow{\partial_r^M} & \text{IT}(A) \end{array}$$

By Theorem 3.1, this also applies to the Kleisli interpretation. From this connection to the interpretation in the cartesian closed bicategory $\mathbf{Thin}_!$, we immediately get the following corollary:

COROLLARY 3.6. *Consider $\Gamma \vdash M, M' : A$ simply-typed λ -term, s.t. $M \rightarrow_\beta M'$. Then, there is a weak iso of spans $\text{IT}(M) \cong \text{IT}(M')$.*

This shows that although rigid intersection types do not enjoy subject reduction as observed in the introduction, the interpretation in $\mathbf{Thin}_!$ associates to every β -reduction $M \rightarrow_\beta M'$ a bijective *transport* between derivations of M and M' “correcting” the error, up to some residual symmetries in the groupoids for Γ and A .

3.4 Rigid Resource Calculus

As derivations are somewhat heavy, it seems helpful to remark that they can be equivalently presented as certain *rigid resource terms*.

3.4.1 *Resource terms.* The grammar for **rigid resource terms** is:

$$\begin{aligned} m, n, \dots &::= x^\alpha \mid \lambda x. m \mid m \bar{n} \\ \vec{m}, \vec{n}, \dots, m_k &::= \langle m_1, \dots, m_k \rangle, \end{aligned}$$

where x^α is the data of a variable $x \in \text{Var}$ and of a **labelling** intersection type α . Our resource terms depart from standard resource terms [11] in two significant ways. Firstly, as in [25] our calculus is *rigid*: argument subterms are sequences rather than finite multisets. Secondly, we label variable occurrences with intersection types, so as to guarantee the correspondence with derivations.

3.4.2 *Approximation relations.* Those resource terms are already implicitly present in our derivations. To formalize that, we introduce the **linear and multilinear approximation judgements**

$$\Theta \triangleleft \Gamma \vdash m \triangleleft M : \alpha \triangleleft A \quad \text{and} \quad \Theta \triangleleft \Gamma \vdash \vec{m} \triangleleft M : \vec{\alpha} \triangleleft A$$

which are defined by the (full) rules of Figure 1. We have a canonical forgetful function U mapping a derivation π of $\Theta \triangleleft \Gamma \vdash m \triangleleft M : \alpha \triangleleft A$ to the corresponding derivation $U(\pi)$ of $\Theta \triangleleft \Gamma \vdash M : \alpha \triangleleft A$ and similarly for multilinear judgements. We easily check that:

PROPOSITION 3.7. *The following two properties hold:*

- Given a term $\Gamma \vdash M : A$ and resource term m , there is at most one (Θ, α, π) with π a derivation of $\Theta \triangleleft \Gamma \vdash m \triangleleft M : \alpha \triangleleft A$,
- For a derivation π of $\Theta \triangleleft \Gamma \vdash M : \alpha \triangleleft A$, there is a unique $(u, \tilde{\pi})$ s.t. $\tilde{\pi}$ is a derivation of $\Theta \triangleleft \Gamma \vdash m \triangleleft M : \alpha \triangleleft A$ and $U(\tilde{\pi}) = \pi$.

For a term $\Gamma \vdash M : A$, we write $\text{Res}(M)$ for the set of resource terms m such that $\Theta \triangleleft \Gamma \vdash m \triangleleft M : \alpha \triangleleft A$ is derivable, for some rigid intersection types / contexts α, Θ . The proposition above gives

$$\text{Res}(M) \simeq \text{Ob}(\text{IT}(M))$$

a bijection showing that up to isomorphism, $\mathbf{Thin}_!$ interprets a simply-typed λ -term as a set of rigid resource terms.

$$\begin{array}{c}
\frac{(\phi :: \alpha \Rightarrow \alpha' \triangleleft A_i)}{\dots, x_i : (\text{id}_{\{1\}}, \langle \phi \rangle) :: \langle \alpha \rangle \Rightarrow \langle \alpha' \rangle \triangleleft A_i, \dots \vdash x_i : \phi :: \alpha \Rightarrow \alpha' \triangleleft A_i} \quad \frac{\exists \vdash M : (\tilde{\phi} \multimap \psi) :: (\tilde{\alpha} \multimap \beta) \Rightarrow (\tilde{\alpha}' \multimap \beta') \triangleleft A \rightarrow B \quad \exists' \vdash N : \tilde{\phi} :: \tilde{\alpha} \Rightarrow \tilde{\alpha}' \triangleleft A}{\exists \bullet \exists' \vdash MN : \psi :: \beta \Rightarrow \beta' \triangleleft B} \\
\frac{\exists, x : \tilde{\phi} :: \tilde{\alpha} \Rightarrow \tilde{\alpha}' \triangleleft A \vdash M : \psi :: \beta \Rightarrow \beta' \triangleleft B}{\exists \vdash \lambda x. M : (\tilde{\phi} \multimap \psi) :: (\tilde{\alpha} \multimap \beta) \Rightarrow (\tilde{\alpha}' \multimap \beta') \triangleleft A \rightarrow B} \quad \frac{n \in \mathbb{N} \quad \sigma \in \mathcal{S}_n \quad \forall i \in \{1, \dots, n\}, \exists_i \vdash M : \phi_i :: \alpha_i \Rightarrow \alpha'_i \triangleleft A}{\sigma \otimes (\exists_i)_{1 \leq i \leq n} \vdash M : (\sigma, \langle \phi_1, \dots, \phi_n \rangle) :: \langle \alpha_1, \dots, \alpha_n \rangle \Rightarrow \langle \alpha'_{\sigma^{-1}(1)}, \dots, \alpha'_{\sigma^{-1}(n)} \rangle \triangleleft A}
\end{array}$$

Figure 2: The rules for rigid intersection type morphisms

3.4.3 *Resource terms and reduction.* This representation lets us examine the action of the interpretation of reduction steps given by Corollary 3.6. Consider a β -redex $\vdash (\lambda x. M) N$. There is an iso

$$\llbracket (\lambda x. M) N \rightarrow_{\beta} M[N/x] \rrbracket : \llbracket (\lambda x. M) N \rrbracket \cong \llbracket M[N/x] \rrbracket$$

obtained via the cartesian closed bicategorical structure of **Thin**_! [13], and through our results it yields a bijection $\Omega : \mathbf{Res}(\llbracket (\lambda x. M) N \rrbracket) \simeq \mathbf{Res}(M[N/x])$ which we can compute. Considering a resource term $u = (\lambda x. m) \langle n_1, \dots, n_k \rangle \in \mathbf{Res}(\llbracket (\lambda x. M) N \rrbracket)$ for $m \triangleleft M, \vec{n} \triangleleft N$, we get

$$\Omega((\lambda x. m) \langle n_1, \dots, n_k \rangle) = m[n_1/x_1, \dots, n_k/x_k] \quad (4)$$

where x_1, \dots, x_k are the occurrences of x in m , in order from left to right – there must indeed be k occurrences with the right intersection types, because u matches an intersection type derivation.

But this apparent simplicity for top-level β -reductions is misleading: **Thin**_! interprets reduction as *weak* span isos. If we have

$$\Theta \triangleleft \Gamma \vdash m \triangleleft M : \alpha \triangleleft A,$$

for $\Gamma \vdash M : A$ with $M \rightarrow_{\beta} M'$, then we do not have $\Theta \triangleleft \Gamma \vdash M' : \alpha \triangleleft A$ but only $\Theta' \triangleleft \Gamma \vdash M' : \alpha' \triangleleft A$ for $\Theta' \cong_{\Gamma}^{-} \Theta$ and $\alpha' \cong_A^{+} \alpha$; so we cannot directly perform (4) deep within m as the resulting resource term would fail to typecheck in our rigid intersection type system.

Thin_! does provide some $m' = \llbracket M \rightarrow_{\beta} M' \rrbracket(m)$, obtained through an interactive reindexing of all components of m , correcting the typing mismatches. But its construction fully exploits the bicategorical structure of **Thin**_!, and in particular the horizontal composition of 2-cells (via the uniqueness property of Lemma 2.4) and it does not seem to have a simple syntactic presentation.

3.4.4 *Link with multiset resource terms.* To conclude this section, we show how our rigid resource terms do not have a self-contained rewriting theory; however we show here how they can be used as representatives for more standard (multiset-based) resource terms.

We consider **multiset resource terms** generated by the grammar:

$$\begin{array}{ll}
\mathbf{u}, \mathbf{v}, \dots & ::= x^{\alpha} \mid \lambda x. \mathbf{u} \mid \mathbf{u} \mathbf{v}^* \\
\mathbf{u}^*, \mathbf{v}^*, \dots & ::= [\mathbf{u}_1, \dots, \mathbf{u}_n]
\end{array}$$

using the (multiset) non-idempotent intersection types defined by

$$\begin{array}{ll}
\alpha, \beta, \dots & ::= \star \mid \alpha^* \multimap \beta \\
\alpha^*, \beta^*, \dots & ::= [\alpha_1, \dots, \alpha_n] \quad (n \in \mathbb{N})
\end{array}$$

where, as expected, we use multisets $[\dots]$ instead of sequences $\langle \dots \rangle$. Given a rigid intersection type α , one can obtain a multiset intersection type $\bar{\alpha}$ by replacing inductively the sequences $\langle \dots \rangle$ with multisets $[\dots]$. Similarly, given a rigid resource term m , one obtains a multiset resource term \bar{m} with the same operation. Then:

PROPOSITION 3.8. *Take β -normal $\Gamma \vdash M : A$, and $m, n \in \mathbf{Res}(M)$. Then, $m \cong n$ if and only if $\bar{m} = \bar{n}$.*

This is direct by induction – here $m \cong n$ is defined via the correspondence with derivations. This shows that standard resource terms fit in the theory of thin spans of groupoids as symmetry classes in the interpretation of terms, albeit for β -normal terms. For non-normal terms this correspondence fails: we have

$$(\lambda y. x y y) \langle z, w \rangle \neq (\lambda y. x y y) \langle w, z \rangle$$

though they both map to $(\lambda y. x y y) [w, z]$ – in rigid resource terms, β -redexes explicitly match variable occurrences and resources in the argument sequence, while usual resource terms do not.

4 THIN SPANS AND RELATIONAL MODELS

Now, we relate thin spans and other extensions of the relational model. This shall let us re-interpret what these compute in terms of rigid resource terms and symmetries of rigid intersection types.

4.1 The Relational Model

First of all, we start by describing the relationship between thin spans of groupoids and the relational model [16]. It is fairly straightforward, but is hopefully helpful for the generalizations to come.

4.1.1 *Introducing the relational model.* The relational model builds on the category **Rel** of *sets* and *relations*. **Rel** has a symmetric monoidal structure, obtained by defining the tensor $A \otimes B = A \times B$ as the cartesian product of sets – the unit is any singleton set. **Rel** is actually compact closed: the *dual* A^* of a set A is itself, and there are a unit $I \rightarrow A \otimes A^*$ and co-unit $A^* \otimes A \rightarrow I$ given by the obvious diagonal relations. This turns **Rel** into a symmetric monoidal closed category, and as such a model of the linear λ -calculus – in particular, it supports a linear arrow defined as $A \multimap B = A \times B$.

But **Rel** also has an *exponential modality*, given by $!A = \mathcal{M}(A)$ the set of finite multisets of elements of A . This extends to a comonad $!$ on **Rel** and for each A, B there is an isomorphism $!(A \& B) \cong !A \otimes !B$, the *Seely isomorphism*. Together with additional coherence conditions [23], this makes **Rel** a *Seely category*, a model of intuitionistic linear logic, and the Kleisli category **Rel**_! is cartesian closed.

4.1.2 *From Thin to Rel.* It seems clear how to relate **Thin** and **Rel**: on objects, simply send a thin groupoid A to $|A| = A/\cong$ its **symmetry classes** (or connected components) – clearly, $|\mathbf{Sym}(A)| = \mathcal{M}(|A|)$. Likewise, given a thin span $A \leftarrow S \rightarrow B$, we can obtain

$$|S| = \{(\overline{s_A}, \overline{s_B}) \mid s \in S\} \in \mathbf{Rel}[|A|, |B|]$$

called its **relational collapse**, for $\overline{(-)}$ the equivalence class. Then:

PROPOSITION 4.1. *This yields a functor $|-| : \mathbf{Thin} \rightarrow \mathbf{Rel}$.*

PROOF. This requires us to compose witnesses *up to symmetry*, which we do thanks to Lemma 2.4 – see Appendix D.1. \square

4.1.3 *Preservation of further structure.* From the definition, it is straightforward that we have bijection yielding isos in **Rel**:

$$\begin{aligned} t_{A,B}^{\otimes} &: |A| \otimes |B| \cong |A \otimes B| \\ t_{A,B}^{\&} &: |A| \& |B| \cong |A \& B| \\ t_A^! &: !|A| \cong !|A| \end{aligned}$$

for A and B thin groupoids; in particular the third amounts to $!|A| \cong \mathcal{M}(|A|)$ for A any thin groupoid. It is a routine verification that these components satisfy the coherence conditions required to make $|-| : \mathbf{Thin} \rightarrow \mathbf{Rel}$ a Seely functor (see Appendix E), so that:

THEOREM 4.2. *Setting, for any $!A \leftarrow S \rightarrow B$ in $\mathbf{Thin}_!$, $[A, B]$,*

$$|S|_! = |S| \circ t_A^! \in \mathbf{Rel}_! [|A|, |B|],$$

yields $|-| : \mathbf{Thin}_! \rightarrow \mathbf{Rel}_!$ a cartesian closed functor.

It follows that this preserves the interpretation of the simply-typed λ -calculus: for every simple type A there is a bijection $t_A : \llbracket A \rrbracket_{\mathbf{Rel}_!} \simeq |(A)|$ – and likewise for contexts – so that if $\Gamma \vdash M : A$, $\gamma \in \llbracket \Gamma \rrbracket_{\mathbf{Rel}_!}$, $a \in \llbracket A \rrbracket_{\mathbf{Rel}_!}$, $(\gamma, a) \in \llbracket M \rrbracket_{\mathbf{Rel}_!}$ iff $(t_\Gamma \gamma, t_A a) \in |(M)|_!$.

4.2 Weighted Relations

The *weighted relational model* is due to Larmarche [20], though its application to semantics was fleshed out by Laird *et al.* [19]. In full generality, its construction is parametrized by a complete semiring; but for the purposes of this paper we will only work with the semiring $\mathbb{N}_\infty = \mathbb{N} \cup \{+\infty\}$ of completed natural numbers.

4.2.1 *The weighted relational model.* Rather than merely collecting the completed executions, the weighted relational assigns a *weight* – here, an element of \mathbb{N}_∞ – to any execution. In other words, a weighted relation from set A to set B is a function $A \times B \rightarrow \mathbb{N}_\infty$.

This lets us *count* properties of execution: for instance, it is shown in [19] how the relational model weighted by \mathbb{N}_∞ counts how many distinct executions may lead to a given result at ground type, for a non-deterministic extension of PCF. But even for purely deterministic programs (in fact, simply-typed λ -terms), the weighted relational model computes non-trivial coefficients.

Example 4.3. Considering the simply-typed λ -term

$$f : o \rightarrow o \rightarrow o, x : o, y : o \vdash f(fyx)(fxy) : o,$$

then the point of the web written in intersection type notation as

$$f : \llbracket [\star] \multimap [] \multimap \star, [] \multimap [\star] \multimap \star \rrbracket, x : [\star], y : [] \vdash \star$$

has a weight of 2 in the weighted relational model – this reflects the fact that this point can be realized in two distinct ways, depending on which occurrence of f calls which argument; seemingly corresponding to two distinct normal resource terms:

$$f[f[] [x^\star]][] \quad f[] [f[x^\star]][],$$

or (via Section 3.4.4) to two symmetry classes of rigid terms.

This suggests that, maybe, the weighted relational model counts the number of resource terms inhabiting a certain intersection type. But that is not actually the case, as illustrated by this next example.

Example 4.4. Considering now the simply-typed λ -term

$$f : o \rightarrow o, g : o \rightarrow o, y : o \vdash f(gy) : o,$$

then the point of the web written in intersection type notation as

$$f : \llbracket [\star, \star] \multimap \star \rrbracket, g : \llbracket [] \multimap \star, [\star] \multimap \star \rrbracket, y : [\star] \vdash \star$$

is *also* assigned a weight of 2 by the weighted relational model, even though the reader can check that there is only one resource term inhabiting that type. Clearly here we are somehow accounting for the *symmetries* of this resource term – but which symmetries?

4.2.2 *Categorical structure.* The weighted relational model is structured around the category **WRel**: its objects are sets, and a morphism from A to B is $\alpha \in \mathbb{N}_\infty^{A \times B}$ – for $a \in A$ and $b \in B$, we write $\alpha_{a,b} \in \mathbb{N}_\infty$ for $\alpha(a, b)$. Identity is $(\text{id}_A)_{a,a'} = \delta_{a,a'}$. Composition is

$$(\beta \circ \alpha)_{a,c} = \sum_{b \in B} \alpha_{a,b} \cdot \beta_{b,c}$$

for $\alpha \in \mathbf{WRel}[A, B]$, $\beta \in \mathbf{WRel}[B, C]$, $a \in A$ and $c \in C$. This potentially infinite sum always “converges” because our set of weights \mathbb{N}_∞ includes the infinity. Just like **Rel**, **WRel** is a compact closed category with biproducts, see [19] for details.

Finally, there is an exponential modality $!A = \mathcal{M}(A)$ on sets. On morphisms, the critical definition is that of *functorial promotion*:

$$(!\alpha)_{\mu, [b_1, \dots, b_n]} = \sum_{\substack{(a_1, \dots, a_n) \\ \text{s.t. } \mu = [a_1, \dots, a_n]}} \prod_{i=1}^n \alpha_{a_i, b_i}.$$

Altogether, just like **Rel**, **WRel** is a Seely category, and thus the associated Kleisli category **WRel**_! is cartesian closed.

4.2.3 *Positive witnesses.* We must make the functor of Section 4.1.2 quantitative – from a thin span $A \leftarrow S \rightarrow B$ and symmetry classes $\mathbf{a} \in |A|$, $\mathbf{b} \in |B|$, we must assign a number $|S|_{\mathbf{a}, \mathbf{b}} \in \mathbb{N}_\infty$. We naturally expect this number to be the cardinal of a set of *witnesses*

$$|S|_{\mathbf{a}, \mathbf{b}} = \# \text{wit}_S(\mathbf{a}, \mathbf{b}),$$

thus our question boils down to the following: what is the adequate notion of witnesses, in a thin span, for symmetry classes \mathbf{a}, \mathbf{b} ? It is tempting to count symmetry classes in S , however we have seen in Section 3.4.4 that (for normal terms) those correspond to resource terms, and Example 4.4 shows that it is not what the weighted relational model counts; in fact we shall see it accounts for

$$f \langle \lambda x. g \langle y \rangle, \lambda x g \langle \rangle \rangle, \quad f \langle \lambda x. g \langle \rangle, \lambda x. g \langle y \rangle \rangle, \quad (5)$$

the *two* rigid resource terms that intuitively inhabit the intersection type of Example 4.4 – even though the two are symmetric. But it is not the case that we are simply counting rigid resource terms! If we were to replace y with x in Example 4.4, then the weight given by **WRel** becomes one and thus the two rigid resource terms displayed in (5) with x instead of y should suddenly just account for one...

Thin will help sort this out. Assume that all groupoids interpreting types come equipped with a function $(-)$ associating to each symmetry class $\mathbf{a} \in |A|$ a representative $\underline{\mathbf{a}} \in \mathbf{a}$. Then we set

$$\text{wit}_S^+(\mathbf{a}, \mathbf{b}) = \{s \in S \mid \underline{\mathbf{a}} \cong_A^- s_A \& s_B \cong_B^+ \underline{\mathbf{b}}\} \quad (6)$$

where $a \cong_A^+ a'$ means there is $\theta^+ \in A_+ [a, a']$ and likewise for \cong_A^- ; we call those the **positive witnesses** of \mathbf{a} and \mathbf{b} in S . This depends

on a choice of representatives for symmetry classes – our development will apply for thin groupoids equipped with representatives:

Definition 4.5. A **representation** for a thin groupoid A is a function $(-): (a \in |A|) \rightarrow \underline{a}$ such that for all $a \in A$, \underline{a} is **canonical**, in the sense that for all $\theta \in A[\underline{a}, \underline{a}]$, the unique factorization $\theta = \theta^- \circ \theta^+$ given by Lemma 2.3 satisfies $\theta^- \in A_-[\underline{a}, \underline{a}]$ and $\theta^+ \in A_+[\underline{a}, \underline{a}]$.

If A is a thin groupoid with a representation and $a \in A$, we write $m(a) = \#A(\underline{a}, \underline{a})$ the **symmetry degree** of a . Likewise, we write $m_+(a) = \#A_+(\underline{a}, \underline{a})$ (resp. $\#A_-(\underline{a}, \underline{a})$) the **positive symmetry degree** (resp. negative) of a . From Definition 4.5, we then have

$$m(a) = m_+(a) \cdot m_-(a) \quad (7)$$

reflecting quantitatively the factorization of Lemma 2.3.

One can build a representation for all constructions on thin groupoids so far. The non-trivial case is the exponential: if we have canonical $a_1, \dots, a_n \in A$, then so is $\langle a_1, \dots, a_n \rangle \in !A$, provided that whenever $a_i \cong_A a_j$ then $a_i = a_j$. Thus given $\mathbf{a} = [a_1, \dots, a_n] \in !A$ we first consider $[\underline{a}_1, \dots, \underline{a}_n]$, which we present in a sequential ordering, following some total order on objects of A that we assume globally fixed in advance. From now on, we consider all thin groupoids equipped with a canonical representation.

Summing up, to any thin span $A \leftarrow S \rightarrow B$ we associate $|S|_{\mathbf{a}, \mathbf{b}} = \#\text{wit}_S^+(\mathbf{a}, \mathbf{b})$, and we now aim to prove that this extends to a functor.

4.2.4 Functoriality. Preservation of the identity is obvious by the factorization property of Lemma 2.3. Composition is more subtle. Naturally, for $A \leftarrow S \rightarrow B$ and $B \leftarrow T \rightarrow C$ we expect a bijection

$$\text{wit}_{T \circ S}^+(\mathbf{a}, \mathbf{c}) \simeq \sum_{\mathbf{b} \in |B|} \text{wit}_S^+(\mathbf{a}, \mathbf{b}) \times \text{wit}_T^+(\mathbf{b}, \mathbf{c}), \quad (8)$$

and while our results imply that such a bijection exists for cardinality reasons, it is not actually what we shall build directly. In fact, there appears to be no natural function from the right-hand side to the left-hand side. We must assemble $s \in \text{wit}_S^+(\mathbf{a}, \mathbf{b})$ and $t \in \text{wit}_T^+(\mathbf{b}, \mathbf{c})$ into an element of $\text{wit}_{T \circ S}^+(\mathbf{a}, \mathbf{c})$ but we cannot do that directly, as we only have $s_B \cong_B t_B$ and not $s_B = t_B$. We can, as in the proof of Proposition 4.14, compose s and t via any symmetry $\theta_B : s_B \cong_B t_B$ to obtain an element of $\text{wit}_{T \circ S}^+(\mathbf{a}, \mathbf{c})$; but this does not yield a function as the result depends on the choice of θ_B .

To address this dependency in the undetermined mediating symmetry, we consider instead the composition of witnesses carrying explicit symmetries: the \sim -**witnesses** from \mathbf{a} to \mathbf{b} are triples

$$\sim\text{-wit}_S^+(\mathbf{a}, \mathbf{b}) = \{(\theta_A^-, s, \theta_B^+) \mid \theta_A^- : \underline{a} \cong_A^- s_A \ \& \ s_B \cong_B^+ \underline{b}\};$$

so $(\theta_A^-, s, \theta_B^+) \in \sim\text{-wit}_S^+(\mathbf{a}, \mathbf{b})$ and $(\theta_B^-, t, \theta_C^+) \in \sim\text{-wit}_T^+(\mathbf{b}, \mathbf{c})$ providing $\theta_B^- \circ \theta_B^+$ used to compose s and t via Lemma 2.4.

While in a thin span $A \leftarrow S \rightarrow B$ the display $S \rightarrow A \times B$ is not a fibration, \sim -witnesses do enjoy a fibration-like property:

PROPOSITION 4.6. Consider $A \leftarrow S \rightarrow B$ a thin span, $s \in S$, and

$$\theta_A^- : a \cong_A^- s_A \quad \theta_B^+ : s_B \cong_B^+ b.$$

For $\Omega_A : a' \cong_A a$ and $\Omega_B : b \cong_B b'$, there are unique $\varphi^S : s \cong_S s'$ and $\vartheta_A^- : a' \cong_A^- s'_A$, $\vartheta_B^+ : s'_B \cong_B^+ b'$ s.t. the diagrams commute:

$$\begin{array}{ccc} a & \xrightarrow{\theta_A^-} & s_A \\ \Omega_A \uparrow & & \downarrow \varphi_A^S \\ a' & \xrightarrow{\vartheta_A^-} & s'_A \end{array} \quad \begin{array}{ccc} s_B & \xrightarrow{\theta_B^+} & b \\ \varphi_B^S \downarrow & & \downarrow \Omega_B \\ s'_B & \xrightarrow{\vartheta_B^+} & b' \end{array}$$

This follows from Lemma 2.5. We can now establish the bijection patching (8). Consider $A \leftarrow S \rightarrow B$ and $B \leftarrow T \rightarrow C$, $\mathbf{a} \in |A|$, $\mathbf{b} \in |B|$ and $\mathbf{c} \in |C|$, we write $\sim\text{-wit}_{S,T}^+(\mathbf{a}, \mathbf{b}, \mathbf{c})$ for the \sim -**interaction witnesses**, i.e. tuples $(\theta_A^-, s, \theta, t, \theta_C^+)$ where $\theta_A^- : \underline{a} \cong_A^- s_A$, $s_B = t_B = b$ and $\theta_C^+ : t_C \cong_C^+ \underline{c}$ so that $(s, t) \in T \circ S$; and $\theta : \underline{b} \cong_B b$.

PROPOSITION 4.7. For $S, T, \mathbf{a}, \mathbf{b}, \mathbf{c}$ as above, there is a bijection

$$\Upsilon : \sim\text{-wit}_S^+(\mathbf{a}, \mathbf{b}) \times \sim\text{-wit}_T^+(\mathbf{b}, \mathbf{c}) \simeq \sim\text{-wit}_{S,T}^+(\mathbf{a}, \mathbf{b}, \mathbf{c})$$

s.t. for any $\Upsilon((\theta_A^-, s, \theta_B^+), (\theta_B^-, t, \theta_C^+)) = (\psi_A^-, s', \Theta, t', \psi_C^+)$, there are unique $\omega^S : s \cong_S s'$ and $v^T : t \cong_T t'$ making the diagrams commute:

$$\begin{array}{ccccc} \underline{a} & \begin{array}{c} \xrightarrow{\theta_A^-} s_A \\ \downarrow \psi_A^- \\ s'_A \end{array} & \begin{array}{c} s_B \xrightarrow{\theta_B^+} \underline{b} \\ \downarrow \omega_B^S \\ s'_B \end{array} & \begin{array}{c} \xrightarrow{\theta_B^-} t_B \\ \downarrow \Theta \\ t'_B \end{array} & \begin{array}{c} t_C \xrightarrow{\theta_C^+} \underline{c} \\ \downarrow v_C^T \\ t'_C \end{array} \\ & & & & \end{array}$$

This is direct from Lemma 2.4 and Proposition 4.6, see App. D.2.

We now have a bijection that somewhat looks like (8), but we must sum over all symmetry classes in B and check that the cardinality of added symmetries cancels out. Indeed it is easy that

$$\#\sim\text{-wit}_S^+(\mathbf{a}, \mathbf{b}) = m_-(\mathbf{a}) \cdot \#\text{wit}_S^+(\mathbf{a}, \mathbf{b}) \cdot m_+(\mathbf{b});$$

from the definition, and since \sim -interaction witnesses carry a symmetry class in B and an endo-symmetry, it is also direct that

$$\#\sim\text{-wit}_{T \circ S}^+(\mathbf{a}, \mathbf{c}) = \sum_{\mathbf{b} \in |B|} \frac{1}{m(\mathbf{b})} \cdot \#\sim\text{-wit}_{S,T}^+(\mathbf{a}, \mathbf{b}, \mathbf{c}).$$

From there and (7), (8) follows from a simple computation. So:

COROLLARY 4.8. This yields a functor $|-| : \mathbf{Thin} \rightarrow \mathbf{WRel}$.

4.2.5 Exponential. The crucial point remaining is that the functorial action of $!$ is preserved. For this section, we adopt notations inlining the bijections of Section 4.1.3: in particular, we write elements of $!A$ as finite multisets of elements of $|A|$. We must give

$$\text{wit}_S^+(\boldsymbol{\mu}, [\mathbf{b}_1, \dots, \mathbf{b}_n]) \simeq \sum_{\langle \mathbf{a}_1, \dots, \mathbf{a}_n \rangle} \prod_{i=1}^n \text{wit}_S^+(\mathbf{a}_i, \mathbf{b}_i) \quad (9)$$

s.t. $[\mathbf{a}_1, \dots, \mathbf{a}_n] = \boldsymbol{\mu}$

a bijection, for any thin span $A \leftarrow S \rightarrow B$.

From left to right, recall that writing $\boldsymbol{\nu} = [\mathbf{b}_1, \dots, \mathbf{b}_n]$, $\text{wit}_S^+(\boldsymbol{\mu}, \boldsymbol{\nu})$ comprises those \vec{s} such that $\underline{\boldsymbol{\mu}} \cong_A^- \vec{s}_A$ and $\vec{s}_B \cong_B^+ \underline{\boldsymbol{\nu}}$. Let us write $\underline{\boldsymbol{\nu}} = \langle b_1, \dots, b_n \rangle$. On the right-hand side, as positive symmetries cannot exchange elements of a sequence, we have $\vec{s} = \langle s^1, \dots, s^n \rangle$ where $s_B^i \cong_B^+ b_i$. However on the left-hand side symmetries can exchange elements, so that there must exist an (unspecified) permutation $\sigma \in \zeta(n)$ such that $\underline{\boldsymbol{\mu}}_{\sigma(i)} \cong_A^- s_A^i$, informing $\langle \mathbf{a}_{\sigma(1)}, \dots, \mathbf{a}_{\sigma(n)} \rangle$ satisfying $[\mathbf{a}_{\sigma(1)}, \dots, \mathbf{a}_{\sigma(n)}] = \boldsymbol{\mu}$ as needed. Reciprocally, it is clear that data on the right-hand side can be assembled into an element of $\text{wit}_S^+(\boldsymbol{\mu}, \boldsymbol{\nu})$ and that those operations are inverse of one another.

This shows that modulo the bijection $t_A^!$ of Section 4.1.3, the functorial action of $!$ is preserved. The other bijections of Section 4.1.3 still yield isomorphisms in \mathbf{WRel} – for which, by a slight abuse, we keep the same notation. All necessary coherence conditions are satisfied, so that this operation lifts to the Kleisli (bi)categories.

THEOREM 4.9. *We have $|-| : \mathbf{Thin}_! \rightarrow \mathbf{WRel}_!$ cartesian closed.*

4.2.6 Consequences. Since a cartesian closed functor preserves the interpretation of the simply-typed λ -calculus, this gives us a combinatorial description of the coefficients computed by $\mathbf{WRel}_!$:

COROLLARY 4.10. *Consider $\Gamma \vdash M : A$ a simply-typed λ -term.*

For every $\gamma \in \llbracket \Gamma \rrbracket_{\mathbf{WRel}_!}$ and $\mathbf{a} \in \llbracket A \rrbracket_{\mathbf{WRel}_!}$, we have

$$(\llbracket M \rrbracket_{\mathbf{WRel}_!})_{\gamma, \mathbf{a}} = \# \text{wit}_{\langle M \rangle}^+(t_\Gamma \gamma, t_A \mathbf{a}).$$

By the results in Section 3.2, this is also the number of derivations $\Theta \triangleleft \Gamma \vdash M : \alpha \triangleleft A$ (or their representations as rigid resource terms) where Θ is negatively symmetric (*resp.* α is positively symmetric) to the intersection type matching a chosen canonical rigid representative for γ (*resp.* for \mathbf{a}). Note that we can also derive:

PROPOSITION 4.11. *Consider $\Gamma \vdash M : A$ a simply-typed λ -term.*

For every $\gamma \in \llbracket \Gamma \rrbracket_{\mathbf{WRel}_!}$ and $\mathbf{a} \in \llbracket A \rrbracket_{\mathbf{WRel}_!}$, we have

$$(\llbracket M \rrbracket_{\mathbf{WRel}_!})_{\gamma, \mathbf{a}} = \sum_{\mathbf{w} \in W} \frac{m_+(t_\Gamma \gamma) \cdot m_-(t_A \mathbf{a})}{m(\mathbf{w})}$$

where W is the set of symmetry classes in $\langle M \rangle$ mapping to $(t_\Gamma \gamma, t_A \mathbf{a})$, and $m(\mathbf{w})$ is the size of the group of symmetries on \mathbf{w} .

This is because to each symmetry class \mathbf{w} correspond a number of positive witnesses equal to the negative symmetries of the matching rigid intersection type, divided by the symmetries of \mathbf{w} – the proof appears in Appendix A. Thus, one can obtain the right coefficient from symmetry classes (and therefore for normal standard resource terms following Section 3.4.4), but the weight of each symmetry class must be corrected suitably accounting for symmetries.

4.3 Distributors and Generalized Species

We now establish a link between thin spans and the bicategory of distributors (*i.e.* profunctors). We keep this section succinct; to a large extent, it is a simplification of the construction in [4].

4.3.1 The bicategory of groupoids and distributors. A **distributor** from groupoid A to B (*a.k.a.* *profunctor*) is a functor $\alpha : A^{\text{op}} \times B \rightarrow \mathbf{Set}$ giving, for all $a \in A, b \in B$, a set $\alpha(a, b)$ of *witnesses*, along with an action of symmetries: if $x \in \alpha(a, b)$ and $\theta \in B(b, b')$, we write $\theta \cdot x$ for the functorial action $\alpha(\text{id}, \theta)(x) \in \alpha(a, b')$. Similarly, if $\vartheta \in A(a', a)$, we write $x \cdot \vartheta \in \alpha(a', b)$ for $\alpha(\vartheta, \text{id})$.

The bicategory \mathbf{Dist} has groupoids as objects, distributors as morphisms, and natural transformations as 2-cells. The **identity distributor** on A is the hom-set functor $\text{id}_A = A[-, -] : A^{\text{op}} \times A \rightarrow \mathbf{Set}$. The **composition** of two distributors $\alpha : A^{\text{op}} \times B \rightarrow \mathbf{Set}$ and $\beta : B^{\text{op}} \times C \rightarrow \mathbf{Set}$ is defined in terms of the coend formula:

$$(\beta \bullet \alpha)(a, c) = \int^{b \in B} \alpha(a, b) \times \beta(b, c),$$

meaning that concretely, $(\beta \bullet \alpha)(a, c)$ consists in pairs (x, y) , where $x \in \alpha(a, b)$ and $y \in \beta(b, c)$ for some $b \in B$, quotiented by $(g \cdot x, y) \sim (x, y \cdot g)$ for $x \in \alpha(a, b), g \in B(b, b')$ and $y \in \beta(b', c)$. The bicategory \mathbf{Dist} has cartesian products given by the disjoint union $A + B$.

4.3.2 Extracting distributors from thin spans. On objects, we send a thin groupoid (A, A_-, A_+, U_A, T_A) to its underlying groupoid A .

On morphisms, given a thin span $A \leftarrow S \rightarrow B$, for all $a \in A$ and $b \in B$ we must specify a set $\llbracket S \rrbracket(a, b)$. It is tempting to set simply the pre-image $(\partial^S)^{-1}(a, b)$, but there is no functorial action

$$\llbracket S \rrbracket(\theta_A, \theta_B) : \llbracket S \rrbracket(a, b) \rightarrow \llbracket S \rrbracket(a', b')$$

for $\theta_A \in A(a', a)$ and $\theta_B \in B(b, b')$ as ∂^S is not a fibration. We need a finer symmetry lifting property of thin spans – and we have one, seen in Proposition 4.6. Thus, we set instead $\llbracket S \rrbracket(a, b)$ as the set $\sim\text{-wit}_S^+(a, b)$ of $\sim\text{-witnesses}$ of (a, b) in S , *i.e.* triples $(\theta_A^-, s, \theta_B^+)$ *s.t.* $s \in S, \theta_A^- \in A_-(a, s_A)$ and $\theta_B^+ \in B_+(s_B, b)$. Though we keep the same terminology and notation as in Section 4.2.4, those are $\sim\text{-witnesses}$ of *specific* objects of the groupoids A and B , not symmetry classes.

We get a functorial action by setting $\llbracket S \rrbracket(\Omega_A, \Omega_B)(\theta_A^-, s, \theta_B^+)$ as the positive witness $(\vartheta_A^-, s', \vartheta_B^+)$ as in the statement of Proposition 4.6, yielding a distributor for every thin span $A \leftarrow S \rightarrow B$:

PROPOSITION 4.12. *We have a distributor $\llbracket S \rrbracket : A^{\text{op}} \times B \rightarrow \mathbf{Set}$.*

4.3.3 Constructing natural transformations. Consider S, T thin spans from A to B , and $(F, F^A, F^B) : S \rightarrow T$ a positive morphism; consisting for each $s \in S$ of $F_s^A \in A_-(s_A, (Ft)_A)$ and $F_s^B \in B_+(s_B, (Fs)_B)$.

To each $\mathbf{w} = (\theta_A^-, s, \theta_B^+) \in \llbracket S \rrbracket(a, b)$, we set $\llbracket S \rrbracket(F, F^A, F^B)(\mathbf{w})$ to

$$(a \xrightarrow{\theta_A^-} s_A \xrightarrow{F_s^A} (Ft)_A, \quad Ft, \quad (Ft)_B \xrightarrow{F_s^B} s_B \xrightarrow{\theta_B^+} b)$$

which by the uniqueness property of Proposition 4.6 can be easily verified to give a natural transformation from $\llbracket S \rrbracket$ to $\llbracket T \rrbracket$.

4.3.4 Further components. To complete the pseudofunctor, we need two natural isomorphisms, the *unitor* and the *compositor*.

PROPOSITION 4.13. *Given a thin span A , there is a natural iso*

$$\text{pid}^A : \llbracket \text{id}_A \rrbracket \xrightarrow{\cong} A[-, -] : A^{\text{op}} \times A \rightarrow \mathbf{Set}.$$

This is straightforward from the factorization result of Lemma 2.3. Now, we focus on the preservation of composition. For two thin spans $A \leftarrow S \rightarrow B$ and $B \leftarrow T \rightarrow C$, we have the **compositor**:

PROPOSITION 4.14. *There is a natural isomorphism:*

$$\text{pcomp}^{S, T} : \llbracket T \circ S \rrbracket \Rightarrow \llbracket T \rrbracket \bullet \llbracket S \rrbracket : A^{\text{op}} \times B \rightarrow \mathbf{Set}.$$

PROOF. The map $\text{pcomp}_{a, c}^{S, T}$ sends $(\theta_A^-, (s, t), \theta_C^+) \in \llbracket T \circ S \rrbracket(a, c)$ (with $s_B = t_B = b$) to (the equivalence class of) the pair

$$((\theta_A^-, s, \text{id}_b), (\text{id}_b, t, \theta_C^+)) \in (\llbracket T \rrbracket \bullet \llbracket S \rrbracket)(a, c).$$

For each $a \in A$ and $c \in C$, this forms a bijection. Consider indeed

$$\mathbf{w}^S = (\theta_A^-, s, \theta_B^+) \in \llbracket S \rrbracket(a, b) \quad \mathbf{w}^T = (\theta_B^-, t, \theta_C^+) \in \llbracket T \rrbracket(b, c)$$

composable witnesses. By Lemma 2.4 we compose s and t through $\theta_B^- \circ \theta_B^+$, yielding unique $\varphi^S \in S[s, s'], \varphi^T \in T[t, t'], \vartheta_A^-, \vartheta_C^+$ *s.t.*:

$$\begin{array}{ccccc} \theta_A^- & \xrightarrow{s_A} & s_B & \xrightarrow{\theta_B^-} & t_B & \xrightarrow{t_C} & \theta_C^+ \\ a & \searrow & \downarrow \varphi_B^S & & \downarrow \varphi_B^T & \varphi_C^T \downarrow & \nearrow c \\ \vartheta_A^- & \xrightarrow{s'_A} & s'_B = b' = t'_B & & t'_C & \nearrow \vartheta_C^+ \end{array}$$

which, writing $\Theta_B = \varphi_B^S \circ \theta_B^{+1} = \varphi_B^T \circ \theta_B^-$, entails

$$\begin{aligned} v^S &= (\vartheta_{A'}^-, s', \text{id}_{y_B}) = \Theta_B \cdot (\theta_{A'}^-, s, \theta_B^+) \\ v^T &= (\text{id}_{y_B}, t', \vartheta_C^+) = (\theta_B^-, t, \theta_C^+) \cdot \Theta_B \end{aligned}$$

so $(v^S, v^T) = (\Theta_B \cdot w^S, v^T) \sim (w^S, v^T \cdot \Theta_B) = (w^S, w^T)$. Now $(v^S, v^T) = \text{pcomp}^{S,T}(\vartheta_{A'}^-, t' \circ s', \vartheta_C^+)$, showing surjectivity – injectivity also follows from the uniqueness clause in Lemma 2.4. \square

The naturality and coherence requirements hold, and altogether:

THEOREM 4.15. *This yields a pseudofunctor $\| - \| : \mathbf{Thin} \rightarrow \mathbf{Dist}$.*

4.3.5 Lifting to Kleisli bicategories. Recall that \mathbf{Esp} is the Kleisli bicategory $\mathbf{Dist}_{\mathbf{Sym}}$. Composition of $F : \mathbf{Sym}(A)^{\text{op}} \times B \rightarrow \mathbf{Set}$ and $G : \mathbf{Sym}(B)^{\text{op}} \times C \rightarrow \mathbf{Set}$ is $G \bullet F^{\mathbf{Sym}}$, where the **promotion** is

$$F^{\mathbf{Sym}}(\vec{a}, \langle b_1, \dots, b_n \rangle) = \int^{\vec{a}'_1, \dots, \vec{a}'_n} A[\vec{a}, \vec{a}'_1 \dots \vec{a}'_n] \times \prod_{i=1}^n F(\vec{a}'_i, b_i)$$

comprising a morphism in $A[\vec{a}, \vec{a}'_1, \dots, \vec{a}'_n]$ along with a family in $\prod_{i=1}^n F(\vec{a}'_i, b_i)$, quotiented by an equivalence relation.

Likewise, the promotion $S^{\mathbf{Sym}}$ of a thin span, constructed as

$$\mathbf{Sym}(A) \leftarrow \mathbf{Sym}(\mathbf{Sym}(A)) \leftarrow \mathbf{Sym}(S) \rightarrow \mathbf{Sym}(B),$$

yields by $\| - \|$ the distributor associating to $\vec{a}, \langle b_1, \dots, b_n \rangle$ triples

$$(\theta_{\mathbf{Sym}(A)}^-, \langle s_1, \dots, s_n \rangle, \theta_{\mathbf{Sym}(B)}^+) \in \|S\|^{\mathbf{Sym}}\|\vec{a}, \vec{b}\|, \quad (10)$$

but $\theta_{\mathbf{Sym}(B)}^+$ is positive, so cannot reindex the b_i s and must be $(\text{id}_{1\dots n}, (\theta_i^+)_{1 \leq i \leq n})$ for θ_i^+ is positive in B . So we map (10) to

$$(\theta_{\mathbf{Sym}(A)}^-, \langle (\text{id}, s_i, \theta_i^+) \mid 1 \leq i \leq n \rangle) \in \|S\|^{\mathbf{Sym}}\|\vec{a}, \vec{b}\|$$

inducing a natural bijection $\|S\|^{\mathbf{Sym}}\|\vec{a}, \vec{b}\| \simeq \|S\|_{\vec{a}, \vec{b}}^{\mathbf{Sym}}$.

Combined with $\text{pcomp}^{S,T}$ this provides a natural iso for preservation of Kleisli composition. Together with a straightforward natural isomorphism for Kleisli identity laws and lengthy verifications for coherence, we obtain a pseudofunctor $\| - \| : \mathbf{Thin}_! \rightarrow \mathbf{Esp}$.

4.3.6 A cartesian closed pseudofunctor. We check that this extends to a *cc-pseudofunctor* [13]. First, $\| - \|$ preserves constructions on objects strictly. The notion of a *fp-pseudofunctor* [13] requires that for each $(A_i)_{1 \leq i \leq n}$, $\langle \|\pi_1\|, \dots, \|\pi_n\| \rangle$ is part of an adjoint equivalence

$$\begin{array}{ccc} \prod_{i=1}^n A_i & \xleftarrow{\langle \|\pi_1\|, \dots, \|\pi_n\| \rangle} & \prod_{i=1}^n A_i \\ & \perp & \\ & \xrightarrow{q_A^\times} & \end{array}$$

in \mathbf{Esp} : here q^\times can be taken to be the identity in \mathbf{Esp} , completed to an adjoint equivalence in the obvious way. On top of that, the definition of a *cc-pseudofunctor* [13] then additionally requires that $e_{A,B} = \Lambda(\|\text{ev}_{A,B}\| \bullet_{\mathbf{Sym}} q^\times) : A \Rightarrow B \rightarrow A \Rightarrow B$ is also part of

$$\begin{array}{ccc} A \Rightarrow B & \xrightarrow{e_{A,B}} & A \Rightarrow B \\ & \perp & \\ & \xrightarrow{q_{A,B}} & \end{array}$$

an adjoint equivalence. But $e_{A,B}$ can be computed to be naturally isomorphic to the identity on $A \Rightarrow B$ in \mathbf{Esp} ; constructing the adjoint equivalence is then straightforward. Altogether:

THEOREM 4.16. $\| - \| : \mathbf{Thin}_! \rightarrow \mathbf{Esp}$ is a *cc-pseudofunctor*.

4.3.7 Consequences. Fix a simply-typed λ -term $\Gamma \vdash M : A$.

By Theorem 4.16, we have a natural isomorphism $I : \llbracket M \rrbracket_{\mathbf{Esp}} \cong \llbracket \langle M \rangle \rrbracket$ showing that up to iso, generalized species of structure compute positive witnesses in the sense of thin spans of groupoids.

By the results of Section 3, this can be reformulated as:

COROLLARY 4.17. *For $\gamma \in (\|\Gamma\|)$ and $a \in (\|A\|)$, we have a bijection*

$$\llbracket M \rrbracket_{\mathbf{Esp}}(\gamma, a) \cong \left\{ \begin{array}{l} \theta_{\Gamma}^- \in \mathbf{IT}_-(\Gamma)[K_{\Gamma}(s_{\Gamma}), \Theta], \\ \mathbf{w} \in \mathbf{IT}(M)_{\Theta, \alpha}, \\ \theta_A^+ \in \mathbf{IT}_+(A)[\alpha, K_A a] \end{array} \right\}.$$

This captures the interpretation of simply-typed λ -terms in \mathbf{Esp} syntactically. This is analogous to results by Tsukada *et al.* [27] and Olimpieri [24], except our derivations are simpler, without quotient.

Finally, altogether, the isomorphism I and Corollary 4.10 entail:

COROLLARY 4.18. *For any $\gamma \in \llbracket \Gamma \rrbracket_{\mathbf{WRel}}$ and $a \in \llbracket A \rrbracket_{\mathbf{WRel}}$,*

$$(\llbracket M \rrbracket_{\mathbf{WRel}})_{\gamma, a} = \frac{\#\llbracket M \rrbracket_{\mathbf{Esp}}(t_{\Gamma} \gamma, t_A a)}{m_-(t_{\Gamma} \gamma) \cdot m_+(t_A a)}$$

where $\#\llbracket M \rrbracket_{\mathbf{Esp}}(t_{\Gamma} \gamma, t_A a)$ is defined for any representative.

This is independent of $\mathbf{Thin}_!$, though it does require the positive and negative symmetries – this shows that these are fundamental in quantitative semantics, independently of their role in \mathbf{Thin} .

5 CONCLUSION

We have illustrated our results on the simply-typed λ -calculus for the economy of presentation and since it already features the phenomena of interest, but \mathbf{Thin} readily supports non-determinism and can be easily extended with quantitative (probabilistic and quantum) primitives, for which we expect our results still hold.

Our results show that the interpretation of the simply-typed λ -calculus in \mathbf{Thin} can be regarded as a rigid Taylor expansion. Section 3.4.4 then suggests a link with the standard Taylor expansion of λ -terms which may illuminate the coefficients appearing there; however we could not find an exposition of the simply-typed Taylor expansion in the literature, so we had to omit this by lack of space. Detailing that, and the untyped case, is left for future work.

REFERENCES

- [1] Samson Abramsky, Radha Jagadeesan, and Pasquale Malacaria. 2000. Full Abstraction for PCF. *Inf. Comput.* 163, 2 (2000), 409–470. <https://doi.org/10.1006/inco.2000.2930>
- [2] Simon Castellan, Pierre Clairambault, and Glynn Winskel. 2019. Thin Games with Symmetry and Concurrent Hyland-Ong Games. *Log. Methods Comput. Sci.* 15, 1 (2019). [https://doi.org/10.23638/LMCS-15\(1:18\)2019](https://doi.org/10.23638/LMCS-15(1:18)2019)
- [3] Pierre Clairambault and Simon Forest. 2023. The Cartesian Closed Bicategory of Thin Spans of Groupoids. In *LICS*. 1–13. <https://doi.org/10.1109/LICS56636.2023.10175754>
- [4] Pierre Clairambault, Federico Olimpieri, and Hugo Paquet. 2023. From Thin Concurrent Games to Generalized Species of Structures. In *LICS*. 1–14. <https://doi.org/10.1109/LICS56636.2023.10175681>
- [5] Daniel De Carvalho. 2007. *Sémantiques de la logique linéaire et temps de calcul*. Ph. D. Dissertation.
- [6] Daniel de Carvalho. 2018. Execution time of λ -terms via denotational semantics and intersection types. *Math. Struct. Comput. Sci.* 28, 7 (2018), 1169–1203. <https://doi.org/10.1017/S0960129516000396>
- [7] Daniel de Carvalho, Michele Pagani, and Lorenzo Tortora de Falco. 2011. A semantic measure of the execution time in linear logic. *Theor. Comput. Sci.* 412, 20 (2011), 1884–1902. <https://doi.org/10.1016/j.tcs.2010.12.017>
- [8] Thomas Ehrhard. 2002. On Köthe Sequence Spaces and Linear Logic. *Math. Struct. Comput. Sci.* 12, 5 (2002), 579–623.
- [9] Thomas Ehrhard, Michele Pagani, and Christine Tasson. 2018. Full Abstraction for Probabilistic PCF. *J. ACM* 65, 4 (2018), 23:1–23:44.

- [10] Thomas Ehrhard and Laurent Regnier. 2003. The differential lambda-calculus. *Theor. Comput. Sci.* 309, 1-3 (2003), 1–41. [https://doi.org/10.1016/S0304-3975\(03\)00392-X](https://doi.org/10.1016/S0304-3975(03)00392-X)
- [11] Thomas Ehrhard and Laurent Regnier. 2008. Uniformity and the Taylor expansion of ordinary lambda-terms. *Theoretical Computer Science* 403, 2-3 (2008), 347–372.
- [12] Marcelo Fiore, Nicola Gambino, Martin Hyland, and Glynn Winskel. 2008. The cartesian closed bicategory of generalised species of structures. *Journal of the London Mathematical Society* 77, 1 (2008), 203–220.
- [13] Marcelo Fiore and Philip Saville. 2021. Coherence for bicategorical cartesian closed structure. *Math. Struct. Comput. Sci.* 31, 7 (2021), 822–849. <https://doi.org/10.1017/S0960129521000281>
- [14] Philippa Gardner. 1994. Discovering Needed Reductions Using Type Theory. In *TACS (Lecture Notes in Computer Science, Vol. 789)*. Springer, 555–574.
- [15] Jean-Yves Girard. 1987. Linear Logic. *Theor. Comput. Sci.* 50 (1987), 1–102. [https://doi.org/10.1016/0304-3975\(87\)90045-4](https://doi.org/10.1016/0304-3975(87)90045-4)
- [16] Jean-Yves Girard. 1988. Normal functors, power series and λ -calculus. *Ann. Pure Appl. Log.* 37, 2 (1988), 129–177.
- [17] J. M. E. Hyland and C.-H. Luke Ong. 2000. On Full Abstraction for PCF: I, II, and III. *Inf. Comput.* 163, 2 (2000), 285–408. <https://doi.org/10.1006/inco.2000.2917>
- [18] Martin Hyland. 1997. Game Semantics. *Semantics and Logics of Computation* 14 (1997), 131.
- [19] Jim Laird, Giulio Manzonetto, Guy McCusker, and Michele Pagani. 2013. Weighted Relational Models of Typed Lambda-Calculi. In *28th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2013, New Orleans, LA, USA, June 25-28, 2013*. IEEE Computer Society, 301–310. <https://doi.org/10.1109/LICS.2013.36>
- [20] François Lamarche. 1992. Quantitative Domains and Infinitary Algebras. *Theor. Comput. Sci.* 94, 1 (1992), 37–62.
- [21] Paul-André Mellies. 2019. Template games and differential linear logic. In *LICS. IEEE*, 1–13.
- [22] Paul-André Mellies. 2003. Asynchronous games 1: Uniformity by group invariance.
- [23] Paul-André Mellies. 2009. Categorical semantics of linear logic. *Panoramas et synthèses* 27 (2009), 15–215.
- [24] Federico Olimpieri. 2021. Intersection Type Distributors. In *36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 - July 2, 2021*. IEEE, 1–15. <https://doi.org/10.1109/LICS52264.2021.9470617>
- [25] Federico Olimpieri and Lionel Vaux Auclair. 2022. On the Taylor expansion of λ -terms and the groupoid structure of their rigid approximants. *Log. Methods Comput. Sci.* 18, 1 (2022).
- [26] Michele Pagani, Peter Selinger, and Benoît Valiron. 2014. Applying quantitative semantics to higher-order quantum computing. In *POPL. ACM*, 647–658.
- [27] Takeshi Tsukada, Kazuyuki Asada, and C.-H. Luke Ong. 2017. Generalised species of rigid resource terms. In *32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, Reykjavik, Iceland, June 20-23, 2017*. IEEE Computer Society, 1–12. <https://doi.org/10.1109/LICS.2017.8005093>
- [28] Takeshi Tsukada, Kazuyuki Asada, and C.-H. Luke Ong. 2018. Species, Profunctors and Taylor Expansion Weighted by SMCC: A Unified Framework for Modelling Nondeterministic, Probabilistic and Quantum Programs. In *LICS. ACM*, 889–898.

Appendix

A ADDITIONAL PROPERTIES IN **Thin**

Here, we provide the proofs of properties of thin spans of groupoids that this paper need, which were not provided in [3].

A.1 Reindexing by a symmetry

Here, we show the detailed proof of Lemma 2.5, which expresses how **Thin** lets us reindex witnesses by symmetries.

LEMMA 2.5. *Consider $A \leftarrow S \rightarrow B$ a thin span, $s \in S$, with $\theta_A : a \cong_A s_A$ and $\theta_B : s_B \cong_B b$. Then, there are unique $s' \in S$, $\varphi : s \cong_S s'$, ϑ_A^- and ϑ_B^+ such that the two triangles commute:*

$$\begin{array}{ccc} a & \xrightarrow{\theta_A} & s_A \\ & \searrow \vartheta_A^- & \downarrow \varphi_A \\ & & s'_A \end{array} \quad \begin{array}{ccc} s_B & \xrightarrow{\theta_B} & b \\ \varphi_B \downarrow & & \nearrow \vartheta_B^+ \\ s'_B & & \end{array}$$

PROOF. We show this ignoring the left-hand side, for $S \in \mathbf{T}_B$, $s \in S$, $\theta_B : s_B \cong_B b$; the general case follows by applying this to $A^\perp \mathcal{Y} B$.

Existence. By hypothesis, we know that $S \in \mathbf{T}_B \subseteq \mathbf{U}_B$. By definition of thin groupoids, we know that $(B_+, \text{id}_B^+) \in \mathbf{T}_B^\perp \subseteq \mathbf{U}_B^\perp$, so that $S \perp (B_+, \text{id}_B^+)$. Hence, the pullback

$$\begin{array}{ccc} & \begin{array}{c} l \quad \cdot \quad r \\ \swarrow \quad \downarrow \quad \searrow \\ S \quad \quad B_+ \end{array} & \\ \vartheta^s \searrow & & \swarrow \text{id}_B^+ \\ & B & \end{array}$$

is a bipullback. By our concrete characterisation of bipullbacks in **Gpd**, applying this to $s \in S$, $b \in B_+$ and $\theta_B : s_B \cong_B b$, this gives us $\varphi : s \cong_S s'$ and $\vartheta_B^+ : s'_B \cong_B^+ b$ such that $\theta_B = \vartheta_B^+ \circ \varphi_B$ as required.

Uniqueness. Consider another solution, comprising $\psi : s \cong_S s''$ and $\nu_B^+ : s''_B \cong_B^+ b$ such that $\nu_B^+ \circ \psi_B = \theta_B$. Then, $\nu_B^+ \circ \psi_B = \vartheta_B^+ \circ \varphi_B$, so

$$(\psi \circ \varphi^{-1})_B = \psi_B \circ \varphi_B^{-1} = (\vartheta_B^+)^{-1} \circ \nu_B^+$$

a positive morphism. But by [3, Lem. 3], a morphism $\psi \circ \varphi^{-1}$ in S which maps to a positive morphism in B must be an identity; hence $s' = s''$ and $\psi \circ \varphi^{-1} = \text{id}_{s'}$, so that $\varphi = \psi$. Additionally, $\vartheta_B^+ = \theta_B \circ (\varphi_B)^{-1} = \theta_B \circ (\psi_B)^{-1} = \nu_B^+$ as desired, concluding the proof. \square

A.2 Counting symmetry classes

Our aim here is to provide a characterisation of the number of positive witnesses inhabiting a given symmetry class; providing the missing brick for the proof of Proposition 4.11. For this section, let us fix a thin groupoid A and some $S \in \mathbf{T}_A$; we shall derive the two-sided version of the result by simply applying it to $A^\perp \mathcal{Y} B$.

First, we show that any symmetry class in S has a representative that is positively symmetric to (the chosen representative) of the corresponding symmetry class in A :

LEMMA A.1. *Consider $\mathfrak{s} \in S/\cong_S$, and consider a its display.*

Then, there is $s \in \mathfrak{s}$ such that $s_A \cong_A^+ \mathfrak{a}$.

PROOF. Consider first any $s \in \mathfrak{s}$. By hypothesis, there is $\theta_A : s_A \cong_A \mathfrak{a}$. It might not be positive, but by Lemma 2.3 (applied to θ_A^{-1}) it factors uniquely as $s_A \cong_A^{\theta_A^-} a \cong_A^{\theta_A^+} \mathfrak{a}$ and now, by Proposition 4.6, there are unique $\varphi : s \cong_S s'$ and $\vartheta_A^+ : s'_A \cong_A^+ a$ such that $\theta_A^- = \vartheta_A^+ \circ \varphi_A$. But then $s' \in \mathfrak{s}$ and $s'_A \cong_A^+ a \cong_A^+ \mathfrak{a}$. \square

So, for each $\mathfrak{s} \in S/\cong_S$, we choose a representative $\underline{\mathfrak{s}} \in \mathfrak{s}$ such that $(\underline{\mathfrak{s}})_A \cong_A^+ \mathfrak{a}$; and we also choose a ‘‘reference’’ positive symmetry $\theta_{\underline{\mathfrak{s}}}^+ : (\underline{\mathfrak{s}})_A \cong_A^+ \mathfrak{a}$. Finally, for every $s \in \mathfrak{s}$ we choose some $\kappa_s : \underline{\mathfrak{s}} \cong_S s$.

Our aim is, for a fixed $\mathfrak{a} \in |A|$ and for every symmetry class \mathfrak{s} such that $\mathfrak{s}_A = \mathfrak{a}$, to count the number of concrete positive witnesses in \mathfrak{s} . We introduce some notations for this set – let us write

$$\begin{aligned} \text{wit}_S^+(\mathfrak{s}) &= \{s \in \mathfrak{s} \mid s_A \cong_A^+ \mathfrak{a}\} \\ \sim\text{-wit}_S^+(\mathfrak{s}) &= \{(s, \theta_A^+) \mid s \in \mathfrak{s} \ \& \ \theta_A^+ : s_A \cong_A^+ \mathfrak{a}\} \end{aligned}$$

for the concrete witnesses (resp. \sim^+ -witnesses) within a symmetry class \mathfrak{s} for \mathfrak{a} .

Then, we prove the following bijection, for $\mathcal{S}(\mathbf{s}) = S[\underline{\mathbf{s}}, \underline{\mathbf{s}}]$ and $\mathcal{S}(\mathbf{a}) = A[\underline{\mathbf{a}}, \underline{\mathbf{a}}]$.

PROPOSITION A.2. *There is a bijection $\sim\text{-wit}_\sigma^+[\mathbf{s}] \times \mathcal{S}(\mathbf{s}) \simeq \mathcal{S}(\mathbf{a})$.*

PROOF. First we show that for every $(s, \vartheta_A^+) \in \sim\text{-wit}_\sigma^+(\mathbf{a})$ and $\varphi \in \mathcal{S}(\mathbf{s})$, there is a unique $\psi_A \in \mathcal{S}(\mathbf{a})$ such that the following diagram commutes:

$$\begin{array}{ccc} (\underline{\mathbf{s}})_A & \xrightarrow{\theta_s^+} & \underline{\mathbf{a}} \\ (\kappa_s \circ \varphi)_A \downarrow & & \downarrow \psi_A \\ s_A & \xrightarrow{\vartheta_A^+} & \underline{\mathbf{a}} \end{array}$$

but this is obvious, as ψ_A is determined by composition from the other components.

Reciprocally, we show that for all $\psi_A \in \mathcal{S}(\mathbf{a})$, there are unique $(s, \vartheta_A^+) \in \sim\text{-wit}_\sigma^+(\mathbf{a})$ and $\varphi \in \mathcal{S}(\mathbf{s})$ such that the same diagram above commutes. First, by canonicity of $\underline{\mathbf{a}}$, ψ_A factors as $\psi_A = \psi_A^+ \circ \psi_A^-$ for $\psi_A^- \in \mathcal{S}(\mathbf{a})$ negative and $\psi_A^+ \in \mathcal{S}(\mathbf{a})$ positive. By Proposition 4.6, there are unique $(s', \omega_A^+) \in \sim\text{-wit}_\sigma^+(\mathbf{a})$ and $\phi : \underline{\mathbf{s}} \cong_S s'$ such that the following diagram commutes:

$$\begin{array}{ccc} (\underline{\mathbf{s}})_A & \xrightarrow{\theta_s^+} & \underline{\mathbf{a}} \\ \phi_A \downarrow & & \downarrow \psi_A^- \\ s'_A & \xrightarrow{\omega_A^+} & \underline{\mathbf{a}} \end{array}$$

We may then define $s := s'$, $\vartheta_A^+ := \psi_A^+ \circ \omega_A^+$, and $\varphi := (\kappa_{s'})^{-1} \circ \phi$ and the diagram is obviously satisfied. It remains to prove uniqueness, so assume we have $(t, \nu_A^+) \in \sim\text{-wit}_\sigma^+(\mathbf{a})$ and $\xi \in \mathcal{S}(\mathbf{s})$ such that the following diagram commutes:

$$\begin{array}{ccc} (\underline{\mathbf{s}})_A & \xrightarrow{\theta_s^+} & \underline{\mathbf{a}} \\ (\kappa_t \circ \xi)_A \downarrow & & \downarrow \psi_A \\ t_A & \xrightarrow{\nu_A^+} & \underline{\mathbf{a}} \end{array}$$

But then $(\kappa_t \circ \xi) \circ (\kappa_s \circ \varphi)^{-1}$ is a symmetry in S displaying to a positive symmetry in A , so must be an identity by [3, Lem. 3]. Thus $s = t$, $\xi = \varphi$, and so also $\vartheta_A^+ = \nu_A^+$ as it is uniquely determined from the other components by the diagram. This gives constructions in both directions, and that they are inverses follows directly from the uniqueness properties. \square

From that bijection, we may conclude the following result:

THEOREM A.3. *Consider A a thin groupoid, $S \in \mathbf{T}_A$ and $\mathbf{a} \in |A|$, s displaying to \mathbf{a} . Then,*

$$\#\text{wit}_S^+[\mathbf{s}] = \frac{\#\mathcal{S}_-(\mathbf{a})}{\#\mathcal{S}(\mathbf{s})}$$

where $\mathcal{S}_-(\mathbf{a})$ is the group of negative symmetries on $\underline{\mathbf{a}}$.

PROOF. By Proposition A.2, we have $\#\sim\text{-wit}_\sigma^+[\mathbf{s}] \times \#\mathcal{S}(\mathbf{s}) = \#\mathcal{S}(\mathbf{a})$, so we have

$$\#\mathcal{S}_+(\mathbf{a}) \times \#\text{wit}_\sigma^+[\mathbf{s}] \times \#\mathcal{S}(\mathbf{s}) = \#\mathcal{S}_+(\mathbf{a}) \times \#\mathcal{S}_-(\mathbf{a})$$

via the easy fact that $\#\sim\text{-wit}_\sigma^+[\mathbf{s}] = \#\mathcal{S}_+(\mathbf{a}) \times \#\text{wit}_\sigma^+[\mathbf{s}]$ and canonicity of $\underline{\mathbf{a}}$. The identity follows. \square

And now, we can finally deduce:

COROLLARY A.4. *Consider A, B thin groupoids, $A \leftarrow S \rightarrow B$ a thin span, $\mathbf{a} \in |A|$ and $\mathbf{b} \in |B|$. Then,*

$$|S|_{\mathbf{a}, \mathbf{b}} = \sum_{s \in W} \frac{m_+(\mathbf{a}) \cdot m_-(\mathbf{b})}{m(\mathbf{s})}$$

for W the set of symmetry classes in S mapping to \mathbf{a}, \mathbf{b} .

PROOF. We calculate:

$$\begin{aligned} |S|_{\mathbf{a},\mathbf{b}} &= \#\text{wit}_S^+(\mathbf{a}, \mathbf{b}) \\ &= \sum_{s \in W} \#\text{wit}_S^+[s] \\ &= \sum_{s \in W} \frac{m_+(\mathbf{a}) \cdot m_-(\mathbf{b})}{m(s)} \end{aligned}$$

using the definition, then partitioning the positive witnesses by symmetry class, and by Theorem A.3. \square

Proposition 4.11 immediately follows from Corollary 4.10 in combination with this.

B THE Sym PSEUDOCOMONAD

Here, we give additional details about the definition of the **Sym** pseudocomonad on **Thin**, derived from the **Sym** monad on **Gpd**. Notably, we reuse the general results presented in [3, App. G], which were developed to show that the **Fam** pseudomonad on **Gpd** lifted to a pseudocomonad, and that we recall below.

B.1 General definitions and results

A functor $F: \mathbf{Gpd} \rightarrow \mathbf{Gpd}$ is called *bicartesian* when it preserves pullbacks and sends pullbacks that are bipullbacks to bipullbacks. A \pm -functor is a tuple $(\mathbf{H}, \mathbf{H}^+, \iota)$ with \mathbf{H}, \mathbf{H}^+ being functors $\mathbf{Gpd} \rightarrow \mathbf{Gpd}$ where \mathbf{H} and \mathbf{H}^+ are bicartesian and preserve functors (between groupoids) that are bijective on objects (of the groupoids), and such that \mathbf{H}^+ preserves discrete groupoids, and $\iota: \mathbf{H}^+ \Rightarrow \mathbf{H}$ being a natural transformation which is pointwise monomorphic (that is, such that each ι_X is a monomorphism) and surjective on objects of the groupoids, satisfying moreover that it is *bicartesian*, meaning that its naturality squares are both pullbacks and bipullbacks.

Given two \pm -functors $\mathbf{H} = (\mathbf{H}, \mathbf{H}^+, \iota)$ and $\mathbf{K} = (\mathbf{K}, \mathbf{K}^+, \kappa)$, a \pm -transformation between \mathbf{H} and \mathbf{K} is a pair (α, α^+) of natural transformations where $\alpha: \mathbf{H} \Rightarrow \mathbf{K}$ and $\alpha^+: \mathbf{H}^+ \Rightarrow \mathbf{K}^+$ are such that α is bicartesian and

$$\kappa \circ \alpha^+ = \alpha \circ \iota. \quad (11)$$

LEMMA B.1. *Given a \pm -transformation (α, α^+) , α^+ is bicartesian.*

PROOF. The bicartesianness of α^+ can be deduced using standard properties of rectangles of pullbacks and their adaptation to rectangles of pullbacks that are bipullbacks [3, Lemma 5]. \square

Now, a \pm -modification between two such \pm -transformations (α, α^+) and (β, β^+) is the data of a modification $m: \alpha \Rightarrow \beta$ in the 3-category of 2-categories.

Definition B.2. We write $\pm\text{-Funct}$ for the 3-category with one object, \pm -functors as 1-morphisms, \pm -transformations as 2-morphisms, and \pm -modifications as 3-morphisms.

Given a \pm -functor $(\mathbf{H}, \mathbf{H}^+, \iota)$ and a thin groupoid A , there is a canonical thin groupoid $\mathbf{H}A$ whose class of uniform strategies is $\mathbf{U}_{\mathbf{H}A} = \{\mathbf{H}S \mid S \in \mathbf{U}_A\}^{\perp\perp}$, whose class of thin prestrategies is $\mathbf{T}_{\mathbf{H}A} = \{\mathbf{H}S \mid S \in \mathbf{T}_A\}^{\perp\perp}$, and whose negative and positive sub-groupoids are $(\mathbf{H}A)_- = \mathbf{H}A_-$ and $(\mathbf{H}A)_+ = \mathbf{H}^+A_+$ with embeddings given by the compositions

$$\mathbf{H}A_- \xrightarrow{\mathbf{H}(\text{id}_A^-)} \mathbf{H}A \quad \text{and} \quad \mathbf{H}^+A_+ \xrightarrow{\mathbf{H}^+(\text{id}_A^+)} \mathbf{H}^+A \xrightarrow{\iota_A} \mathbf{H}A.$$

By the conditions of \pm -functors, they can be shown to be elements of $\mathbf{T}_{\mathbf{H}A}$ and $\mathbf{T}_{\mathbf{H}A}^{\perp\perp}$ as required.

The mapping $A \mapsto \mathbf{H}A$ can be extended to a pseudofunctor $\check{\mathbf{H}}: \mathbf{Thin} \rightarrow \mathbf{Thin}$ by mapping a thin span $A \leftarrow \partial_r^S - S - \partial_r^S \rightarrow B$ to the thin span $\mathbf{H}A \leftarrow \mathbf{H}(\partial_r^S) - \mathbf{H}S - \mathbf{H}(\partial_r^S) \rightarrow \mathbf{H}B$, and by mapping weak morphisms to their image by \mathbf{H} .

Similarly, given a \pm -transformation $\alpha = (\alpha, \alpha^+)$ between two \pm -functors \mathbf{H} and \mathbf{K} , one can define a pseudonatural transformation $\check{\alpha}$ between $\check{\mathbf{K}}$ and $\check{\mathbf{H}}$ by putting

$$\check{\alpha}_A = \mathbf{K}A \leftarrow \alpha_A - \mathbf{H}A - \text{id}_{\mathbf{H}A} \rightarrow \mathbf{H}A$$

and given a \pm -modification between two \pm -transformations α and β , one can define a modification \check{m} between $\check{\alpha}$ and $\check{\beta}$ the expected way. By checking all the details, we get that

PROPOSITION B.3 ([3, PROPOSITION 20]). *Considering $\pm\text{-Funct}$ as a strict 2-category by forgetting the dimension 0, $(\check{-})$ induces a pseudofunctor*

$$(\check{-}) : \pm\text{-Funct}^{\text{co}} \rightarrow \text{Bicat}(\text{Thin}, \text{Thin})$$

between bicategories.

Now, one can define the notion of monad (or even pseudomonad) in $\pm\text{-Funct}$ as particular instances of the general notion of monad (or pseudomonad) expressed in $\pm\text{-Funct}$ seen as an abstract 3-category. While [3] considered **Fam** which was a pseudomonad in **Gpd**, this work is concerned with **Sym**, which is a monad on **Gpd**, so that we only require results for the monadic case. In this regard, we have

PROPOSITION B.4. *The functor $(\check{-})$ send a monad on $\pm\text{-Funct}$ to a pseudocomonad on **Thin**.*

PROOF. By a direct adaptation of the proof of [3, Theorem 3], which shows that **Fam**, seen as a pseudomonad on $\pm\text{-Funct}$, is sent to a pseudocomonad on **Thin**. Indeed, the proof is not specific to **Fam** and can be specialized to the case of a monad on $\pm\text{-Funct}$. \square

B.2 Permutations and the multiplication of Sym

The full definition of the multiplication μ of the monad **Sym** on groupoids relies on operations on permutations that we introduce below. In the following, given $n \in \mathbb{N}$, we write $[n]$ for the set $\{1, \dots, n\}$.

The category \mathbb{P} of *permutations* is defined as the category whose objects are the natural numbers $n \in \mathbb{N}$, and whose morphisms from n to n are the elements of the symmetric group \mathcal{S}_n , that is, the bijections from $[n]$ to itself, and with no morphisms from m to n for $m \neq n$; composition of morphisms is given by the composition of the underlying functions, and the identity morphism on n is the identity function on $[n]$. We will often write \mathbb{P}_n for $\mathbb{P}(n, n)$.

The category \mathbb{P} is equipped with a tensor product \oplus defined by putting $m \oplus n = m + n$ for $m, n \in \mathbb{N}$ and, for $\lambda \in \mathbb{P}_m$ and $\rho \in \mathbb{P}_n$, by defining $\lambda \oplus \rho$ as the bijection $v: [m+n] \rightarrow [m+n]$ such that $v(i) = \lambda(i)$ for $i \in [m]$, and $v(m+i) = m + \rho(i)$ for $i \in [n]$. More generally, given $k \in \mathbb{N}$, $\vec{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$ and bijections $v_i: n_i \rightarrow n_i$, we write $\oplus_{1 \leq i \leq k} v_i$ for the bijection $(\dots (v_1 \oplus v_2) \oplus \dots) \oplus v_k$. The natural number 0 is the unit object for this tensor product, making \oplus a strict monoidal category. It is even a strict symmetric monoidal category: for $m, n \in \mathbb{N}$, one defines a bijection $\sigma_{m,n}: [m+n] \rightarrow [n+m]$ by putting $\sigma_{m,n}(i) = n+i$ for $i \in [m]$ and $\sigma_{m,n}(m+i) = i$ for $i \in [n]$, and one readily verifies that it gives an adequate symmetry for the monoidal structure. More generally, given $k \in \mathbb{N}$, $\vec{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$ and $\rho \in \mathcal{S}_k$, one can define a k -ary symmetry $\sigma_{\vec{n}, \rho}: [n_1 + \dots + n_k] \rightarrow [n_{\rho(1)} + \dots + n_{\rho(k)}]$ by putting

$$\sigma_{\vec{n}, \rho}(n_1 + \dots + n_{l-1} + i) = n_{\rho^{-1}(1)} + \dots + n_{\rho^{-1}(\rho(l)-1)} + i$$

for $l \in [k]$ and $i \in [n_l]$. Given permutations $\tau_l \in \mathbb{P}_{n_l}$ for every $l \in [k]$, we will often write $\rho \otimes (\tau_l)_{l \in [k]}$ for the composite $\sigma_{\vec{n}, \rho} \circ (\oplus_{l \in [k]} \tau_l)$. One easily verifies the following property:

LEMMA B.5. *Given $l \in \mathbb{N}$, $(m_i)_{i \in [l]} \in \mathbb{N}^l$ and $(n_{i,j})_{i \in [l], j \in [m_i]}$, permutations $\rho_{i,j} \in \mathbb{P}_{n_{i,j}}$ for $i \in [l], j \in [m_i]$ and $\sigma_i \in \mathcal{S}_{m_i}$ for $i \in [l]$ and $\tau \in \mathcal{S}_l$, we have*

$$\rho \otimes (\sigma_i \otimes (\tau_{i,j})_j)_i = (\tau \otimes (\sigma_i)_i) \otimes (\tau_{i,j})_{i,j}.$$

We can now use the definition of $(-)\otimes(-)$ to give a precise definition of μ : it is the natural transformation

$$\mu: \text{Sym} \circ \text{Sym} \Rightarrow \text{Sym}$$

defined on a groupoid A as the functor μ_A , defined as follows. Given $\langle \langle a_{i,j} \rangle_{j \in [n_i]} \rangle_{i \in [m]} \in \text{Sym}(\text{Sym}(A))$, we have

$$\mu_A(\langle \langle a_{i,j} \rangle_{j \in [n_i]} \rangle_{i \in [m]}) = \langle a_{i,j} \rangle_{i \in [m], j \in [n_i]}$$

and, given a morphism $u = (\sigma, \langle \langle \tau_i, \langle f_{i,j} \rangle_{j \in [n_i]} \rangle \rangle_{i \in [m]}) \in \text{Sym}(\text{Sym}(A))$, we have

$$\mu_A(u) = (\sigma \otimes (\tau_i)_{i \in [m]}, \langle f_{i,j} \rangle_{i \in [m], j \in [n_i]}).$$

One can then use Lemma B.5 to show that μ is associative in the monadic sense.

B.3 The Sym monad on $\pm\text{-Funct}$

In order to show that the **Sym** monad on **Gpd** induces an adequate comonad ! on **Thin**, we just need to lift the monadic structure on **Gpd** to a monadic structure on $\pm\text{-Funct}$, and then conclude by Proposition B.4. We first show that the **Sym** endofunctor on **Gpd** can be lifted to a 1-morphism of $\pm\text{-Funct}$.

PROPOSITION B.6. *Sym preserves pullbacks and sends pullbacks that are bipullbacks to bipullbacks.*

PROOF. Consider a pullback

$$\begin{array}{ccc} P & \xrightarrow{r} & R \\ i \downarrow & \lrcorner & \downarrow f^R \\ L & \xrightarrow{f^L} & M \end{array} \quad (12)$$

in \mathbf{Gpd} . In order to show that this pullback is preserved by \mathbf{Sym} , we just need to show that a pair of morphisms $(\sigma^L, (u_i^L)_{1 \leq i \leq n^L}) \in \mathbf{Sym}L$ and $(\sigma^R, (u_i^R)_{1 \leq i \leq n^R}) \in \mathbf{Sym}R$ which are projected to the same morphism in $\mathbf{Sym}P$. But it is quite immediate, since the common projection on $\mathbf{Sym}M$ implies that $n^L = n^R$, $\sigma^L = \sigma^R$ and that $f^L(u_i^L) = f^R(u_i^R)$ for every $i \in \{1, \dots, n^L\}$. Thus, \mathbf{Sym} preserves pullbacks.

Now, assuming that (12) is moreover a bipullback, we are required to show that its image by \mathbf{Sym} is also a bipullback. For this, we use the criterion given by [3, Proposition 9]. Let $\vec{a} = \langle a_1, \dots, a_n \rangle \in \mathbf{Sym}L$ and $\vec{b} = \langle b_1, \dots, b_m \rangle \in \mathbf{Sym}R$, and a morphism $v = (\sigma, (v_i)_{1 \leq i \leq n})$ between $\mathbf{Sym}(f^L)(\vec{a})$ and $\mathbf{Sym}(f^R)(\vec{b})$. We need to show that $v = \mathbf{Sym}(f^R)(u^R) \circ \mathbf{Sym}(f^L)(u^L)$ for some $u^L \in \mathbf{Sym}L$ and $u^R \in \mathbf{Sym}R$. Since $v = (\sigma, (\text{id})_{1 \leq i \leq n}) \circ (\text{id}, (v_i)_{1 \leq i \leq n})$ and that $(\sigma, (\text{id})_{1 \leq i \leq n})$ is in the image of $\mathbf{Sym}R$, we may assume that $\sigma = \text{id}$. Since (12) is a bipullback, we have that $v_i = f^R(u_i^R) \circ f^L(u_i^L)$ for some $u_i^L: a_i \rightarrow a'_i \in L$ and $u_i^R: b'_i \rightarrow b_i \in R$ for every i . By taking $u^L = (\text{id}, \langle u_i^L \rangle_i)$ and $u^R = (\text{id}, \langle u_i^R \rangle_i)$, we have $v = \mathbf{Sym}(f^R)(u^R) \circ \mathbf{Sym}(f^L)(u^L)$ as wanted. Thus, by [3, Proposition 9], the image of (12) by \mathbf{Sym} is a bipullback. \square

Moreover, it is immediate to check that \mathbf{Sym} preserves functor $f: A \rightarrow B \in \mathbf{Gpd}$ that are bijective on objects.

We now define \mathbf{Sym}^+ as the functor $\mathbf{Gpd} \rightarrow \mathbf{Gpd}$ mapping a groupoid $X \in \mathbf{Gpd}$ to the subgroupoid of $\mathbf{Sym}X$ with the same objects but whose morphisms are restricted to be the ones of the form $(\text{id}, \langle u_i \rangle_i)$ in $\mathbf{Sym}X$, and with the evident image of functors $X \rightarrow Y$. This functor comes with a canonical embedding natural transformation $\iota: \mathbf{Sym}^+ \Rightarrow \mathbf{Sym}$. Just like for \mathbf{Sym} , we have

PROPOSITION B.7. \mathbf{Sym}^+ preserves pullbacks and sends pullbacks that are bipullbacks to bipullbacks.

PROOF. The proof for \mathbf{Sym} of Proposition B.6 directly adapts to the case of \mathbf{Sym}^+ . \square

PROPOSITION B.8. The natural transformation $\iota: \mathbf{Sym}^+ \Rightarrow \mathbf{Sym}$ is bicartesian.

PROOF. Given $F: A \rightarrow B \in \mathbf{Gpd}$, consider the natural square

$$\begin{array}{ccc} \mathbf{Sym}^+ A & \xrightarrow{\mathbf{Sym}^+(F)} & \mathbf{Sym}^+ B \\ \iota_A \downarrow & & \downarrow \iota_B \\ \mathbf{Sym} A & \xrightarrow{\mathbf{Sym}(F)} & \mathbf{Sym} B \end{array} \quad (13)$$

and consider a pair of morphisms $u = (\rho, (u_i)_{1 \leq i \leq m}) \in \mathbf{Sym}A$ and $v = (\sigma, (v_i)_{1 \leq i \leq n}) \in \mathbf{Sym}^+ B$ which are projected to the same morphism in $\mathbf{Sym}B$. For the square (13) to be a pullback, we need to show that this pair can be lifted to a unique morphism of $\mathbf{Sym}^+ A$. From the common projection on $\mathbf{Sym}B$, we get that $\rho = \sigma = \text{id}$ and that $v_i = F(u_i)$ for every i . Thus, u actually lifts to $\mathbf{Sym}^+ A$ through ι_A , which is the required unique lifting. Thus, (13) is a pullback.

We now show that (13) is a bipullback. Let $\vec{a} = \langle a_1, \dots, a_n \rangle \in \mathbf{Sym}A$ and $\vec{b} = \langle b_1, \dots, b_m \rangle \in \mathbf{Sym}^+ B$, and a morphism $v = (\sigma, (v_i)_{1 \leq i \leq n})$ between $\mathbf{Sym}(F)(\vec{a})$ and $\iota_B(\vec{b}) = \vec{b}$. We need to show that $v = \iota_B(u^R) \circ \mathbf{Sym}(F)(u^L)$ for some $u^L \in \mathbf{Sym}A$ and $u^R \in \mathbf{Sym}^+ B$. Since $v = (\text{id}, (v_{\sigma^{-1}(i)})_{1 \leq i \leq n}) \circ (\sigma, (\text{id})_{1 \leq i \leq n})$ and that $(\sigma, (\text{id})_{1 \leq i \leq n})$ is in the image of $\mathbf{Sym}(F)$, we may assume that $\sigma = \text{id}$. But then, v is in the image of ι_B , so that, by [3, Proposition 9], (13) is a bipullback. \square

We thus get that

PROPOSITION B.9. $(\mathbf{Sym}, \mathbf{Sym}^+, \iota)$ is a \pm -functor.

PROOF. The other conditions for being a \pm -functor are readily verified. \square

We now provide liftings in $\pm\text{-Funct}$ for the natural transformations $\eta: \text{id}_{\mathbf{Gpd}} \Rightarrow \mathbf{Sym}$ and $\mu: \mathbf{Sym} \circ \mathbf{Sym} \Rightarrow \mathbf{Sym}$. For this, we provide $\eta^+: \text{id}_{\mathbf{Gpd}} \Rightarrow \mathbf{Sym}^+$ and $\mu^+: \mathbf{Sym}^+ \circ \mathbf{Sym}^+ \Rightarrow \mathbf{Sym}^+$ so that (η, η^+) and (μ, μ^+) define \pm -transformations. Actually, this is easy: by the equation (11), η^+ is essentially η and μ_X^+ is the adequate restriction of μ_X to the subgroupoid $\mathbf{Sym}^+(\mathbf{Sym}^+ X)$ of $\mathbf{Sym}(\mathbf{Sym}X)$.

PROPOSITION B.10. *We have \pm -transformations*

$$(\eta, \eta^+): (\text{id}_{\mathbf{Gpd}}, \text{id}_{\mathbf{Gpd}}, \text{id}_{\text{id}_{\mathbf{Gpd}}}) \Rightarrow (\text{Sym}, \text{Sym}^+, \iota)$$

and

$$(\mu, \mu^+): (\text{SymSym}, \text{Sym}^+\text{Sym}^+, \iota) \Rightarrow (\text{Sym}, \text{Sym}^+, \iota).$$

PROOF. By Lemma B.1, we just need to show that η, μ are bicartesian natural transformations. We only give the proof for μ , since the bicartesianness of η is quite easy to show.

Consider a functor $F: A \rightarrow B \in \mathbf{Gpd}$ and the natural square

$$\begin{array}{ccc} \text{Sym}(\text{Sym}A) & \xrightarrow{\text{Sym}(\text{Sym}(F))} & \text{Sym}(\text{Sym}B) \\ \mu_A \downarrow & & \downarrow \mu_B \\ \text{Sym}A & \xrightarrow{\text{Sym}(F)} & \text{Sym}B. \end{array} \quad (14)$$

In order to show that it is a pullback, we consider a pair of morphisms $u = (\sigma, \langle u_i \rangle_{1 \leq i \leq n}) \in \text{Sym}A$ and $v = (\rho, \langle \langle \rho'_i, \langle v_{i,j} \rangle_{1 \leq j \leq m_i} \rangle \rangle_{1 \leq i \leq l}) \in \text{Sym}(\text{Sym}B)$ that project on the same morphism of $\text{Sym}B$, and show that it can be lifted to a unique morphism of $\text{Sym}(\text{Sym}A)$ adequately projecting on u and v . By the common projection on $\text{Sym}B$, we have that $\sigma = \rho \circ (\rho'_i)_i$, so that u can be written $\mu_A(\tilde{u})$ for $\tilde{u} = (\rho, \langle \langle \rho'_i, \langle \tilde{u}_{i,j} \rangle_{1 \leq j \leq m_i} \rangle \rangle_{1 \leq i \leq l})$ for some adequate morphisms $\tilde{u}_{i,j}$. We moreover have that $F(\tilde{u}_{i,j}) = v_{i,j}$, so that \tilde{u} is a lift for the pair (u, v) , and it can be easily proved to be unique, so that the square (14) is a pullback.

We now show that it is a bipullback. So consider $\vec{a} = \langle a_i \rangle_{1 \leq i \leq n} \in \text{Sym}A$ and $\vec{b} = \langle \langle b_{i,j} \rangle_{1 \leq j \leq m_i} \rangle_{1 \leq i \leq l} \in \text{Sym}(\text{Sym}B)$ and a morphism $w = (\tau, \langle w_i \rangle_{1 \leq i \leq n}): \text{Sym}(F)(\vec{a}) \rightarrow \mu_B(\vec{b}) \in \text{Sym}B$. We have to show that $w = \mu_B(v) \circ \text{Sym}(F)(u)$ for some $u: \vec{a} \rightarrow \vec{a}' \in \text{Sym}A$ and $v: \vec{b}' \rightarrow \vec{b} \in \text{Sym}(\text{Sym}B)$. Since $w = (\text{id}, \langle w_{\tau^{-1}(i)} \rangle_{1 \leq i \leq n}) \circ (\tau, \langle \text{id} \rangle_{1 \leq i \leq n})$ and that $(\tau, \langle \text{id} \rangle_{1 \leq i \leq n})$ is in the image of $\text{Sym}(F)$, we may assume that $\tau = \text{id}$. But then, it is clear that we may find $v: \vec{b}' \rightarrow \vec{b}$ such that $w = \mu_B(v)$, so that (14) is a bipullback by [3, Proposition 9]. \square

PROPOSITION B.11. *The triple $((\text{Sym}, \text{Sym}^+, \iota), (\eta, \eta^+), (\mu, \mu^+))$ defines a monad on $\pm\text{-Funct}$.*

PROOF. One just need to check the monad axioms for this triple. But they directly follow from the monad axioms satisfied by the monad (Sym, η, μ) on \mathbf{Gpd} . \square

We may now conclude that

PROPOSITION B.12. *The Sym functor defines a pseudocomonad $\check{\text{Sym}}$ on \mathbf{Thin} , with $\check{\eta}$ as counit and $\check{\mu}$ as comultiplication.*

PROOF. This is a consequence of Propositions B.4 and B.11. \square

C DETAILS ON INTERSECTION TYPES

C.1 The correspondence for the objects

PROOF OF PROPOSITION 3.2. We prove the statement by induction on A . If $A = o$, then $\llbracket A \rrbracket = (1, \dots)$ (the unique thin groupoid with the terminal groupoid as underlying groupoid) so that we can take $K_A = \text{id}_1$. If $A = B \rightarrow C$, then K_A is defined as the composite

$$\begin{aligned} \text{Ob}(\llbracket B \rightarrow C \rrbracket) &= \text{Ob}(!\llbracket B \rrbracket) \times \text{Ob}(\llbracket C \rrbracket) \\ &\xrightarrow{K_B^! \times K_C} \{ \langle \beta_1, \dots, \beta_n \rangle \mid \beta_i \triangleleft B \text{ for } i \in \{1, \dots, n\} \} \times \{ \gamma \mid \gamma \triangleleft C \} \\ &\xrightarrow{\sim} \{ \delta \mid \delta \triangleleft (B \rightarrow C) \}. \end{aligned}$$

Finally, we put $K_A^! = !K_A$, assuming the same encoding of sequences for the $!$ construction and the multilinear refinement types, for simplicity. \square

PROOF OF PROPOSITION 3.3. Let us first give some precisions on the “and similarly for $\llbracket M \rrbracket^!$ ” part. By that, we mean that, given a derivation $\Gamma \vdash M : A$, for every $\gamma \in \llbracket \Gamma \rrbracket$ and $\vec{a} \in !(A)$, we have a bijection

$$\llbracket M \rrbracket_{\gamma, \vec{a}}^! \simeq \{ \pi \mid \pi \text{ is a derivation of } K_\Gamma(\gamma) \triangleleft \Gamma \vdash M : K_A^!(\vec{a}) \triangleleft A \}.$$

We then prove the property by induction on the derivations $\Gamma \vdash M : A$ and $\Gamma \vdash M : A$:

- in the case of the variable typing rule $x_1 : A_1, \dots, x_n : A_n \vdash x_i : A_i$, if $\gamma = (\langle \rangle, \dots, \langle \rangle, \langle a \rangle, \langle \rangle, \dots, \langle \rangle)$ then

$\text{Ob}(\llbracket x_i \rrbracket_{\gamma, a}) = \{a\} \xrightarrow{\sim} \{\pi \mid \pi \text{ is a derivation of}$

$$x_1 : \langle \rangle \triangleleft A_1, \dots, x_i : \langle K_{A_i}(a) \rangle \triangleleft A_i, \dots, x_n : \langle \rangle \triangleleft A_n \triangleleft \Gamma \vdash x : K_{A_i}(a) \triangleleft A\}$$

$$\xrightarrow{\sim} \{\pi \mid \pi \text{ is a derivation of } K_{\Gamma}(\gamma) \triangleleft \Gamma \vdash x : K_{A_i}(a) \triangleleft A_i\}$$

which is canonically isomorphic by K_{A_i} to $\{\alpha \mid \alpha \triangleleft A_i\}$. Otherwise, for other γ 's, we have $\text{Ob}(\llbracket M \rrbracket_{\gamma, a}) = \emptyset$ and no derivations for $K_{\Gamma}(\gamma) \triangleleft \Gamma \vdash x : K_{A_i}(a) \triangleleft A_i$;

- in the case of an intersection typing $\Gamma \vdash M : A$ derived from a proof of $\Gamma \vdash M : A$, and given $\vec{a} = \langle a_1, \dots, a_l \rangle \in \llbracket A \rrbracket$, we have

$$\begin{aligned} \llbracket M \rrbracket_{\gamma, \vec{a}}^! &\simeq \bigsqcup_{\substack{\vec{\gamma} = \langle \gamma_1, \dots, \gamma_l \rangle \in \llbracket \Gamma \rrbracket \\ \gamma_1 \bullet \dots \bullet \gamma_l = \gamma}} \prod_{i=1}^l \llbracket M \rrbracket_{\gamma_i, a_i} \\ &\simeq \bigsqcup_{\substack{\vec{\gamma} = \langle \gamma_1, \dots, \gamma_l \rangle \in \llbracket \Gamma \rrbracket \\ \gamma_1 \bullet \dots \bullet \gamma_l = \gamma}} \prod_{i=1}^l \{\pi \mid \pi \text{ derivation of } K_{\Gamma}(\gamma_i) \triangleleft \Gamma \vdash M : K_A(a_i) \triangleleft A\} \\ &\simeq \{\pi \mid \pi \text{ derivation of } K_{\Gamma}(\gamma) \triangleleft \Gamma \vdash M : K_A^!(\vec{a}) \triangleleft A\}; \end{aligned}$$

- in the case of an application $\Gamma \vdash MN : B$ constructed from two derivations of $\Gamma \vdash M : A \rightarrow B$ and $\Gamma \vdash N : A$ for some unique simple type A , considering the definition $\llbracket MN \rrbracket$, we have that

$$\begin{aligned} \llbracket MN \rrbracket_{\gamma, b} &\simeq \bigsqcup_{\substack{\gamma_1, \gamma_2 \in \llbracket \Gamma \rrbracket \\ \gamma_1 \bullet \gamma_2 = \gamma}} \bigsqcup_{\vec{a} \in \llbracket A \rrbracket} \llbracket M \rrbracket_{\gamma_1, (\vec{a}, b)} \times \llbracket N \rrbracket_{\gamma_2, \vec{a}}^! \\ &\simeq \bigsqcup_{\substack{\gamma_1, \gamma_2 \in \llbracket \Gamma \rrbracket \\ \gamma_1 \bullet \gamma_2 = \gamma}} \bigsqcup_{\vec{a} \in \llbracket A \rrbracket} \{\pi_1 \mid \pi_1 \text{ derivation of } K_{\Gamma}(\gamma_1) \triangleleft \Gamma \vdash M : K_{A \rightarrow B}(\vec{a}, b) \triangleleft A \rightarrow B\} \\ &\quad \times \{\pi_2 \mid \pi_2 \text{ derivation of } K_{\Gamma}(\gamma_2) \triangleleft \Gamma \vdash N : K_A^!(\vec{a}) \triangleleft A\} \\ &\simeq \{\pi \mid \pi \text{ derivation of } K_{\Gamma}(\gamma) \triangleleft \Gamma \vdash MN : K_B(b) \triangleleft B\}; \end{aligned}$$

- in case of a lambda-abstraction $\Gamma \vdash \lambda x. M : A \rightarrow B$, given $\gamma \in \llbracket \Gamma \rrbracket$ and $(\kappa, \beta) \in \llbracket A \rightarrow B \rrbracket$, we have

$$\begin{aligned} \llbracket \lambda x. M \rrbracket_{\gamma, (\kappa, \beta)} &\simeq \llbracket M \rrbracket_{\gamma; \kappa, \beta} \\ &\simeq \{\pi \mid \pi \text{ derivation of } K_{(\Gamma, x:A)}(\gamma :: \kappa) \triangleleft (\Gamma, x : A) \vdash M : K_B(\beta) \triangleleft B\} \\ &\simeq \{\pi \mid \pi \text{ derivation of } K_{\Gamma}(\gamma) \triangleleft \Gamma \vdash \lambda x. M : K_{A \rightarrow B}((\kappa, \beta)) \triangleleft A \rightarrow B\} \end{aligned}$$

where, for $\gamma = (\gamma_1, \dots, \gamma_n) \in \llbracket \Gamma \rrbracket$, we write $\gamma :: \kappa$ for $(\kappa_1, \dots, \kappa_n, \kappa) \in \llbracket \Gamma, x : A \rrbracket$. \square

C.2 The groupoids of intersection types

Here, we give details about the structure of the groupoids $\text{IT}(A)$ (and its multilinear version) of intersection types and the associated morphisms that refine a simple type A .

Given a simple type A , we first define $\text{Ob}(\text{IT}(A))$ as the set of linear intersection types α such that $\alpha \triangleleft A$. We then define the symmetries $\alpha \rightarrow \alpha'$ of $\text{IT}(A)$ as the linear intersection type morphisms ϕ such that $\phi :: \alpha \Rightarrow \alpha' \triangleleft A$. We have the convenient property that

LEMMA C.1. *Given a linear intersection type morphism ϕ , there is at most one pair (α, α') of linear intersection types such that $\phi :: \alpha \Rightarrow \alpha' \triangleleft A$, and similarly for multilinear intersection type morphisms θ .*

PROOF. By a simple induction on ϕ and θ . \square

Thus, given a morphism ϕ of $\text{IT}(A)$, we may write $\partial^-(\phi)$ and $\partial^+(\phi)$ for the unique α and α' such that $\phi :: \alpha \Rightarrow \alpha' \triangleleft A$.

We may define similarly (the beginning of) a groupoid $\text{IT}_l(A)$, whose objects are the multilinear intersection types $\vec{\alpha}$ such that $\vec{\alpha} \triangleleft A$, and whose morphisms $\vec{\alpha} \rightarrow \vec{\alpha}'$ are the multilinear intersection type morphisms are the $\vec{\phi}$'s such that $\vec{\phi} :: \vec{\alpha} \Rightarrow \vec{\alpha}' \triangleleft A$. By Lemma C.1, we may write $\partial^-(\vec{\phi})$ and $\partial^+(\vec{\phi})$ for these unique $\vec{\alpha}$ and $\vec{\alpha}'$.

Given $\alpha \in \text{Ob}(\text{IT}(A))$ and $\vec{\alpha} \in \text{Ob}(\text{IT}_l(A))$, we define $\text{id}_{\alpha} \in \text{Ar}(\text{IT}(A))$ and $\text{id}_{\vec{\alpha}} \in \text{Ar}(\text{IT}_l(A))$ such that $\text{id}_{\alpha} :: \alpha \Rightarrow \alpha \triangleleft A$ and $\text{id}_{\vec{\alpha}} :: \vec{\alpha} \Rightarrow \vec{\alpha} \triangleleft A$ by mutual induction on the derivations of $\alpha \triangleleft A$ and $\vec{\alpha} \triangleleft A$:

- in the case of the axiom $\star \triangleleft o$, we put $\text{id}_\star = \text{id}_\star$. We then have $\text{id}_\star :: \star \Rightarrow \star \triangleleft A$ by corresponding rule for morphisms;
- in the case of the linear arrow $(\vec{\beta} \multimap \gamma) \triangleleft (B \rightarrow C)$, by induction hypothesis, we get $\text{id}_{\vec{\beta}} \in \text{Ar}(\text{IT}_!(B))$ and $\text{id}_\gamma \in \text{Ar}(\text{IT}(C))$ such that $\text{id}_{\vec{\beta}} :: \vec{\beta} \Rightarrow \vec{\beta} \triangleleft B$ and $\text{id}_\gamma :: \gamma \Rightarrow \gamma \triangleleft C$. We then put $\text{id}_{\vec{\beta} \multimap \gamma}$ to be $\text{id}_{\vec{\beta}} \multimap \text{id}_\gamma$, for which we are able to derive $\text{id}_{\vec{\beta} \multimap \gamma} :: (\vec{\beta} \multimap \gamma) \Rightarrow (\vec{\beta} \multimap \gamma) \triangleleft B \rightarrow C$ using the corresponding rule for morphisms;
- in the case of the multilinear formation $\vec{\alpha} \triangleleft A$ for $\vec{\alpha} = \langle \alpha_1, \dots, \alpha_n \rangle$, by induction hypothesis, we get $\text{id}_{\alpha_i} \in \text{Ar}(\text{IT}(A))$ such that $\text{id}_{\alpha_i} :: \alpha_i \Rightarrow \alpha_i \triangleleft A$ for $i \in \{1, \dots, n\}$. We then put $\text{id}_{\vec{\alpha}} = (\text{id}, \langle \text{id}_{\alpha_1}, \dots, \text{id}_{\alpha_n} \rangle)$, for which we can easily derive that $\text{id}_{\vec{\alpha}} :: \vec{\alpha} \Rightarrow \vec{\alpha} \triangleleft A$ using the corresponding rule for morphisms.

Given $\phi_1, \phi_2 \in \text{Ar}(\text{IT}(A))$ (resp. $\tilde{\phi}_1, \tilde{\phi}_2 \in \text{Ar}(\text{IT}_!(A))$), we say that they are *composable* when $\partial^+(\phi_1) = \partial^-(\phi_2)$ (resp. $\partial^+(\tilde{\phi}_1) = \partial^-(\tilde{\phi}_2)$). It happens that two composable intersection type morphisms are very “similar” in their construction:

LEMMA C.2. *Given $\phi_1, \phi_2 \in \text{Ar}(\text{IT}(A))$ such that they are composable, we have that:*

- if $\phi_1 = \text{id}_\star$, then $A = o$ and $\phi_2 = \text{id}_\star$;
- if $\phi_1 = \tilde{\phi}_1 \multimap \psi_1$, then
 - $A = B \rightarrow C$ for some unique simple types B and C ,
 - $\tilde{\phi}_1 \in \text{Ar}(\text{IT}_!(B))$ and $\psi_1 \in \text{Ar}(\text{IT}(C))$,
 - $\phi_2 = \tilde{\phi}_2 \multimap \psi_2$ for some unique $\tilde{\phi}_2 \in \text{Ar}(\text{IT}_!(B))$ and $\psi_2 \in \text{Ar}(\text{IT}(C))$,
 - and $\tilde{\phi}_1, \tilde{\phi}_2$ (resp. ψ_1, ψ_2) are composable.

Similarly, given $\tilde{\phi}_1 = (\sigma_1, \langle \phi_{1,1}, \dots, \phi_{1,n_1} \rangle)$ and $\tilde{\phi}_2 = (\sigma_2, \langle \phi_{2,1}, \dots, \phi_{2,n_2} \rangle)$ in $\text{Ar}(\text{IT}_!(A))$ for some $n_i \in \mathbb{N}$, $\sigma_i \in \mathcal{S}_{n_i}$ and intersection type morphisms $\phi_{i,1}, \dots, \phi_{i,n_i}$ for $i \in \{1, 2\}$, such that $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are composable, we have that $n_1 = n_2$ and $\phi_{1,j}$ is composable with $\phi_{2,\sigma_1(j)}$ for $j \in \{1, \dots, n_1\}$.

PROOF. By mutual induction on the derivations of ϕ_1 and $\tilde{\phi}_1$. □

Given a simple type A and two composable linear (resp. multilinear) intersection type morphisms $\phi_1, \phi_2 \in \text{Ar}(\text{IT}(A))$ (resp. $\tilde{\phi}_1, \tilde{\phi}_2 \in \text{Ar}(\text{IT}_!(A))$), we now define their *composition* $\phi_2 \circ \phi_1$ (resp. $\tilde{\phi}_2 \circ \tilde{\phi}_1$) by mutual induction. We use Lemma C.2 to give a complete definition with a minimal case analysis:

- we put $\text{id}_\star \circ \text{id}_\star = \text{id}_\star$;
- given composable $\tilde{\phi}_1 = \zeta_1 \multimap \psi_1$ and $\tilde{\phi}_2 = \zeta_2 \multimap \psi_2$, we put $\tilde{\phi}_2 \circ \tilde{\phi}_1 = (\zeta_2 \circ \zeta_1) \multimap (\psi_2 \circ \psi_1)$;
- given composable $\tilde{\phi}_1 = (\sigma_1, \langle \psi_{1,1}, \dots, \psi_{1,n} \rangle)$ and $\tilde{\phi}_2 = (\sigma_2, \langle \psi_{2,1}, \dots, \psi_{2,n} \rangle)$, we put

$$\tilde{\phi}_2 \circ \tilde{\phi}_1 = (\sigma_2 \circ \sigma_1, \langle \psi_{2,\sigma_1(1)} \circ \psi_{1,1}, \dots, \psi_{2,\sigma_1(n)} \circ \psi_{1,n} \rangle).$$

Given a simple type A and a composable linear (resp. multilinear) intersection type morphism $\phi \in \text{IT}(A)$ (resp. $\tilde{\phi} \in \text{Ar}(\text{IT}_!(A))$), we now define its *inverse* ϕ^{-1} (resp. $\tilde{\phi}^{-1}$) by induction on the derivations:

- we put $\text{id}_\star^{-1} = \text{id}_\star$;
- given $\phi = \tilde{\phi} \multimap \psi$, we put $\phi^{-1} = \tilde{\phi}^{-1} \multimap \psi^{-1}$;
- given $\tilde{\phi} = (\sigma, \langle \psi_1, \dots, \psi_n \rangle)$, we put $\tilde{\phi}^{-1} = (\sigma^{-1}, \langle \psi_{\sigma^{-1}(1)}, \dots, \psi_{\sigma^{-1}(n)} \rangle)$.

The above operations assemble into a groupoidal structure:

LEMMA C.3. *Given a simple type A , we have that*

- \circ is an associative composition operation on $\text{IT}(A)$ (resp. $\text{IT}_!(A)$) with id as unit, making $\text{IT}(A)$ (resp. $\text{IT}_!(A)$) a category;
- for every $\phi \in \text{Ar}(\text{IT}(A))$ (resp. $\tilde{\phi} \in \text{Ar}(\text{IT}_!(A))$), we have $\phi^{-1} \circ \phi = \text{id}_{\partial^-(\phi)}$ and $\phi \circ \phi^{-1} = \text{id}_{\partial^+(\phi)}$ (resp. $\tilde{\phi}^{-1} \circ \tilde{\phi} = \text{id}_{\partial^-(\tilde{\phi})}$ and $\tilde{\phi} \circ \tilde{\phi}^{-1} = \text{id}_{\partial^+(\tilde{\phi})}$), so that $\text{IT}(A)$ (resp. $\text{IT}_!(A)$) is a groupoid.

PROOF. By simple inductions. □

PROOF OF PROPOSITION 3.4. The functors K_A and $K_A^!$ are built as the direct extensions to symmetries of the ones built in the proof of Proposition 3.2, since the definition of multilinear intersection type morphisms closely follows the definition of the action of $!$ on groupoids and their symmetries. □

C.3 Resource context groupoid

We recall that we only consider contexts, resource contexts and resource morphism contexts that are well-formed, so that we will often omit to precise that such contexts are well-formed for conciseness.

Given a (well-formed) context $\Gamma = (x_1 : A_1, \dots, x_n : A_n)$, we give some details about the definition of the groupoid $\mathbf{IT}(\Gamma)$: its objects are the resource contexts $\Theta = (x_1 : \kappa_1 \triangleleft A_1, \dots, x_n : \kappa_n \triangleleft A_n)$, and its morphisms of type $\Theta \rightarrow \Theta'$, for another resource context $\Theta' = (x_1 : \kappa'_1 \triangleleft A_1, \dots, x_n : \kappa'_n \triangleleft A_n)$, are the resource morphism contexts $\Xi = (x_1 : \theta_1 :: \kappa_1 \Rightarrow \kappa'_1 \triangleleft A_1, \dots, x_n : \theta_n :: \kappa_n \Rightarrow \kappa'_n \triangleleft A_n)$. We then write $\partial^-(\Xi)$ for Θ and $\partial^+(\Xi)$ for Θ' . Two resource morphism contexts Ξ_1 and Ξ_2 of $\mathbf{IT}(\Gamma)$ defined by

$$\Xi_i = (x_1 : \theta_{i,1} :: \kappa_{i,1} \Rightarrow \kappa'_{i,1} \triangleleft A_1, \dots, x_n : \theta_{i,n} :: \kappa_{i,n} \Rightarrow \kappa'_{i,n} \triangleleft A_n)$$

are *composable* when $\partial^+(\Xi) = \partial^-(\Xi')$. In this case, we define their composite as

$$\Xi_2 \circ^{\text{ctxt}} \Xi_1 = (x_1 : \theta_{2,1} \circ \theta_{1,1} :: \kappa_{1,1} \Rightarrow \kappa'_{2,1} \triangleleft A_1, \dots, x_n : \theta_{2,n} \circ \theta_{1,n} :: \kappa_{1,n} \Rightarrow \kappa'_{2,n} \triangleleft A_n).$$

Moreover, given Ξ as above, there is a resource morphism context Ξ^{-1} defined by

$$\Xi^{-1} = (x_1 : \theta_1^{-1} :: \kappa'_1 \Rightarrow \kappa_1 \triangleleft A_1, \dots, x_n : \theta_n^{-1} :: \kappa'_n \Rightarrow \kappa_n \triangleleft A_n)$$

and which is the *inverse* of Ξ . Given a resource context $\Theta = (x_1 : \kappa_1 \triangleleft A_1, \dots, x_n : \kappa_n \triangleleft A_n)$, there is an *identity resource morphism context* $\text{id}_{\Theta}^{\text{ctxt}}$ defined by

$$\text{id}_{\Theta}^{\text{ctxt}} = (x_1 : \text{id}_{\kappa_1} :: \kappa_1 \Rightarrow \kappa_1 \triangleleft A_1, \dots, x_n : \text{id}_{\kappa_n} :: \kappa_n \Rightarrow \kappa_n \triangleleft A_n).$$

Following what was done in the previous section, we readily have that

PROPOSITION C.4. *$\mathbf{IT}(\Gamma)$ has a structure of groupoid and, as such, it is isomorphic to the groupoid $\mathbf{IT}_1(A_1) \times \dots \times \mathbf{IT}_1(A_n)$.*

C.4 Morphisms between derivations

Here, we give more details about the definition of the $\rho \otimes (-)$ operation on families of multilinear intersection type morphisms and resource morphism contexts.

Let $m \in \mathbb{N}, \rho \in \mathcal{S}_m$. Given a family of morphisms $(\tilde{\phi}_j :: \tilde{\alpha}_j \Rightarrow \tilde{\alpha}'_j \triangleleft A)_{1 \leq j \leq m}$ where $\tilde{\phi}_j = (\sigma_j, \langle \phi_{j,k} \rangle_{1 \leq k \leq l_j})$ with l_j the length of $\tilde{\alpha}_j$ (and $\tilde{\alpha}'_j$) for every $j \in \{1, \dots, m\}$, we write $\rho \otimes (\tilde{\phi}_j)_{1 \leq j \leq m}$ for the multilinear intersection type morphism $\tilde{\phi}$ defined by

$$\tilde{\phi} = (\rho \otimes (\sigma_j)_{1 \leq j \leq m}, \langle \phi_{1,k} \rangle_{1 \leq k \leq l_1} \bullet \dots \bullet \langle \phi_{m,k} \rangle_{1 \leq k \leq l_m})$$

where $\langle \phi_{1,k} \rangle_{1 \leq k \leq l_1} \bullet \dots \bullet \langle \phi_{m,k} \rangle_{1 \leq k \leq l_m}$ is the mere concatenation of the sequences of morphisms. Note that we then have the refinement

$$\tilde{\phi} :: \tilde{\alpha}_1 \bullet \dots \bullet \tilde{\alpha}_m \Rightarrow \tilde{\alpha}'_{\rho^{-1}(1)} \bullet \dots \bullet \tilde{\alpha}'_{\rho^{-1}(m)} \triangleleft A.$$

Now, given a family

$$(\Xi_j)_{1 \leq j \leq m} = ((x_i : \tilde{\phi}_{i,j} :: \tilde{\alpha}_{i,j} \Rightarrow \tilde{\alpha}'_{i,j} \triangleleft A_i)_{1 \leq i \leq n})_{1 \leq j \leq m}$$

of resource morphism contexts, all refining a common context $\Gamma = x_1 : A_1, \dots, x_n : A_n$, we define $\rho \otimes (\Xi_j)_{1 \leq j \leq m}$ as the resource morphism context

$$\rho \otimes (\Xi_j)_{1 \leq j \leq m} = (x_i : \rho \otimes (\tilde{\phi}_{i,j})_{1 \leq j \leq m} :: \tilde{\alpha}_{i,1} \bullet \dots \bullet \tilde{\alpha}_{i,m} \Rightarrow \tilde{\alpha}_{i,\rho^{-1}(1)} \bullet \dots \bullet \tilde{\alpha}_{i,\rho^{-1}(m)} \triangleleft A_i)_{1 \leq i \leq n}.$$

C.5 The intersection type groupoid for a term

We now give some details about the definition of the groupoid $\mathbf{IT}(M)$ for a well-typed λ -term $\Gamma \vdash M : A$. Given a derivation π of $\Xi \triangleleft \Gamma \vdash M : \phi :: \alpha \Rightarrow \alpha' \triangleleft A$, one can define a derivation $\partial^-(\pi)$ of $\text{dom}(\Xi) \triangleleft \Gamma \vdash M : \alpha \triangleleft A$ and a derivation $\partial^+(\pi)$ of $\text{cod}(\Xi) \triangleleft \Gamma \vdash M : \alpha' \triangleleft A$ by induction on π (and similarly for multilinear judgements). We only give the definition of $\partial^-(\pi)$ in Figure 3 since the one of $\partial^+(\pi)$ is similar. The correction of this definition relies on the following easy compatibility property between resource contexts and the dom operation:

PROPOSITION C.5. *Let Γ be a context. We have:*

- (a) *given two resource morphism contexts $\Xi, \Xi' \triangleleft \Gamma$, we have $\text{dom}(\Xi \bullet \Xi') = \text{dom}(\Xi) \bullet \text{dom}(\Xi')$;*
- (b) *given resource morphism contexts $\Xi_1, \dots, \Xi_n \triangleleft \Gamma$ and \mathcal{S}_n , we have*

$$\text{dom}(\sigma \otimes (\Xi_i)_i) = \text{dom}(\Xi_1) \bullet \dots \bullet \text{dom}(\Xi_n)$$

and similarly for cod.

$$\begin{aligned}
 \partial^- \left(\frac{(\phi :: \alpha \Rightarrow \alpha' \triangleleft A_i)}{\dots, x_i : (\text{id}_{\{1\}}, \langle \phi \rangle) :: (\alpha \Rightarrow \langle \alpha' \rangle \triangleleft A_i, \dots \vdash x_i : \phi :: \alpha \Rightarrow \alpha' \triangleleft A_i)} \right) &= \frac{(\alpha \triangleleft A_i)}{\dots, x_i : (\alpha \triangleleft A_i, \dots \triangleleft x_1 : A_1, \dots, x_n : A_n \vdash x_i : \alpha \triangleleft A_i)} \\
 \partial^- \left(\frac{\frac{\pi_1}{\Xi \vdash M : (\tilde{\phi} \multimap \psi) :: (\tilde{\alpha} \multimap \beta) \Rightarrow (\tilde{\alpha}' \multimap \beta')} \triangleleft A \rightarrow B \quad \frac{\pi_2}{\Xi' \vdash N : \tilde{\phi} :: \tilde{\alpha} \Rightarrow \tilde{\alpha}' \triangleleft A}}{\Xi \bullet \Xi' \vdash MN : \psi :: \beta \Rightarrow \beta' \triangleleft B}} &= \frac{\frac{\partial^-(\pi_1)}{\text{dom}(\Xi) \triangleleft \vdash M : (\tilde{\alpha} \multimap \beta) \triangleleft A \rightarrow B} \quad \frac{\partial^-(\pi_2)}{\text{dom}(\Xi') \triangleleft \Gamma \vdash N : \tilde{\alpha} \triangleleft A}}{\text{dom}(\Xi \bullet \Xi') \triangleleft \Gamma \vdash MN : \beta \triangleleft B}} \\
 \partial^- \left(\frac{\frac{\pi'}{\Xi, x : \tilde{\phi} :: \tilde{\alpha} \Rightarrow \tilde{\alpha}' \triangleleft A \vdash M : \psi :: \beta \Rightarrow \beta' \triangleleft B}}{\Xi \vdash \lambda x. M : (\tilde{\phi} \multimap \psi) :: (\tilde{\alpha} \multimap \beta) \Rightarrow (\tilde{\alpha}' \multimap \beta')} \triangleleft A \rightarrow B} \right) &= \frac{\frac{\partial^-(\pi')}{\text{dom}(\Xi, x : \tilde{\alpha} \triangleleft A \triangleleft \Gamma, x : A \vdash M : \beta \triangleleft B}}{\text{dom}(\Xi) \triangleleft \Gamma \vdash \lambda x. M : (\tilde{\alpha} \multimap \beta) \triangleleft A \rightarrow B}} \\
 \partial^- \left(\frac{\frac{\pi_i}{n \in \mathbb{N} \quad \sigma \in \mathcal{S}_n \quad \forall i \in \{1, \dots, n\}, \Xi_i \vdash M : \phi_i :: \alpha_i \Rightarrow \alpha'_i \triangleleft A}}{\sigma \otimes (\Xi_i)_{1 \leq i \leq n} \vdash M : (\sigma, \langle \phi_1, \dots, \phi_n \rangle) :: \langle \alpha_1, \dots, \alpha_n \rangle \Rightarrow \langle \alpha'_{\sigma^{-1}(1)}, \dots, \alpha'_{\sigma^{-1}(n)} \rangle \triangleleft A}} \right) &= \frac{\frac{\partial^-(\pi_i)}{\forall i \in \{1, \dots, k\}, \text{dom}(\Xi_i) \triangleleft \Gamma \vdash M : \alpha_i \triangleleft A}}{\text{dom}(\sigma \otimes (\Xi_i)_{1 \leq i \leq n}) \triangleleft \Gamma \vdash M : \langle \alpha_1, \dots, \alpha_k \rangle \triangleleft A}}
 \end{aligned}$$

 Figure 3: The definition of $\partial^-(\pi)$

$$\begin{aligned}
 &\frac{(\phi_2 :: \alpha' \Rightarrow \alpha'' \triangleleft A_i)}{\dots, x_i : (\text{id}_{\{1\}}, \langle \phi_2 \rangle) :: (\alpha' \Rightarrow \langle \alpha'' \rangle \triangleleft A_i, \dots \vdash x_i : \phi_2 :: \alpha' \Rightarrow \alpha'' \triangleleft A_i)} \circ \frac{(\phi_1 :: \alpha \Rightarrow \alpha' \triangleleft A_i)}{\dots, x_i : (\text{id}_{\{1\}}, \langle \phi_1 \rangle) :: (\alpha \Rightarrow \langle \alpha' \rangle \triangleleft A_i, \dots \vdash x_i : \phi_1 :: \alpha \Rightarrow \alpha' \triangleleft A_i)} \\
 &= \frac{(\phi_2 \circ \phi_1 :: \alpha \Rightarrow \alpha'' \triangleleft A_i)}{\dots, x_i : (\text{id}_{\{1\}}, \langle \phi_2 \circ \phi_1 \rangle) :: (\alpha \Rightarrow \langle \alpha'' \rangle \triangleleft A_i, \dots \vdash x_i : \phi_2 \circ \phi_1 :: \alpha \Rightarrow \alpha'' \triangleleft A_i)} \\
 &\frac{\frac{\pi_{2,1}}{\Xi_2 \vdash M : (\tilde{\phi}_2 \multimap \psi_2) :: (\tilde{\alpha}' \multimap \beta') \Rightarrow (\tilde{\alpha}'' \multimap \beta'')} \triangleleft A \rightarrow B \quad \frac{\pi_{2,2}}{\Xi'_2 \vdash N : \tilde{\phi}_2 :: \tilde{\alpha}' \Rightarrow \tilde{\alpha}'' \triangleleft A}}{\Xi_2 \bullet \Xi'_2 \vdash MN : \psi_2 :: \beta' \Rightarrow \beta'' \triangleleft B}} \circ \frac{\frac{\pi_{1,1}}{\Xi_1 \vdash M : (\tilde{\phi}_1 \multimap \psi_1) :: (\tilde{\alpha} \multimap \beta) \Rightarrow (\tilde{\alpha}' \multimap \beta')} \triangleleft A \rightarrow B \quad \frac{\pi_{1,2}}{\Xi'_1 \vdash N : \tilde{\phi}_1 :: \tilde{\alpha} \Rightarrow \tilde{\alpha}' \triangleleft A}}{\Xi_1 \bullet \Xi'_1 \vdash MN : \psi_1 :: \beta \Rightarrow \beta' \triangleleft B}} \\
 &= \frac{\frac{\pi_{2,1} \circ \pi_{1,1}}{\Xi_2 \circ \Xi_1 \vdash M : (\tilde{\phi}_2 \multimap \psi_2) \circ (\tilde{\phi}_1 \multimap \psi_1) :: (\tilde{\alpha} \multimap \beta) \Rightarrow (\tilde{\alpha}'' \multimap \beta'')} \triangleleft A \rightarrow B \quad \frac{\pi_{2,2} \circ \pi_{1,2}}{\Xi'_2 \circ \Xi'_1 \vdash N : \tilde{\phi}_2 \circ \tilde{\phi}_1 :: \tilde{\alpha} \Rightarrow \tilde{\alpha}'' \triangleleft A}}{(\Xi_2 \bullet \Xi'_2) \circ (\Xi_1 \bullet \Xi'_1) \vdash MN : \psi_2 \circ \psi_1 :: \beta \Rightarrow \beta'' \triangleleft B}} \\
 &\frac{\sigma_2 \in \mathcal{S}_n \quad \forall i \in \{1, \dots, n\}, \Xi_{2,i} \vdash M : \phi_{2,i} :: \alpha'_{\sigma_2^{-1}(i)} \Rightarrow \alpha''_{\sigma_2^{-1}(i)} \triangleleft A}{\sigma_2 \otimes (\Xi_{2,i})_{1 \leq i \leq n} \vdash M : (\sigma_2, \langle \phi_{2,1}, \dots, \phi_{2,n} \rangle) :: \langle \alpha'_{\sigma_2^{-1}(1)}, \dots, \alpha'_{\sigma_2^{-1}(n)} \rangle \Rightarrow \langle \alpha''_{\sigma_2^{-1}(1)}, \dots, \alpha''_{\sigma_2^{-1}(n)} \rangle \triangleleft A}} \circ \frac{n \in \mathbb{N} \quad \sigma_1 \in \mathcal{S}_n \quad \forall i \in \{1, \dots, n\}, \Xi_{1,i} \vdash M : \phi_{1,i} :: \alpha_i \Rightarrow \alpha'_i \triangleleft A}{\sigma_1 \otimes (\Xi_{1,i})_{1 \leq i \leq n} \vdash M : (\sigma_1, \langle \phi_{1,1}, \dots, \phi_{1,n} \rangle) :: \langle \alpha_1, \dots, \alpha_n \rangle \Rightarrow \langle \alpha'_{\sigma_1^{-1}(1)}, \dots, \alpha'_{\sigma_1^{-1}(n)} \rangle \triangleleft A}} \\
 &= \frac{\frac{\pi_{2,\sigma_1(i)} \circ \pi_{1,i}}{\forall i \in \{1, \dots, n\}, \Xi_{2,\sigma_1(i)} \circ \Xi_{1,i} \vdash M : \phi_{2,\sigma_1(i)} \circ \phi_{1,i} :: \alpha_i \Rightarrow \alpha'_i \triangleleft A}}{(\sigma_2 \otimes (\Xi_{2,i})_{1 \leq i \leq n}) \circ (\sigma_1 \otimes (\Xi_{1,i})_{1 \leq i \leq n}) \vdash M : (\sigma_2, \langle \phi_{2,1}, \dots, \phi_{2,n} \rangle) \circ (\sigma_1, \langle \phi_{1,1}, \dots, \phi_{1,n} \rangle) :: \langle \alpha_1, \dots, \alpha_n \rangle \Rightarrow \langle \alpha'_{\sigma_2^{-1}(\sigma_2^{-1}(1))}, \dots, \alpha'_{\sigma_2^{-1}(\sigma_2^{-1}(n))} \rangle \triangleleft A}} \\
 &\frac{\frac{\pi'_2}{\Xi_2, x : \tilde{\phi}_2 :: \tilde{\alpha}' \Rightarrow \tilde{\alpha}'' \triangleleft A \vdash M : \psi_2 :: \beta' \Rightarrow \beta'' \triangleleft B}}{\Xi_2 \vdash \lambda x. M : (\tilde{\phi}_2 \multimap \psi_2) :: (\tilde{\alpha}' \multimap \beta') \Rightarrow (\tilde{\alpha}'' \multimap \beta'')} \triangleleft A \rightarrow B \quad \frac{\frac{\pi'_1}{\Xi_1, x : \tilde{\phi}_1 :: \tilde{\alpha} \Rightarrow \tilde{\alpha}' \triangleleft A \vdash M : \psi_1 :: \beta \Rightarrow \beta' \triangleleft B}}{\Xi_1 \vdash \lambda x. M : (\tilde{\phi}_1 \multimap \psi_1) :: (\tilde{\alpha} \multimap \beta) \Rightarrow (\tilde{\alpha}' \multimap \beta')} \triangleleft A \rightarrow B}}{\frac{\pi'_2 \circ \pi'_1}{\Xi_2 \circ \Xi_1, x : \tilde{\phi}_2 \circ \tilde{\phi}_1 :: \tilde{\alpha} \Rightarrow \tilde{\alpha}'' \triangleleft A \vdash M : \psi_2 \circ \psi_1 :: \beta \Rightarrow \beta'' \triangleleft B}}{\Xi_2 \circ \Xi_1 \vdash \lambda x. M : (\tilde{\phi}_2 \multimap \psi_2) \circ (\tilde{\phi}_1 \multimap \psi_1) :: (\tilde{\alpha} \multimap \beta) \Rightarrow (\tilde{\alpha}'' \multimap \beta'')} \triangleleft A \rightarrow B}
 \end{aligned}$$

Figure 4: The definition of composition of intersection type morphism derivations

PROOF. By direct computation. \square

We can now start the definition of $\text{IT}(M)$. Its objects are the derivations π of $\Theta \triangleleft \Gamma \vdash M : \alpha \triangleleft A$ and its morphisms between two objects

$$\pi_s : \Theta \triangleleft \Gamma \vdash M : \alpha \triangleleft A \quad \text{and} \quad \pi_t : \Theta' \triangleleft \Gamma \vdash M : \alpha' \triangleleft A$$

are the derivations π of $\Xi \triangleleft \Gamma \vdash M : \phi :: \alpha \Rightarrow \alpha' \triangleleft A$ such that $\partial^-(\pi) = \pi_s$ and $\partial^+(\pi) = \pi_t$. Given two composable morphisms $\pi_1 : \Xi_1 \triangleleft \Gamma \vdash M : \phi_1 :: \alpha \Rightarrow \alpha' \triangleleft A$ and $\pi_2 : \Xi_2 \triangleleft \Gamma \vdash M : \phi_2 :: \alpha' \Rightarrow \alpha'' \triangleleft A$, their composition $\pi_2 \circ \pi_1$ using the rules of Figure 4. Note that these are the only required rules, since, when π_1 and π_2 are composable, they “have the same shape”, because they are derivations for the same term M , and the fact that $\partial^+(\pi_1) = \partial^-(\pi_2)$ allows one to infer other constraints. The rules produce derivations of the adequate type since we have:

PROPOSITION C.6. *The followings hold:*

(a) given composable Ξ_1, Ξ_2 and composable Ξ'_1, Ξ'_2 that all refine a context Γ , we have

$$(\Xi_2 \bullet \Xi'_2) \circ (\Xi_1 \bullet \Xi'_1) = (\Xi_2 \circ \Xi_1) \bullet (\Xi'_2 \circ \Xi'_1);$$

(b) given $n \in \mathbb{N}$ and $\sigma_1, \sigma_2 \in \mathcal{S}_n$ and resource morphism contexts $\Xi_{1,1}, \dots, \Xi_{1,n}$ and $\Xi_{2,1}, \dots, \Xi_{2,n}$ such that they all refine a context Γ and $\Xi_{1,i}$ is composable with $\Xi_{2,\sigma_1(i)}$ for every $i \in \{1, \dots, n\}$, we have

$$(\sigma_2 \otimes (\Xi_{2,i})_{1 \leq i \leq n}) \circ (\sigma_1 \otimes (\Xi_{1,i})_{1 \leq i \leq n}) = (\sigma_2 \circ \sigma_1) \otimes (\Xi_{2,\sigma(i)} \circ \Xi_{1,i})_{1 \leq i \leq n}.$$

PROOF. By direct computation. \square

By a similar inductive definition, we can define the identity id_π of a derivation π of a judgement $\Theta \triangleleft \Gamma \vdash M : \alpha \triangleleft A$. Moreover, following the definition of inverses for intersection type morphisms, we can define the inverse π^{-1} of a derivation π of a judgement $\Xi \triangleleft \Gamma \vdash M : \phi :: \alpha \Rightarrow \alpha' \triangleleft A$. It is then routine to check that

PROPOSITION C.7. *The above operations equip $\mathbf{IT}(M)$ with a structure of groupoid.*

C.6 The correspondence interpretation/derivation correspondence

PROOF OF THEOREM 3.5. In fact, we prove the following stronger statement:

Let Γ be a well-typed context. Given a derivation of $\Gamma \vdash M : A$, there is a canonical morphism of groupoid $K_M : \llbracket M \rrbracket \rightarrow \mathbf{IT}(M)$ (resp. $K_M^! : \llbracket M \rrbracket^! \rightarrow \mathbf{IT}_!(M)$) making the squares of the following diagram commute:

$$\begin{array}{ccc} \llbracket \Gamma \rrbracket & \xleftarrow{\partial_r^{\llbracket M \rrbracket}} & \llbracket M \rrbracket & \xrightarrow{\partial_r^{\llbracket M \rrbracket}} & \llbracket A \rrbracket \\ K_\Gamma \downarrow & & \downarrow K_M & & \downarrow K_A \\ \mathbf{IT}(\Gamma) & \xleftarrow{\partial_r^M} & \mathbf{IT}(M) & \xrightarrow{\partial_r^M} & \mathbf{IT}(A) \end{array} \quad \left(\text{resp. } \begin{array}{ccc} \llbracket \Gamma \rrbracket^! & \xleftarrow{\partial_r^{\llbracket M \rrbracket^!}} & \llbracket M \rrbracket^! & \xrightarrow{\partial_r^{\llbracket M \rrbracket^!}} & \llbracket A \rrbracket^! \\ K_\Gamma \downarrow & & \downarrow K_M^! & & \downarrow K_A^! \\ \mathbf{IT}(\Gamma) & \xleftarrow{\partial_r^{M,!}} & \mathbf{IT}_!(M) & \xrightarrow{\partial_r^{M,!}} & \mathbf{IT}_!(A) \end{array} \right). \quad (15)$$

We prove it by induction on a derivation of $\Gamma \vdash M : A$:

- in the case of the variable typing rule $x_1 : A_1, \dots, x_n : A_n \vdash x_i : A_i$, K_M is defined as the functor sending $u : a \rightarrow a' \in \llbracket A \rrbracket = \llbracket x_i \rrbracket$ to the unique derivation of

$$\begin{aligned} (x_1 : \langle \rangle :: \langle \rangle \Rightarrow \langle \rangle \triangleleft A_1, \dots, x_i : (\text{id}_{\{1\}}, \langle K_{A_i}(u) \rangle) :: \langle K_{A_i}(a) \rangle \Rightarrow \langle K_{A_i}(a') \rangle \triangleleft A_i, \\ \dots, x_n : \langle \rangle :: \langle \rangle \Rightarrow \langle \rangle \triangleleft A_n) \triangleleft \Gamma \vdash x : K_{A_i}(u) :: K_{A_i}(a) \Rightarrow \langle K_{A_i}(a') \rangle \triangleleft A_i. \end{aligned}$$

It is immediate that the squares of (15) commute for this definition;

- in the case of an application $\Gamma \vdash MN : B$, we have that there exists a unique simple type A such that $\Gamma \vdash N : A$, so that the typing of the application MN is constructed from two derivations $\Gamma \vdash M : A \rightarrow B$ and $\Gamma \vdash N : A$. We then have by the rules for intersection type morphism judgements that $\mathbf{IT}(MN)$ is the pullback

$$\begin{array}{ccc} \mathbf{IT}(MN) & \overset{P_N}{\dashrightarrow} & \mathbf{IT}_!(N) \\ P_M \downarrow & \lrcorner & \downarrow \partial_r^N \\ \mathbf{IT}(M) & \xrightarrow{\partial_r^M} \mathbf{IT}(A \rightarrow B) \xrightarrow{P_A^{A \rightarrow B}} & \mathbf{IT}_!(A). \end{array}$$

where $P_A^{A \rightarrow B}$ is the functor projecting a derivation of $\tilde{\phi} \dashv \psi :: \tilde{\alpha} \dashv \beta \Rightarrow \tilde{\alpha}' \dashv \beta \triangleleft A \rightarrow B$ to the associated derivation $\tilde{\phi} :: \tilde{\alpha} \Rightarrow \tilde{\alpha}' \triangleleft A$. Similarly, by considering again the definition of the groupoid $\llbracket MN \rrbracket$, we see that it can be alternatively expressed as the pullback

$$\begin{array}{ccc} \llbracket MN \rrbracket & \overset{P'_N}{\dashrightarrow} & \llbracket N \rrbracket^! \\ P'_M \downarrow & \lrcorner & \downarrow \partial_r^{\llbracket N \rrbracket^!} \\ \llbracket M \rrbracket & \xrightarrow{\partial_r^{\llbracket M \rrbracket}} \llbracket A \rrbracket \times \llbracket B \rrbracket \xrightarrow{t} & \llbracket A \rrbracket. \end{array}$$

Then, using $K_A^!$ and the inductively defined $K_M, K_N^!$, we build an isomorphism between the underlying cospans of these pullbacks, so that we get a factorizing isomorphism $K_{MN} : \llbracket MN \rrbracket \rightarrow \mathbf{IT}(MN)$.

Concerning the commutativity condition we have the diagram

$$\begin{array}{ccccccc}
 \llbracket \Gamma \rrbracket & \xleftarrow{(\bullet)} & \llbracket \Gamma \rrbracket \times \llbracket \Gamma \rrbracket & \xleftarrow{\partial_i^{\llbracket M \rrbracket} \times \partial_i^{\llbracket N \rrbracket}!} & \llbracket M \rrbracket \times \llbracket N \rrbracket! & \xleftarrow{(P'_M, P'_N)} & \llbracket MN \rrbracket \\
 K_\Gamma \downarrow & & K_\Gamma \times K_\Gamma \downarrow & & K_M \times K_N \downarrow & & \downarrow K_{MN} \\
 \mathbf{IT}(\Gamma) & \xleftarrow{(\bullet)} & \mathbf{IT}(\Gamma) \times \mathbf{IT}(\Gamma) & \xleftarrow{\partial_i^M \times \partial_i^{N,!}} & \mathbf{IT}(M) \times \mathbf{IT}_!(N) & \xleftarrow{(P_M, P_N)} & \mathbf{IT}(MN)
 \end{array}$$

where each rectangle commutes and the top row is precisely $\partial_i^{\llbracket MN \rrbracket}$ and the bottom row ∂_i^{MN} . On the side of B , we have the diagram

$$\begin{array}{ccccccc}
 \llbracket MN \rrbracket & \xrightarrow{P'_M} & \llbracket M \rrbracket & \xrightarrow{\partial_r^{\llbracket M \rrbracket}} & !\llbracket A \rrbracket \times \llbracket B \rrbracket & \xrightarrow{r} & \llbracket B \rrbracket \\
 K_{MN} \downarrow & & K_M \downarrow & & K_{A \rightarrow B} \downarrow & & \downarrow K_B \\
 \mathbf{IT}(MN) & \xrightarrow{P_M} & \mathbf{IT}(M) & \xrightarrow{\partial_r^M} & \mathbf{IT}(A \rightarrow B) & \xrightarrow{P_B^{A \rightarrow B}} & \mathbf{IT}(B)
 \end{array}$$

where $P_B^{A \rightarrow B}$ is defined like $P_A^{A \rightarrow B}$, where every rectangle commutes, where the top row is $\partial_r^{\llbracket MN \rrbracket}$ and the bottom row is ∂_r^{MN} . Which concludes the proof of the commutativity conditions;

- in the case of a λ -abstraction $\Gamma \vdash \lambda x.M : A \rightarrow B$, we get by induction the commutative diagram

$$\begin{array}{ccc}
 \llbracket \Gamma, x : A \rrbracket & \xleftarrow{\partial_i^{\llbracket M \rrbracket}} & \llbracket M \rrbracket \xrightarrow{\partial_r^{\llbracket M \rrbracket}} \llbracket B \rrbracket \\
 K_{(\Gamma, x:A)} \downarrow & & K_M \downarrow \quad \downarrow K_B \\
 \mathbf{IT}(\Gamma, x : A) & \xleftarrow{\partial_i^M} & \mathbf{IT}(M) \xrightarrow{\partial_r^M} \mathbf{IT}(B)
 \end{array}$$

so that, using the isomorphisms $\llbracket \Gamma, x : A \rrbracket \cong \llbracket \Gamma \rrbracket \times !\llbracket A \rrbracket$, $\mathbf{IT}(\Gamma, x : A) \cong \mathbf{IT}(\Gamma) \times \mathbf{IT}_!(A)$, $!\llbracket A \rrbracket \times \llbracket B \rrbracket \cong \llbracket A \rightarrow B \rrbracket$ and $\mathbf{IT}_!(A) \times \mathbf{IT}(B) \cong \mathbf{IT}(A \rightarrow B)$, we are able to get a similar commutative diagram for $\lambda x.M$;

- finally, we define the multilinear interpretation of a judgement $\Gamma \vdash M : A$ from the above cases: we have a commutative diagram

$$\begin{array}{ccccccc}
 \llbracket \Gamma \rrbracket & \xleftarrow{\partial_i^{\llbracket M \rrbracket}!} & \llbracket M \rrbracket! & \xrightarrow{\partial_r^{\llbracket M \rrbracket}!} & !\llbracket A \rrbracket & & \\
 \parallel & & \parallel & & \parallel & & \\
 \llbracket \Gamma \rrbracket & \xleftarrow{\tilde{\mu}_\Gamma} & !\llbracket \Gamma \rrbracket & \xleftarrow{!\partial_i^{\llbracket M \rrbracket}} & !\llbracket M \rrbracket & \xrightarrow{!\partial_r^{\llbracket M \rrbracket}} & !\llbracket A \rrbracket \\
 \downarrow K_\Gamma & & \downarrow !K_\Gamma & & \downarrow !K_M & & \downarrow !K_A \\
 \mathbf{IT}(\Gamma) & \xleftarrow{\tilde{\mu}_\Gamma^{\text{it}}} & \mathbf{IT}(\Gamma) & \xleftarrow{!\partial_i^M} & \mathbf{IT}(M) & \xrightarrow{!\partial_r^M} & \mathbf{IT}(A) \\
 \parallel & & \parallel & & \parallel & & \\
 \mathbf{IT}(\Gamma) & \xleftarrow{\partial_i^{M,!}} & \mathbf{IT}_!(M) & \xrightarrow{\partial_r^{M,!}} & \mathbf{IT}_!(A) & &
 \end{array}$$

where the morphism $\mathbf{IT}(M) \xrightarrow{\sim} \mathbf{IT}_!(M)$ is basically the multilinear introduction rule—a sequence $\langle \pi_i \rangle_{1 \leq i \leq n}$ of derivations π_i of $\Theta_i \triangleleft \Gamma \vdash M : \alpha_i \triangleleft A$ is mapped to the derivation of $\Theta_1 \bullet \dots \bullet \Theta_n \triangleleft \Gamma \vdash M : \langle \alpha_1, \dots, \alpha_n \rangle \triangleleft A$ and similarly for sequences of morphism derivations—and the morphism $\mathbf{IT}(A) \xrightarrow{\sim} \mathbf{IT}_!(A)$ is similarly the multilinear refinement introduction rule, and $\tilde{\mu}^{\text{it}}$ is the functor mapping a sequence of resource contexts $\langle \Theta_1, \dots, \Theta_n \rangle \in \mathbf{Ob}(\mathbf{IT}(\Gamma))$ to $\Theta_1 \bullet \dots \bullet \Theta_n$ and a morphism $(\sigma, \langle \Xi_i \rangle_{1 \leq i \leq n}) \in \mathbf{Ar}(\mathbf{IT}(\Gamma))$ between two sequences $\langle \Theta_1, \dots, \Theta_n \rangle$ and $\langle \Theta'_1, \dots, \Theta'_n \rangle$ to $\sigma \otimes \langle \Xi_i \rangle_{1 \leq i \leq n}$. It is quite immediate to check that every rectangle commutes. \square

D POSTPONED PROOFS FOR RELATIONAL COLLAPSES

D.1 Functoriality of the collapse to Rel

We give the postponed proof of the following result:

PROPOSITION 4.1. *This yields a functor $|-| : \mathbf{Thin} \rightarrow \mathbf{Rel}$.*

PROOF. It is obvious that the identity $\text{span } A \leftarrow A \rightarrow A$ is sent to the identity relation on $|A|$. For functoriality, it is obvious by definition that $|T \circ S| \subseteq |T| \circ |S|$ – but the other direction is not, since composition in **Thin** is more constrained than in **Rel**.

So consider $(\mathbf{a}, \mathbf{b}) \in |S|$ and $(\mathbf{b}, \mathbf{c}) \in |T|$. By definition, there are $s \in S$ such that $\mathbf{a} = \overline{s_A}$ and $\mathbf{b} = \overline{s_B}$, and $t \in T$ such that $\mathbf{b} = \overline{t_B}$ and $\mathbf{c} = \overline{t_C}$. Since $\overline{s_B} = \overline{t_B}$ those two are symmetric, but they might not be equal, meaning that the pair (s, t) may not be a valid element of $T \circ S$. However, by Lemma 2.4 there must be $\varphi^S \in S[s, s']$ and $\varphi^T \in T[t, t']$ such that $s'_B = t'_B$ and we can now form $(s', t') \in T \circ S$ with $\overline{(s', t')_A} = s'_A = \mathbf{a}$ and $\overline{(s', t')_C} = t'_C = \mathbf{c}$ as required. \square

D.2 Bijection for the quantitative collapse

PROPOSITION 4.7. For $S, T, \mathbf{a}, \mathbf{b}, \mathbf{c}$ as above, there is a bijection

$$\Upsilon : \sim\text{-wit}_S^+(\mathbf{a}, \mathbf{b}) \times \sim\text{-wit}_T^+(\mathbf{b}, \mathbf{c}) \simeq \sim\text{-wit}_{S,T}^+(\mathbf{a}, \mathbf{c})$$

s.t. for any $\Upsilon((\theta_A^-, s, \theta_B^+), (\Omega_B^-, t, \Omega_C^+)) = (\psi_A^-, s', \Theta, t', \psi_C^+)$, there are unique $\omega^S : s \cong_S s'$ and $v^T : t \cong_T t'$ making the diagrams commute:

$$\begin{array}{ccccc} \theta_A^- \nearrow s_A & s_B \xrightarrow{\theta_B^+} \mathbf{b} & \xrightarrow{\Omega_B^-} t_B & t_C \xrightarrow{\Omega_C^+} \mathbf{c} & \\ \downarrow \omega_A^S & \omega_B^S \downarrow & \downarrow \Theta & \downarrow v_B^T & v_C^T \downarrow \\ \mathbf{a} & \downarrow \omega_A^S & s'_B = b = t'_B & \downarrow v_B^T & t'_C \xrightarrow{\psi_C^+} \mathbf{c} \\ \psi_A^- \searrow s'_A & & & & \end{array}$$

PROOF. From $(\theta_A^-, s, \theta_B^+)$ and $(\Omega_B^-, t, \Omega_C^+)$, we can apply Lemma 2.4 and compose s and t via $\Omega_B^- \circ \theta_B^+$, giving us unique ω^S, v^T such that the big rectangle commutes, ω_A^S negative and v_C^T positive. We get Θ as either path around the rectangle, and ψ_A^-, ψ_C^+ by composition. Reciprocally, from $(\psi_A^-, s', \Theta, t', \psi_C^+)$ we obtain uniquely the remaining data by Proposition 4.6. \square

E SEELY FUNCTORS AND THEIR KLEISLI LIFTING

Here we include a few folklore results that we required regarding the adequate definition of morphisms between Seely categories, along with the fact that they admit a lifting to cartesian closed functors between the Kleisli categories.

Definition E.1. Consider \mathcal{C} and \mathcal{D} two Seely categories.

A **Seely functor** $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor, additionally equipped with isomorphisms

$$\begin{array}{lll} t_A^! & : & !FA \rightarrow F!A \\ t_{A,B}^\otimes & : & FA \otimes FB \rightarrow F(A \otimes B) \\ t_{A,B}^\& & : & FA \& FB \rightarrow F(A \& B) \\ t_{A,B}^\circ & : & FA \multimap FB \rightarrow F(A \multimap B) \end{array}$$

such that $t_A^!$ is natural in A and $t_{A,B}^\otimes$ is natural in A and B , and subject to the following coherence conditions:

$$\begin{array}{ccc} \begin{array}{ccc} !FA & \xrightarrow{t_A^!} & F!A \\ \delta_{FA}^{\mathcal{D}} \swarrow & & \searrow F\delta_A^{\mathcal{C}} \\ !!FA & \xrightarrow{!t_A^!} & !F!A \end{array} & & \begin{array}{ccc} !FA & \xrightarrow{t_A^!} & F!A \\ \epsilon_{FA}^{\mathcal{D}} \swarrow & & \searrow F\epsilon_A^{\mathcal{C}} \\ FA & & FA \end{array} \\ \\ \begin{array}{ccc} !FA \otimes !FB & \xrightarrow{t_A^! \otimes t_B^!} & F!A \otimes F!B \\ \text{see}_{FA,FB}^{\mathcal{D}} \swarrow & & \searrow \\ !(FA \& FB) & \xrightarrow{!t_{A,B}^\&} & !F(A \& B) \end{array} & & \begin{array}{ccc} FA \& FB & \xrightarrow{t_{A,B}^\&} & F(A \& B) \\ \pi_1^{\mathcal{D}} \swarrow & & \searrow \pi_2^{\mathcal{D}} \\ FA & & FB \\ F\pi_1^{\mathcal{C}} \swarrow & & \searrow F\pi_2^{\mathcal{C}} \\ F(A \& B) & & F(A \& B) \end{array} \\ \\ \begin{array}{ccc} !F(A \& B) & \xrightarrow{!t_{A\&B}} & F!(A \& B) \\ & & \searrow F\text{see}_{A,B}^{\mathcal{C}} \\ & & F!(A \& B) \end{array} \end{array}$$

$$\begin{array}{ccc}
 (FA \multimap FB) \otimes FA & \xrightarrow{t_{A,B}^{\multimap} \otimes FA} & F(A \multimap B) \otimes FA & \xrightarrow{t_{A \multimap B, A}^{\otimes}} & F((A \multimap B) \otimes A) \\
 & \searrow \text{ev}_{FA, FB}^{\mathcal{D}} & & \swarrow \text{Fev}_{A, B}^{\mathcal{C}} & \\
 & & FB & &
 \end{array}$$

The main interest of those is that they lift to cartesian closed functors between the Kleisli categories:

THEOREM E.2. *Consider \mathcal{C} and \mathcal{D} two Seely categories, and $F : \mathcal{C} \rightarrow \mathcal{D}$ a Seely functor. Then, defining $F_!(A) = F(A)$ on objects and $F_!(f) = Ff \circ t_A^!$ for $f \in \mathcal{C}[!A, B]$, we get*

$$F_! : \mathcal{C}_! \rightarrow \mathcal{D}_!$$

a cartesian closed functor.

PROOF. We must show that products and arrows are preserved up to (canonical) isomorphism. For that, we construct the following morphisms in \mathcal{D} :

$$\begin{aligned}
 k_{A, B}^{\&} &= t_{A, B}^{\&} \circ \epsilon_{FA \& FB}^{\mathcal{D}} & : & \quad !(FA \& FB) \rightarrow F(A \& B) \\
 k_{A, B}^{\Rightarrow} &= t_{!A, B}^{\multimap} \circ ((t_A^!)^{-1} \multimap B) \circ \epsilon_{!FA \multimap FB}^{\mathcal{D}} & : & \quad !(!FA \multimap FB) \rightarrow F(!A \multimap B),
 \end{aligned}$$

which we regard as $k_{A, B}^{\&} \in \mathcal{D}_![FA \& FB, F(A \& B)]$ and $k_{A, B}^{\Rightarrow} \in \mathcal{D}_![FA \Rightarrow FB, F(A \Rightarrow B)]$ where $A \Rightarrow B = !A \multimap B$. By construction those are isomorphisms, and to show canonicity we must prove (in $\mathcal{D}_!$) the diagrams corresponding to the last two diagrams of Definition E.1, which is a lengthy diagram chase. \square