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# A RELLICH TYPE THEOREM FOR DISCRETE MAXWELL OPERATORS

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ABSTRACT. We study the Rellich type theorem (RT) for the Maxwell operator  $\hat{H}^D = \hat{D}\hat{H}_0$  on  $\mathbf{Z}^3$  with constant anisotropic medium, i.e. the permittivity and permeability of which are non-scalar constant diagonal matrices. We also study the unique continuation theorem for the perturbed Maxwell operator  $\hat{H}^{D_p} = \hat{D}_p\hat{H}_0$  on  $\mathbf{Z}^3$  where the permittivity and permeability are locally perturbed from a constant matrix on a compact set in  $\mathbf{Z}^3$ . It then implies that, if  $\hat{H}^D - \lambda$  satisfies (RT), then all distributions  $\hat{u}$  in Besov space  $\mathcal{B}_0^*(\mathbf{Z}^3; \mathbf{C}^3)$  satisfying the equation  $(\hat{H}^{D_p} - \lambda)\hat{u} = 0$  outside a compact set vanish near infinity.

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*Keywords.* Maxwell equations, Scattering theory of linear operators, Quantum mechanics on lattices.

## 1. INTRODUCTION

**1.1. Presentation and Main results.** We are interested in the Rellich type theorem (RT) for the discrete Maxwell operator  $\hat{H}^D$  defined on the square lattice  $\mathbf{Z}^3$ . Let  $\varepsilon$  and  $\mu$  be the permittivity and the permeability in the ambient space  $\mathbf{Z}^3$ , which are the  $3 \times 3$  constant diagonal matrices with diagonal elements,  $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in (0, \infty)$  for  $\varepsilon$ , and  $\mu_1, \mu_2, \mu_3 \in (0, \infty)$  for  $\mu$ . Letting  $\hat{H}_0$  be the isotropic discrete Maxwell operator with  $\varepsilon = \mu = I_{3 \times 3}$ , the anisotropic Maxwell operator is defined by

$$\hat{H}^D = \hat{D}\hat{H}_0,$$

where we set

$$\hat{D} = \begin{pmatrix} \varepsilon & 0_{3 \times 3} \\ 0_{3 \times 3} & \mu \end{pmatrix}.$$

We define the Besov type space

$$\mathcal{B}_0^*(\mathbf{Z}^d; \mathbf{C}^m) := \{\hat{u} : \mathbf{Z}^d \rightarrow \mathbf{C}^m \mid \lim_{R \rightarrow \infty} \frac{1}{R} \sum_{n: |n| < R} |\hat{u}(n)|^2 = 0\}.$$

We say that (RT) holds for  $\hat{H}^D$  at  $\lambda \in \mathbf{R}$  if  $\hat{u} \in \mathcal{B}_0^*(\mathbf{Z}^3; \mathbf{C}^6)$  satisfying the equation

$$(1.1) \quad (\hat{H}^D - \lambda)\hat{u}(n) = 0$$

outside a compact set in  $\mathbf{Z}^3$  vanishes on  $\{n \in \mathbf{Z}^3 \mid |n| > R\}$  for some constant  $R > 0$ . (RT) is usually combined with the unique continuation property (UCP) in exterior

domains of  $\mathbf{Z}^3$ . Let us consider the perturbed Maxwell operator  $\hat{H}^{D_p} = \hat{D}_p \hat{H}_0$  in an exterior domain  $K_{ext}$ , where  $\hat{D}_p$  is a perturbation of  $\hat{D}$ , i.e.,

$$\hat{D}_p := \begin{pmatrix} \tilde{\varepsilon} & 0_{3 \times 3} \\ 0_{3 \times 3} & \tilde{\boldsymbol{\mu}} \end{pmatrix},$$

$\tilde{\varepsilon}_j(n) > 0$ ,  $\tilde{\boldsymbol{\mu}}_j(n) > 0$  for  $1 \leq j \leq 3$  and  $n \in \mathbf{Z}^3$ , and  $\hat{V} := \hat{D}_p - \hat{D}$  has compact support. The exterior domain has the form  $K_{ext} := \mathbf{Z}^3 \setminus K_{int}$  where  $K_{int} \subset \mathbf{Z}^3$  is bounded. We say that (UCP) holds for the operator  $(\hat{H}^{D_p} - \lambda)$  in  $K_{ext}$  if for all  $\hat{u}$  with compact support in  $\mathbf{Z}^3$  satisfying  $(\hat{H}^{D_p} - \lambda)\hat{u} = 0$  in  $K_{ext}$ , we have  $\hat{u} = 0$  in  $K_{ext}$ .

A first interest of (RT) appears in the spectral analysis of the self-adjoint operator  $\hat{H}^D$ . The point spectrum  $\sigma_p(\hat{H}^D)$  is known to be  $\{0\}$  by the conjugate operator method in [5], however, an alternative proof can be obtained by using the simpler tools of (RT) and (UCP). A second interest of (RT) is found in the application to inverse problems similar to those studied in [1] for the Schrödinger operator on lattices.

Following the methods in [4], [2], we pass to the Fourier series and consider  $H^D = \hat{D}H_0$ , which is a multiplication operator on the real torus  $\mathbb{T}^3 \approx (\mathbf{R}/(2\pi\mathbf{Z}))^3$ . The operator  $H^D$  is then represented by the analytically fibered self-adjoint operator  $\mathbb{T}^3 \ni x \mapsto H^D(x) = \hat{D}H_0(x)$  where  $H_0(x)$  is a  $6 \times 6$  real symmetric matrix depending on the variable  $y = (y_1, y_2, y_3)$ ,  $y_i = \sin x_i$  only (see (2.1)). Equation (1.1) is then equivalent to

$$(1.2) \quad (H^D - \lambda)u = f,$$

where  $f$  is a trigonometric polynomial of the variable  $x \in \mathbb{T}^3$ , and where (1.2) has to be taken in the sense of distributions. Using the correspondence  $\hat{u} \in \mathcal{B}_0^*(\mathbf{Z}^3) \iff u \in \mathcal{B}_0^*(\mathbb{T}^3)$ , the problem is reduced to the Besov space in terms of the variable  $x$ . Multiplying (1.2) by the cofactor matrix of  $H^D(x) - \lambda$  leads to the equation

$$(1.3) \quad q(\cdot; \lambda)u = g \quad \text{on } \mathbb{T}^3,$$

where  $q(x; \lambda) := \det(H^D(x) - \lambda)$  and  $g$  is a trigonometric polynomial. Let us recall the factorization of  $q(x; \lambda)$  and the spectral analysis of  $H^D(x)$  (see [5]). The spectrum  $\sigma(H^D(x))$  of  $H^D(x)$  depends on  $z = (z_1, z_2, z_3)$ ,  $z_i = \sin^2 x_i$  only, and  $q(x; \lambda)$  is a polynomial of  $z$  and  $\lambda$ , which we denote by  $p(z; \lambda)$ . We write the diagonal matrices  $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ ,  $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$  and put:

$$(1.4) \quad \begin{aligned} \boldsymbol{\beta} &:= \boldsymbol{\varepsilon} \times \boldsymbol{\mu} = (\beta_1, \beta_2, \beta_3) \\ \alpha_i &:= (\varepsilon_j \mu_k + \varepsilon_k \mu_j)/2, \\ \gamma_i &:= \varepsilon_j \varepsilon_k \mu_j \mu_k, \end{aligned}$$

where  $(i, j, k)$  ranges over the cyclic permutations of  $(1, 2, 3)$ , and  $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \alpha_3)$ . We then have

$$(1.5) \quad q(x; \lambda) = p(z; \lambda) = \lambda^2 (\lambda^4 - 2\lambda^2 \Psi_0(z) + \Psi_0^2(z) - K_0(z)),$$

where we put

$$(1.6) \quad \begin{aligned} K_0(z) &:= \frac{1}{4} \sum_{i=1}^3 (\beta_i z_i)^2 - \frac{1}{2} \sum_{1 \leq i < j \leq 3} \beta_i \beta_j z_i z_j, \\ \Psi_0(z) &:= \alpha \cdot z = \alpha_1 z_1 + \text{c.p.} \end{aligned}$$

Then,  $p(z; \lambda)$  is rewritten as:

$$p(z; \lambda) = \lambda^2 (\tau^+(z) - \lambda^2) (\tau^-(z) - \lambda^2),$$

where we put:

$$(1.7) \quad \tau^\pm(z) := \Psi_0(z) \pm \sqrt{K_0(z)}.$$

In [5], it is shown that  $\tau^-(z) \geq 0$  for all  $z \in [0, +\infty)^3$ , and that

$$\sigma(H^D(x)) = \{-\sqrt{\tau^+(z)}, -\sqrt{\tau^-(z)}, 0, \sqrt{\tau^-(z)}, \sqrt{\tau^+(z)}\}, \quad x \in \mathbb{T}^3.$$

We put

$$\lambda_\pm = \max_{[0,1]^3} \sqrt{\tau^\pm(z)}.$$

Note that  $\lambda_+ \geq \lambda_- > 0$ . We put

$$\begin{aligned} \mathbf{B}_0 &:= \{(0, 0, 0)\}, \\ \mathbf{B}_3 &:= \{\boldsymbol{\beta} \in \mathbf{R}^3 \mid \beta_1 \beta_2 \beta_3 \neq 0\}, \\ \mathbf{B}_{12} &:= \mathbf{R}^3 \setminus (\mathbf{B}_0 \cup \mathbf{B}_3). \end{aligned}$$

Our main result is stated in terms of the parameter  $\boldsymbol{\beta} = \boldsymbol{\varepsilon} \times \boldsymbol{\mu}$  and  $\lambda_\pm$ .

**Theorem 1.1.** *(RT) for  $\hat{H}^D$  holds at  $\lambda$  in each of the following cases:*

- (1)  $\boldsymbol{\beta} \in \mathbf{B}_0$  and  $0 < |\lambda| < \lambda_+$ ,
- (2)  $\boldsymbol{\beta} \in \mathbf{B}_3$  and  $0 < |\lambda| < \lambda_+$ ,
- (3)  $\boldsymbol{\beta} \in \mathbf{B}_{12}$  and  $0 < |\lambda| < \lambda_-$ .

Since  $\lambda_- < \lambda_+$  if  $\boldsymbol{\beta} \in \mathbf{B}_3$ , the case (3) in Theorem 1.1 is critical.

**Theorem 1.2.** *Assume  $\boldsymbol{\beta} \in \mathbf{B}_{12}$  and  $|\lambda| > \lambda_-$ . Then (RT) for  $\hat{H}^D$  fails at  $\lambda$ .*

**Remark 1.1.** *Even in the situation of Theorem 1.2 the following fact holds. Let  $\lambda \in (\lambda_-, \lambda_+)$ , and  $u \in \mathcal{B}_0^*(\mathbb{T}^3)$  be such that  $(H^D - \lambda)u$  is a trigonometric polynomial of  $x \in \mathbb{T}^3$ . Then  $(\tau^-(z) - \lambda^2)u(x)$  is a trigonometric polynomial of  $x \in \mathbb{T}^3$ . We consequently obtain the following result.*

**Proposition 1.3.** *Assume  $\boldsymbol{\beta} \in \mathbf{B}_{12}$  and  $|\lambda| > \lambda_-$ . Let  $\hat{u} \in \mathcal{B}_0^*(\mathbf{Z}^3)$  such that  $(\hat{H}^{D_p} - \lambda)\hat{u} = 0$  and  $\hat{u} \neq 0$ . Then,  $\hat{u} \in \ell^2(\mathbf{Z}^3)$  is an eigenvector of  $\hat{H}^D$  associated with the eigenvalue  $\lambda$ .*

The results of [5] show that  $\sigma_p(\hat{H}^D) = \{0\}$ ; hence, if  $\lambda \neq 0$  then (UCP) holds for the unperturbed operator  $\hat{H}^D - \lambda$  in the whole  $\mathbf{Z}^3$ . Nevertheless (UCP) does not hold for  $\hat{H}^D$  in  $\mathbf{Z}^3$  since the kernel of  $\hat{H}^D$  contains trigonometric polynomials. To consider the exterior domain, we restrict ourselves to the situation where  $K_{int}$  is a compact convex set.

**Theorem 1.4.** *Assume  $\lambda \neq 0$  and let  $K_{int}$  be a compact convex set of  $\mathbf{Z}^3$ . Then (UCP) holds for the operator  $\hat{H}^{D_p} - \lambda$  in  $K_{ext} := \mathbf{Z}^3 \setminus K_{int}$ .*

**Corollary 1.5.** *Let  $\lambda \neq 0$  and  $K_{ext}$  be as in Theorem 1.4. Assume that (RT) for  $\hat{H}^D$  holds at  $\lambda$ . Let  $\hat{u} \in \mathcal{B}_0^*(\mathbf{Z}^3)$  be a solution of  $(\hat{H}^{D_p} - \lambda)\hat{u} = 0$  in  $K_{ext}$ . Then  $\hat{u}$  vanishes in  $K_{ext}$ .*

**Remark 1.2.** *The proof of Theorem 1.4 shows that Theorem 1.4 and so Corollary 1.5 extend to the case where  $\hat{H}^{D_p}$  is anisotropic, the perturbation  $\hat{D}_p - \hat{D}$  being any operator of multiplication by real or complex valued function with compact support such that, for all  $n$ ,  $\hat{D}_p(n)$  is a self-adjoint positive matrix. Nevertheless, it is difficult to extend Corollary 1.5 to the operator  $(\hat{H}^D + \hat{V} - \lambda)$ , where  $\hat{V}$  is an operator of multiplication on  $\mathbf{Z}^3$  with compact support, since (UCP) fails for  $\hat{H}_0$ . In fact, the trigonometric polynomial,  $\phi^+(x) := (y, 0_{\mathbf{R}^3})$  (we can choose also  $\phi^-(x) := (0_{\mathbf{R}^3}, y)$ )  $\in \mathbf{R}^6$  is an eigenmode for  $H_0$  (and so for  $H^D$ ) associated with the eigenvalue 0, then (UCP) obviously fails for  $\hat{H}_0$  in the whole space  $\mathbf{Z}^3$ . We then have the following general negative result.*

*Let  $\lambda \in \mathbf{R}$ . Then there exists a compactly supported multiplication operator  $\hat{V}$  and a compactly supported eigenmode  $\hat{u}$  of  $\hat{H}_0 + \hat{V}$  associated with the eigenvalue  $\lambda$ , i.e.,*

$$(\hat{H}_0 + \hat{V} - \lambda)\hat{u} = 0.$$

*In fact, setting  $u = \phi^+$  and  $\hat{V} = \lambda \cdot 1_{\text{supp } \hat{u}}$ , the conclusion follows.*

**1.2. Plan of the paper.** In §2 we give the definition of the discrete Maxwell operator, and recall its main spectral properties. In §3 we state Theorem 3.1 which is a very slight extension of results in [4], [2]. We then give a proof of Theorem 3.1. In §4 we describe the Fermi surface  $M^{\mathbf{C}}(q(\cdot; \lambda))$ . The analysis begins with the description of the Fermi surface  $M^{\mathbf{C}}(p(\cdot; \lambda))$  which is simpler since the function  $p(\cdot; \lambda)$  is a polynomial of second order of the variable  $z$ . Then, we obtain the description of  $M^{\mathbf{C}}(q(\cdot; \lambda))$ , thanks to the key-lemma Lemma 4.8 notably. In §5 we give the proofs of Theorems 1.1, 1.2, 1.4, Proposition 1.3 and Corollary 1.5.

**Notations.** We use the following notations. For  $n = (n_1, \dots, n_d) \in \mathbf{Z}^d$  we set  $|n| = \sum_{j=1}^d |n_j|$ . For a map  $f : \mathbf{R} \rightarrow \mathbf{R}$  (respect., a function from  $\mathbb{T}^1$  into  $\mathbf{R}^1$ ), we denote by  $f$  again the mapping  $\mathbf{R}^d$  (respectively,  $\mathbb{T}^d$ )  $\ni x \mapsto (f(x_1), \dots, f(x_d)) \in \mathbf{R}^d$ . If  $E \subset \mathbb{T}^d$  or  $E \subset \mathbf{R}^d$  we denote by  $f(E)$  the set  $\{f(x) \mid x \in E\}$ . In particular we set, for  $x \in \mathbb{T}_{\mathbf{C}}^d \cup \mathbf{C}^d$ ,

$$(1.8) \quad \begin{aligned} y &:= \sin x = (\sin x_1, \dots, \sin x_d), \\ z &:= (\sin^2 x_1, \dots, \sin^2 x_d). \end{aligned}$$

We use the abbreviated expression ”+ c.p.” in a computation to indicate that the value is the addition of the two other similar terms obtained by two successive permutations of  $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ . For example, an expression written as  $p_1 + q_{23} +$

c.p. is the sum equal to  $p_1 + q_{23} + p_2 + q_{31} + p_3 + q_{12}$ . For  $\tilde{E} \subset \mathbb{T}_{\mathbf{C}}^d \cup \mathbf{C}^d$ , we set  $\sin^2(\tilde{E}) = \{z \in \mathbf{C}^d \mid x \in \tilde{E}\}$ , and letting  $E \subset \mathbf{C}^d$  we set  $(\sin_{\mathbf{C}}^2)^{-1}(E) = \{x \in \mathbf{C}^d \mid z \in E\}$  and  $(\sin_{\mathbb{T}}^2)^{-1}(E) = \{x \in \mathbb{T}_{\mathbf{C}}^d \mid z \in E\}$ . We denote by  $C_c^\infty(E)$  the space of real  $C^\infty$  functions defined in  $E$  with compact support in  $E$ . For  $m \geq 1$  we denote by  $(e_1, \dots, e_m)$  the standard basis of  $\mathbf{R}^m$  or of  $\mathbf{C}^m$ . The spaces  $\mathbf{R}^m$ ,  $\mathbf{C}^m$  are equipped with the inner product

$$\begin{aligned} \langle f, g \rangle_{\mathbf{R}^m} &:= \sum_{j=1}^m f_j g_j, \quad f = (f_j)_{1 \leq j \leq m}, \quad g = (g_j)_{1 \leq j \leq m}, \\ \langle f, g \rangle_{\mathbf{C}^m} &:= \sum_{j=1}^m f_j \bar{g}_j, \quad f = (f_j)_{1 \leq j \leq m}, \quad g = (g_j)_{1 \leq j \leq m}. \end{aligned}$$

(The formula for  $\langle f, g \rangle_{\mathbf{R}^m}$  extends to the case where  $f_j$  and  $g_j$  are complex valued.) If  $E \subset \mathbf{R}^m$  (respect.,  $E \subset \mathbf{C}^m$ ) then we denote  $E^\perp := \{g \in \mathbf{R}^m \mid \langle f, g \rangle_{\mathbf{R}^m} = 0, \forall f \in E\}$  (respect.,  $E^{\perp \mathbf{C}} := \{g \in \mathbf{C}^m \mid \langle f, g \rangle_{\mathbf{C}^m} = 0, \forall f \in E\}$ ). The Besov space  $\mathcal{B}_0^*(\mathbb{T}^d; \mathbf{C}^m)$ , simply denoted by  $\mathcal{B}_0^*(\mathbb{T}^d)$ , is the space of tempered distributions  $u \in \mathcal{S}'(\mathbb{T}^d; \mathbf{C}^m)$  such that  $\hat{u} \in \mathcal{B}_0^*(\mathbf{Z}^d)$ .

If  $\mathcal{A}$  is a  $n \times p$  matrix with real coefficients we denote  $\text{Im} \mathcal{A} := \mathcal{A}(\mathbf{R}^p) \subset \mathbf{R}^n$ ,  $\text{Im}_{\mathbf{C}} \mathcal{A} := \mathcal{A}(\mathbf{C}^p) \subset \mathbf{C}^n$ ,  $r = \text{rank}(\mathcal{A}) := \dim(\text{Im} \mathcal{A})$ .

## 2. THE DISCRETE-MAXWELL OPERATOR

**2.1. Description by Fourier series.** Let  $\mathbf{Z}^3 = \{n = (n_1, n_2, n_3) \mid n_j \in \mathbf{Z}\}$ ,  $\mathbb{T}^3 \approx (\mathbf{R}/(2\pi\mathbf{Z}))^3$  and  $U$  be the unitary transform between  $L^2(\mathbb{T}^3)$  and  $l^2(\mathbf{Z}^3)$ :

$$(Uf)(n) := \hat{f}(n) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{T}^3} e^{inx} f(x) dx, \quad n \in \mathbf{Z}^3,$$

so that any  $f \in L^2(\mathbb{T}^3)$  can be written

$$f(x) = (U^* \hat{f})(x) := (2\pi)^{-\frac{3}{2}} \sum_{n \in \mathbf{Z}^3} e^{-inx} \hat{f}(n), \quad x \in \mathbb{T}^3.$$

The isotropic discrete Maxwell operator is a bounded operator  $H_0$  on  $\mathcal{H} = (L^2(\mathbb{T}^3; \mathbf{C}))^6$  defined by

$$\hat{H}_0 = UH_0U^*,$$

where  $H_0$  is the operator of multiplication by the real symmetric  $6 \times 6$  matrix:

$$(2.1) \quad H_0(x) := \begin{pmatrix} 0_{3 \times 3} & \tilde{M}(y) \\ -\tilde{M}(y) & 0_{3 \times 3} \end{pmatrix} \in \mathbf{R}^6, \quad x \in \mathbb{T}^3,$$

where  $y = \sin x$  (see (1.8)) and  $\tilde{M}(y)$  is the real anti-symmetric  $3 \times 3$  matrix:

$$\tilde{M}(y) := \begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix}, \quad y \in \mathbf{R}^3.$$

Then  $H_0$  is a bounded self-adjoint operator on  $\mathcal{H} = L^2(\mathbb{T}^3, dx; \mathbf{C}^6)$  with the inner product

$$(u, v) := \int_{\mathbb{T}^3} \langle u(x), v(x) \rangle_{\mathbf{C}^6} dx = \int_{\mathbb{T}^3} \sum_{j=1}^6 u_j(x) \overline{v_j(x)} dx.$$

The anisotropic discrete-Maxwell operator is defined by

$$\hat{H}^D := \hat{D} \hat{H}_0,$$

with

$$\hat{D} := \begin{pmatrix} \varepsilon & 0_{3 \times 3} \\ 0_{3 \times 3} & \boldsymbol{\mu} \end{pmatrix}.$$

We set  $H^D := U^* \hat{H}^D U$  so, since  $\hat{D}$  is constant,

$$H^D = U^*(\hat{D} \hat{H}_0)U = \hat{D} U^* \hat{H}_0 U = \hat{D} H_0.$$

The relation

$$(\hat{D}^{-1} H^D u, v)_{\mathcal{H}} = (H_0 u, v)_{\mathcal{H}}$$

shows that the operator  $H^D$  is bounded and self-adjoint on the space  $\mathcal{H}^D = \mathcal{H}$  equipped with the hilbertian product

$$(u, v)_{\mathcal{H}^D} := (\hat{D}^{-1} u, v) = \int_{\mathbb{T}^3} \langle \hat{D}^{-1} u(x), v(x) \rangle_{\mathbf{C}^6} dx.$$

We write

$$H^D = \int_{\mathbb{T}^3}^{\oplus} H^D(x) dx,$$

where  $H^D(x)$  is self-adjoint on  $\mathbf{C}^6$  equipped with the inner product

$$\langle u(x), v(x) \rangle_{\mathbf{C}^6, D} := \langle \hat{D}^{-1} u(x), v(x) \rangle_{\mathbf{C}^6}.$$

**2.2. Spectrum of  $H^D(x)$ .** Since  $H^D(x)$  depends only on the variable  $y = \sin x$  we write  $H^D(x) = h^D(y)$ . In this part we consider a more general case  $y \in \mathbf{R}^3$ , so  $z \in [0, +\infty)^3$  where  $z_j = y_j^2$ . We recall some results in [5] using the notations  $\boldsymbol{\beta}$ ,  $\boldsymbol{\alpha}$  in (1.4). Since  $\boldsymbol{\beta} \cdot \boldsymbol{\varepsilon} = 0$  and  $\varepsilon_i > 0$  for all  $1 \leq i \leq 3$  then there exists  $1 \leq j \leq 3$  such that  $\beta_j \beta_i \leq 0$  and  $\beta_k \beta_i \geq 0$  for  $i, k \neq j$ . If two of the  $\beta_j$ 's vanish then  $\boldsymbol{\beta}$  vanishes. Moreover  $\boldsymbol{\beta}$  is replaced by  $-\boldsymbol{\beta}$  if  $\boldsymbol{\varepsilon}$  and  $\boldsymbol{\mu}$  are exchanged, which involves the same analysis. Hence if  $\boldsymbol{\beta} \neq 0$  then we can assume without loss of generality that

**(A0):**  $\beta_1 \geq \beta_2 > 0 > \beta_3$  or  $\beta_1 > \beta_2 = 0 > \beta_3$ .

Let us observe that  $\tau^\pm(z)$  defined by (1.7) satisfy

$$\tau^+(z) \geq \tau^-(z) > 0 \quad z \in [0, +\infty)^3 \setminus \{0_{\mathbf{R}^3}\}.$$

**Proposition 2.1.** (Spectrum of  $h^D(y)$ .) If  $y = 0_{\mathbf{R}^3}$  then  $h^D(y) = 0_{6 \times 6}$ .

Let  $y \in \mathbf{R}^3 \setminus \{0_{\mathbf{R}^3}\}$ . Then  $0 \in \sigma(h^D(y))$  has multiplicity two with eigenvectors  $(y_1, y_2, y_3, 0, 0, 0) = y \otimes 0_{\mathbf{C}^3}$  and  $(0, 0, 0, y_1, y_2, y_3) = 0_{\mathbf{C}^3} \otimes y$ .

1) Assume  $\boldsymbol{\beta} = 0$  and  $y \in \mathbf{R}^3 \setminus \{0_{\mathbf{R}^3}\}$ . Then,  $K_0 \equiv 0$  and all the eigenvalues have multiplicity two. Moreover, the nonzero eigenvalues of  $h^D(y)$  are

$$\pm \sqrt{\Psi_0(z)} = \pm \sqrt{\varepsilon_2 \mu_3 z_1 + \varepsilon_3 \mu_1 z_2 + \varepsilon_1 \mu_2 z_3},$$

where  $z_j := y_j^2$  hence  $z \in [0, \infty)^3 \setminus \{0_{\mathbf{R}^3}\}$ .

2) Assume  $\beta \neq 0$  (so (A0) holds) and  $y \in \mathbf{R}^3 \setminus \{0_{\mathbf{R}^3}\}$ . Then the nonzero eigenvalues of  $h^D(y)$  are

- $\sqrt{\tau^+(z)}$  and  $-\sqrt{\tau^+(z)}$ , simple iff  $K_0(z) \neq 0$ ,
- $\sqrt{\tau^-(z)}$  and  $-\sqrt{\tau^-(z)}$ , simple iff  $K_0(z) \neq 0$ .
- $\sqrt{\tau^+(z)} = \sqrt{\tau^-(z)}$  and  $-\sqrt{\tau^+(z)} = -\sqrt{\tau^-(z)}$ , double iff  $K_0(z) = 0$ .

If, in addition,  $\beta_2 = 0$ , then,  $\tau^+$  and  $\tau^-$  are linear with respect to  $z$ :

$$\begin{aligned}\tau^+(z) &= \varepsilon_2 \mu_3 z_1 + \varepsilon_3 \mu_1 z_2 + \varepsilon_2 \mu_1 z_3, \\ \tau^-(z) &= \varepsilon_3 \mu_2 z_1 + \varepsilon_3 \mu_1 z_2 + \varepsilon_1 \mu_2 z_3.\end{aligned}$$

By Proposition 2.1, we observe that: If  $(z_1, z_3) \neq 0_{\mathbf{R}^2}$ , then the nonzero eigenvalues of  $h^D(y)$  are  $\pm\sqrt{\tau^+(z)}$ , simple. If  $(z_1, z_3) = 0_{\mathbf{R}^2}$  and  $z_2 \neq 0$ , then the nonzero eigenvalues of  $h^D(y)$  are

$$\pm\sqrt{\tau^+(z)} = \pm\sqrt{\tau^-(z)} = \pm\sqrt{\alpha_2}|y_2| = \pm\sqrt{\varepsilon_3 \mu_1}|y_2|,$$

2.3. **Spectrum of  $H^D$ .** We set

$$\lambda_{\pm} := \max_{[0,1]^3} \sqrt{\tau^{\pm}(z)}.$$

The following result is proved in [5].

**Proposition 2.2.** (1) The spectrum of  $H^D$  is  $\sigma(H^D) = [-\lambda_+, \lambda_+]$ .

(2) The pure point spectrum is  $\sigma_{pp}(\hat{H}^D) = \{0\}$  and the eigenvalue 0 of  $\hat{H}^D$  has infinite multiplicity.

(3) The singular continuous spectrum of  $\hat{H}^D$  is empty.

**Remark 2.1.** We have, in addition,

(1)  $\lambda_+ = \sqrt{\tau^+(1, 1, 1)}$ ,  $\lambda_- = \max\{\sqrt{\tau^-(1, 1, 1)}, \sqrt{\tau^-(1, 1, 0)}\}$ .

(2) If  $\beta \neq 0$  then  $\lambda_+ > \lambda_-$ .

### 3. A GENERAL TOOL FOR THE PROOF OF RELLICH'S PROPERTY

We apply to (1.3) the following general result.

**Theorem 3.1.** Let  $m \geq 1$ ,  $Q$  a scalar trigonometric polynomial of  $x \in \mathbb{T}^d$ ,  $d \geq 2$  satisfying  $\overline{Q(x)} = Q(x)$ ,  $x \in \mathbb{T}^d$ . Consider the complex  $d$ -dimensional torus  $\mathbb{T}_{\mathbf{C}}^d$ , extend analytically  $Q$  to  $\mathbb{T}_{\mathbf{C}}^d$ , and set

$$\begin{aligned}M^{\mathbf{C}}(Q) &:= \{x \in \mathbb{T}_{\mathbf{C}}^d \mid Q(x) = 0\}, \\ M_{sgn}^{\mathbf{C}}(Q) &:= \{x \in \mathbb{T}_{\mathbf{C}}^d \mid Q(x) = 0, \nabla Q(x) = 0\}, \\ M_{reg}^{\mathbf{C}}(Q) &:= M^{\mathbf{C}}(Q) \setminus M_{sgn}^{\mathbf{C}}(Q).\end{aligned}$$

Assume

(A-1-1)( $Q$ ):  $M_{sgn}^{\mathbf{C}}(Q) \subset \mathbb{T}_{\mathbf{C}}^d \approx (\mathbb{T} \times \mathbf{R})^d$  has Hausdorff  $(2d - 1)$ -measure zero,

(A-1-2)( $Q$ ):  $M_{sgn}^{\mathbf{C}}(Q) \cap \mathbb{T}^d$  is discrete,

(A-2)( $Q$ ): Each connected component of  $M_{reg}^{\mathbf{C}}(Q)$  intersects  $\mathbb{T}^d$  and the intersection is a  $(d - 1)$ -dimensional real analytic submanifold of  $\mathbb{T}^d$ .



Let  $u \in \mathcal{B}_0^*(\mathbb{T}^d)$  be a solution of the equation

$$(3.1) \quad Qu = g \quad \text{on} \quad \mathbb{T}^d,$$

where  $g$  is a trigonometric polynomial of  $x \in \mathbb{T}^d$  with values in  $\mathbf{C}^m$ . Then  $u$  is a trigonometric polynomial.

*Proof.* Lemma 5.2 of [2] shows that  $u \in C^\infty(\mathbb{T}^d \setminus M_{sgn}^{\mathbf{C}}(Q))$  and  $g = 0$  on  $M_{reg}^{\mathbf{C}}(Q) \cap \mathbb{T}^d$ . Lemma 5.3 of [2] shows that  $g = 0$  on  $M_{reg}^{\mathbf{C}}(Q)$ . Let us define the meromorphic function  $v(x) := g(x)/p(x)$  for  $x \in \mathbb{T}_{\mathbf{C}}^d \setminus M^{\mathbf{C}}(Q)$ . The proof of [2, Lemma 5.4] shows that  $v$  extends continuously to  $\mathbb{T}_{\mathbf{C}}^d$ . In fact this proof is based on Assumption (A-1)'( $Q$ ) (i.e.,  $M_{sgn}^{\mathbf{C}}(Q)$  is discrete) which is stronger than (A-1-1)-(A-1-2)( $Q$ ). However [2, Lemma 5.3], which does not use Assumption (A-1)'( $Q$ ), implies that  $v$  is analytic near  $M_{reg}^{\mathbf{C}}(Q)$  and the singularities of  $v$  are localized in  $M_{sgn}^{\mathbf{C}}(Q)$ . Thanks to (A-1-1), the closed set  $M_{sgn}^{\mathbf{C}}(Q)$  is negligible in the sense of Shiffman in [6] and [7], which means that  $v$  extends analytically to  $\mathbb{T}_{\mathbf{C}}^d$  (see also [3] and [8]).

Let us set  $u' = u - v|_{\mathbb{T}^d}$ . Thus,  $u'$  belongs to  $\mathcal{B}_0^*(\mathbb{T}^d)$  and satisfies  $\text{supp } u' \subset \mathbb{T}^d \cap M_{sgn}^{\mathbf{C}}(Q)$ . Therefore, thanks to Assumption (A-1-2),  $\text{supp } u'$  is discrete. Let us prove  $u' = 0$ . In fact, if  $u' \neq 0$ , then there exists  $x^* \in \text{supp } u'$ . Let  $\chi$  be a smooth cut-off function on  $\mathbb{T}^d$  such that  $\text{supp } (\chi u') \subset \{x^*\}$  and  $\chi(x^*) \neq 0$ . We then use the following equivalent characterization of  $\mathcal{B}_0^*(\mathbb{T}^d)$  (see [4]). Take a  $C^\infty$ -partition of unity  $\{\chi_l\}_{1 \leq l \leq L}$  on  $\mathbb{T}^d$  where the support of  $\chi_l$  is sufficiently small. Then

$$\mathcal{B}_0^*(\mathbb{T}^d; \mathbf{C}^m) := \{w \in \mathcal{S}'(\mathbb{T}^d; \mathbf{C}^m) \mid \chi_l w \in \mathcal{B}_0^*(\mathbf{R}^d; \mathbf{C}^m)\},$$

where

$$\mathcal{B}_0^*(\mathbf{R}^d; \mathbf{C}^m) := \{w \in \mathcal{S}'(\mathbf{R}^d; \mathbf{C}^m) \mid \lim_{R \rightarrow +\infty} \frac{1}{R} \int_{|\xi| < R} |w(\xi)|^2 d\xi = 0\}.$$

Let  $l$  be such that  $\chi_l(x^*) \neq 0$ . Thus  $u'' := \chi_l \chi u'$  belongs to  $\mathcal{B}_0^*(\mathbf{R}^d; \mathbf{C}^m)$  and is a finite linear combination of derivatives of Dirac distribution  $\delta_{x^*}$ :

$$u'' = \sum_{|\alpha| \leq N} b_\alpha \partial_\alpha \delta_{x^*},$$

where  $N \geq 0$  and  $\{b_\alpha\}_{|\alpha|=N} \neq \{0\}$ . Let  $\tilde{u}''_N(\xi)$  be the Fourier transform of  $u''_N := \sum_{|\alpha|=N} b_\alpha \partial^\alpha \delta_{x^*}$ . It satisfies

$$\tilde{u}''_N(\xi) := (2\pi)^{-d/2} \sum_{|\alpha|=N} b_\alpha \xi^\alpha e^{i\xi x^*}, \quad \xi \in \mathbf{R}^d.$$

Putting  $\xi = |\xi|\omega$  with  $\omega \in \mathbb{S}^{d-1}$ , we get

$$\begin{aligned} \frac{1}{R} \int_{|\xi| < R} |\tilde{u}''_N(\xi)|^2 d\xi &= (2\pi)^{-d/2} \frac{1}{R} \int_0^R \left( \int_{\mathbb{S}^{d-1}} \left| \sum_{|\alpha|=N} b_\alpha \omega^\alpha \right|^2 d\omega \right) t^{2N+d-1} dt \\ &= (2\pi)^{-d/2} R^{2N+d-2} \int_{\mathbb{S}^{d-1}} \left| \sum_{|\alpha|=N} b_\alpha \omega^\alpha \right|^2 d\omega \\ &= CR^{2N+d-2} \end{aligned}$$

with  $C > 0$ , since  $\{b_\alpha\}_{|\alpha|=N} \neq \{0\}$ . A similar calculation shows that

$$\frac{1}{R} \int_{|\xi| < R} |\tilde{u}''(\xi) - \tilde{u}''_N(\xi)|^2 d\xi = O(R^{2N+d-4}).$$

We thus have

$$\frac{1}{R} \int_{|\xi| < R} |\tilde{u}''(\xi)|^2 d\xi \approx R^{2N+d-2}.$$

It contradicts  $u'' \in \mathcal{B}_0^*(\mathbf{R}^d; \mathbf{C}^m)$ . Hence  $u' = 0$  and  $u = v|_{\mathbb{T}^d}$  is a trigonometric polynomial.  $\square$

**Remark 3.1.** 1) Theorem 3.1 is a slight improvement of results in [2] and [4], where Assumption (A-1-1)-(A-1-2) is replaced by: (A-1)'(Q):  $M_{sgn}^{\mathbf{C}}(Q)$  is discrete.

2) Theorem 1.1 for the second case follows from Conditions (A-1-1)-(A-1-2) by taking  $Q$  to be  $q(\cdot; \lambda)$  in Theorem 3.1 but not from Condition (A-1)'( $q(\cdot; \lambda)$ ) of [2] and [4], which is not fullfield.

#### 4. PROPERTIES OF THE COMPLEX FERMI VARIETY

**4.1. Properties in the complex variable  $z$ .** We consider  $\lambda \in \mathbf{R}^*$ . Thanks to (1.5) we write

$$(4.1) \quad p(z; \lambda)/\lambda^2 = Az \cdot z + \mathbf{b} \cdot z + c \quad z \in \mathbf{C}^3,$$

where we put

$$(4.2) \quad A := \begin{pmatrix} \gamma_1 & \varepsilon_3 \mu_3 \alpha_3 & \varepsilon_2 \mu_2 \alpha_2 \\ \varepsilon_3 \mu_3 \alpha_3 & \gamma_2 & \varepsilon_1 \mu_1 \alpha_1 \\ \varepsilon_2 \mu_2 \alpha_2 & \varepsilon_1 \mu_1 \alpha_1 & \gamma_3 \end{pmatrix}, \quad \mathbf{b} := -2\lambda^2 \boldsymbol{\alpha}, \quad c := \lambda^4,$$

and  $\gamma_i$  is defined by (1.4).

We state analytic properties of the variety defined by the polynomial  $p(\cdot; \lambda)$  in a slightly general form. We consider a polynomial of the form  $P(z) = \mathcal{A}z \cdot z + \mathbf{b} \cdot z + c$ ,  $z \in \mathbf{C}^d$ , where  $\mathcal{A}$  is a real symmetric  $d \times d$  matrix,  $\mathbf{b} \in \mathbf{R}^d$ ,  $c \in \mathbf{R}$ . We set the complex analytic variety  $M^{\mathbf{C}}(P) := \{z \in \mathbf{C}^d \mid P(z) = 0\}$ ,  $M_{sgn}^{\mathbf{C}}(P) := \{z \in M^{\mathbf{C}}(P) \mid \nabla P(z) = 0\}$  the singular part and  $M_{reg}^{\mathbf{C}}(P) := M^{\mathbf{C}}(P) \setminus M_{sgn}^{\mathbf{C}}(P)$  the regular part.

**Lemma 4.1.** (1) Assume  $\mathcal{A} = 0$ ,  $\mathbf{b} \neq 0$ . Then  $M_{sgn}^{\mathbf{C}}(P) = \emptyset$  and  $M_{reg}^{\mathbf{C}}(P) = M^{\mathbf{C}}(P)$  is a  $(d-1)$ -dimensional connected analytic submanifold of  $\mathbf{C}^d$ . In addition,  $M_{reg}^{\mathbf{C}}(P) \cap \mathbf{R}^d$  is a  $(d-1)$ -dimensional real analytic submanifold of  $\mathbf{R}^d$ .

(2) Assume  $\mathcal{A} \neq 0$ . Then:

(2-1) Assume  $\mathbf{b} \notin \text{Im} \mathcal{A}$ . Then,  $M_{sgn}^{\mathbf{C}}(P) = \emptyset$ ,  $M^{\mathbf{C}}(P) = M_{reg}^{\mathbf{C}}(P)$  is an analytic connected manifold of dimension  $d-1$  and the intersection  $M_{reg}^{\mathbf{C}}(P) \cap \mathbf{R}^d$  is a  $(d-1)$ -dimensional real analytic submanifold of  $\mathbf{R}^d$ .

(2-2) Assume  $\mathbf{b} \in \text{Im} \mathcal{A}$ . Then there exists a unique vector  $z^0 \in \text{Im}_{\mathbf{C}} \mathcal{A}$  such that  $\mathcal{A}z^0 = -\mathbf{b}/2$ . In addition,  $z^0 \in \mathbf{R}^d$ . Furthermore, setting  $c^* := \mathcal{A}z^0 \cdot z^0 = -\mathbf{b}z^0/2$  and  $c' := c - c^*$ , the following assertions (2-2-1) and (2-2-2) hold.

(2-2-1) Assume  $c \neq c^*$ . Then  $M_{sgn}^{\mathbf{C}}(P)$  is void and  $M^{\mathbf{C}}(P) = M_{reg}^{\mathbf{C}}(P)$ . Moreover,

- (1) Assume  $0 < s = r$  and  $c > c^*$ , or  $0 = s < r$  and  $c < c^*$ . Then,  $M^{\mathbf{C}}(P)$  does not intersect  $\mathbf{R}^d$ .
- (2) Assume  $0 \leq s < r$  and  $c > c^*$ , or  $0 < s \leq r$  and  $c < c^*$ .
- Assume  $r \geq 2$ . Then  $M^{\mathbf{C}}(P)$  is a  $(d-1)$ -dimensional connected complex analytic submanifold of  $\mathbf{C}^d$  and  $M^{\mathbf{C}}(P) \cap \mathbf{R}^d$  is a  $(d-1)$ -dimensional real analytic submanifold of  $\mathbf{R}^d$ .
  - Assume  $r = 1$ . Assume  $s = 0$  and  $c > c^*$  or  $s = 1$  and  $c < c^*$ . Then  $M^{\mathbf{C}}(P)$  has two connected components which are  $(d-1)$ -dimensional connected complex analytic submanifolds of  $\mathbf{C}^d$  whose intersection with  $\mathbf{R}^d$  are  $(d-1)$ -dimensional real analytic submanifolds of  $\mathbf{R}^d$ .
- (2-2-2) Assume  $c = c^*$ . Then,  $M_{sgn}^{\mathbf{C}}(P)$  is a  $(d-r)$ -dimensional complex analytic submanifold of  $\mathbf{C}^d$  and the intersection  $M_{sgn}^{\mathbf{C}}(P) \cap \mathbf{R}^d$  is a  $(d-r)$ -dimensional real analytic submanifold of  $\mathbf{R}^d$ . Moreover,
- Assume  $r = d$ . Then  $M_{sgn}^{\mathbf{C}}(P) = \{0_{\mathbf{C}^d}\}$  and  $M_{reg}^{\mathbf{C}}(P)$  is a connected  $(d-1)$ -dimensional submanifold of  $\mathbf{C}^d$ .
- (1) Assume  $s = 0$  (i.e.  $\mathcal{A} < 0$ ) or  $s = r$  (i.e.  $\mathcal{A} > 0$ ). Then  $M_{reg}^{\mathbf{C}}(P) \cap \mathbf{R}^d = \emptyset$ .
- (2) Assume  $1 \leq s \leq d-1$  (and  $r = d$ ). Then  $M_{reg}^{\mathbf{C}}(P) \cap \mathbf{R}^d$  is a  $(d-1)$ -dimensional real analytic submanifold of  $\mathbf{R}^d$ .
- Assume  $r = d-1$ .
- (1) Assume  $s = 0$  (i.e.  $\mathcal{A} \leq 0$ ) or  $s = r$  (i.e.  $\mathcal{A} \geq 0$ ). Then  $M_{reg}^{\mathbf{C}}(P) \cap \mathbf{R}^d = \emptyset$ .
- (2) Assume  $1 \leq s \leq d-1$  (and  $r = d-1$ ). If  $d > 3$  then  $M_{reg}^{\mathbf{C}}(P)$  is a connected  $(d-1)$ -dimensional submanifold of  $\mathbf{C}^d$  and  $M_{reg}^{\mathbf{C}}(P) \cap \mathbf{R}^d$  is a  $(d-1)$ -dimensional real analytic submanifold of  $\mathbf{R}^d$ . If  $d = 3$  then  $M_{reg}^{\mathbf{C}}(P)$  admits two connected components:  $M_{reg}^{\mathbf{C}}(P) = M^{\mathbf{C}}(P^+) \cup M^{\mathbf{C}}(P^-)$ , where each  $M^{\mathbf{C}}(P^\pm)$  is a two-dimensional submanifold of  $\mathbf{C}^3$  and each  $M^{\mathbf{C}}(P^\pm) \cap \mathbf{R}^3$  is a two-dimensional real analytic submanifold of  $\mathbf{R}^3$ .

*Proof.* (1) (Case  $\mathcal{A} = 0$ ,  $\mathbf{b} \neq 0$ .) This is obvious.

(2) (Case  $\mathcal{A} \neq 0$ .) We observe that if it is not void, then  $M_{reg}^{\mathbf{C}}(P)$  is an analytic manifold of dimension  $d-1$ . If  $M_{sgn}^{\mathbf{C}}(P) \neq \emptyset$ , then  $M_{sgn}^{\mathbf{C}}(P)$  is a smooth manifold, since its singular part,

$$M_{sgn,sgn}^{\mathbf{C}}(P) = \{z \in \mathbf{C}^d \mid P(z) = 0, \nabla P(z) = 0, D^2P(z) = 0\}$$

is necessarily void (we have  $z \in M_{sgn,sgn}^{\mathbf{C}}(P)$  iff  $\mathcal{A} = 0$ ,  $\mathbf{b} = 0$ ,  $c = 0$ , which is forbidden). Moreover,  $M_{sgn}^{\mathbf{C}}(P)$  is characterized by the equations  $\mathcal{A}z = -\mathbf{b}/2$ ,  $\mathbf{b} \cdot z/2 + c = 0$ . Let  $(s, r-s)$  be the signature of the quadratic form  $z \mapsto \mathcal{A}z \cdot z$ . Gauss's reduction provides a *real* linear change of coordinates on  $z$ , so we can assume that  $\mathcal{A}$  is diagonal and  $\mathcal{A}z \cdot z = \sum_{j=1}^s z_j^2 - \sum_{j=s+1}^r z_j^2$ . The equation of  $M^{\mathbf{C}}(P)$  is then

$$(4.3) \quad M^{\mathbf{C}}(P) : \sum_{j=1}^d a_j z_j^2 + \sum_{j=1}^d b_j z_j + c = 0,$$

with  $a_j = 1$  if  $j \leq s$ ,  $a_j = -1$  if  $s < j \leq r$ ,  $a_j = 0$  if  $j > r$ . The equations of  $M_{sgn}^{\mathbf{C}}$  are:

$$M_{sgn}^{\mathbf{C}}(P) : \sum_{j=1}^d a_j z_j^2 + \sum_{j=1}^d b_j z_j + c = 0, \quad \text{and} \quad 2a_j z_j + b_j = 0 \quad \forall j.$$

(2-1) (Case  $\mathbf{b} \notin \text{Im} \mathcal{A}$ .) Hence,  $r < d$  and there exists  $j_0 > r$  such that  $b_{j_0} \neq 0$ . Thus  $M_{sgn}^{\mathbf{C}}(P) = \emptyset$  and  $M^{\mathbf{C}}(P) = M_{reg}^{\mathbf{C}}(P)$  is an analytic manifold of dimension  $d - 1$ . Moreover the equation of  $M^{\mathbf{C}}(P)$  becomes

$$z_{j_0} = \frac{-1}{b_{j_0}} \left( \sum_{j=1}^r z_j^2 + \sum_{j=r+1, j \neq j_0}^d b_j z_j + c \right),$$

which shows that  $M^{\mathbf{C}}(P)$  is connected and that  $M^{\mathbf{C}}(P) \cap \mathbf{R}^d$  has dimension  $d - 1$ . This proves the assertion.

(2-2) (Case  $\mathbf{b} \in \text{Im} \mathcal{A}$ .) We denote  $\text{Im}_{\mathbf{C}} \mathcal{A} = \{\mathcal{A}z \mid z \in \mathbf{C}^d\}$ ,  $\ker_{\mathbf{C}} \mathcal{A} = \{z \in \mathbf{C}^d \mid \mathcal{A}z = 0\}$  and  $\ker \mathcal{A} = \ker_{\mathbf{C}} \mathcal{A} \cap \mathbf{R}^d$ . We recall that, since  $\mathcal{A}$  is real symmetric, we then have  $\text{Im} \mathcal{A} := \{\mathcal{A}z \mid z \in \mathbf{R}^d\} = (\ker \mathcal{A})^\perp$ ,  $\text{Im}_{\mathbf{C}} \mathcal{A} = (\ker_{\mathbf{C}} \mathcal{A})^{\perp \mathbf{C}}$ , and

$$\mathbf{R}^d = \text{Im} \mathcal{A} \oplus \ker \mathcal{A}, \quad \mathbf{C}^d = \text{Im}_{\mathbf{C}} \mathcal{A} \oplus \ker_{\mathbf{C}} \mathcal{A}.$$

Hence,  $\mathcal{A}$  is an automorphism on  $\text{Im}_{\mathbf{C}} \mathcal{A}$ . It implies the existence and uniqueness of a vector  $z^0 \in \text{Im}_{\mathbf{C}} \mathcal{A}$  such that  $\mathcal{A}z^0 = -\mathbf{b}/2$ . Since  $\mathbf{b}$  is a real vector so is  $z^0$ . Set  $c^* := \mathcal{A}z^0 \cdot z^0 = -\mathbf{b}z^0/2$ ,  $c' := c - c^*$ , and consider the translation  $z' := z - z^0$ . Then (4.3) becomes

$$M^{\mathbf{C}}(P) : \sum_{j=1}^s z_j'^2 - \sum_{j=s+1}^r z_j'^2 + c' = 0,$$

and the equations of  $M_{sgn}^{\mathbf{C}}(P)$  become

$$M_{sgn}^{\mathbf{C}}(P) : \sum_{j=1}^s z_j'^2 - \sum_{j=s+1}^r z_j'^2 + c' = 0 \quad \text{and} \quad z_j' = 0 \quad \text{for} \quad 1 \leq j \leq r,$$

i.e.,

$$M_{sgn}^{\mathbf{C}}(P) : \quad c' = 0 \quad \text{and} \quad z_j' = 0 \quad \text{for} \quad 1 \leq j \leq r.$$

(2-2-1) (Case  $c \neq c^*$ , i.e.,  $c' \neq 0$ .) We assume  $c > c^*$ , the other case being similar. Then, obviously,  $M_{sgn}^{\mathbf{C}}(P)$  is void and  $M^{\mathbf{C}}(P) = M_{reg}^{\mathbf{C}}(P)$ .

(1) Assume  $0 < s = r$ , i.e.,  $\mathcal{A} \geq 0$ . Obviously,  $M^{\mathbf{C}}(P)$  does not intersect  $\mathbf{R}^d$ .

(2) Assume  $0 \leq s < r$ .

- Assume  $r \geq 2$ . Since  $M^{\mathbf{C}}(P)$  is an irreducible and regular algebraic variety then it is  $(d - 1)$ -dimensional and connected. In addition  $M^{\mathbf{C}}(P)$  intersects  $\mathbf{R}^d$ ; it contains the points  $z \in \mathbf{R}^d$  such that  $z_r' = \pm \sqrt{\sum_{j=1}^s z_j'^2 - \sum_{j=s+1}^{r-1} z_j'^2 + c'}$  and  $\sqrt{\sum_{j=1}^{r-1} |z_j'|^2} < c'$ . Hence  $M^{\mathbf{C}}(P)$  is a  $(d - 1)$ -dimensional real analytic submanifold of  $\mathbf{R}^d$ .

- Assume  $s = 0$ ,  $r = 1$ . Then

$$M^{\mathbf{C}}(P) = M_+^{\mathbf{C}} \cup M_-^{\mathbf{C}}, \quad M_{\pm}^{\mathbf{C}} := \{z \in \mathbf{C}^d \mid z_1' = \pm \sqrt{c'}\}.$$

Clearly,  $M_+^{\mathbf{C}}$  and  $M_-^{\mathbf{C}}$  are the connected components of  $M^{\mathbf{C}}(P)$  and their intersection with  $\mathbf{R}^d$  is  $(d-1)$ -dimensional.

(2-2-2) (Case  $c = c^*$ , i.e.,  $c' = 0$ .) Then,  $M_{sgn}^{\mathbf{C}}(P) = \{z \in \mathbf{C}^d \mid z'_1 = \dots = z'_r = 0\}$  and  $M_{sgn}^{\mathbf{C}}(P) \cap \mathbf{R}^d = \{z \in \mathbf{R}^d \mid z'_1 = \dots = z'_r = 0\}$  are affine spaces with same dimension  $d-r$ .

- Assume  $r = d$ . Then  $M_{sgn}^{\mathbf{C}}(P) = \{0_{\mathbf{C}^d}\}$ . Since  $d \geq 3$  then  $M_{reg}^{\mathbf{C}}(P)$  is a irreducible and regular algebraic variety and so it is a  $(d-1)$ -dimensional connected submanifold of  $\mathbf{C}^d$ . Clearly,  $M_{reg}^{\mathbf{C}}(P) \cap \mathbf{R}^d$  is a regular subset of  $\mathbf{R}^d$ , but may be void.

- (1) Assume  $s = 0$  (i.e.,  $\mathcal{A} < 0$ ) or  $s = r$  (i.e.,  $\mathcal{A} > 0$ ). Then  $M_{reg}^{\mathbf{C}}(P) \cap \mathbf{R}^d = \emptyset$ .
- (2) Assume  $1 \leq s \leq d-1$  (and  $r = d$ ). Then  $M_{reg}^{\mathbf{C}}(P)$  intersects  $\mathbf{R}^d$  since it contains the points  $z \in \mathbf{R}^d$  such that  $z'_d = \pm \sqrt{\sum_{j=1}^s z_j'^2 - \sum_{j=s+1}^{d-1} z_j'^2}$  and  $|z'_1| > \sqrt{\sum_{j=s+1}^{d-1} z_j'^2}$ . Hence  $M_{reg}^{\mathbf{C}}(P) \cap \mathbf{R}^d$  is a  $(d-1)$ -dimensional real analytic submanifold of  $\mathbf{R}^d$ .

- Assume  $r = d-1$ .

- (1) Case  $s = 0$  or  $s = r$  (i.e.,  $\mathcal{A} \geq 0$ ): the result is obvious.
- (2) Assume  $1 \leq s \leq d-1$  (and  $r = d-1$ ).
  - Assume  $d > 3$ . The proof is similar to the proof of case (ii)(B)( $r = d$ ).
  - Assume  $d = 3$  so  $r = 2$ ,  $s = 1$ . We have  $M_{reg}^{\mathbf{C}}(P) = M_+^{\mathbf{C}}(P)' \cup M_-^{\mathbf{C}}(P)'$  with  $M^{\mathbf{C}}(P^\pm)' := M^{\mathbf{C}}(P^\pm) \setminus M_{sgn}^{\mathbf{C}}(P)$ ,  $M^{\mathbf{C}}(P^\pm) := \{z \in \mathbf{C}^3 \mid z'_1 = \pm z'_2\}$ . Hence  $M^{\mathbf{C}}(P^\pm)' = \{z \in \mathbf{C}^3 \mid z'_1 = \pm z'_2 \text{ and } z'_2 \neq 0\}$ . Clearly,  $M^{\mathbf{C}}(P^+)'$  and  $M^{\mathbf{C}}(P^-)'$  are the connected components of the complex variety  $M^{\mathbf{C}}(P)$  and each intersection  $M^{\mathbf{C}}(P^\pm)'$  with  $\mathbf{R}^3$  is a two-dimensional open set of the plane  $\{z \in \mathbf{R}^3 \mid z'_1 = \pm z'_2\}$ .

We have thus completed the proof of Lemma 4.1.  $\square$

We put

$$(4.4) \quad \mathbf{g} := (\varepsilon_1 \mu_1 \beta_1, \varepsilon_2 \mu_2 \beta_2, \varepsilon_3 \mu_3 \beta_3).$$

**Lemma 4.2.** *We have the following relation:*

$$(4.5) \quad \boldsymbol{\alpha} \cdot \mathbf{g} = \sum_{i=1}^3 \varepsilon_i \mu_i \alpha_i \beta_i = \frac{1}{2} \beta_1 \beta_2 \beta_3.$$

*Proof.* We easily observe that the following relations and their cyclic changes hold:

$$(4.6) \quad \begin{aligned} \alpha_1^2 - \gamma_1 &= \frac{1}{4} \beta_1^2, \\ \varepsilon_1 \mu_1 \alpha_1 - \alpha_2 \alpha_3 &= \frac{1}{4} \beta_2 \beta_3, \\ \alpha_3 \beta_2 + \alpha_2 \beta_3 &= -\varepsilon_1 \mu_1 \beta_1. \end{aligned}$$

Then, we obtain

$$\begin{aligned}
\boldsymbol{\alpha} \cdot \mathbf{g} &= \sum_{i=1}^3 \varepsilon_i \mu_i \alpha_i \beta_i = -(\alpha_3 \beta_2 + \alpha_2 \beta_3) \alpha_1 + \varepsilon_2 \mu_2 \alpha_2 \beta_2 + \varepsilon_3 \mu_3 \alpha_3 \beta_3 \\
&= (\varepsilon_2 \mu_2 \alpha_2 - \alpha_1 \alpha_3) \beta_2 + (\varepsilon_3 \mu_3 \alpha_3 - \alpha_1 \alpha_2) \beta_3 = \frac{1}{4} \beta_1 \beta_3 \beta_2 + \frac{1}{4} \beta_1 \beta_2 \beta_3 \\
&= \frac{1}{2} \beta_1 \beta_2 \beta_3.
\end{aligned}$$

□

**Lemma 4.3.** *Let  $A$  be defined by (4.2) and  $\mathbf{g}$  by (4.4).*

(1) *Let  $\{a_1, a_2, a_3\}$  be the spectrum of  $A$ . Then, we have*

(4.7)

$$\det A = a_1 a_2 a_3 = 0, \quad \operatorname{tr} A = a_1 + a_2 + a_3 = \gamma_1 + \gamma_2 + \gamma_3 > 0, \quad \prod_{i \neq j} a_i a_j = -\mathbf{g}^2/4 \leq 0.$$

(2) *We have  $\boldsymbol{\alpha} \in \operatorname{Im} A$  iff  $\beta_1 \beta_2 \beta_3 = 0$ .*

(3) *Assume  $\boldsymbol{\beta} = 0$ . Then  $A = \boldsymbol{\alpha} \boldsymbol{\alpha}^T$ ,  $\operatorname{rank}(A) = 1$ ,  $\operatorname{Im} A = \operatorname{span}(\boldsymbol{\alpha})$  and  $\ker A$  is the complex plane orthogonal to  $\boldsymbol{\alpha}$ .*

(4) *Assume  $\boldsymbol{\beta} \neq 0$ . Then  $\operatorname{rank}(A) = 2$  and  $\ker A = \operatorname{span}(\mathbf{g})$ . Moreover,  $A$  has one positive eigenvalue and one negative eigenvalue.*

(5) *Assume  $\beta_1 \beta_2 \beta_3 = 0$  (and  $\boldsymbol{\beta} \neq 0$ ), so we assume (A0) with  $\beta_1 > 0 = \beta_2 > \beta_3$ . Then there exists a unique vector  $z^* \in \operatorname{Im} A$  such that  $A z^* = \boldsymbol{\alpha}$ . We have  $z^* = (0, 1/\sqrt{\gamma_2}, 0)$  and*

$$(4.8) \quad \boldsymbol{\alpha} \cdot z^* = 1.$$

*Proof.* (1) A simple computation shows that  $A \cdot \mathbf{g} = 0$  and, if  $\boldsymbol{\beta} = 0$ ,  $A \cdot (1, 1, 1) = 0$ . Hence  $\det A = 0$ . Moreover, we have

$$\begin{aligned}
\prod_{i \neq j} a_i a_j &= \gamma_2 \gamma_3 - \varepsilon_1^2 \mu_1^2 \alpha_1^2 + \text{c.p.} = \varepsilon_1^2 \mu_1^2 (\varepsilon_2 \varepsilon_3 \mu_2 \mu_3 - \alpha_1^2) + \text{c.p.} \\
&= \varepsilon_1^2 \mu_1^2 (\varepsilon_2 \varepsilon_3 \mu_2 \mu_3 - (\varepsilon_2 \mu_3 + \varepsilon_3 \mu_2)^2/4) + \text{c.p.} \\
&= -\varepsilon_1^2 \mu_1^2 \beta_1^2/4 + \text{c.p.} = -\mathbf{g}^2/4 \leq 0.
\end{aligned}$$

(3) From (1.6), (1.5) and (4.1) we get  $A = \boldsymbol{\alpha} \boldsymbol{\alpha}^T$ .

(2) Assume  $\boldsymbol{\beta} = 0$ . Since  $A = \boldsymbol{\alpha} \boldsymbol{\alpha}^T$  (see (3)), then  $\boldsymbol{\alpha} \in \operatorname{Im} A$ .

Assume  $\boldsymbol{\beta} \neq 0$ . Since  $A$  is real symmetric then  $\operatorname{Im} A = \ker A^T = \mathbf{g}^\perp$ , so, thanks to (4.5),  $\boldsymbol{\alpha} \in \operatorname{Im} A \iff \boldsymbol{\alpha} \cdot \mathbf{g} = 0 \iff \prod_{j=1}^3 \beta_j = 0$ .

(4) (Case  $\boldsymbol{\beta} \neq 0$ .) Thanks to (4.7) and since  $\mathbf{g} \neq 0$  then  $\operatorname{rank}(A) = 2$  and  $\ker A = \operatorname{span}(\mathbf{g})$ , and (2) follows from (4.7).

(5) The proof of Lemma 4.1 (see case (2-2)) with  $(\mathcal{A}, \mathbf{b}, d, r, s) = (A, -2\boldsymbol{\alpha}, 3, 2, 1)$  shows the existence and uniqueness of such a  $z^*$ . Put  $z = (0, 1/\alpha_2, 0)$  and observe that  $\alpha_2 = \sqrt{\gamma_2}$ . Since  $\mathbf{g} = (\varepsilon_1 \mu_1 \beta_1, 0, \varepsilon_3 \mu_3 \beta_3)$ , then  $z \perp \mathbf{g}$  and so  $z \in \operatorname{Im} A$ . In addition, we have  $(Az)^T = (\varepsilon_3 \mu_3 \alpha_3/\alpha_2, \alpha_2, \varepsilon_1 \mu_1 \alpha_1/\alpha_2)$ . Thanks to (4.6), we obtain  $Az = \boldsymbol{\alpha}$ . This shows that  $z^* = z$ . Finally, (4.8) is obvious and (5) is proved. □

Let  $\lambda \in \mathbf{R}^*$ . We use the notations of Lemma 4.1 and apply Lemma 4.1 to  $P := \frac{1}{\lambda^2}p(\cdot; \lambda)$ , so  $(\mathcal{A}, \mathbf{b}, c, d) = (A, -2\lambda^2\boldsymbol{\alpha}, \lambda^4, 3)$ . In view of Lemmas 4.1 and 4.3 we obtain

**Lemma 4.4.** (1) Assume  $\boldsymbol{\beta} = 0$ . Then  $b \in \text{Im}A$ ,  $z^0 = \frac{\lambda^2}{|\boldsymbol{\alpha}|^2}\boldsymbol{\alpha}$ ,  $c^* = c$ , and  $M_{\text{sgn}}^{\mathbf{C}}(P)$  is a two-dimensional complex submanifold of  $\mathbf{C}^3$ .

(2) Assume  $\boldsymbol{\beta} \neq 0$ .

(2-1) Assume  $\prod_{j=1}^3 \beta_j \neq 0$ . Then  $\mathbf{b} \notin \text{Im}A$ ,  $M_{\text{sgn}}^{\mathbf{C}}(P) = \emptyset$ ,  $M^{\mathbf{C}}(P) = M_{\text{reg}}^{\mathbf{C}}(P)$  is an analytic connected manifold of dimension two and its intersection with  $\mathbf{R}^3$  is a real analytic manifold of dimension two.

(2-2) Assume  $\prod_{j=1}^3 \beta_j = 0$ . Then  $\mathbf{b} \in \text{Im}A$ ,  $z^0 = \lambda^2 z^*$ ,  $c^* = c$ ,  $M_{\text{sgn}}^{\mathbf{C}}(P)$  is a straight line of  $\mathbf{C}^3$ , and the intersection  $M_{\text{sgn}}^{\mathbf{C}}(P) \cap \mathbf{R}^3$  is a straight line of  $\mathbf{R}^3$ . In addition,  $M_{\text{reg}}^{\mathbf{C}}(P)$  has two connected components,  $M^{\mathbf{C}}(P^+)'$  and  $M^{\mathbf{C}}(P^-)'$ , and each  $M^{\mathbf{C}}(P^\pm)' \cap \mathbf{R}^3$  is a two-dimensional real analytic submanifold of  $\mathbf{R}^3$ .

**Remark 4.1.** Let us supplement (5) of Lemma 4.3 and (2-2) of Lemma 4.4. We assume  $\beta_1 > 0 = \beta_2 > \beta_3$ . Then each  $\tau^\pm$  is linear and we have

$$A(z - z^0)(z - z^0) = \lambda^{-2}p(z; \lambda) = (\tau^+ - \lambda^2)(\tau^- - \lambda^2),$$

where  $z^0 = \lambda^2 z^*$  is the corresponding point of Lemmas 4.1 and 4.3. Set  $P = p(\cdot; \lambda)$  and the linear functions  $P^\pm := \tau^\pm - \lambda^2$ . Then each  $M_{\text{sgn}}^{\mathbf{C}}(P^\pm)$  is void and, since  $P = \lambda^2 P^+ P^-$ , we have

$$(4.9) \quad M_{\text{sgn}}^{\mathbf{C}}(P) = M^{\mathbf{C}}(P^+) \cap M^{\mathbf{C}}(P^-) = \{z \in \mathbf{C}^3 \mid K_0(z) = \Psi_0(z) - \lambda^2 = 0\}$$

is a complex analytic variety of dimension one. Then,

$$M_{\text{reg}}^{\mathbf{C}}(P) = M^{\mathbf{C}}(P^+) \cup M^{\mathbf{C}}(P^-),$$

where  $M^{\mathbf{C}}(P^\pm)' := M^{\mathbf{C}}(P^\pm) \setminus M_{\text{sgn}}^{\mathbf{C}}(P)$ , as in the proof of Lemma 4.1. Thanks to Lemma 4.1,  $M^{\mathbf{C}}(P^+)'$  and  $M^{\mathbf{C}}(P^-)'$  are the two connected component of  $M_{\text{reg}}^{\mathbf{C}}(P)$ ; they are two-dimensional submanifolds of  $\mathbf{C}^3$ .

Lemma 4.4 shows that we can't apply Theorem 3.1 directly to prove Rellich's properties if  $\boldsymbol{\beta} = 0$  since Assumptions (A-1-1)( $p(\cdot; \lambda)$ ) fails.

**Lemma 4.5.** Assume  $\boldsymbol{\beta} = 0$ . Set  $P = \Psi_0 - \lambda^2$ . Then  $M_{\text{reg}}^{\mathbf{C}}(P) = M^{\mathbf{C}}(P)$  and  $M^{\mathbf{C}}(P) \cap [0, 1]^3$  is a closed convex set. In addition, we have

(1) Let  $|\lambda| \in (0, \lambda_+)$ . Then  $M_{\text{reg}}^{\mathbf{C}}(P) \cap (0, 1)^3$  has dimension two.

(2) Let  $|\lambda| = \lambda_+$ . Then  $M_{\text{reg}}^{\mathbf{C}}(P) \cap [0, 1]^3 = \{(1, 1, 1)\}$ .

*Proof.* Obviously,  $M_{\text{sgn}}^{\mathbf{C}}(P) = \emptyset$  and  $M^{\mathbf{C}}(P) \cap [0, 1]^3$  is a closed convex set as intersection of the plane  $\boldsymbol{\alpha} \cdot z = \lambda^2$  of  $\mathbf{R}^3$  and the cube  $[0, 1]^3$ .

(1) Let  $|\lambda| \in (0, \lambda_+)$ . Set  $z_\lambda = \frac{|\lambda|}{\lambda_+}(1, 1, 1) \in (0, 1)^3$ , so  $\Psi_0(z_\lambda) = \tau^\pm(z_\lambda) = \lambda^2$ , since we have  $\lambda_+ = \Psi_0(1, 1, 1) = \sum_{j=1}^3 \alpha_j$ . Thus,  $z_\lambda \in M^{\mathbf{C}}(P) \cap (0, 1)^3$ , and the conclusion follows.

(2) Assume  $|\lambda| = \lambda_+$ . Since  $\partial_{z_j} \Psi_0(z) = \alpha_j > 0$  for all  $j$  then  $|\Psi_0(z)| < \lambda_+ =$

$\Psi_0(1, 1, 1)$  for all  $z \in [0, 1]^3 \setminus \{(1, 1, 1)\}$ . Hence  $\{(1, 1, 1)\} = M^{\mathbf{C}}(P) \cap [0, 1]^3$ , which proves (2).  $\square$

**Lemma 4.6.** *Under the assumptions in Remark 4.1 with the same notations (hence we assume  $\beta \neq 0$  and  $\prod_{j=1}^3 \beta_j = 0$  and we have  $P = p(\cdot; \lambda)$ ,  $P^\pm := \tau^\pm - \lambda^2$ ), we have*

- (1) *The intersection  $M_{sgn}^{\mathbf{C}}(P) \cap [0, 1]^3$  has one point at most.*
- (2) *Let  $|\lambda| \in (0, \lambda_-)$ . Each  $M^{\mathbf{C}}(P^\pm)' \cap (0, 1)^3$  has dimension two.*
- (3-1) *Let  $|\lambda| \in (0, \lambda_+)$ . Then  $M^{\mathbf{C}}(P^+) \cap (0, 1)^3$  has dimension two.*
- (3-2) *Let  $|\lambda| \in (0, \lambda_-)$ . Then  $M^{\mathbf{C}}(P^-)' \cap (0, 1)^3$  has dimension two.*
- (3-3) *Let  $|\lambda| \geq \lambda_-$ . Then  $M^{\mathbf{C}}(P^-)' \cap [0, 1]^3$  has at most one point.*

*Proof.* 1) Thanks to (4.9) and since  $\beta_1 > 0 > \beta_2$ , we have the equivalence  $(K_0(z) = 0$  and  $z \in [0, 1]^3)$  iff  $z_1 = z_3 = 0$ . Hence

$$M_{sgn}^{\mathbf{C}}(P) \cap [0, 1]^3 = \{z = (0, z_2, 0) \mid z_2 \in [0, 1] \text{ and } z_2 = \lambda^2/\alpha_2\} \subset \{z^0\},$$

with  $z^0 = (0, \lambda^2/\alpha_2, 0)$ . This proves (1).

(2), (3-1) and (3-2). It is similar to the proof of Lemma 4.5.

(3-3) If  $|\lambda| > \lambda_- = \sup_{[0, 1]^3} \tau^-$  then, obviously,  $M^{\mathbf{C}}(P^-) \cap \mathbb{T}^3$  is void. If  $|\lambda| = \lambda_- = \sqrt{\tau^-(z)}$  and  $z \in [0, 1]^3$ , then  $z = (1, 1, 1)$ , so  $M^{\mathbf{C}}(P^-) \cap [0, 1]^3 = \{(1, 1, 1)\}$ . The conclusion follows.  $\square$

**Lemma 4.7.** *Assume  $\prod_{j=1}^3 \beta_j \neq 0$ . Let  $\lambda \neq 0$  and set  $P = \lambda^{-2}p(\cdot; \lambda)$ , so  $M_{sgn}^{\mathbf{C}}(P) = \emptyset$ .*

- (1) *Let  $|\lambda| \in (0, \lambda_+)$ . Then  $M^{\mathbf{C}}(P) \cap (0, 1)^3$  has dimension two.*
- (2) *Let  $|\lambda| = \lambda_+$ . Then  $M^{\mathbf{C}}(P) \cap [0, 1]^3 = \{(1, 1, 1)\}$ .*

*Proof.* We have  $P(z) = (\tau^+(z) - \lambda^2)(\tau^-(z) - \lambda^2)$  so  $M^{\mathbf{C}}(P) \cap (0, 1)^3 = \{z \in (0, 1)^3 \mid \tau^+(z) = \lambda^2\} \cup \{z \in (0, 1)^3 \mid \tau^-(z) = \lambda^2\}$ . Since  $\nabla P(z) \neq 0$  for all  $z \in [0, 1]^3$  then  $M^{\mathbf{C}}(P) \cap (0, 1)^3$  has dimension is 2 iff it is non empty.

- (1) Let  $|\lambda| \in (0, \lambda_+)$  and set  $z(t) = t(1, 1, 1)$ . Since  $\tau^+(z(0)) = 0$  and  $\tau^+(z(1)) = \lambda_+^2$  then there exists  $t \in (0, 1)$  such that  $\tau^+(z(t)) = \lambda^2$ . Hence  $z(t) \in M^{\mathbf{C}}(P) \cap (0, 1)^3$ .
- (2) Let  $|\lambda| \geq \lambda_+$ . Since  $\lambda_+ < \lambda_-$  then  $\tau^-(z) < \lambda^2$  for all  $z \in [0, 1]^3$ . Assume  $|\lambda| > \lambda_+$ . Then  $\tau^+(z) < \lambda^2$  for all  $z \in [0, 1]^3$  so  $M^{\mathbf{C}}(P) = \emptyset$ . Assume  $|\lambda| = \lambda_+$ . Since we have  $(\tau^+(z) = \lambda_+^2$  and  $z \in [0, 1]^3)$  iff  $z = (1, 1, 1)$ , the conclusion then follows.  $\square$

**4.2. The complex Fermi variety in the  $x$ -variable.** We denote  $\sin_{\mathbb{T}} : \mathbb{T}_{\mathbf{C}}^d \ni x \mapsto z = \sin^2 x \in \mathbf{C}^d$  and set

$$X_{0,1} := (\sin_{\mathbb{T}}^2)^{-1}(\{0, 1\}^3) = \{x \in \mathbb{T}^3 \mid z \in \{0, 1\}^3\}.$$

**Lemma 4.8.** *Let  $d \geq r \geq 1$  and  $E \subset \mathbf{C}^d$  a connected set. Assume that  $\tilde{E} := (\sin_{\mathbb{T}}^2)^{-1}(E)$  is a  $r$ -dimensional smooth submanifold of  $\mathbb{T}_{\mathbf{C}}^d$ . Let  $C_x$  a connected component of  $\tilde{E}$ . Then  $C_x$  is open and closed and  $\sin^2(C_x) = E$ .*



*Proof.* Let us consider the topological set  $E$  with the topology induced by those of  $\mathbf{C}^d$  and the topological set  $\tilde{E}$  with the topology induced by those of  $\mathbb{T}_{\mathbf{C}}^d$ . Then, it is easy to see that  $\phi_E: \tilde{E} \ni x \mapsto z \in E$  is continuous. Since  $\tilde{E}$  is a  $r$ -dimensional smooth submanifold of  $\mathbb{T}_{\mathbf{C}}^d$ , near any  $x^0 \in \tilde{E}$ , there exists an open ball  $B(x^0, r) \subset \mathbb{T}_{\mathbf{C}}^d$  and a smooth function  $f: B(x^0, r) \mapsto \mathbf{C}$  such that  $\nabla f(x) \neq 0$  for all  $x \in B(x^0, r)$ . Then,  $\tilde{E} \cap B(x^0, r) = f^{-1}(0_{\mathbf{C}})$ . Hence  $\tilde{E}$  is locally connected and each of its connected components is open and closed. We now prove that the map  $\phi_E$  is both open and closed. It then implies that the set  $\phi_E(C_x)$  is open and closed hence equals to  $E$ , since  $E$  is connected. Since  $\sin^2$  is a non-constant holomorphic function from  $\mathbf{C}^d$  into itself then it is an open map and  $\sin_{\mathbb{T}}^2: \mathbb{T}_{\mathbf{C}}^d = \mathbf{C}^d / (2\pi\mathbf{Z})^d \mapsto \mathbf{C}^d$  is also open. Let  $V$  an open set of  $\tilde{E}$  so  $V = \tilde{E} \cap V'$  where  $V'$  is an open set of  $\mathbb{T}_{\mathbf{C}}^d$ . Then  $\phi_E(V) = \phi_E((\sin_{\mathbb{T}}^2)^{-1}(E) \cap V') = E \cap \sin_{\mathbb{T}}^2(V')$ . Since  $\sin_{\mathbb{T}}^2$  is an open map then  $\sin_{\mathbb{T}}^2(V')$  is open. Thus  $\phi_E(V)$  is an open set of  $E$ . Hence  $\phi_E$  is an open map. Let us prove that  $\phi_E$  is a closed map. We observe that  $|\sin^2(\Re x + i\Im x)| = \sinh^2(\Im x) + \sin^2(\Re x)$  so a set of the form  $\sin_{\mathbb{T}}^2(A)$ ,  $A \subset \mathbb{T}_{\mathbf{C}}^d$ , is unbounded iff  $A$  is unbounded. (In fact  $\sin_{\mathbb{T}}^2$  is proper.) Let  $F$  a closed subset of  $\tilde{E}$ . Let  $z_n$  a sequence of values in  $\phi_E(F)$  which tends to some  $z \in E$ . Then  $z_n = \phi_E(x_n)$  tends to  $z$ ,  $x_n \in F$ , so, by the above observation, the sequence  $(x_n) \subset F^{\mathbf{N}}$  is bounded. Let  $x'$  a subsequential limit of  $x_n$ . Then  $z = \phi_E(x')$  so  $z \in \phi_E(F)$ . Hence  $\phi_E(F)$  is closed.

The conclusion then follows. In fact, since  $\phi_E$  is an open and closed map and  $C_x$  is open and closed then  $\phi_E(C_x)$  is both open and closed in the connected space  $E$ , so it coincides with  $E$ .  $\square$

**Lemma 4.9.** *Assume  $\beta = 0$ . Set  $Q(x) = \Psi_0(z) - \lambda^2$  with  $z = \sin^2 x$ . Then  $M_{sgn}^{\mathbf{C}}(Q)$  is discrete. In addition, assume  $|\lambda| \in (0, \lambda_+)$ . Then, each connected component of  $M_{reg}^{\mathbf{C}}(Q)$  intersects  $\mathbb{T}^3$  and the intersection is a two-dimensional real manifold.*

*Proof.* Since  $\partial_{z_j} \Psi_0(z) \neq 0$  for all  $z \in \mathbf{C}^3$ , and since  $\partial_{x_j} z = \sin(2x_j)e_j$  vanishes iff  $z_j \in \{0, 1\}$ , then  $\nabla Q(x) = 0$  iff  $z \in \{0, 1\}^3$ . Hence  $M_{sgn}^{\mathbf{C}}(Q) = X_{0,1}$  is discrete. Let  $C_x$  be a connected component of  $M_{reg}^{\mathbf{C}}(Q)$ . Set  $P = \Psi_0 - \lambda^2$ . We have  $M_{reg}^{\mathbf{C}}(Q) = (\sin_{\mathbb{T}}^2)^{-1}(E)$  with  $E := M_{reg}^{\mathbf{C}}(P) \setminus \{0, 1\}^3$ . Since  $M_{reg}^{\mathbf{C}}(P)$  is a connected two-dimensional complex manifold, then so is  $E$ . Since  $E$  is a two-dimensional complex manifold, so is  $C_x$ . Thanks to Lemma 4.8, we obtain  $\sin^2(C_x) = E$ . Since  $E \subset M_{reg}^{\mathbf{C}}(P)$  and  $M_{reg}^{\mathbf{C}}(P)$  intersects  $(0, 1)^3$ ,  $\sin^2(C_x)$  intersects  $(0, 1)^3$ . Hence  $C_x$  intersects  $\mathbb{T}^3$ . Finally, since the jacobian of  $\sin^2|_{\mathbb{T}^3}$  does not vanish on  $(\sin_{\mathbb{T}}^2)^{-1}((0, 1)^3) (\subset \mathbb{T}^3)$ , then  $C_x \cap \mathbb{T}^3$  is a two-dimensional real manifold.  $\square$

**Lemma 4.10.** *Assume  $\beta \neq 0$  and  $\prod_{j=1}^3 \beta_j = 0$ . Set  $Q(x) = q(x; \lambda)$ .*

*(1) The analytic variety  $M_{sgn}^{\mathbf{C}}(Q)$  has Hausdorff (5)-measure zero and  $M_{sgn}^{\mathbf{C}}(Q) \cap \mathbb{T}^3$  is finite.*

(2) We have  $M_{reg}^{\mathbf{C}}(Q) = M_{\mathbf{C}}^+ \cup M_{\mathbf{C}}^-$  where each  $M_{\mathbf{C}}^{\pm}$ , defined by

$$M_{\mathbf{C}}^{\pm} := \{x \in \mathbb{T}_{\mathbf{C}}^3 \mid z \notin \{0, 1\}^3, \tau^{\pm}(z) = \lambda^2 \neq \tau^{\mp}(z)\}$$

is a two-dimensional submanifold of  $\mathbb{T}_{\mathbf{C}}^3$ .

(3) If  $|\lambda| \in (0, \lambda_-)$ , each connected component of  $M_{\mathbf{C}}^{\pm}$  intersects  $\mathbb{T}^3$  and the intersection is a two-dimensional real manifold.

(4) If  $|\lambda| \in [\lambda_-, \lambda_+)$ , each connected component of  $M_{\mathbf{C}}^+$  intersects  $\mathbb{T}^3$  and the intersection is a two-dimensional real manifold. However, the set  $M_{\mathbf{C}}^- \cap \mathbb{T}^3$  is finite.

*Proof.* We use the notations in Lemma 4.6, so  $Q(x) = P(z)$ , and we set  $Q^{\pm}(x) = \tau^{\pm}(z) - \lambda^2 = P^{\pm}(z)$ .

(1) We have

$$\begin{aligned} M_{sgn}^{\mathbf{C}}(Q) &= (M^{\mathbf{C}}(Q) \cap X_{0,1}) \cup (M^{\mathbf{C}}(Q^+) \cap M^{\mathbf{C}}(Q^-)) \\ (4.10) \quad &= (M^{\mathbf{C}}(Q) \cap X_{0,1}) \cup (\sin_{\mathbb{T}}^2)^{-1}(M_{sgn}^{\mathbf{C}}(P)). \end{aligned}$$

The set  $M^{\mathbf{C}}(Q) \cap X_{0,1}$  is finite since  $X_{0,1}$  is finite. In addition, the set  $(\sin_{\mathbb{T}}^2)^{-1}(M_{sgn}^{\mathbf{C}}(P))$  has Hausdorff  $k$ -measure zero for all  $k \geq 3$ , since  $M_{sgn}^{\mathbf{C}}(P)$  is a complex straight line (see Lemmas 4.4, 4.6) and  $\sin_{\mathbb{T}}^2$  is a local smooth diffeomorphism except on a finite set of  $\mathbf{C}^3$ . Hence  $M_{sgn}^{\mathbf{C}}(Q)$  has Hausdorff  $(2d - 1 = 5)$ -measure zero. Since  $M_{sgn}^{\mathbf{C}}(P) \cap [0, 1]^3$  has at most one point (see Lemma 4.6), (4.10) shows that  $M_{sgn}^{\mathbf{C}}(Q) \cap \mathbb{T}^3$  is finite. This proves (1).

(2) The relation  $M_{reg}^{\mathbf{C}}(Q) = M_{\mathbf{C}}^+ \cup M_{\mathbf{C}}^-$  is then obvious. In addition, we have  $M_{\mathbf{C}}^{\pm} = (\sin_{\mathbb{T}}^2)^{-1}(M^{\mathbf{C}}(P^{\pm})) \setminus M_{sgn}^{\mathbf{C}}(Q)$ . Since  $M^{\mathbf{C}}(P^{\pm})$  is a two-dimensional submanifold of  $\mathbf{C}^3$  and  $M_{sgn}^{\mathbf{C}}(Q)$  has dimension one,  $M_{\mathbf{C}}^{\pm}$  is a two-dimensional submanifold of  $\mathbb{T}_{\mathbf{C}}^3$ .

(3) and (4) for  $M_{\mathbf{C}}^+$ . It is similar to the corresponding assertion in Lemma 4.9.

(4) for  $M_{\mathbf{C}}^-$ . This is a direct consequence of (3-3), Lemma 4.6.  $\square$

**Lemma 4.11.** *Assume  $\prod_{j=1}^3 \beta_j \neq 0$ . Let  $|\lambda| \in (0, \lambda_+)$  and set  $Q(x) = \lambda^{-2}p(z; \lambda)$ .*

(1) *The set  $M_{sgn}^{\mathbf{C}}(Q)$  is finite.*

(2) *Assume  $|\lambda| \in (0, \lambda_+)$ . Then, each connected component of  $M_{reg}^{\mathbf{C}}(Q)$  intersects  $\mathbb{T}^3$ , and the intersection is a two-dimensional real manifold.*

*Proof.* Put  $P(z) := Q(x)$  so  $\partial_{x_j} Q(x) = 0$  iff  $\partial_j P(z) = 0$  or  $z_j \in \{0, 1\}$ . Hence  $M_{sgn}^{\mathbf{C}}(Q) = (\sin_{\mathbb{T}}^2)^{-1}(F)$  with

$$\begin{aligned} F &:= M_{sgn}^{\mathbf{C}}(P) \cup_{j=1}^3 F_j \cup_{j \neq k} F_{j,k} \cup (\{0, 1\}^3 \cap M^{\mathbf{C}}(P)), \\ F_j &:= \{z \in M^{\mathbf{C}}(P) \mid z_j \in \{0, 1\}^3, \partial_{z_k} P(z) = 0 \quad k \neq j\}, \\ F_{j,k} &:= \{z \in M^{\mathbf{C}}(P) \mid z_j, z_k \in \{0, 1\}^3, \partial_{z_l} P(z) = 0 \quad l \neq j, k\}. \end{aligned}$$

(1) Thanks to Lemma 4.7 we have  $M_{sgn}^{\mathbf{C}}(P) = \emptyset$ . We denote by  $A_j$  the  $j^{\text{th}}$  column of the matrix  $A$  defined by (4.2), by  $E_j$  the column of the coefficients of  $e_j$  in the canonical basis  $(e_1, e_2, e_3)$ , by  $\tilde{A}_j$  the  $3 \times 3$  matrix obtained from  $A$  by replacing the column  $A_j$  by  $E_j$  and by  $\tilde{A}_{j,k}$  (with  $j \neq k$ ) the  $3 \times 3$  matrix obtained from  $A$  by replacing the columns  $A_j$  and  $A_k$  respectively by the column  $E_j$  and  $E_k$ . Since

$P(z) = Az \cdot z + \mathbf{b} \cdot z + c$ , then  $\partial_j P(z) = 2A_j \cdot z + \mathbf{b}_j$ .

Let us consider  $F_1$ . We have  $F_1 = F_1(0) \cup F_1(1)$ , where  $F_1(\xi)$  is the intersection of the three hyperplanes  $z_1 = \xi$ ,  $2A_k \cdot z + \mathbf{b}_k = 0$ ,  $k = 2, 3$ . The matrix of the above system is  $B_1 := (E_1|A_2|A_3)$  where  $E_1 := (1 \ 0 \ 0)^T$ . Its determinant is

$$\det(B_1) = \gamma_2 \gamma_3 - (\varepsilon_1 \mu_1 \alpha_1)^2 = (\gamma_1 - \alpha_1^2)(\varepsilon_1 \mu_1)^2 = -\frac{1}{4} \beta_1^2 (\varepsilon_1 \mu_1)^2 < 0.$$

Hence  $F_1(\xi)$  is reduced to one point, and then  $F_1$  is a couple of points. Similarly, each  $F_j$  is reduced to two points. Let us consider  $F_{1,2}$ . We have  $F_{1,2} = \cup_{\xi, \xi' \in \{0,1\}} F_{1,2}(\xi, \xi')$  where  $F_{1,2}(\xi, \xi')$  is the intersection of the three hyperplanes  $z_1 = \xi$ ,  $z_2 = \xi'$ ,  $2A_3 \cdot z + \mathbf{b}_3 = 0$ . The matrix of the above system is  $B_{1,2} := (E_1|E_2|A_3)$  where  $E_2 := (0 \ 1 \ 0)^T$ . Its determinant is  $\det(B_{1,2}) = \gamma_3 \neq 0$ . Hence  $F_{1,2}$  is finite. Then,  $F$  is finite. Now we observe that, for any finite set  $E \subset \mathbf{C}^3$ ,  $(\sin_{\mathbb{T}})^{-1}(E)$  is a finite subset of  $\mathbb{T}_{\mathbf{C}}^3$ . Consequently,  $M_{sgn}^{\mathbf{C}}(Q)$  is finite.

(2) We have  $M_{reg}^{\mathbf{C}}(Q) = (\sin_{\mathbb{T}}^2)^{-1}(E)$  with  $E := M_{reg}^{\mathbf{C}}(P) \setminus F$ . Thanks to Lemma 4.7,  $M_{reg}^{\mathbf{C}}(P)$  is a connected two-dimensional submanifold of  $\mathbf{C}^3$ . So, since  $\dim F \leq 1$ , then  $E$  is also a connected two-dimensional submanifold of  $\mathbf{C}^3$ . Thus  $M_{reg}^{\mathbf{C}}(Q)$  is a two-dimensional submanifold of  $\mathbb{T}_{\mathbf{C}}^3$ . Let  $C_x$  be a connected component of  $M_{reg}^{\mathbf{C}}(Q)$ . Lemma 4.8 says that  $\sin_{\mathbb{T}}^2(C_x) = E$ . We have  $E \cap (0, 1)^3 = M_{reg}^{\mathbf{C}}(P) \cap (0, 1)^3$ , since  $F \cap (0, 1)^3$  is empty. Thanks to Lemma 4.7,  $M_{reg}^{\mathbf{C}}(P) \cap (0, 1)^3$  has dimension two. Thus  $E$  intersects  $(0, 1)^3$ . The end of the proof is similar to the end of the proof of Lemma 4.9.  $\square$

## 5. PROOF OF THE MAIN THEOREMS

**5.1. Proof of Theorem 1.1.** Theorem 1.1 is a straight consequence of Theorem 3.1, Lemmas 4.9, 4.10, 4.11.

**5.2. Proof of Theorem 1.2.**

**Theorem 5.1.** *Assume  $\beta \neq 0$  and  $\beta_1 \beta_2 \beta_3 = 0$ . Let  $\lambda \neq 0$ .*

(1) *The set  $M_{reg}^{\mathbf{C}}(q(\cdot; \lambda))$  is a disjoint union of the following two complex manifolds of dimension 2:*

$$M_{\mathbf{C}}^{\pm} := \{x \in \mathbb{T}_{\mathbf{C}}^3 \mid z \notin \{0, 1\}^3, \tau^{\pm}(z) = \lambda^2 \neq \tau^{\mp}(z)\}.$$

(2) *Assume  $|\lambda| > \lambda_-$ . Then,  $M_{\mathbf{C}}^- \cap \mathbb{T}^3$  is finite,*

The assertions (1) and (2) of Theorem 5.1 are a straight consequence of Lemma 4.10.

Set

$$B(x) = -\boldsymbol{\mu} \tilde{M}(y) \varepsilon \tilde{M}(y),$$

and  $u = (u_E, u_H)$  with

$$u_E(x) := \frac{1}{\lambda} \varepsilon \tilde{M}(y) u_H(x), \quad u_H(x) := (\tau^-(z) - \lambda^2)^{-1} \text{comat}(B(x) - \lambda^2) v_H,$$

where  $v_H$  is a non-null constant column-vector of length 3. Since  $u_H \in \mathcal{C}^\infty(\mathbb{T}^3; \mathbf{C}^3)$  so does  $u_E$ . Thus  $\hat{u} \in L^2(\mathbf{Z}^3) \subset \mathcal{B}_0^*(\mathbf{Z}^3)$ . Then,  $(B(x) - \lambda^2)u_H(x) = \lambda^2(\tau^+(z) - \lambda^2)v_H$  is a trigonometric polynomial, and so is  $(H^D - \lambda)u = (0, \lambda^{-1}(B - \lambda^2)u_H)$ . Let us prove that we can choose  $v_H$  such that  $u$  is not a trigonometric polynomial. We remember that the family of eigenvalues of the symmetric matrix  $-\varepsilon \tilde{M}(y) \mu \tilde{M}(y)$  is  $(0, \tau^+(z), \tau^-(z))$  and we denote by  $\Pi^0(y)$ ,  $\Pi^+(y)$ ,  $\Pi^-(y)$ , respectively, the associated spectral eigenprojectors. Thus, we have

$$\begin{aligned} \text{comat}(B(x) - t) &= -t(\tau^+(z) - t)\Pi^-(y) - t(\tau^-(z) - t)\Pi^+(y) \\ &\quad + (\tau^+(z) - t)(\tau^-(z) - t)\Pi^0(y), \end{aligned}$$

therefore,

$$(5.1) \quad \text{comat}(B(x) - \tau^-(z)) = -2\tau^-(z)\sqrt{K_0(z)}\Pi^-(y).$$

Hence,  $\text{comat}(B(x) - \tau^-(z))$  has rank one at any  $z \in \mathbf{C}^3$  such that  $K_0(z) \neq 0$ . If each coefficient  $c_{j,k}(y)$  of  $\text{comat}(B(x) - \lambda^2)$  were reducible by  $(\tau^-(z) - \lambda^2)$ , i.e. if  $(\tau^-(z) - \lambda^2)^{-1}c_{j,k}(y)$  were a trigonometrical polynomial, then  $\text{comat}(B(x) - \lambda^2)$  would vanish at any  $z \in \mathbf{C}^3$  such that  $\tau^-(z) - \lambda^2 = 0$ . This is in contradiction with (5.1). Hence there exists  $v_H$  such that  $u$  is not a trigonometrical polynomial.

We complete this section by proving the assertion of Remark 1.1. Set  $Q^\pm(x) = (\tau^\pm(z) - \lambda^2)v(x)$ ,  $v := Q^-u$ . Since  $\tau^-$  is linear,  $Q^-$  is then smooth in the  $x$  variable, so  $v \in \mathcal{B}_0^*(\mathbb{T}^3)$ . In addition we have  $Q^+ = \lambda^{-2}q(\cdot; \lambda)u$  which is a trigonometric polynomial. Since  $Q^+$  satisfies Conditions (A-1)-(A-2) of Theorem 3.1, the result follows.

**5.3. Proof of Theorem 1.4.** I. We put

$$G^* := \{\xi \in \mathbb{S}^2 \mid |\xi_i| > |\xi_j| \geq |\xi_l| \text{ for some permutation } (i, j, l) \text{ of } (1, 2, 3)\},$$

where  $\mathbb{S}^2$  denotes the euclidian unit sphere of  $\mathbf{R}^3$ . By definition we have  $K_{int} = \Omega \cap \mathbf{Z}^3$  where  $\Omega$  is a bounded convex set of  $\mathbf{R}^3$ . Then, there exists a (non void) finite family  $F \subset G^* \times \mathbf{R}$  such that

$$(5.2) \quad K_{int} = \bigcap_{(\xi, l) \in F} \mathcal{P}_{disc}^-(\xi, l), \quad \mathcal{P}_{disc}^-(\xi, l) := \{n \in \mathbf{Z}^3 \mid \xi \cdot n \leq l\}.$$

The characterization by (5.2) is well-known with  $G^*$  replaced by  $\mathbb{S}^2$ . Let us prove it with  $G^*$ . Actually, the bounded convex set  $\Omega \subset \mathbf{R}^3$  can be assumed closed and written  $\Omega = \bigcap_{(\xi, r) \in F_0} \mathcal{P}^-(\xi, r)$ , where  $\mathcal{P}^-(\xi, r) := \{\nu \in \mathbf{R}^3 \mid \nu \cdot \xi \leq r\}$  and  $F_0 \subset \mathbb{S}^2 \times \mathbf{R}$ . Since  $K_{int}$  is bounded then  $K_{int} \subset K_1$ ,  $K_1 := [-l_0, l_0]^3 \cap \mathbf{Z}^3$  for some  $l_0 > 0$ . Since  $K_{int}$  is finite then we have

$$K_{int} = \bigcap_{(\xi, r) \in F_1} \mathcal{P}^-(\xi, r) \cap K_1,$$

where  $F_1 \subset F_0$  is a finite family. Obviously, we can assume that  $F_1$  is non void and that  $K_1 \not\subset \mathcal{P}^-(\nu, r)$ ,  $(\nu, r) \in F_1$ . (If  $K_1 \subset \mathcal{P}^-(\nu, r)$  we suppress  $(\nu, r)$  in  $F_1$ , that is, we replace  $F_1$  by  $F_1 \setminus \{(\nu, r)\}$ .) Fix  $(\nu, r) \in F_1$ . Since  $K_1 \setminus \mathcal{P}^-(\nu, r)$  is finite and non void, we put  $\delta := \sup\{\nu \cdot n - r \mid n \in K_1 \setminus \mathcal{P}^-(\nu, r)\} > 0$ . We then have

$$\mathcal{P}^-(\nu, r) \cap K_1 = \mathcal{P}^-(\nu, r + \delta/2) \cap K_1,$$

$$K_1 \setminus \mathcal{P}^-(\nu, r) = K_1 \setminus \mathcal{P}^-(\nu, r - \delta/2).$$

Since  $G^*$  is dense in  $\mathbb{S}^2$  and since  $K_1$  is finite, there then exists  $\xi \in G^*$  sufficiently closed to  $\nu$  such that

$$\begin{aligned} \mathcal{P}^-(\xi, r) \cap K_1 &\subset \mathcal{P}^-(\nu, r + \delta/2) \cap K_1, \\ K_1 \setminus \mathcal{P}^-(\xi, r) &\subset K_1 \setminus \mathcal{P}^-(\nu, r - \delta/2). \end{aligned}$$

Then we obtain

$$\mathcal{P}^-(\nu, r) \cap K_1 = \mathcal{P}^-(\xi, r) \cap K_1 = \mathcal{P}_{disc}^-(\xi, r) \cap K_1.$$

Hence, (5.2) holds with  $F = F_1 \cup_{j=1}^3 (e_j, l_0) \cup_{j=1}^3 (-e_j, l_0)$ .

II. Letting  $\hat{v} : \mathbf{Z}^3 \ni n \mapsto \hat{v}(n) \in \mathbf{C}^d$  be a sequence with compact support, we set

$$\hat{v}_S(n) = \overline{\hat{v}(n)},$$

so  $v_S(x) = \overline{v(-x)}$ . In addition, letting  $\xi \in \mathbf{R}^3$ ,  $\xi \neq 0$ , we define

$$N_\xi^{max}(v) = N_\xi^{max}(\hat{v}) := \max\{n \cdot \xi \mid \hat{v}(n) \neq 0\},$$

with the convention  $\max(\emptyset) = -\infty$ . We then observe that if  $v$  and  $w$  are two trigonometric polynomials, then  $N_\xi^{max}(vw) \leq N_\xi^{max}(v) + N_\xi^{max}(w)$  holds true and if, in addition,  $v$  or  $w$  is scalar-valued, then  $N_\xi^{max}(vw) = N_\xi^{max}(v) + N_\xi^{max}(w)$  holds true. Furthermore, we have

$$N_\xi^{max}(v_S) = N_\xi^{max}(v),$$

$$N_\xi^{max}(\langle v, \overline{v_S(x)} \rangle_{\mathbf{C}^m}) = 2N_\xi^{max}(v),$$

since  $\langle v(x), \overline{v_S(x)} \rangle_{\mathbf{C}^d} = (2\pi)^{-3} \sum_{n,m} e^{-i(n+m)x} \hat{v}(n) \overline{\hat{v}(m)}$  is a trigonometric polynomial.

III. We put  $L_t(x) = L(x; t) := \text{comat}(H^D(x) - t)$ , and  $q_t(x) := \det(H^D(x) - t)$ . The symmetric real matrix  $L(x; t)$  has size  $5 \times 5$  and  $L$  is a homogeneous polynomial of degree 5 in the variables  $t, y_1, y_2, y_3$ . Since  $H^D(x)$  has rank  $\leq 4$  then,  $L_0 = 0$  and, consequently,  $L_t$  is a polynomial of the variables  $y_1, y_2, y_3$  with degree  $\leq 4$ . Hence  $L_t$  has the form

$$(5.3) \quad L_t(x) = tB_1(x) + t^2B_2(x; t),$$

where  $B_1(x) = -\Phi_0(z)\Pi_0(y)$ ,  $\Pi_0(y)$  is the orthogonal projection on  $\ker H^D(x)$ ,  $\Phi_0 := \Psi_0^2 - K_0$  is a homogeneous polynomial of degree 2 of the variables  $z_1, z_2, z_3$ , and  $B_2(\cdot; t)$  is a trigonometric polynomial of degree 3. Thus, if  $t \neq 0$ , then  $L_t$  is a trigonometric polynomial of degree 4. Since  $B_1(x) = \lim_{t \rightarrow 0} t^{-1}L_t(x)$  we then see that  $B_1$  is also a trigonometric polynomial of degree 4.

We put  $|\xi|_\infty := \max(|\xi_j|; j \in \{1, 2, 3\})$ , where the  $\xi_l$ 's are the usual coordinates of  $\xi$ . We have

**Lemma 5.2.** *Let  $t \neq 0$  and  $\xi \in G^*$ . We then have*

$$N_\xi^{max}(L_t) = N_\xi^{max}(B_1) = N_\xi^{max}(q_t) = 4|\xi|_\infty > 3|\xi|_\infty \geq N_\xi^{max}(B_2(t)).$$

*Proof.* Without loss of generality we assume  $\xi_1 > |\xi_j|$ ,  $j \in \{2, 3\}$ . We have

$$N_\xi^{max}(e^{ix_j}) = N_\xi^{max}(\delta_{e_j}) = \xi_j,$$

then,

$$N_\xi^{max}(e^{ix_1}) = \xi_1 > N_\xi^{max}(e^{\pm ix_j}) \in \{-\xi_j, \xi_j\}, \quad j = 2, 3.$$

Since  $z_j = (-4)^{-1}(e^{2ix_j} + e^{-2ix_j} - 2)$  we then have

$$\begin{aligned} N_\xi^{max}(z_1) &= 2\xi_1 > N_\xi^{max}(z_j) = 2|\xi_j|, & j \neq 1, \\ N_\xi^{max}(z_1^2) &= 4\xi_1 > N_\xi^{max}(z_j z_k) = 2|\xi_j| + 2|\xi_k|, & (j, k) \neq (1, 1). \end{aligned}$$

Thanks to (4.1) and (4.2) we have

$$t^{-2}q_t(x) = \gamma_1 z_1^2 + \sum_{(j,l) \neq (1,1)} A_{j,l} z_j z_l + \mathbf{b} \cdot z + t^2,$$

with  $t\gamma_1 \neq 0$ . Thus,

$$N_\xi^{max}(q_t) = N_\xi^{max}(z_1^2) = 4\xi_1.$$

Since  $q_t I = (H^D - t)L_t$ , we then have

$$(5.4) \quad 4\xi_1 = N_\xi^{max}(q_t) \leq N_\xi^{max}(H^D) + N_\xi^{max}(L_t).$$

Since  $H^D - t$  (respect.,  $L_t$ ) is a polynomial of degree 1 (respect., of degree 4) of the variables  $y_j = (2i)^{-1}(e^{ix_j} - e^{-ix_j})$ , then  $N_\xi^{max}(H^D - t) \leq \xi_1$  and  $N_\xi^{max}(L_t) \leq 4\xi_1$ . Hence the inequalities in (5.4) are equalities. Finally, since  $B_2(t)$  is a polynomial of degree  $\leq 3$  of the variables  $y_j = (2i)^{-1}(e^{ix_j} - e^{-ix_j})$ , then  $N_\xi^{max}(B_2(t)) \leq 3\xi_1$ .  $\square$

IV. We have the following

**Lemma 5.3.** *Let  $\lambda \neq 0$ . Let  $\hat{u}$  be compactly supported in  $\mathbf{Z}^3$  such that  $(\hat{H}^{D_p} - \lambda)\hat{u}(n) = 0$  in  $\{n \in \mathbf{Z}^3 \mid \xi \cdot n > 0\}$ . Let  $\xi \in G^*$ . Then  $\hat{u}(n) = 0$  in  $\{n \in \mathbf{Z}^3 \mid \xi \cdot n > 0\}$ .*

*Proof.* The sequence  $\hat{u}$  satisfies

$$(5.5) \quad (H^{D_p} - \lambda)u = f,$$

where  $\hat{f}$  is compactly supported in  $\{n \in \mathbf{Z}^3 \mid \xi \cdot n \leq 0\}$ . Thus,  $N_\xi^{max}(f) \leq 0$ . The operator  $\hat{D}(\hat{D}_p)^{-1}$  is bounded and invertible in  $l^2(\mathbf{Z}^3)$  and is an operator of multiplication by a diagonal matrix of the form  $I + \hat{K}$ , so  $\hat{K} = \hat{D}(\hat{D}_p)^{-1} - I$  is an operator of multiplication by a diagonal matrix and has compact support. We then have

$$(5.6) \quad (H^D - \lambda)u = \lambda K u + g,$$

where  $\hat{g} := \hat{D}(\hat{D}_p)^{-1}\hat{f}$  vanishes in  $\{n \in \mathbf{Z}^3 \mid n \cdot \xi > 0\}$ , so  $N_\xi^{max}(g) \leq 0$ .

Since  $\hat{D}_p$  is real-valued we have

$$D_p v_S(x) = (2\pi)^{-3/2} \sum_{n \in \mathbf{Z}^3} \hat{D}_p(n) \overline{\hat{v}(n)} e^{-inx} = \overline{D_p v(-x)}.$$

In addition, since  $\overline{H^D(-x)} = H^D(-x) = -H^D(x)$ , then (5.5) implies

$$-D_p H_0 u_S = \lambda u_S + f_S.$$

Since  $B_1(y) = -\Phi_0(z)\Pi_0(y)$  we have  $B_1H_0 = 0$ . Then, multiplying (5.3) from the left by  $B_1D_p^{-1}$  and observing that  $\Pi_0H_0 = 0$ , we get

$$(5.7) \quad B_1D_p^{-1}u_S = -\lambda^{-1}B_1D_p^{-1}f_S.$$

Multiplying (5.6) from the left by  $L_\lambda$  and using (5.3), we get

$$q_\lambda u = \lambda L_\lambda K u + L_\lambda g = \lambda^2 B_1 K u + R_0,$$

where

$$R_0 := \lambda^3 B_2(\cdot; \lambda) K u + L_\lambda g.$$

Put  $T(x) := \sum_{n \in \mathbf{Z}^3} \hat{D}_p^{-1}(n) |\hat{u}(n)|^2 e^{2inx}$ . On one hand, since  $\hat{D}_p^{-1}(n) |\hat{u}(n)|^2 \neq 0$  if  $\hat{u}(n) \neq 0$ , then

$$(5.8) \quad N_\xi^{max}(T) = 2N_\xi^{max}(u).$$

On the other hand, we have

$$T(x) = \langle D_p^{-1}u_S(x), u(x) \rangle_{\mathbf{R}^6}.$$

Thus,

$$\begin{aligned} q_\lambda(x)T(x) &= \langle D_p^{-1}u_S(x), q_\lambda u(x) \rangle_{\mathbf{R}^6} \\ &= \lambda^2 \langle D_p^{-1}u_S(x), B_1(x)K u(x) \rangle_{\mathbf{R}^6} + R(x), \end{aligned}$$

where  $R(x) := \langle D_p^{-1}u_S(x), R_0(x) \rangle_{\mathbf{R}^6}$ . Then, since  $B_1(x)$  and  $D$  are real-symmetric matrices, by virtue of (5.7),

$$(5.9) \quad \begin{aligned} q_\lambda(x)T(x) &= \lambda^2 \langle B_1D_p^{-1}u_S(x), K u(x) \rangle_{\mathbf{R}^6} + R(x) \\ &= -\lambda \langle B_1D_p^{-1}f_S(x), K u(x) \rangle_{\mathbf{R}^6} + R(x). \end{aligned}$$

We have, in view of (5.4),

$$(5.10) \quad \begin{aligned} N_\xi^{max}(R) &\leq N_\xi^{max}(D_p^{-1}u_S) + \max(N_\xi^{max}(B_2(\cdot; \lambda)K u), N_\xi^{max}(L_\lambda g)) \\ &\leq N_\xi^{max}(u) + \max(3|\xi|_\infty + N_\xi^{max}(u), 4|\xi|_\infty), \end{aligned}$$

and, in addition,

$$(5.11) \quad N_\xi^{max}(\langle B_1D_p^{-1}f_S, K u \rangle_{\mathbf{R}^6}) \leq 4|\xi|_\infty + N_\xi^{max}(u).$$

Thus, in view of (5.9), (5.10), (5.11),

$$N_\xi^{max}(q_\lambda T) \leq \max(4|\xi|_\infty + N_\xi^{max}(u), 3|\xi|_\infty + 2N_\xi^{max}(u)).$$

Using (5.8) and Lemma 5.2, we obtain

$$(5.12) \quad \begin{aligned} N_\xi^{max}(T) &= N_\xi^{max}(q_\lambda T) - N_\xi^{max}(q_\lambda) \\ &\leq \max(N_\xi^{max}(u), -|\xi|_\infty + 2N_\xi^{max}(u)). \end{aligned}$$

Thus, (5.12) and (5.8) show that

$$N_\xi^{max}(u) = N_\xi^{max}(T) - N_\xi^{max}(u) \leq 0.$$

□

V. (Last step.) Theorem 1.4 is a direct consequence of Lemma 5.3. Actually, by translation and symmetries we can extend the above lemma by replacing the set  $H(\xi, 0) = \{n \in \mathbf{Z}^3 \mid \xi \cdot n > 0\}$  by the more general half space  $H(\xi, l) := \{n \in \mathbf{Z}^3 \mid \xi \cdot n > l\} = \mathbf{Z}^3 \setminus \mathcal{P}_{disc}^-(\xi, l)$ ,  $l \in \mathbf{R}$ . Hence if  $u$  satisfies the conditions of Theorem 1.4, then  $\hat{u}(n) = 0$  in  $\cup_{(\xi, l) \in F} H(\xi, l) = \mathbf{Z}^3 \setminus K_{int} = K_{ext}$ .  $\square$

5.4. **Proof of Proposition 1.3.** The distribution  $u$  satisfies the above relation (5.6) with  $g = 0$  and where  $\hat{K} = \hat{D}(\hat{D}_p)^{-1} - I$  has compact support. Hence, the distribution  $v \in \mathcal{B}_0^*(\mathbb{T}^3)$  defined by  $v(x) = (\tau^-(z) - \lambda^2)u(x)$  is a trigonometric polynomial. By observing that  $\tau^-(z) - \lambda^2 \leq \lambda_-^2 - \lambda^2 < 0$ , we thus obtain  $u(x) = (\tau^-(z) - \lambda^2)^{-1}v(x)$ , so  $u \in L^2(\mathbb{T}^3, \mathbf{C}^6)$ .

5.5. **Proof of Corollary 1.5.** Put  $f := (H^{D_p} - \lambda)u$ . Then  $\hat{f}$  has support in  $K_{int}$ . In addition, we have  $(\hat{H}^D - \lambda)\hat{u} = \lambda\hat{K}\hat{u} + \hat{D}^{-1}\hat{D}_p\hat{f}$ , where  $\hat{K} := \hat{D}(\hat{D}_p)^{-1} - I$  has compact support. Since (RT) for  $\hat{H}^D$  holds at  $\lambda$ , then  $\hat{u}$  has compact support. Consequently, thanks to Theorem 1.4,  $\hat{u}$  vanishes in  $K_{ext}$ .  $\square$

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