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► **To cite this version:**

Baptiste Cerclé, Nathan Huguenin. BOUNDARY TODA CONFORMAL FIELD THEORY FROM THE PATH INTEGRAL. 2024. hal-04634623

HAL Id: hal-04634623

<https://amu.hal.science/hal-04634623>

Preprint submitted on 4 Jul 2024

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BOUNDARY TODA CONFORMAL FIELD THEORY FROM THE PATH INTEGRAL

BAPTISTE CERCLÉ AND NATHAN HUGUENIN

ABSTRACT. Toda Conformal Field Theories (CFTs hereafter) are generalizations of Liouville CFT where the underlying field is no longer scalar but takes values in a finite-dimensional vector space. The algebra of symmetry of such models is given by W -algebras, which contain the Virasoro algebra as a subalgebra. In contrast with Liouville CFT, and in the presence of a boundary, there may exist non-trivial automorphisms of the associated W -algebra, corresponding to different boundary conditions for the field of the theory.

Based on this particular feature, we provide in this document a probabilistic construction of Toda CFTs on a Riemann surface with or without boundary. To be more specific we define different classes of models in relation with the different types of boundary conditions suggested by the non-triviality of the automorphism group of the W -algebra. To do so, we rely on a probabilistic framework based on Gaussian Free Fields and Gaussian Multiplicative Chaos, and make sense of “Cardy’s doubling trick” to construct the underlying field of the theory.

1. INTRODUCTION

1.1. Toda Conformal Field Theories.

1.1.1. *Liouville theory on closed surfaces.* The probabilistic framework designed to provide a mathematical study of Liouville Conformal Field Theory (CFT hereafter) has led to major breakthroughs in the understanding of this model of two-dimensional random geometry since its introduction by David-Guillarmou-Kupiainen-Rhodes-Vargas [17]. Among other achievements are the probabilistic derivation of the celebrated DOZZ formula [18, 43] in [31] and a mathematically rigorous implementation of the conformal bootstrap procedure [24] as well as Segal’s axioms [41, 25]. In this perspective the probabilistic framework has proved to be key in the mathematical formulation and derivation of predictions from the physics literature.

Indeed, before becoming a mathematical topic of research Liouville CFT was initially introduced in the physics literature in the groundbreaking work of Polyakov [33] where a notion of two-dimensional random geometry, natural in the setting of string theory and quantum gravity, was designed. It has then become a key object of study in the understanding of two-dimensional CFT following the introduction of the *conformal bootstrap method* by Belavin-Polyakov-Zamolodchikov [7] (BPZ hereafter) since (among other things) it provides a canonical model in which this procedure can be implemented. In the language of physics Liouville CFT is defined using a path integral, which means in the present context that one defines a measure on a functional space $\mathcal{F} \subset L^2(\Sigma, g)$ with Σ a Riemann surface (to start with assumed to be closed, that is without boundary) and g a Riemannian metric on Σ .

This measure is formally defined so that the average of a functional F over \mathcal{F} takes the form

$$(1.1) \quad \langle F(\Phi) \rangle_{\Sigma, g} := \frac{1}{\mathcal{Z}} \int_{\phi \in \mathcal{F}} F(\phi) e^{-S_L(\phi, g)} D\phi$$

where \mathcal{Z} is a normalization factor and $D\phi$ stands for a “uniform measure” over \mathcal{F} , while $S_L : \mathcal{F} \rightarrow \mathbb{R}$ is the so-called *Liouville action*. The latter is given by

$$(1.2) \quad S_L(\phi, g) := \frac{1}{4\pi} \int_{\Sigma} \left(|d_g \phi|^2 + Q R_g \phi + 4\pi \mu e^{\gamma \phi} \right) dv_g$$

where R_g is the scalar curvature, d_g the gradient and dv_g the volume form, all taken in the metric g . The action also features a coupling constant $\gamma \in (0, 2)$ and the background charge $Q := \frac{\gamma}{2} + \frac{2}{\gamma}$, as well as a cosmological constant $\mu \geq 0$. Based on this heuristic, the correlation functions of Liouville CFT are given by considering special observables F in Equation (1.1), given by products of *Vertex Operators* $V_{\alpha}(z) = e^{\alpha \Phi(z)}$ where $\alpha \in \mathbb{R}$ is a *weight* and $z \in \Sigma$ an *insertion*. Hence, the correlation functions are the quantities formally defined by

$$(1.3) \quad \left\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \right\rangle_{\Sigma, g} := \int_{\phi \in \mathcal{F}} \prod_{k=1}^N e^{\alpha_k \phi(z_k)} e^{-S_L(\phi)} D\phi.$$

However this approach used in the physics does not really make sense mathematically speaking; in this perspective one of the key achievements of David-Kupiainen-Rhodes-Vargas [17] has been to provide a rigorous probabilistic framework to give a meaning to the above, hence translating this heuristic into a well-defined object of study.

1.1.2. Toda CFTs on closed Riemann surfaces. The conformal bootstrap method envisioned by BPZ [7] has been successfully implemented in the physics literature, among other examples, for a large family of models of statistical physics at criticality: this is the case for instance for *minimal models* such as the planar Ising model at criticality. However, there is a whole class of CFTs enjoying in addition to conformal symmetry an enhanced level of symmetry: the critical three-states Potts model is a prototypical example of such a phenomenon in that it admits a *higher-spin symmetry*. In order to provide a suitable generalization of the conformal bootstrap procedure to the setting of models enjoying such a higher level of symmetry, Zamolodchikov introduced in 1985 [44] the notion of *W-algebra*, which allowed him to extend the method of BPZ to models featuring *higher-spin* or *W-symmetries*.

In this perspective the algebra of symmetry of Liouville CFT, the Virasoro algebra, corresponds to the special case where the *W-algebra* is associated to the simple and complex Lie algebra \mathfrak{sl}_2 . Toda CFTs allow one to extend this construction by generalizing Liouville theory to a family of CFTs indexed by simple and complex Lie algebras \mathfrak{g} and for which the algebras of symmetry are given by simple *W-algebras*. Despite playing a role similar to Liouville CFT in this generalized setting and therefore having being studied in depth in the physics literature, due to their complexity many features of Toda CFTs remain to be understood.

In contrast with Liouville theory, the field in Toda CFTs has values in a r -dimensional vector space, where r stands for the rank of the underlying Lie algebra. The problem then becomes the one of defining a measure μ_T on a functional space $\mathcal{F} \subset L^2(\Sigma \rightarrow \mathfrak{a}, g)$ where

$\mathfrak{a} \simeq \mathbb{R}^r$ corresponds to the (real part of the) Cartan subalgebra of \mathfrak{g} . This measure on \mathcal{F} is formally defined by

$$(1.4) \quad \mu_T(D\phi) := e^{-S_T(\phi, g)} D\phi$$

where $D\phi$ stands for the hypothetical uniform measure on this space of functions. Put differently, making sense of the formal measure (1.4) is tantamount to defining for F a suitable functional over \mathcal{F}

$$(1.5) \quad \langle F \rangle_{\Sigma, g} := \frac{1}{\mathcal{Z}} \int F(\phi) e^{-S(\phi, g)} D\phi.$$

In these expressions the Toda action is given on a Riemann surface (Σ, g) without boundary by the following Lagrangian:

$$(1.6) \quad S_T(\phi, g) = \frac{1}{4\pi} \int_{\Sigma} \left(|d_g \phi|_g^2 + R_g \langle Q, \phi \rangle + \sum_{i=1}^r \mu_i e^{\langle \gamma e_i, \phi \rangle} \right) dv_g$$

where:

- $\gamma \in (0, \sqrt{2})^1$ is the coupling constant and $Q = \gamma\rho + \frac{2}{\gamma}\rho^\vee \in \mathfrak{a}$ (see Subsection 2.3) is the background charge;
- the μ_i , $1 \leq i \leq r$, are the cosmological constants;
- the $(e_i)_{1 \leq i \leq r}$ are the *simple roots* of \mathfrak{g} ; they form a basis of \mathfrak{a} . Likewise $\langle \cdot, \cdot \rangle$ is a scalar product inherited from the Lie algebra structure.

A probabilistic definition for quantities given by Equation (1.5) has been proposed in [15] in the case where the underlying Riemann surface is the two-dimensional sphere \mathbb{S}^2 . To do so, the authors proposed a generalization of the framework designed in [17] to the setting of Toda CFTs. This in particular allowed to recover predictions made in the physics literature concerning the symmetries of such models in [14], a key ingredient in the derivation [13, 12] of the Fateev-Litvinov formula [21] for a family of structure constants of the theory.

In this document we extend the construction proposed there by making sense of the path integral (1.5) in several different cases, and to start with for hyperbolic Riemann surfaces without a boundary. But the main contribution of the present document is to initiate the mathematical study of Toda CFTs on surfaces with a non-empty boundary, which makes apparent some key notions in the study of CFT in the presence of a boundary. The program developed to address the case of Toda CFTs on the sphere should in theory admits a counterpart, though less easy to implement, in the setting we introduce here. This in principle should lead us to unveil the symmetries of the model defined in the present document and hopefully would allow us to compute some of the structure constants of the theory.

¹This range of values for the coupling constant differs from the usual one in that the simple roots are normalized so that the longest ones have norm $\sqrt{2}$. To recover the standard convention one therefore needs to rescale γ by $\sqrt{2}$.

1.1.3. *Boundary conditions and Cardy’s doubling trick.* Namely, up to now we have considered Liouville or Toda CFTs defined on a Riemann surface without boundary. It is therefore rather natural to try to extend these models to a boundary CFT that would sit on a Riemann surface whose boundary is non-empty. This task is actually non-trivial in that one needs to impose boundary conditions, leading to different boundary CFTs based on the same bulk theory. Let us (informally) discuss this issue in the following paragraphs.

It is understood in the physics that any boundary condition cannot arbitrarily be chosen, and the task of classifying which boundary conditions are admissible (in the sense that they preserve conformal symmetry among other things) is a classical question, that goes back to Cardy [9, 10]. For instance a first constraint related to this type of invariance is to impose boundary values for the stress-energy tensor \mathbf{T} of the theory, an observable that encodes its conformal symmetry. Actually the stress-energy tensor should satisfy a more restrictive assumption, related to the so-called “doubling trick”, in order to preserve conformal covariance.

Indeed, a remarkable observation due to Cardy [9, Section 4] and now referred to as the “doubling trick” allows one to relate the boundary theory and the bulk one. Namely, let us assume that we work on the upper-half plane \mathbb{H} and consider a correlation function $\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle_{\mathbb{H}}$ of bulk Vertex Operators. Then viewed as a function of $(z_1, \bar{z}_1, \dots, z_N, \bar{z}_N)$ this correlation function satisfies the same set of differential equations (such as *Ward identities*) as the corresponding correlation function on the whole plane \mathbb{C} , where the bulk Vertex Operators are reflected over the real line: $\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \prod_{k=1}^N V_{\alpha_k}(\bar{z}_k) \rangle_{\mathbb{C}}$, see Figure 1 below. This translates to an equality which holds at the level of *conformal blocks*, and in particular for the stress-energy tensor of the theory: the latter should formally satisfy the equality $\mathbf{T}(z) = \bar{\mathbf{T}}(\bar{z})$.

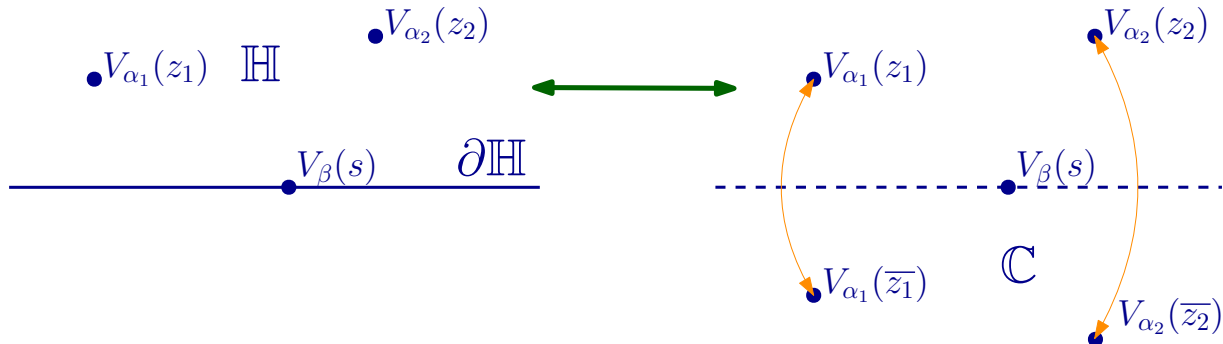


FIGURE 1. Illustration of Cardy’s doubling trick

In relation with this property, the definition of boundary observables is also subject to consistency assumptions known as the *Cardy’s conditions* [11] that translate compatibility with fusion rules. From a similar perspective and to quote Cardy [11], “juxtaposition of different (conformally invariant) boundary conditions is equivalent to the insertion of boundary operators”. All these features make the study of boundary CFT all the more interesting.

1.1.4. *Boundary Liouville CFT.* The above discussion on admissible boundary conditions has been kept at a very informal level; the probabilistic study of boundary Liouville CFT

naturally arises within this setting in that it provides a solid mathematical framework to investigate such properties. Like before the Liouville CFT is defined via a path integral similar to Equation (1.5), though the boundary action has to be adapted in order to take into account the presence of a boundary. Namely, if (Σ, g) is a Riemann surface with non-empty boundary $\partial\Sigma$ we set

$$(1.7) \quad S_L(\phi, g) = \frac{1}{4\pi} \int_{\Sigma} (|d_g\phi|_g^2 + R_g Q\phi + \mu e^{\gamma\phi}) dv_g + \frac{1}{2\pi} \int_{\partial\Sigma} (k_g Q\phi + \mu_{\partial} e^{\frac{\gamma}{2}\phi}) d\lambda_g$$

where λ_g is the line element on $\partial\Sigma$ and k_g the geodesic curvature, while $\mu_{\partial} \in \mathbb{C}$ is the boundary cosmological constant. A key manifestation of the boundary phenomena that have been discussed above is that in the presence of boundary Vertex Operators, one can actually choose this boundary cosmological constant μ_{∂} to take different values between the different boundary components delimited by the boundary insertions, so that μ_{∂} is generalized to a piecewise constant function.

The mathematical study of boundary Liouville CFT has made major breakthroughs over the past few years. Initially introduced in [29] and later extended to other geometries [36, 42], its properties related to integrability of the theory have then become a key problem, starting from the derivation of the so-called Fyodorov-Bouchaud formula [37] to the computation of all structure constants in [4] building on BPZ-type differential equations [2]. The process of computing all these structure constants has been carried out through several intermediate steps corresponding to some special structure constants [38, 3] and allowed to recover the formulas predicted in the physics literature [20, 34, 28].

1.2. Boundary Toda CFTs. However, the probabilistic study of boundary Liouville CFT does not actually exhibit all the features related to the structure of admissible boundary conditions. Indeed, the mathematical models considered in the aforementioned works on boundary Liouville CFT are based on Neumann boundary conditions for the underlying field, and apart from the degree of freedom provided by the form of the boundary cosmological constants (in relation with the insertion of boundary Vertex Operators) there is actually only one boundary theory constructed from its bulk counterpart.

The probabilistic study of boundary Toda CFTs allows one to address this issue in that several boundary CFTs can actually be defined based on the same bulk one, in providing a well-defined mathematical object of study to investigate properties related to the structure of possible boundary conditions. In analogy with the definition of boundary Liouville theory (1.7) and of the bulk Toda CFT (1.6), it is natural to define boundary Toda CFTs by introducing the following generalization of Equation (1.6) in the setting where (Σ, g) has non-empty boundary $\partial\Sigma$:

$$(1.8) \quad S(\phi, g) = \frac{1}{4\pi} \int_{\Sigma} \left(|d_g\phi|_g^2 + R_g \langle Q, \phi \rangle + \sum_{i=1}^r \mu_i e^{\langle \gamma e_i, \phi \rangle} \right) dv_g + \frac{1}{2\pi} \int_{\partial\Sigma} \left(k_g \langle Q, \phi \rangle + \sum_{i=1}^r \mu_i^{\partial} e^{\langle \frac{\gamma e_i}{2}, \phi \rangle} \right) d\lambda_g.$$

The boundary Toda CFT should then formally be defined via a path integral similar to Equation (1.5), which in principle could be interpreted using probability theory.

Let us mention that the existence of a path integral for boundary Toda CFTs and the form it should take are two questions not completely clear from the perspective of the physics literature. Indeed, in the paper [22] it is predicted that no path integral corresponds to the non-trivial automorphism of the W_3 algebra as soon as the underlying field has free boundary conditions (we do not treat this case in this document). Likewise in the case where the trivial automorphism is considered the degree of freedom under which the boundary cosmological constants can be chosen should be rather restricted.

We now discuss the issue of the boundary conditions for the Toda CFTs in relation with automorphisms of W -algebras and like before keep the discussion at a heuristic level. More details will be given in Subsection 2.4.

1.2.1. Automorphisms of W -algebras. One key difference between boundary Liouville and Toda CFTs lies in the non-trivial structure of the “outer automorphism group” of the algebra of symmetry. Indeed, in the case of the Virasoro algebra this group reduces to one element, the identity, while for instance for the W_3 algebra this group contains in addition one element of order two. This translates at the level of the currents of the theory in the following way: for Liouville CFT the conformal symmetry is encoded by the stress-energy tensor \mathbf{T} , but for the \mathfrak{sl}_3 Toda CFT there is, in addition to \mathbf{T} , a higher-spin tensor \mathbf{W} associated to higher-spin symmetry. Then the two elements of the outer automorphism group of the W_3 algebra have the effect of transforming (\mathbf{T}, \mathbf{W}) to (\mathbf{T}, \mathbf{W}) for the identity, but for the order two element we have $(\mathbf{T}, \mathbf{W}) \rightarrow (\mathbf{T}, -\mathbf{W})$. For general Lie algebras \mathfrak{g} similar transformations hold.

This property has strong implications on the admissible boundary conditions for the corresponding boundary theory since there are now two models whose symmetries differ; to each such automorphism of the W -algebra should correspond non-trivial boundary conditions for the underlying field. In this perspective it is suggested in the physics literature [27] that associated to the order two element “twisted” [10] boundary conditions can be adopted, but one could alternatively consider the orbifold model associated if one projects on the states invariant under this involution. There can therefore be several boundary CFTs based on the same model for closed surfaces, and for which the conformal symmetry is preserved while the higher-spin symmetry is transformed as described above.

This picture naturally arises in the realm of statistical physics. Indeed, if one considers the three-states Potts model at critical temperature, it is predicted in the physics that it can be studied based on the W_3 algebra and that twisted boundary conditions (in the sense of [10]) can naturally be imposed in addition to zero or free ones, see for instance [27]. Under such boundary conditions, the stress-energy tensor still satisfies the property that $\mathbf{T}(z) = \overline{\mathbf{T}}(\bar{z})$ but this is no longer the case for the higher-spin tensor \mathbf{W} , for which we now have $\mathbf{W}(z) = -\overline{\mathbf{W}}(\bar{z})$, where the latter is defined like before based on the doubling trick.

This feature is present in boundary Toda CFTs, which means that different models can be constructed to take into account these different boundary conditions. As we will now justify,

thanks to Cardy’s doubling trick we are able in some cases to translate them as assumptions on the field of the theory.

1.2.2. *Cardy’s doubling trick and the field of the theory.* In agreement with [23], there is a one-to-one correspondence between “outer automorphisms” of a W -algebra and outer automorphisms of the corresponding complex and simple Lie algebra \mathfrak{g} , that is to say an element τ of the group $\text{Out}(\mathfrak{g}) = \text{Aut}(\mathfrak{g})/\text{Inn}(\mathfrak{g})$ (see Subsection 2.3 for more details). Assuming that the underlying surface is the upper half-plane \mathbb{H} , a heuristic inspired from Cardy’s doubling trick is then that the bulk Vertex Operator defined from $V_\alpha(z)$ (with $\alpha \in \mathfrak{a}$ and $z \in \mathbb{H}$) in the boundary theory will still correspond to two Vertex Operators on the plane \mathbb{C} , but the “reflected” operator is then given by $V_{\tau\alpha}(\bar{z})$ rather than $V_\alpha(\bar{z})$. In terms of the field of the theory, our goal is therefore to construct a random function $\Phi : \bar{\mathbb{H}} \rightarrow \mathfrak{a}$ such that, formally speaking, the *boundary* correlation function $\langle \prod_k e^{\langle \alpha_k, \Phi(z_k) \rangle} \rangle_{\mathbb{H}}$ has the same features (in terms of symmetries/Ward identities) as the *bulk* correlation function $\langle \prod_k e^{\langle \alpha_k, \phi(z_k) \rangle} \prod_k e^{\langle \tau\alpha_k, \phi(\bar{z}_k) \rangle} \rangle_{\mathbb{C}}$ where $\phi : \mathbb{C} \rightarrow \mathfrak{a}$ represents the field of the theory on \mathbb{C} as illustrated by Figure 2 below.

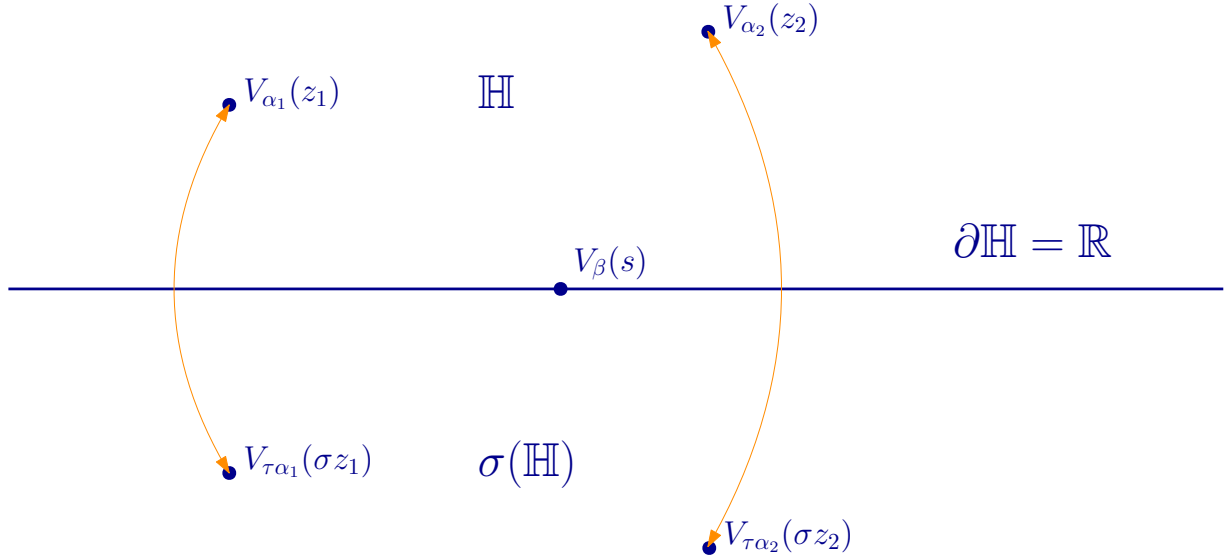


FIGURE 2. The doubling trick for non-trivial automorphisms

A natural way of satisfying this assumption is by considering the following “twisted” field

$$\Phi(z) = \frac{\phi(z) + \tau\phi(\bar{z})}{\sqrt{2}}.$$

In the case of Liouville CFT we have $\tau = \text{Id}$ so that the construction simply defines a field with Neumann boundary conditions. But for Toda CFTs for which the Lie algebra \mathfrak{g} contains a non-trivial outer automorphism of order 2 the field thus defined has mixed boundary conditions: one component has Neumann boundary conditions while the other one has Dirichlet boundary conditions.

Note that this construction is valid provided that τ , like the conjugacy operator $z \rightarrow \bar{z}$, is an involution. When τ is an automorphism of order 3 (which occurs only at triality, that is for $\mathfrak{g} = D_4$) the construction needs to be suitably adapted.

1.2.3. *Organization of the paper.* So far, we have highlighted some key concepts entering the consideration of Toda CFTs in the presence of a boundary. The ideas described in the previous paragraphs should be understood as guiding lines when one wants to investigate these questions based on a mathematical setting. Inspired by these issues, we initiate in the present article a rigorous mathematical study of these models.

Our first result corresponds to the probabilistic construction of Toda CFTs on closed Riemann surfaces with negative Euler characteristic: $\chi(\Sigma) < 0$. Using Gaussian Free Fields and the theory of Gaussian Multiplicative Chaos we make sense of the path integral of the theory and show that it satisfies basic assumptions related to conformal symmetry. We also provide a mathematical definition of the correlation functions $\langle \prod_{k=1}^N e^{\langle \alpha_k, \Phi(z_k) \rangle} \rangle$ for suitable values of the weights $(\alpha_k)_{1 \leq k \leq N} \in \mathfrak{a}^N$ and distinct insertions $(z_k)_{1 \leq k \leq N} \in \Sigma^N$. More specifically Theorem 3.14 in Section 3 shows that:

Theorem 1.1. *Assume that (Σ, g) is closed and of hyperbolic type and take any simple and complex finite-dimensional Lie algebra \mathfrak{g} . Then for any $\gamma \in (0, \sqrt{2})$, the correlation functions exist as soon as:*

$$(1.9) \quad \left\langle \sum_{k=1}^N \alpha_k - Q\chi(\Sigma), \omega_i^\vee \right\rangle > 0 \quad \text{for all } i = 1, \dots, r;$$

$$(1.10) \quad \langle \alpha_k - Q, e_i \rangle < 0 \quad \text{for all } i = 1, \dots, r \text{ and } k = 1, \dots, N.$$

We then proceed to the construction of boundary Toda CFT and distinguish between the different types of boundary conditions, or put differently, the element of the outer automorphism group τ of the underlying Lie algebra. More precisely we first define in Section 4 the boundary Toda CFT associated to the more standard assumption, which corresponds to $\tau = \text{Id}$. We then discuss the construction of a model related to non-trivial automorphisms in Section 5, where we define a CFT satisfying the desired properties, based on a field with mixed boundary conditions. Namely the statement of Theorem 5.5 allows us to define correlation functions for boundary Toda CFTs under the assumption that τ is the outer automorphism of order two of the underlying Lie algebra \mathfrak{g} . In that case in addition to the bulk Vertex Operators $e^{\langle \alpha_k, \Phi(z_k) \rangle}$ with $\alpha_k \in \mathfrak{a}$ and $z_k \in \Sigma \setminus \partial\Sigma$ there also exist boundary Vertex Operators $e^{\langle \beta_l, \Phi(s_l) \rangle}$ with $\beta_l \in \mathfrak{a}$ but now $s_l \in \partial\Sigma$. Hence a general correlation function takes the form

$$\left\langle \prod_{k=1}^N e^{\langle \alpha_k, \Phi(z_k) \rangle} \prod_{l=1}^M e^{\langle \beta_l, \Phi(s_l) \rangle} \right\rangle.$$

Theorem 1.2. *Assume that (Σ, g) has non-empty boundary and is of hyperbolic type, and that the underlying Lie algebra \mathfrak{g} has an order two outer automorphism τ . Then for any $\gamma \in$*

$(0, \sqrt{2})$, one can define correlation functions based on a “twisted” field. These correlation functions exist as soon as for all $i = 1, \dots, r$:

$$(1.11) \quad \left\langle \sum_{k=1}^N \alpha_k + \sum_{l=1}^M \frac{\beta_l}{2} - Q\chi(\Sigma), \omega_i^\vee \right\rangle > 0;$$

$$(1.12) \quad \langle \alpha_k - Q, e_i \rangle < 0 \quad \text{for all } k = 1, \dots, N \quad \text{and} \quad \langle \beta_l - Q, e_i \rangle < 0 \quad \text{for all } l = 1, \dots, M.$$

The precise meaning of this statement as well as the rigorous definition of the underlying field can be found in Section 5. We stress that Theorem 4.6 provides the counterpart statement with a field that is not “twisted” (that is admits Neumann boundary conditions).

Finally, Section 2 provides basic notions that we will use in the document, in relation with two-dimensional conformal geometry, Lie algebras (this topic being more detailed in Appendix A) and outer automorphism groups. Section 6 gathers some future perspectives of work related to the symmetries of the model as well as its integrability.

1.3. Outlook and perspectives. In the present work we have defined several conformal field theories based on a probabilistic framework and extensively relying on Cardy’s doubling trick. This construction has been made so as to take into account the non-triviality of the automorphism group of the underlying algebra of symmetry.

1.3.1. Symmetries of the model. The most natural question is thus the following: to what extent does the models thus defined *actually* enjoy the desired properties? In a work in progress we aim to unveil under which conditions the theories constructed in this document do enjoy the level of symmetry that they are expected to satisfy. We will focus on the case where $\mathfrak{g} = \mathfrak{sl}_3$ since it represents the simplest instance of Toda CFT distinct from the well-known Liouville CFT.

To do so we will first derive *Ward identities* associated to the models. These Ward identities encode the symmetries of the theory and showing that they are valid would amount to showing that the two theories constructed here have the desired features. Based on these Ward identities we will be able to define the descendant fields of the Vertex Operators.

1.3.2. Higher equations of motion and BPZ differential equations. The next step in the understanding of the \mathfrak{sl}_3 boundary Toda CFT is the derivation of *higher equations of motion*, which corresponds to the study of its null vectors. In the case of Liouville theory they have been proved to be valid in [5, 16], recovering predictions from the physics literature [6].

However, in the case of Toda CFTs the form under which such higher equations of motion are supposed to hold is not known at the time being. Likewise and apart from the differential equation considered in [22] (based on a boundary condition not treated in this document), it is not clear under which assumptions —related to the boundary cosmological constants in particular— are to be made in order to derive BPZ-type differential equations for boundary Toda CFTs.

We investigate this question in a forthcoming work based on the explicit expression of the descendant fields obtained from the Ward identities. The method developed in [16] should allow us to address this issue.

1.3.3. *Integrability of the theory.* The derivation of a BPZ-type differential equation is a major step in the understanding of features of the theory related to its integrability. For instance in the case of boundary Liouville CFT these differential equations are at the heart of the computation of the structure constants [4], while for the closed \mathfrak{sl}_3 Toda CFT they enter the derivation [13, 12] of the Fateev-Litvinov formula [21] for a family of three-point correlation functions.

For boundary Toda CFTs there are no known expression proposed for the structure constants of the theories considered in this article, apart from the formula for a one-point function proposed in [22, Equation (4.29)] (that should correspond to our model with mixed boundary conditions though the precise connection is not completely clear at the time being) — the formula proposed in [22, Equation (4.26)] corresponding to a model not constructed here. Being able to compute an expression for some structure constants would be a major improvement in the understanding of Toda CFTs.

Note that in both the closed and boundary cases, it is not known at the time being whether it would be possible or not to compute all the structure constants of the theory. Nonetheless there is good hope that the probabilistic framework introduced in the present document would provide a solid framework to investigate this issue. In particular we expect to be able to compute some of the structure constants associated to boundary Toda CFTs, in the two cases that we have constructed here. Additional details on this issue are gathered in Section 6.

1.3.4. *Definition of all the models.* In this document we have constructed several models associated to boundary Toda CFT, in relation with the element of the outer automorphism group considered. However when this element is different from the identity we have defined a CFT that satisfies the assumptions related to higher-spin symmetry and boundary conditions, but it is assumed in the physics that it is possible to define other models (though no path integral should be valid). It would be interesting to investigate to what extent the probabilistic tools introduced in the present document would allow one to make sense of this theory, and understand to what extent this phenomenon is related to orbifolding the theory.

Acknowledgements: B.C. would like to thank the Aix-Marseille University where part of this work has been undertaken and acknowledges support from the Eccellenza grant 194648 of the Swiss National Science Foundation; he is a member of NCCR SwissMAP. N.H. is grateful to Rémi Rhodes for having introduced him to conformal field theory and for many enlightening discussions.

2. BACKGROUND AND NOTATIONS

The purpose of this section is to provide the reader with the necessary tools to make sense of Toda conformal field theories. We recall some basic notions related to conformal geometry in dimension two as well as semi-simple and complex Lie algebras.

2.1. Compact Riemann surfaces. If Σ is a compact Riemann surface with boundary $\partial\Sigma$, we denote by $\mathbf{g}(\Sigma)$ its genus and by $\mathbf{k}(\Sigma)$ the number of connected components of $\partial\Sigma$. The Euler characteristic is then defined by $\chi(\Sigma) = 2(1 - \mathbf{g}(\Sigma)) - \mathbf{k}(\Sigma) \leq 2$. The surfaces of Euler characteristic < 0 are called of hyperbolic type. The only (up to biholomorphism) surfaces of Euler characteristic ≥ 0 are the torus, the sphere, the disk and the cylinder. In the construction of Toda CFT we will consider hyperbolic surfaces. The case of the disk (or equivalently the complex upper half-plane) is discussed in section 6. The Gauss-Bonnet theorem for a compact Riemann surface equipped with a smooth Riemannian metric g states that

$$(2.1) \quad \int_{\Sigma} R_g dv_g + 2 \int_{\partial\Sigma} k_g d\lambda_g = 4\pi\chi(\Sigma)$$

where R_g is the Ricci scalar curvature (that equals twice the Gaussian curvature) and k_g the geodesic curvature along the boundary. Two smooth Riemannian metrics g and g' are said to be conformally equivalent if there exists a smooth function φ on Σ such that $g' = e^\varphi g$. The set of all metrics conformally equivalent to a given metric g is called the conformal class of g and is denoted $[g]$. We recall here the following useful identities:

$$(2.2) \quad R_{g'} = e^{-\varphi}(R_g - \Delta_g \varphi) \quad ; \quad k_{g'} = e^{-\varphi/2}(k_g + \partial_{\bar{n}_g} \varphi)$$

where Δ_g is the Laplacian and $\partial_{\bar{n}_g}$ is the outer normal derivative (the Neumann operator). For a surface without boundary, the uniformization theorem implies that there exists a unique metric of constant Gaussian curvature called uniform metric (up to scaling and isometry). For a surface with boundary, it is shown in [32] that there exist:

- a unique uniform metric of type 1, that is with constant Gaussian curvature and 0 geodesic curvature along the boundary;
- a unique uniform metric of type 2, which means that it is flat and such that the geodesic curvature is constant.

Again uniqueness has to be understood up to scaling and isometry.

We now describe the doubling of a hyperbolic surface with boundary. Let Σ be a hyperbolic Riemann surface. We equip Σ with the uniform type 1 metric g . It follows from the Gauss-Bonnet theorem that the metric g is hyperbolic, *i.e.* R_g is a negative constant (that one usually choose equal to -2 but this is not relevant for the present discussion). It is readily seen that there exists a closed surface $\hat{\Sigma} = \Sigma \sqcup \partial\Sigma \sqcup \sigma(\Sigma)$ such that σ is an involution that preserves the boundary: $\sigma \circ \sigma = \text{Id}$ and $\sigma|_{\partial\Sigma} = \text{Id}$. (Intuitively, one "glues" Σ with a copy of itself along the boundary). We naturally define the metric on $\sigma(\Sigma)$ as the pullback of g by σ . Further, since the boundaries of Σ and $\sigma(\Sigma)$ are geodesics of same length, the metric g extends to a metric \hat{g} on $\hat{\Sigma}$ given on $\sigma(\Sigma)$ by the pullback σ^*g . Since g is smooth on Σ , the extended metric \hat{g} is smooth away from $\partial\Sigma$. Near each point x in $\partial\Sigma$, since the metric g is

hyperbolic, there is an isometry from the geodesic ball $B_{\hat{g}}(x, \varepsilon)$ (with $\varepsilon > 0$ small enough) to the hyperbolic ball $B_{g_P}(i, \varepsilon)$ in the complex upper half-plane \mathbb{H} , such that the $\sigma(\Sigma)$ -part of the geodesic ball in $\hat{\Sigma}$ is sent to the left-hand side of the imaginary axis in \mathbb{H} and the Σ -part of the ball is sent to the right of the imaginary axis. This provides local complex coordinates z so that $\hat{g} = g_P = |dz|^2/\text{Im}(z)^2$ near x , showing that \hat{g} is smooth on $\hat{\Sigma}$.

2.2. Green functions. In this paper we will be interested in three different PDE problems on a compact Riemann surface. If the boundary of the surface is empty then we will be interested in what we call the closed Poisson problem, while if the boundary is non-empty we have to specify boundary conditions, that will be either of Neumann type or of Dirichlet type. In this section we describe these problems, then define and study their Green functions.

Given a compact Riemann surface Σ equipped with a Riemannian metric g such that $\partial\Sigma = \emptyset$, the closed Poisson problem on (Σ, g) is

$$(2.3) \quad \begin{cases} \Delta_g u = f - m_g(f) & \text{in } \Sigma \\ m_g(u) = 0 \end{cases}$$

where $f \in H^1(\Sigma)$ and $m_g(u)$ is the average of u over Σ , with respect to the metric g ($m_g(u) = v_g(\Sigma)^{-1} \int_{\Sigma} u dv_g$). The Green function of this problem is defined as the solution G_g of the following weak formulated problem

$$(2.4) \quad \begin{cases} \Delta_g G_g(x, y) = \delta(x - y) - \frac{1}{v_g(\Sigma)} & \text{in } \Sigma \\ m_g(G_g(x, \cdot)) = 0 \end{cases} \quad \text{for all } x \in \Sigma$$

where δ is the Dirac delta function. The Green function G_g enjoys the property that

$$(2.5) \quad - \int_{\Sigma} \Delta_g f(y) G_g(x, y) dv_g(y) = f(x) - m_g(f)$$

for any function f smooth enough. Finally, if $g' = e^{\varphi} g$ is in the conformal class of the metric g then the function

$$(2.6) \quad G_{g'}(x, y) := G_g(x, y) - m_{g'}(G_g(x, \cdot)) - m_{g'}(G_g(\cdot, y)) + \frac{1}{v_{g'}(\Sigma)^2} \iint_{\Sigma \times \Sigma} G_g(x, y) dv_{g'}(x) dv_{g'}(y)$$

solves the problem (2.4) in the metric g' .

We now turn to the case where $\partial\Sigma \neq \emptyset$. We will be interested in two different boundary conditions, namely the Neumann problem

$$(2.7) \quad \begin{cases} \Delta_g u = f - m_g(f) & \text{in } \Sigma \\ \partial_{\vec{n}_g} u = 0 & \text{along } \partial\Sigma \\ m_g(u) = 0 \end{cases}$$

and the Dirichlet problem

$$(2.8) \quad \begin{cases} \Delta_g u = f - m_g(f) & \text{in } \Sigma \\ u = 0 & \text{along } \partial\Sigma \end{cases}$$

where $f \in H_0^1(\Sigma)$. The Neumann and Dirichlet Green functions are defined respectively as the solutions of

$$(2.9) \quad \begin{cases} \Delta_g G_g^N(x, y) = \delta(x - y) - \frac{1}{v_g(\Sigma)} & \text{in } \Sigma \\ \partial_{\bar{n}_g} G_g^N(x, y) = 0 & \text{along } \partial\Sigma \\ m_g(G_g^N(\cdot, y)) = 0 \end{cases}$$

and

$$(2.10) \quad \begin{cases} \Delta_g G_g^D(x, y) = \delta(x - y) & \text{in } \Sigma \\ G_g^D(x, y) = 0 & \text{along } \partial\Sigma \end{cases}$$

The Neumann Green function has the following useful property:

$$(2.11) \quad \int_{\partial\Sigma} \partial_{\bar{n}_g} f(y) G_g^N(x, y) d\lambda_g(y) - \int_{\Sigma} \Delta_g f(y) G_g^N(x, y) dv_g(y) = f(x) - m_g(f)$$

where f is a suitable function and $x \in \Sigma$, while the Dirichlet Green function satisfies (2.5). Both also enjoy the same change of metric formula (2.6) as the closed Green function G_g .

We now provide a link between the closed Green function and the boundary ones, using the doubling of surface described in the previous section.

Lemma 2.1. *Let Σ be a compact surface of hyperbolic type with boundary $\partial\Sigma \neq \emptyset$, equipped with a uniform type 1 metric g . Let $\hat{\Sigma}$ be the doubled surface equipped with the smooth hyperbolic metric \hat{g} and σ bet the canonical involution. Then for $x, y \in \Sigma$ we have*

$$(2.12) \quad G_g^N(x, y) = G_{\hat{g}}(x, y) + G_{\hat{g}}(x, \sigma(y))$$

and

$$(2.13) \quad G_g^D(x, y) = G_{\hat{g}}(x, y) - G_{\hat{g}}(x, \sigma(y))$$

Proof. We first have to show that the right-hand side of (2.12) is a solution to the problem (2.9). We compute the Laplacian

$$\Delta_g(G_{\hat{g}}(x, y) + G_{\hat{g}}(x, \sigma(y))) = \delta(x - y) - \frac{1}{v_{\hat{g}}(\hat{\Sigma})} + \delta(x - \sigma(y)) - \frac{1}{v_{\hat{g}}(\hat{\Sigma})} = \delta(x - y) - \frac{1}{v_g(\Sigma)}.$$

Then we investigate the outer normal derivative along $\partial\Sigma$. Since it holds that $G_{\hat{g}}(x, y) = G_{\hat{g}}(\sigma(x), \sigma(y))$, then

$$\begin{aligned} \partial_{\bar{n}_g}(\hat{G}_g(x, y) + \hat{G}_g(x, \sigma(y))) &= \partial_{\bar{n}_g} \hat{G}_g(x, y) - \partial_{\bar{n}_g} \hat{G}_g(x, y) = 0 \quad \text{and likewise} \\ \int_{\Sigma} (G_{\hat{g}}(x, y) + G_{\hat{g}}(x, \sigma(y))) dv_g(y) &= \int_{\hat{\Sigma}} G_{\hat{g}}(x, y) dv_{\hat{g}}(y) = 0. \end{aligned}$$

Combining these properties with the symmetry of the Green functions implies the result. The proof of (2.13) is immediate once one recalls that $\sigma = \text{Id}$ on the boundary of Σ . \square

We recall here some estimates from [26] on the Green function $G_{\hat{g}}$ on a compact surface Σ without boundary equipped with a hyperbolic metric g .

Proposition 2.2. *On a compact surface Σ without boundary equipped with a hyperbolic metric g , the Green function G_g can be written when x and y are closed to each other as*

$$(2.14) \quad G_g(x, y) = -\frac{1}{2\pi} \log d_g(x, y) + f_g(x, y)$$

with f_g a smooth function on Σ^2 . If g' is another metric in the conformal class of the hyperbolic metric g , then the previous estimate holds near the diagonal with $f_{g'}$ continuous. Moreover, for any x_0 in Σ there are coordinates $z \in B(0, 1) \subset \mathbb{C}$ such that $g = \frac{4|dz|^2}{(1-|z|^2)^2}$ and

$$(2.15) \quad G_g(z, z') = -\frac{1}{2\pi} \log |z - z'| + F(z, z')$$

near x_0 , with F smooth.

2.3. Finite-dimensional simple Lie algebras. The definition of Toda CFTs relies on a Lie algebra structure. We gather in this section some classical facts about finite-dimensional simple and complex Lie algebras, and introduce the notations that will be used throughout the paper. Additional details can be found *e.g.* in the textbook [30].

In virtue of a celebrated classification, it is well-known that every finite-dimensional simple and complex Lie algebra is isomorphic to either one of the following family of classical Lie algebras: A_n for $n \geq 1$ (corresponding to $\mathfrak{sl}_{n+1}(\mathbb{C})$), B_n for $n \geq 2$ (corresponding to $\mathfrak{so}_{2n+1}(\mathbb{C})$), C_n for $n \geq 3$ (corresponding to $\mathfrak{sp}_{2n}(\mathbb{C})$) and D_n for $n \geq 4$ (corresponding to $\mathfrak{so}_{2n}(\mathbb{C})$), or one of the five exceptional simple Lie algebras: E_6 , E_7 , E_8 , F_4 and G_2 . For any finite-dimensional simple complex Lie algebra \mathfrak{g} there is a natural Euclidean space \mathfrak{a} which is such that the Cartan subalgebra of \mathfrak{g} decomposes as $\mathfrak{a} + i\mathfrak{a}$. To this Euclidean space is attached a scalar product $\langle \cdot, \cdot \rangle$ which is proportional to the Killing form of \mathfrak{g} . The Euclidean space $(\mathfrak{a}, \langle \cdot, \cdot \rangle)$ is unique up to isomorphism and can be realized as \mathbb{R}^r equipped with its standard scalar product, where r is the rank of \mathfrak{g} .

The so-called simple roots $(e_i)_{i=1, \dots, r}$ form a particular basis of \mathfrak{a} ; this basis is such that $2\langle e_i, e_j \rangle / \langle e_i, e_i \rangle = A_{i,j}$ where A is the Cartan matrix of \mathfrak{g} . In the sequel we will consider that the scalar product is renormalized in such a way that the longest root has length 2, which is a standard assumption. The dual basis of (e_i) is given by $(\omega_i)_{i=1, \dots, r}$ where the

$$\omega_i = \sum_{j=1}^r (A^{-1})_{i,j} e_j \text{ for } j = 1, \dots, r$$

are called the fundamental weights. They verify

$$\langle e_i^\vee, \omega_j \rangle = \delta_{ij} \quad ; \quad \langle \omega_i, \omega_j \rangle = (A^{-1})_{i,j}$$

where $e_i^\vee = 2 \frac{e_i}{\langle e_i, e_i \rangle}$ is the coroot of e_i . Another important object is the Weyl vector, defined by $\rho = \sum_{i=1}^r \omega_i$, which is such that $\langle \rho, e_i^\vee \rangle = 1$ for all i . We also define the Weyl vector associated to the coroots by $\rho^\vee = \sum_{i=1}^r \omega_i^\vee$ where ω_i^\vee is defined so that $\langle \omega_i^\vee, e_j \rangle = \delta_{ij}$ for $i = 1, \dots, r$.

2.4. The outer automorphism group of simple Lie and W -algebras. There is yet a fundamental notion to be presented in relation with the admissible boundary conditions of the theory as discussed in the introduction, that is a certain type of transformations that would preserve the symmetry of the model. The notion of “outer automorphism” (to use the terminology of [27]) of a W -algebra naturally arises in this setting.

Namely, let us consider \mathfrak{g} to be a finite-dimensional simple and complex Lie algebra like in the previous subsection. Then an automorphism τ of \mathfrak{g} induces an automorphism of the W -algebra on \mathfrak{g} , that is roughly a bijective map preserving the commutation relations. The automorphism group of \mathfrak{g} is a semi-direct product $\text{Aut}(\mathfrak{g}) = \text{Out}(\mathfrak{g}) \ltimes \text{Inn}(\mathfrak{g})$, where elements of $\text{Out}(\mathfrak{g})$ are called outer automorphisms while $\text{Inn}(\mathfrak{g})$ is generated by elements of the form $\text{Ad}(g)$ where g lies in the Lie group with Lie algebra \mathfrak{g} and Ad is the adjoint map. It is however argued in [23, Section 3.5] that the W -algebra on \mathfrak{g} transformed using such an inner automorphism of \mathfrak{g} does not feature twisted boundary conditions.

We are therefore naturally led to considering the outer automorphism group of \mathfrak{g} in order to “twist” our algebra of symmetry: the latter being thus defined as the quotient $\text{Out}(\mathfrak{g}) = \text{Aut}(\mathfrak{g})/\text{Inn}(\mathfrak{g})$. Standard results then show that for a finite-dimensional simple and complex Lie algebra \mathfrak{g} this group is either: trivial, this is the case for $A_1, B_n, C_n, E_7, E_8, F_4$ and G_2 , or isomorphic to the cyclic group of order two in all the other cases, except for D_4 where the outer automorphism group is isomorphic to the group of permutations on three elements (thus of order 6). In all the cases, the non-trivial outer automorphism considered will be denoted by a star \cdot^* .

As an example, let us consider the Lie algebra $A_2(\mathfrak{sl}_3)$. The non-trivial outer automorphism has the effect of swapping the two simple roots, that is, if $u = u_1e_1 + u_2e_2 \in \mathfrak{a}$, then $u^* = u_2e_1 + u_1e_2$. At the level of the W_3 algebra, this transformation translates in terms of the two generators (\mathbf{T}, \mathbf{W}) of the symmetry as the involution $(\mathbf{T}, \mathbf{W}) \rightarrow (\mathbf{T}, -\mathbf{W})$. In general for A_n the non-trivial automorphism is given by $e_i \mapsto e_{n-i}$ and the resulting transformation on the generators of the symmetry is $\left(\mathbf{W}^{(s)}\right)_{2 \leq s \leq n} \rightarrow \left((-1)^s \mathbf{W}^{(s)}\right)_{2 \leq s \leq n}$ where s denotes the spin of the operator. For D_4 the five non-trivial automorphisms are the non-trivial permutations of the three orthogonal simple roots.

An important feature of the outer automorphisms of order two (*i.e.* any non-trivial outer automorphism except the two 3-cycles of $\text{Out}(D_4)$) is that they are self-adjoint with respect to the scalar product on \mathfrak{a} . Any non-trivial outer automorphism of order two thus allows one to decompose the space \mathfrak{a} as the orthogonal sum $\mathfrak{a} = \mathfrak{a}_N \oplus \mathfrak{a}_D$, where $\mathfrak{a}_N = \text{Span}(v + v^*, v \in \mathfrak{a})$ and $\mathfrak{a}_D = \text{Span}(v - v^*, v \in \mathfrak{a})$. For D_4 the situation is different and we discuss this case in Subsection 5.3. We refer to the Appendix A for a more exhaustive treatment of these questions.

3. TCFT ON A SURFACE WITHOUT BOUNDARIES

In this section, we propose a mathematical construction of Toda CFTs based on a Riemann surface without boundary, and associated to any semi-simple and complex Lie algebra \mathfrak{g} . For this purpose we will introduce a probabilistic framework based on Gaussian Free Fields (GFFs hereafter) and Gaussian Multiplicative Chaos (GMC in the sequel). Throughout

this section, Σ will denote a compact Riemann surface without boundary, assumed to be of hyperbolic type in the sense that $\chi(\Sigma) < 0$. We equip Σ with a hyperbolic metric g .

3.1. Gaussian Free Field. The action functional entering the path integral definition of the Toda CFTs (1.5) is made of several terms. Our first goal is to interpret the quadratic component, that is, to make sense of

$$(3.1) \quad \int F(\varphi) e^{-\frac{1}{4\pi} \int_{\Sigma} |d_g \varphi|_g^2 dv_g} D_g \varphi.$$

This is achieved by the introduction of the Gaussian Free Field (GFF hereafter). Indeed, the formal integral (3.1) can be written as

$$(3.2) \quad \int F(\varphi) e^{-\frac{1}{2} \int_{\Sigma} \langle \varphi, -\frac{1}{2\pi} \Delta_g \varphi \rangle_g dv_g} D_g \varphi$$

which describes (formally) the law of a Gaussian field with mean 0 and positive definite covariance kernel $(-\frac{1}{2\pi} \Delta_g)^{-1} = 2\pi G_g$ (where the equality stands in the sense of integral kernels). We thus define the GFF \mathbf{X}_g by setting

$$\mathbb{E}[\langle \langle u, \mathbf{X}^g \rangle, f \rangle_g \langle \langle v, \mathbf{X}^g \rangle, f' \rangle_g] = 2\pi \langle u, v \rangle \int_{\Sigma^2} G_g(x, y) f(x) f(y) dv_g(x) dv_g(y)$$

for all $f \in H^1(\Sigma \rightarrow \mathfrak{a}, g)$ (i.e. all components of f live in $H^1(\Sigma, g)$), where $(\cdot, \cdot)_g$ denotes the usual pairing. For the sake of simplicity we will use the notation

$$(3.3) \quad \mathbb{E}[\langle u, \mathbf{X}^g(x) \rangle \langle v, \mathbf{X}^g(y) \rangle] = 2\pi \langle u, v \rangle G_g(x, y)$$

in the rest of the document. Note that \mathbf{X}^g can be represented as the sum $\mathbf{X}^g = \sum_{i=1}^r X_i^g \mathbf{v}_i$ where the $(X_i^g)_{1 \leq i \leq r}$ are independent, real-valued GFFs over Σ , and $(\mathbf{v}_i)_{1 \leq i \leq r}$ is any orthonormal basis of \mathfrak{a} . The theory of GFF is well-known and we refer for example to [8, 35] for a review.

Remark 3.1. *The GFF is a distribution rather than a well-defined function. As so, it acts on the space of functions $H^1(\Sigma \rightarrow \mathfrak{a}, g)$ through the usual pairing. The definition above, which is the usual one, is valid outside the kernel of the action in (3.1) which is the space of constant functions on Σ . We will take this fact into account in the definition of the full path integral.*

For the picture to be complete, it misses a multiplicative normalization term called the partition function: this corresponds to the total mass of the path integral (3.1). It is possible to make sense of this quantity as a regularized determinant. For details about determinant of Laplacians, we refer to [32] and [1]. We give the definition here.

Definition 3.2. Let us denote by $0 = \lambda_0 < \lambda_1 \leq \dots$ the eigenvalues of the Laplacian Δ_g^1 acting on functions $\Sigma \rightarrow \mathbb{R}$. The partition function of the GFF \mathbf{X}^g is defined by

$$\mathcal{Z}_{\mathbf{X}}(\Sigma, g) = \det' \left(-\frac{\Delta_g^1}{2\pi v_g(\Sigma)} \right)^{-\frac{r}{2}}.$$

where the regularized determinant is given by

$$\det'(-(2\pi)^{-1}\Delta_g^1) = e^{-\zeta'(0)}, \quad \zeta(s) = \sum_{n=1}^{\infty} \left(\frac{\lambda_n}{2\pi}\right)^{-s}.$$

An important property of the GFF is its behavior under conformal changes of metric. Thanks to equation (2.6), \mathbf{X}^g verifies the equality in distribution

$$(3.4) \quad \mathbf{X}^{g'} \stackrel{(\text{law})}{=} \mathbf{X}^g - m_{g'}(\mathbf{X}^g)$$

if $g' \in [g]$. Further, one obtains the following conformal anomaly for the partition function

Proposition 3.3. *Let $g' = e^\varphi g$ be a metric in the conformal class of g . Then we have*

$$(3.5) \quad \mathcal{Z}_{\mathbf{X}}(\Sigma, g') = \mathcal{Z}_{\mathbf{X}}(\Sigma, g) \exp\left(\frac{r}{96\pi} \int_{\Sigma} (|d_g \varphi|_g^2 + 2R_g \varphi) dv_g\right)$$

Proof. This is a straightforward consequence of [32], in which it is proved that

$$(3.6) \quad \log\left(\frac{\det'(-\Delta_{g'})}{v_{g'}(\Sigma)}\right) = \log\left(\frac{\det'(-\Delta_g)}{v_g(\Sigma)}\right) - \frac{1}{48\pi} \int_{\Sigma} (|d_g \varphi|_g^2 + 2R_g \varphi) dv_g.$$

□

Eventually, we recall the following version of the Girsanov (or Cameron-Martin) theorem that will be used throughout the paper:

Theorem 3.4. *Let M be a Riemann surface equipped with a Riemannian metric g , $(X(x))_{x \in M} = (X_1(x), \dots, X_n(x))_{x \in M}$ a family of smooth centered Gaussian fields on M and Z a random variable in the L^2 -closure of $\text{Span}(X(x))_{x \in M}$. Then for any bounded functional F over the space of continuous functions,*

$$\mathbb{E} \left[e^{Z - \frac{1}{2}\mathbb{E}[Z^2]} F(X(x))_{x \in M} \right] = \mathbb{E} [F(X(x) + \mathbb{E}[ZX(x)])_{x \in M}].$$

3.2. Gaussian Multiplicative Chaos. Having given a meaning to the quadratic term that appears in the action (1.6), we now would like to give a rigorous interpretation of the exponential potential that appears there. To be more specific, given a vector $u \in \mathfrak{a}$ we want to define the exponential of the projected field $e^{\langle u, \mathbf{X}^g \rangle} dv_g$. However, since \mathbf{X}^g is far from being a continuous well-defined function we need to introduce a regularization procedure. This method is at the heart of the theory of Gaussian Multiplicative Chaos (GMC hereafter) — for details about the theory of GMC, see for instance [40, 39, 8].

Let X^g be a GFF on Σ . Define the regularized field $X_\varepsilon^g(x)$ as the average of the GFF X^g on the geodesic circle centered at x of small radius $\varepsilon > 0$. It can be shown [19] that the random variable $X_\varepsilon^g(x)$ has an almost surely continuous version that we will consider from now on. We then have the following results (Lemma 3.2 in [26]), in the case where g is hyperbolic.

Lemma 3.5. *Let $C_g(x, \varepsilon)$ be the geodesic circle centered at x of radius $\varepsilon > 0$. Then the regularized field X_ε^g has the following covariance structure:*

$$(3.7) \quad \mathbb{E}[X_\varepsilon^g(x)X_\varepsilon^g(y)] = 2\pi \iint G_g(x', y') d\mu_{x,\varepsilon}(x') d\mu_{x,\varepsilon}(y')$$

where $\mu_{x,\varepsilon}$ is the uniform probability measure on $C_g(x, \varepsilon)$, that is $\int \cdot d\mu_{x,\varepsilon} = \frac{1}{\lambda_g(C_g(x,\varepsilon))} \int_{C_g(x,\varepsilon)} \cdot d\lambda_g$. Moreover, as $\varepsilon \rightarrow 0$,

$$(3.8) \quad \mathbb{E}[X_\varepsilon^g(x)^2] = -\log \varepsilon + W_g(x) + o(1)$$

uniformly in x , with W_g a smooth function on $\hat{\Sigma}$.

We then define the regularized field \mathbf{X}_ε^g by averaging the field \mathbf{X}^g . One can adapt Lemma 3.5 to infer the following:

Lemma 3.6. *In the setting of Lemma 3.5 the field \mathbf{X}^g has the following covariance structure:*

$$(3.9) \quad \mathbb{E}[\langle u, \mathbf{X}_\varepsilon^g(x) \rangle \langle v, \mathbf{X}_\varepsilon^g(y) \rangle] = 2\pi \langle u, v \rangle \iint G_g(x', y') d\mu_{x,\varepsilon}(x') d\mu_{x,\varepsilon}(y')$$

for any $u, v \in \mathfrak{a}$. Besides when $\varepsilon \rightarrow 0$ and uniformly in x

$$(3.10) \quad \mathbb{E}[\langle u, \mathbf{X}_\varepsilon^g(x) \rangle^2] = -|u|^2 \log \varepsilon + |u|^2 W_g(x) + o(1).$$

From Lemma 3.6 we deduce that one needs to rescale the exponential of the regularized field by a power of ε in order to obtain a finite limit. Namely, we define

$$(3.11) \quad e^{\langle u, \mathbf{X}^g(x) \rangle} dv_g(x) := \lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{|u|^2}{2}} e^{\langle u, X_\varepsilon^g(x) \rangle} dv_g(x).$$

Thanks to the theory of GMC it can be shown that, as soon as $|u|^2 < 4$, the above convergence stands in probability in the space of Radon measures equipped with the weak topology. Moreover, the limiting measure is finite and non-trivial.

Lemma 3.6 shows that the limit is the same as

$$(3.12) \quad \lim_{\varepsilon \rightarrow 0} e^{\langle u, \mathbf{X}_\varepsilon^g(x) \rangle - \frac{1}{2} \mathbb{E}[\langle u, \mathbf{X}_\varepsilon^g(x) \rangle^2]} e^{\frac{|u|^2}{2} W_g(x)} dv_g(x)$$

As written in the action (1.6) we only need to consider some particular vectors u in the direction of the simple roots $(e_i)_{1 \leq i \leq r}$, that is we take $u = \gamma e_i$ for $i = 1, \dots, r$. Then $|u|^2 = \gamma^2 \langle e_i, e_i \rangle$: since the longest roots are such that $\langle e_i, e_i \rangle = 2$ this explains the constraint that γ must belong to $(0, \sqrt{2})$.

3.3. Probabilistic interpretation of the path integral. We are now ready to give a probabilistic interpretation of the path integral of Toda CFTs (1.5). Here and below F will stand for any bounded continuous functional on $H^{-s}(\Sigma \rightarrow \mathfrak{a}, g)$, where g is any Riemannian metric. The background charge Q is chosen to be given by

$$(3.13) \quad Q := \gamma \rho + \frac{2}{\gamma} \rho^\vee.$$

This choice is made so as to ensure conformal invariance of the model. The cosmological constants μ_i are non-negative for $1 \leq i \leq r$.

In agreement with the previous discussion, the path integral (1.5) is then defined by considering limits of the form

$$(3.14) \quad \langle F \rangle_g := \lim_{\varepsilon \rightarrow 0} \langle F \rangle_{g,\varepsilon}, \quad \text{where} \quad \langle F \rangle_{g,\varepsilon} := \mathcal{Z}_{\mathbf{X}}(\Sigma, g) \int_{\mathfrak{a}} \mathbb{E} \left[F(\mathbf{X}_\varepsilon^g + \mathbf{c}) e^{-V_g(\mathbf{X}_\varepsilon^g + \mathbf{c})} \right] d\mathbf{c}$$

with the regularized Toda potential given by

$$(3.15) \quad V_g(\mathbf{X}_\varepsilon^g + \mathbf{c}) = \frac{1}{4\pi} \int_{\Sigma} R_g \langle Q, \mathbf{X}_\varepsilon^g + \mathbf{c} \rangle dv_g + \sum_{i=1}^r \mu_i e^{\langle \gamma e_i, \mathbf{c} \rangle} \int_{\Sigma} \varepsilon^{\gamma^2} e^{\langle \gamma e_i, \mathbf{X}_\varepsilon^g \rangle} dv_g.$$

The integration with respect to the zero-mode \mathbf{c} of the field is defined by

$$\int_{\mathfrak{a}} F(\mathbf{c}) d\mathbf{c} = \int_{\mathbb{R}^r} F \left(\sum_{i=1}^r \mathbf{v}_i c_i \right) \prod_{i=1}^r dc_i$$

where $(\mathbf{v}_i)_{1 \leq i \leq r}$ is any orthonormal basis of \mathfrak{a} .

To ensure that the Toda path integral gives rise to a well-defined positive measure we have to show that the limit above exists and belongs to $(0, \infty)$, when F is taken equal to 1. Indeed, in this case the mapping $F \mapsto \langle F \rangle_g / \langle 1 \rangle_g$ defines such a measure on the observables (*i.e.* suitable functionals on $H^{-s}(\Sigma \rightarrow \mathfrak{a}, g)$). The following statement addresses this issue:

Proposition 3.7. *Assume that $\chi(\Sigma) < 0$ and take $F = 1$. Then the limit (3.14) exists and belongs to $(0, \infty)$.*

Proof. Without loss of generality one can choose to work with a metric g_0 conformally equivalent to g with constant Ricci curvature that we suppose equal to 1. This is not a restriction thanks to the conformal anomaly formula (see Proposition 3.9 below). In this setting the partition function writes

$$(3.16) \quad \mathcal{Z}_{\mathbf{X}}(\Sigma, g_0) \int_{\mathfrak{a}} e^{-\langle Q, \mathbf{c} \rangle \chi(\Sigma)} \mathbb{E} \left[\exp \left(- \sum_{i=1}^r \mu_i e^{\langle \gamma e_i, \mathbf{c} \rangle} M_{i,\varepsilon} \right) \right] d\mathbf{c}$$

where we use the shortcut $M_{i,\varepsilon} := \int_{\Sigma} \varepsilon^{\gamma^2} e^{\langle \gamma e_i, \mathbf{X}_\varepsilon^{g_0} \rangle} dv_{g_0}$ for $i = 1, \dots, r$. We then use Fubini to swap the expectation and the integral and make the changes of variables $v_i = \mu_i e^{\langle \gamma e_i, \mathbf{c} \rangle} M_{i,\varepsilon}$, to obtain

$$(3.17) \quad \mathcal{Z}_{\mathbf{X}}(\Sigma, g_0) \prod_{i=1}^r \left(\frac{\Gamma \left(-\frac{\langle Q, \omega_i^\vee \rangle \chi(\Sigma)}{\gamma} \right) \mu_i^{\frac{\langle Q, \omega_i^\vee \rangle \chi(\Sigma)}{\gamma}}}{\gamma} \right) \mathbb{E} \left[\prod_{i=1}^r M_{i,\varepsilon}^{\frac{\langle Q, \omega_i^\vee \rangle \chi(\Sigma)}{\gamma}} \right]$$

where Γ is the Euler Gamma function. The expectation term can be upper-bounded thanks to Hölder's inequality for some $p_i \geq 1$ and such that $\sum_{i=1}^r \frac{1}{p_i} = 1$:

$$(3.18) \quad \mathbb{E} \left[\prod_{i=1}^r M_{i,\varepsilon}^{\frac{q\chi(\Sigma)}{\gamma}} \right] \leq \prod_{i=1}^r \mathbb{E} \left[M_{i,\varepsilon}^{\frac{q\chi(\Sigma)}{p_i \gamma}} \right]^{p_i}$$

Now Proposition 3.6 in [40] ensures that we can take the limit as $\varepsilon \rightarrow 0$ and that the limit is finite, since GMC admits finite negative moments of all order. \square

Remark 3.8. *In the case where $\chi(\Sigma) \geq 0$ the partition function is infinite because of the behavior of the integral when $\langle \mathbf{c}, e_i \rangle \rightarrow -\infty$. We can still define the measure by $F \mapsto \langle F \rangle$ for suitable functionals, although it will have infinite total mass.*

We now study the behavior of the partition function under a conformal change of metric. The following formula is called Weyl anomaly.

Proposition 3.9. *Let g' be a metric in the conformal class of the hyperbolic metric g , with φ denoting the log-conformal factor. Then we have for any bounded functional F ,*

$$\langle F \rangle_{g'} = \langle F(\cdot - \frac{Q}{2}\varphi) \rangle_g \exp\left(\frac{r + 6|Q|^2}{96\pi} \int_{\Sigma} (|d_g\varphi|_g^2 + 2R_g\varphi) dv_g\right)$$

Remark 3.10. *This result allows one to identify the so-called central charge of the theory which is equal to $C = r + 6|Q|^2$. This result is consistent with the one obtained in [15] in the case of the sphere.*

Before actually proving this statement we actually need to collect a few results first. The first one is concerned with the property of conformal covariance of the GFF:

Lemma 3.11. *Let g' be conformally equivalent to the metric g ($g' = e^{\varphi}g$ with φ smooth). Then for all $F \in L^1(H^{-s}(\Sigma, \mathbb{P} \otimes d\mathbf{c}))$,*

$$\int_{\mathbf{a}} \mathbb{E}[F(\mathbf{X}^{g'} + \mathbf{c})] d\mathbf{c} = \int_{\mathbf{a}} \mathbb{E}[F(\mathbf{X}^g + \mathbf{c})] d\mathbf{c}.$$

Proof. Let $\tilde{\mathbf{X}}^g := \mathbf{X}^g - m_{g'}(\mathbf{X}^g)$. We first show that $\tilde{\mathbf{X}}^g$ has the same law as $\mathbf{X}^{g'}$. Indeed, for $u, v \in \mathbf{a}$ and $x, y \in \Sigma$ we have

$$\begin{aligned} \mathbb{E}[\langle u, \tilde{\mathbf{X}}^g(x) \rangle \langle v, \tilde{\mathbf{X}}^g(y) \rangle] &= 2\pi \langle u, v \rangle (G_g(x, y) - m_{g'}(G_g(x, \cdot)) - m_{g'}(G_g(\cdot, y)) \\ &\quad + v_{g'}(\Sigma)^{-2} \iint_{\Sigma^2} G_g(x, y) dv_{g'}(x) dv_{g'}(y)) \end{aligned}$$

which is equal to $2\pi \langle u, v \rangle G_{g'}(x, y)$. Since both fields are centered Gaussian fields this is enough to conclude. Now to prove the lemma we make a change of variables in the zero-mode:

$$\int_{\mathbf{a}} \mathbb{E}[F(\mathbf{X}^{g'} + \mathbf{c})] d\mathbf{c} = \int_{\mathbf{a}} \mathbb{E}[F(\mathbf{X}^g - m_{g'}(\mathbf{X}^g) + \mathbf{c})] d\mathbf{c} = \int_{\mathbf{a}} \mathbb{E}[F(\mathbf{X}^g + \mathbf{c})] d\mathbf{c}.$$

□

We now state the companion result for the GMC:

Lemma 3.12. *Let $g' = e^{\varphi}g$ be a metric in the conformal class of the hyperbolic metric g , and u in \mathbf{a} . Then the following equality holds in distribution:*

$$e^{\langle u, \mathbf{X}^{g'} \rangle} dv_{g'} = e^{(1 + \frac{|u|^2}{4})\varphi} e^{\langle u, \mathbf{X}^g \rangle} dv_g$$

and in particular

$$e^{\langle \gamma e_i, \mathbf{X}^{g'} \rangle} dv_{g'} = e^{\frac{\langle Q, \gamma e_i \rangle}{2}} \varphi e^{\langle \gamma e_i, \mathbf{X}^g \rangle} dv_g.$$

Proof. Adapting the argument in [26, Equations (3.11) and (3.12)] we see that, as $\varepsilon \rightarrow 0$, $\mathbb{E}[\langle u, \mathbf{X}_\varepsilon^{g'} \rangle^2]$ behaves like $-|u|^2 \log \varepsilon + |u|^2 W_g + \frac{|u|^2}{2} \varphi + o(1)$. The result follows from Lemma 3.6 and the fact that the limiting GMC measure is independent from the regularization scheme (a well-known fact, see for example [40]). The case where $u = \gamma e_i$ is handled using the definition of Q in Equation (3.13) which implies that $\langle Q, e_i \rangle = \gamma \frac{|e_i|^2}{2} + \frac{2}{\gamma}$. \square

We are now in position to prove Proposition 3.9.

Proof of Proposition 3.9. Thanks to Lemma 3.11 and the Gauss-Bonnet theorem we can restrict ourselves to

$$\int_{\mathbf{a}} e^{-\langle Q, \mathbf{c} \rangle \chi(\Sigma)} \mathbb{E} \left[F(\mathbf{X}^g + \mathbf{c}) \exp \left(-\frac{1}{4\pi} \int_{\Sigma} R_{g'} \langle Q, \mathbf{X}^g \rangle dv_{g'} - \sum_{i=1}^r \mu_i e^{\langle \gamma e_i, \mathbf{c} \rangle} \int_{\Sigma} e^{\langle \gamma e_i, \mathbf{X}^{g'} \rangle} dv_{g'} \right) \right] d\mathbf{c}.$$

Define $Y := -\frac{1}{4\pi} \int_{\Sigma} R_{g'} \langle Q, \mathbf{X}^g \rangle dv_{g'}$. In order to apply the Girsanov transform to get rid of the e^Y term, we shall compute the variance of Y . Using conformal covariance of the Ricci scalar curvature together with the fact that R_g is constant we see that the latter is equal to

$$\mathbb{E}[Y^2] = \frac{1}{16\pi^2} \mathbb{E} \left[\left(\int_{\Sigma} \Delta_g \varphi \langle Q, \mathbf{X}^g \rangle dv_g \right)^2 \right] = \frac{|Q|^2}{8\pi} \iint_{\Sigma^2} \Delta_g \varphi(x) \Delta_g \varphi(y) G_g(x, y) dv_g(x) dv_g(y)$$

by using Fubini. Thanks to the formula (2.5) for the closed Green function we obtain

$$\mathbb{E}[Y^2] = \frac{|Q|^2}{8\pi} \int_{\Sigma} |d_g \varphi|_g^2 dv_g.$$

In the same way, we obtain $\mathbb{E}[Y \langle u, \mathbf{X}^g \rangle] = -\frac{\langle Q, u \rangle}{2} (\varphi - m_g(\varphi))$ for any $u \in \mathbf{a}$. Using Girsanov transform we thus rewrite the partition function as

$$\int_{\mathbf{a}} e^{\frac{|Q|^2}{16\pi} \int_{\Sigma} |d_g \varphi|_g^2 - \langle Q, \mathbf{c} \rangle \chi(\Sigma)} \mathbb{E} \left[F(\mathbf{c} + \mathbf{X}^g - \frac{Q}{2} (\varphi - m_g(\varphi))) \exp \left(-\sum_{i=1}^r \mu_i e^{\langle \gamma e_i, \mathbf{c} + \frac{Q}{2} m_g(\varphi) \rangle} \int_{\Sigma} e^{-\frac{\langle Q, \gamma e_i \rangle}{2}} \varphi e^{\langle \gamma e_i, \mathbf{X}^{g'} \rangle} dv_{g'} \right) \right] d\mathbf{c}.$$

Now making the change of variables in the zero-mode $\mathbf{c} \leftrightarrow \mathbf{c} - \frac{Q}{2} m_g(\varphi)$ and using Lemma 3.12 we obtain

$$(3.19) \quad \int_{\mathbf{a}} e^{\frac{|Q|^2}{16\pi} \int_{\Sigma} |d_g \varphi|_g^2 - \langle Q, \mathbf{c} \rangle \chi(\Sigma) + \frac{|Q|^2}{2} \chi(\Sigma) m_g(\varphi)} \mathbb{E} \left[F(\mathbf{c} + \mathbf{X}^g - \frac{Q}{2} \varphi) \exp \left(-\sum_{i=1}^r \mu_i e^{\langle \gamma e_i, \mathbf{c} \rangle} \int_{\Sigma} e^{\langle \gamma e_i, \mathbf{X}^{g'} \rangle} dv_{g'} \right) \right] d\mathbf{c}$$

Finally, thanks again to Gauss-Bonnet we can write

$$\langle F \rangle_{g'} = e^{\frac{|Q|^2}{16\pi} \int_{\Sigma} (|d_g \varphi|_g^2 - 2R_g \varphi) dv_g} \mathcal{Z}_{\mathbf{X}}(\Sigma, g') \langle F(\cdot - \frac{Q}{2} \varphi) \rangle_g$$

so the conclusion follows from Equation (3.3). \square

Finally, the fact that the partition function is diffeomorphism invariant directly follows from the fact that all the objects involved are so. Indeed, if $\psi : \Sigma \rightarrow \Sigma$ is an orientation-preserving diffeomorphism, then

$$G_{\psi^*g}(x, y) = G_g(\psi(x), \psi(y)) \quad ; \quad R_{\psi^*g} = R_g \circ \psi \quad ; \quad k_{\psi^*g} = k_g \circ \psi \quad ; \quad \mathbf{X}^{\psi^*g} = \mathbf{X}^g \circ \psi.$$

As a consequence we have the following result:

Proposition 3.13. *Let $\psi : \Sigma \rightarrow \Sigma$ be an orientation-preserving diffeomorphism. Then for any bounded measurable $F : H^{-s}(\Sigma) \rightarrow \mathbb{R}$,*

$$\langle F \rangle_{\psi^*g} = \langle F(\cdot \circ \psi) \rangle_g$$

3.4. Correlation functions. Observables of main interest in Toda theory are the so-called Vertex Operators, formally defined as the exponential moments of the random distribution ϕ under the law described by the path integral (1.5), that is, $\langle e^{\langle \alpha, \phi(z) \rangle} \rangle_g$ for some $\alpha \in \mathfrak{a}$ and $z \in \Sigma$. Averaging products of Vertex Operators with respect to the law defined by the path integral amounts to defining their correlation functions: we study here the existence and some properties of such objects.

Consider a family of $N \in \mathbb{N}$ pairs $(x_k, \alpha_k)_{k=1, \dots, N}$ with $x_k \in \Sigma$ all distinct and $\alpha_k \in \mathfrak{a}$. The regularized Vertex Operator at x_k with weight α_k is defined by

$$(3.20) \quad V_{\alpha_k, \varepsilon}(x_k) := \varepsilon^{\frac{|\alpha_k|^2}{2}} e^{\langle \alpha_k, \mathbf{X}^g(x_k) + \mathbf{c} \rangle}.$$

To simplify the coming computations we also introduce the shortcut $\mathbf{V}_{\alpha, \varepsilon}(\mathbf{x}) = \prod_{k=1}^N V_{\alpha_k, \varepsilon}(x_k)$. The rigorous interpretation of the correlation function of the primary fields is then, if it makes sense, the limit

$$(3.21) \quad \begin{aligned} \left\langle \prod_{k=1}^N V_{\alpha_k}(x_k) \right\rangle_g &:= \lim_{\varepsilon \rightarrow 0} \left\langle \prod_{k=1}^N V_{\alpha_k, \varepsilon}(x_k) \right\rangle_{g, \varepsilon}, \quad \text{where} \\ \left\langle \prod_{k=1}^N V_{\alpha_k, \varepsilon}(x_k) \right\rangle_{g, \varepsilon} &:= \mathcal{Z}_{\mathbf{X}}(\Sigma, g) \int_{\mathfrak{a}} \mathbb{E} \left[\mathbf{V}_{\alpha, \varepsilon}(\mathbf{x}) e^{-V_g(\mathbf{X}_\varepsilon^g + \mathbf{c})} \right] d\mathbf{c}. \end{aligned}$$

Theorem 3.14. *The limit (3.21) exists and belongs to $(0, \infty)$ if and only if the following conditions (referred to as Seiberg bounds) hold:*

$$(3.22) \quad \left\langle \sum_{k=1}^N \alpha_k - Q\chi(\Sigma), \omega_i^\vee \right\rangle > 0 \quad \text{for all } i = 1, \dots, r;$$

$$(3.23) \quad \langle \alpha_k - Q, e_i \rangle < 0 \quad \text{for all } i = 1, \dots, r \text{ and } k = 1, \dots, N.$$

Remark 3.15. *If the Seiberg bounds hold then the mapping $F \mapsto \langle F \mathbf{V}_\alpha(\mathbf{x}) \rangle_g / \langle \mathbf{V}_\alpha(\mathbf{x}) \rangle_g$ defines a measure of probability on $H^{-s}(\Sigma \rightarrow \mathfrak{a}, g)$.*

Proof. We work first with the hyperbolic metric g on Σ . We will generalize later using the conformal anomaly formula for the correlation functions. The proof follows closely that in the sphere case [15]. Since g is hyperbolic the correlation function writes

$$A_\varepsilon := \int_{\mathfrak{a}} e^{-\langle Q, \mathbf{c} \rangle \chi(\Sigma)} \mathbb{E} \left[\left(\prod_{k=1}^N \varepsilon^{\frac{|\alpha_k|^2}{2}} e^{\langle \alpha_k, \mathbf{X}_\varepsilon^g(x_k) + \mathbf{c} \rangle} \right) e^{-\sum_{i=1}^r \mu_i e^{\langle \gamma e_i, \mathbf{c} \rangle} \int_\Sigma \varepsilon^{\gamma^2} e^{\langle \gamma e_i, \mathbf{X}_\varepsilon^g \rangle} dv_g} \right] d\mathbf{c}$$

When ε is small, Lemma 3.6 entails that

$$\varepsilon^{\frac{|\alpha_k|^2}{2}} e^{\langle \alpha_k, \mathbf{X}_\varepsilon^g(x_k) + \mathbf{c} \rangle} = e^{\langle \alpha_k, \mathbf{c} \rangle + \frac{|\alpha_k|^2}{2} W_g(x_k)} e^{\langle \alpha_k, \mathbf{X}^g \varepsilon(x_k) \rangle - \frac{1}{2} \mathbb{E}[\langle \alpha_k, \mathbf{X}_\varepsilon^g(x_k) \rangle^2]} (1 + o(1))$$

so that we can apply Girsanov theorem to $Y := \sum_{k=1}^n \langle \alpha_k, \mathbf{X}_\varepsilon^g(x_k) \rangle$. Its variance is

$$\mathbb{E}[Y^2] = \sum_{k=1}^N \mathbb{E}[\langle \alpha_k, \mathbf{X}_\varepsilon^g(x_k) \rangle^2] + 4\pi \sum_{k \neq l} \langle \alpha_k, \alpha_l \rangle G_{g,\varepsilon}(x_k, x_l)$$

so that $\prod_{k=1}^N e^{\langle \alpha_k, \mathbf{X}^g \varepsilon(x_k) \rangle - \frac{1}{2} \mathbb{E}[\langle \alpha_k, \mathbf{X}_\varepsilon^g(x_k) \rangle^2]} = e^{Y - \frac{1}{2} \mathbb{E}[Y^2]} \prod_{k \neq l} e^{2\pi \langle \alpha_k, \alpha_l \rangle G_{g,\varepsilon}(x_k, x_l)}$ with

$$G_{g,\varepsilon}(x, y) = \lambda_g(C_g(x, \varepsilon))^{-1} \lambda_g(C_g(y, \varepsilon))^{-1} \int_{C_g(x, \varepsilon)} \int_{C_g(y, \varepsilon)} G_g(x', y') d\lambda_g(x) d\lambda_g(y).$$

Furthermore $\mathbb{E}[Y \langle \gamma e_i, \mathbf{X}_\varepsilon^g(x) \rangle] = 2\pi \sum_{k=1}^N \langle \gamma e_i, \alpha_k \rangle G_{g,\varepsilon}(x_k, x)$ so that Girsanov theorem yields

$$A_\varepsilon = \left(\prod_{k \neq l} e^{2\pi \langle \alpha_k, \alpha_l \rangle G_{g,\varepsilon}(x_k, x_l)} \right) \left(\prod_{k=1}^N e^{\frac{|\alpha_k|^2}{2} W_g(x_k)} \right) \int_{\mathfrak{a}} e^{\langle \sum \alpha_k - Q \chi(\Sigma), \mathbf{c} \rangle} \mathbb{E} \left[e^{-\sum_{i=1}^r \mu_i e^{\langle \gamma e_i, \mathbf{c} \rangle} Z_{i,\varepsilon}} \right] d\mathbf{c} (1 + o(1))$$

$$\text{where } Z_{i,\varepsilon} := \int_\Sigma \varepsilon^{\gamma^2} e^{\langle \gamma e_i, \mathbf{X}_\varepsilon^g \rangle} \prod_{k=1}^M e^{2\pi \langle \gamma e_i, \alpha_k \rangle G_{g,\varepsilon}(x, x_k)} dv_g(x).$$

From Proposition 2.2 we see that $\sup_x \prod_{k=1}^N e^{2\pi \langle \gamma e_i, \alpha_k \rangle G_{g,\varepsilon}(x, x_k)} < \infty$, and since the regularized GMC measure is shown to converge to a finite measure we deduce that $\mathbb{E}[Z_{i,\varepsilon}] < \infty$. Hence, it exists a constant A such that $\mathbb{P}(Z_{i,\varepsilon} \leq A) > 0$ for all $i = 1, \dots, r$. We get

$$A_\varepsilon \geq C_{\varepsilon, x, \alpha} \int_{\mathfrak{a}} e^{\langle \sum \alpha_k - Q \chi(\Sigma), \mathbf{c} \rangle} e^{-\sum_{i=1}^r \mu_i e^{\langle \gamma e_i, \mathbf{c} \rangle}} \mathbb{P}(Z_{i,\varepsilon} \leq A \forall i) d\mathbf{c}$$

which is infinite if $\langle \sum \alpha_k - Q \chi(\Sigma), \omega_i^\vee \rangle \leq 0$ for some i . We suppose now that $\langle \sum \alpha_k - Q \chi(\Sigma), \omega_i^\vee \rangle > 0$ for all i . We now use Fubini to exchange the expectation and the integral and make the changes of variables $c_i \leftrightarrow \mu_i e^{\langle \gamma e_i, \mathbf{c} \rangle} Z_{i,\varepsilon}$, to obtain

$$A_\varepsilon = \left(\prod_{k \neq l} e^{2\pi \langle \alpha_k, \alpha_l \rangle G_{g,\varepsilon}(x_k, x_l)} \right) \left(\prod_{k=1}^M e^{\frac{|\alpha_k|^2}{2} W_g(x_k)} \right) \left(\prod_{i=1}^2 \frac{\Gamma(\frac{s_i}{\gamma}) \mu_i^{-\frac{s_i}{\gamma}}}{\gamma} \right) \mathbb{E} \left[\prod_{i=1}^2 Z_{i,\varepsilon}^{-\frac{s_i}{\gamma}} \right]$$

with the notation $s_i = \langle \sum \alpha_k - Q \chi(\Sigma), \omega_i^\vee \rangle$ and where Γ is the Euler gamma function.

It remains to show that, if for all i and k $\langle \alpha_k - Q, e_i \rangle < 0$ then $\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\prod_{i=1}^r Z_{i,\varepsilon}^{-\frac{s_i}{\gamma}} \right] =$

$\mathbb{E} \left[\prod_{i=1}^r Z_i^{-\frac{s_i}{\gamma}} \right] > 0$ (with the obvious notation), and conversely that if for some i or k

$\langle \alpha_k - Q, e_i \rangle \geq 0$ then $\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\prod_{i=1}^2 Z_{i,\varepsilon}^{-\frac{s_i}{\gamma}} \right] = 0$. Let us first assume that for all i and k $\langle \alpha_k - Q, e_i \rangle < 0$. For $r = 2$ Hölder's inequality implies

$$\begin{aligned} \mathbb{E} \left[\left| \prod Z_{i,\varepsilon}^{-\frac{s_i}{\gamma}} - \prod Z_i^{-\frac{s_i}{\gamma}} \right| \right] &\leq \mathbb{E} \left[\left| Z_{1,\varepsilon}^{-\frac{s_1}{\gamma}} - Z_1^{-\frac{s_1}{\gamma}} \right| Z_{2,\varepsilon}^{-\frac{s_2}{\gamma}} \right] + \mathbb{E} \left[\left| Z_{2,\varepsilon}^{-\frac{s_2}{\gamma}} - Z_2^{-\frac{s_2}{\gamma}} \right| Z_1^{-\frac{s_1}{\gamma}} \right] \\ &\leq \mathbb{E} \left[\left| Z_{1,\varepsilon}^{-\frac{s_1}{\gamma}} - Z_1^{-\frac{s_1}{\gamma}} \right|^p \right]^{\frac{1}{p}} \mathbb{E} \left[Z_{2,\varepsilon}^{-p' \frac{s_2}{\gamma}} \right]^{\frac{1}{p'}} + \mathbb{E} \left[\left| Z_{2,\varepsilon}^{-\frac{s_2}{\gamma}} - Z_2^{-\frac{s_2}{\gamma}} \right|^p \right]^{\frac{1}{p}} \mathbb{E} \left[Z_2^{-p' \frac{s_2}{\gamma}} \right]^{\frac{1}{p'}} \end{aligned}$$

The expectation terms without the absolute values are finite because GMC admits negative moments of all order [40] so we only need to show that $\mathbb{E} \left[\left| Z_{i,\varepsilon}^{-\frac{s_i}{\gamma}} - Z_i^{-\frac{s_i}{\gamma}} \right|^p \right]^{\frac{1}{p}} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

The only problem comes from the exponential of the Green function near the insertion points, where it behaves like $d_g(x, x_k)^{-\langle \gamma e_i, \alpha_k \rangle}$ in agreement with Proposition 2.2. This problem is local and is solved by Lemma 3.3 in [17].

Assume now that there exists k or i such that $\langle \alpha_k - Q, e_i \rangle \geq 0$. Since $\langle \gamma e_1, X_\varepsilon^g \rangle$ and $\langle \gamma e_2, X_\varepsilon^g \rangle$ are negatively correlated (due to the fact that $\langle e_i, e_j \rangle \leq 0$ for $i \neq j$) we have $\mathbb{E} \left[\prod Z_{i,\varepsilon}^{-\frac{s_i}{\gamma}} \right] \leq \prod \mathbb{E} \left[Z_{i,\varepsilon}^{-\frac{s_i}{\gamma}} \right]$. Therefore, it is enough to show that $\mathbb{E} \left[Z_{1,\varepsilon}^{-\frac{s_1}{\gamma}} \right] \rightarrow 0$ if for one k it holds that $\langle \alpha_k - Q, e_1 \rangle \geq 0$. This is again part of Lemma 3.3 in [17]. \square

We will from now on assume that the Seiberg bounds hold. From Lemma 3.12 we know that the Wick exponential formally defined for $\alpha \in \mathfrak{a}$ by $e^{\langle \alpha, X^{g'} \rangle}$ has the same law as $e^{\frac{|\alpha|^2}{4} \varphi} e^{\langle \alpha, \mathbf{X}^g \rangle}$ when $g' = e^\varphi g$ is in the conformal class of g . Following the proof of the Weyl anomaly for the partition function (Proposition 3.9) it is easy to see that the following holds:

Proposition 3.16. *Let $g' = e^\varphi g$ be a metric in the conformal class of the hyperbolic metric g . Then for any bounded functional F ,*

$$\langle F \mathbf{V}_\alpha(\mathbf{x}) \rangle_{g'} = \prod_{k=1}^N e^{-\Delta_{\alpha_k} \varphi(x_k)} \exp \left(\frac{r + 6|Q|^2}{96\pi} \int_\Sigma (|d_g \varphi|_g^2 + 2R_g \varphi) dv_g \right) \langle F(\cdot - \frac{Q}{2} \varphi) \mathbf{V}_\alpha(\mathbf{x}) \rangle_g$$

where the numbers $\Delta_{\alpha_k} = \langle \frac{\alpha_k}{2}, Q - \frac{\alpha_k}{2} \rangle$ are called the conformal weights.

Likewise and in analogy with the partition function, we also get immediately diffeomorphism invariance:

Proposition 3.17. *Let $\psi : \Sigma \rightarrow \Sigma$ be an orientation-preserving diffeomorphism. Then for any bounded measurable $F : H^{-s}(\Sigma \rightarrow \mathfrak{a}, g) \rightarrow \mathbb{R}$,*

$$\langle F \mathbf{V}_\alpha(\mathbf{x}) \rangle_{\psi^* g} = \langle F(\cdot \circ \psi) \mathbf{V}_\alpha(\psi(\mathbf{x})) \rangle_g$$

where $\mathbf{V}_\alpha(\psi(\mathbf{x}))$ is a shortcut for the product of the vertex operators $\prod_{k=1}^N V_{\alpha_k}(\psi(z_k))$.

4. BOUNDARY TODA CONFORMAL FIELD THEORY: FREE BOUNDARY CONDITIONS

Having defined Toda CFTs when the underlying surface is a Riemann surface without boundary, we now turn to the case where the compact surface Σ has a boundary $\partial\Sigma \neq \emptyset$. A major difference with the closed case is that the presence of a boundary implies that one needs to specify boundary conditions for the field φ in the Toda action, thus giving rise to several possible constructions for the theory with a boundary. To start with we investigate the definition of the theory with a first type of boundary conditions, Neumann boundary conditions, for which a path integral exists in the literature. We still assume throughout this section that Σ is of hyperbolic type and equip it with a uniform type 1 metric g .

4.1. Neumann Gaussian Free Fields and Gaussian Multiplicative Chaos.

4.1.1. *Background on Neumann GFF.* The Neumann Laplacian on functions from Σ to \mathbb{R} introduced in Section 2.2 admits, as in the closed case, an orthonormal basis of eigenfunctions (e_n^N) together with eigenvalues $0 = \lambda_0^N < \lambda_1^N \leq \dots$. Hence it is possible to define a Gaussian field with covariance structure given by the Neumann Green function G_g^N exactly the same way as in Section 3.1. We denote this field $X^{g,N}$. Due to the presence of the zero-mode, to make sense of the path integral the law of the Neumann GFF has to be tensorized with the Lebesgue measure just as in the closed case. We will take this into account when defining the partition function. The Neumann GFF behaves as the closed one under conformal changes of metric:

$$(4.1) \quad X^{g',N} = X^{g,N} - m_{g'}(X^{g,N}) \quad \text{for } g' \in [g].$$

Lemma 2.1 suggests an alternative way of defining such a random distribution using the doubling trick. Indeed, recall that one can double the surface Σ to obtain a closed surface $\hat{\Sigma}$ equipped with the smooth metric \hat{g} induced by the uniform type 1 metric g on Σ , and let X^g be the GFF on $\hat{\Sigma}$. Then the field $Y^{g,N} := \sqrt{2}^{-1}(X^g + X^g \circ \sigma)$ (where σ is the canonical involution) has the law of the Neumann GFF on Σ . This decomposition will be particularly useful in the case of mixed boundary conditions studied in Section 5.

Recall that we have to define a field with values in \mathfrak{a} . To do so we can rely on the decomposition presented above and set

$$(4.2) \quad \mathbf{Y}^g = \frac{\mathbf{X}^g + \mathbf{X}^g \circ \sigma}{\sqrt{2}}$$

where $\mathbf{X}^g : \hat{\Sigma} \rightarrow \mathfrak{a}$ is the GFF on $\hat{\Sigma}$ defined in the previous section. Like before we can see \mathbf{Y}^g as a vector $\mathbf{Y}^g = \sum_{i=1}^r X_i^{g,N} \mathbf{v}_i$ where $(\mathbf{v}_i)_{1 \leq i \leq r}$ is an orthonormal basis of \mathfrak{a} and with the $X_i^{g,N}$ being independent 1-dimensional Neumann GFFs on Σ . In particular, the partition function of \mathbf{Y}^g is given by

$$(4.3) \quad \mathcal{Z}_{\mathbf{Y}}(\Sigma, g) := \det' \left(-\frac{\Delta_g^N}{2\pi v_g(\Sigma)} \right)^{-\frac{r}{2}} \exp \left(-\frac{r}{16\pi} \int_{\partial\Sigma} k_g d\lambda_g \right)$$

where Δ_g^N is the (one-dimensional) Neumann Laplacian on Σ . The term $\exp \left(-\frac{r}{16\pi} \int_{\partial\Sigma} k_g d\lambda_g \right)$ is added to obtain the right conformal anomaly formula. Indeed, we have the following:

Proposition 4.1. *Let $g' = e^\varphi g$ be a metric in the conformal class of g . Then*

$$(4.4) \quad \mathcal{Z}_{\mathbf{Y}}(\Sigma, g') = \mathcal{Z}_{\mathbf{Y}}(\Sigma, g) \exp \left(\frac{r}{96\pi} \left(\int_{\Sigma} (|d_g \varphi|_g^2 + 2R_g \varphi) dv_g + 4 \int_{\partial \Sigma} k_g \varphi d\lambda_g \right) \right).$$

Proof. This is a consequence of [1, Equation (4.38)], which states that

$$\begin{aligned} & \log \left(\det' \left(-\frac{\Delta_g^N}{2\pi v_g(\Sigma)} \right)^{-\frac{1}{2}} \right) - \log \left(\det' \left(-\frac{\Delta_{g'}^N}{2\pi v_{g'}(\Sigma)} \right)^{-\frac{1}{2}} \right) \\ &= -\frac{1}{96\pi} \left(\int_{\Sigma} (|d_g \varphi|_g^2 + 2R_g \varphi) dv_g + 4 \int_{\partial \Sigma} k_g \varphi d\lambda_g \right) + \frac{1}{16\pi} \int_{\partial \Sigma} \partial_{\bar{n}_g} \varphi d\lambda_g. \end{aligned}$$

□

4.1.2. *Gaussian Multiplicative Chaos.* In order to make sense of the exponential of the GFF we need like before to regularize the field $\mathbf{Y}^g(x)$ by averaging it on the intersection between Σ and the geodesic circle of radius $\varepsilon > 0$ centered at x . Note that this amounts to defining the regularized fields by the formula (4.2) with \mathbf{X}^g replaced by its regularization \mathbf{X}_ε^g . We still denote by $\mu_{x,\varepsilon}$ the uniform probability measure on $\Sigma \cap C_g(x, \varepsilon)$.

Lemma 4.2. *The regularized Neumann GFF \mathbf{Y}_ε^g has the following covariance structure:*

$$(4.5) \quad \mathbb{E}[\langle u, \mathbf{Y}_\varepsilon^g(x) \rangle \langle v, \mathbf{Y}_\varepsilon^g(y) \rangle] = 2\pi |u|^2 \iint G_g^N(x', y') d\mu_{x,\varepsilon}(x') d\mu_{y,\varepsilon}(y').$$

Moreover when $\varepsilon \rightarrow 0$ and uniformly in $x \in \Sigma \cup \partial \Sigma$:

$$(4.6) \quad \mathbb{E}[\langle u, \mathbf{Y}_\varepsilon^g(x) \rangle^2] = \begin{cases} -|u|^2 \log \varepsilon + |u|^2 W_g(x) + 2\pi |u|^2 G_g(x, \sigma(x)) + o(1) & \text{if } x \notin \partial \Sigma \\ -2|u|^2 \log \varepsilon + 2|u|^2 W_g(x) + o(1) & \text{if } x \in \partial \Sigma. \end{cases}$$

We can now define the GMC measures associated to \mathbf{Y}^g , both in the bulk (that is, in Σ) and on the boundary. Namely for $u = \gamma e_i \in \mathfrak{a}$ and $x \in \Sigma$ we define

$$(4.7) \quad \begin{aligned} e^{\langle u, \mathbf{Y}^g(x) \rangle} dv_g(x) &:= \lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{|u|^2}{2}} e^{\langle u, \mathbf{Y}_\varepsilon^g(x) \rangle} dv_g(x) \\ &= \lim_{\varepsilon \rightarrow 0} e^{\langle u, \mathbf{Y}_\varepsilon^g(x) \rangle - \frac{1}{2} \mathbb{E}[\langle u, \mathbf{Y}_\varepsilon^g(x) \rangle^2]} \varepsilon^{\frac{|u|^2}{2} (W_g(x) + 2\pi G_g(x, \sigma(x)))} dv_g(x) \end{aligned}$$

while for $x \in \partial \Sigma$ the GMC measures are given by

$$(4.8) \quad \begin{aligned} e^{\langle u, \mathbf{Y}^g(x) \rangle} d\lambda_g(x) &:= \lim_{\varepsilon \rightarrow 0} \varepsilon^{|u|^2} e^{\langle u, \mathbf{Y}_\varepsilon^g(x) \rangle} d\lambda_g(x) \\ &= \lim_{\varepsilon \rightarrow 0} e^{\langle u, \mathbf{Y}_\varepsilon^g(x) \rangle - \frac{1}{2} \mathbb{E}[\langle u, \mathbf{Y}_\varepsilon^g(x) \rangle^2]} \varepsilon^{|u|^2 W_g(x)} d\lambda_g(x). \end{aligned}$$

All the convergences stand in probability in the space of Radon measures equipped with the weak topology, provided that $|u|^2 < 4$ in the bulk and $|u|^2 < 1$ on the boundary.

4.2. Partition function. The Toda action (1.6) for a Riemann surface with boundary differs from that for a closed Riemann surface in that one needs to add a geodesic curvature term and a boundary potential (depending on boundary cosmological constants) in the expression of the action. In the path integral the quadratic term has been interpreted in terms of a vectorial Neumann GFF. The potential in the action is then defined by the following (regularized) expression:

$$(4.9) \quad V_g^N(\mathbf{Y}_\varepsilon^g + \mathbf{c}) = \frac{1}{4\pi} \int_\Sigma R_g \langle Q, \mathbf{Y}_\varepsilon^g + \mathbf{c} \rangle dv_g + \frac{1}{2\pi} \int_{\partial\Sigma} k_g \langle Q, \mathbf{Y}_\varepsilon^g + \mathbf{c} \rangle d\lambda_g \\ + \sum_{i=1}^r \left(\mu_i e^{\langle \gamma e_i, \mathbf{c} \rangle} \int_\Sigma \varepsilon^{\gamma^2} e^{\langle \gamma e_i, \mathbf{Y}_\varepsilon^g \rangle} dv_g + \mu_i^\partial e^{\langle \frac{\gamma}{2} e_i, \mathbf{c} \rangle} \int_{\partial\Sigma} \varepsilon^{\frac{\gamma^2}{2}} e^{\langle \frac{\gamma}{2} e_i, \mathbf{Y}_\varepsilon^g \rangle} d\lambda_g \right)$$

where the $\mu_i^\partial \geq 0$ are the boundary cosmological constants. The path integral is then defined by the limit, for suitable F ,

$$(4.10) \quad \langle F \rangle_g^N := \lim_{\varepsilon \rightarrow 0} \langle F \rangle_{g,\varepsilon}^N, \quad \langle F \rangle_{g,\varepsilon}^N := \mathcal{Z}_{\mathbf{Y}}(\Sigma, g) \int_{\mathfrak{a}} \mathbb{E} \left[F(\mathbf{Y}_\varepsilon^g + \mathbf{c}) e^{-V_g^N(\mathbf{Y}_\varepsilon^g + \mathbf{c})} \right] d\mathbf{c}.$$

Proposition 4.3. *Assume that $\chi(\Sigma) < 0$ and take $F = 1$. Then the limit (4.10) exists and belongs to $(0, \infty)$.*

Proof. Like before we choose g_0 a uniform metric of type 1 in the conformal class of the metric g so that R_{g_0} is constant. The regularized partition function writes

$$(4.11) \quad \mathcal{Z}_{\mathbf{Y}}(\Sigma, g_0) \int_{\mathfrak{a}} e^{-\langle Q, \mathbf{c} \rangle \chi(\Sigma)} \mathbb{E} \left[\exp \left(- \sum_{i=1}^r \left(\mu_i e^{\langle \gamma e_i, \mathbf{c} \rangle} M_{i,\varepsilon} + \mu_i^\partial e^{\langle \frac{\gamma}{2} e_i, \mathbf{c} \rangle} M_{i,\varepsilon}^\partial \right) \right) \right] d\mathbf{c}$$

where $M_{i,\varepsilon} = \int_\Sigma \varepsilon^{\gamma^2} e^{\langle \gamma e_i, \mathbf{Y}_\varepsilon^{g_0} \rangle} dv_{g_0}$ and $M_{i,\varepsilon}^\partial = \int_{\partial\Sigma} \varepsilon^{\frac{\gamma^2}{2}} e^{\langle \frac{\gamma}{2} e_i, \mathbf{Y}_\varepsilon^{g_0} \rangle} d\lambda_{g_0}$. The partition function can be upper-bounded by

$$(4.12) \quad \mathcal{Z}_{\mathbf{Y}}(\Sigma, g_0) \int_{\mathfrak{a}} e^{-\langle Q, \mathbf{c} \rangle \chi(\Sigma)} \mathbb{E} \left[\exp \left(- \sum_{i=1}^r \mu_i e^{\langle \gamma e_i, \mathbf{c} \rangle} M_{i,\varepsilon} \right) \right] d\mathbf{c}$$

and the very same reasoning as in the closed case (Proposition 3.7) allows one to conclude. \square

Likewise one can adapt the proof of Proposition 3.9 to get the Weyl anomaly formula in the boundary case:

Proposition 4.4. *Let g' be a metric in the conformal class of the uniform type 1 metric g , with φ denoting the log-conformal factor. Then for any bounded functional F ,*

$$\langle F \rangle_g^N = \langle F(\cdot - \frac{Q}{2}\varphi) \rangle_g^N \exp \left(\frac{r + 6|Q|^2}{96\pi} \left(\int_\Sigma (|d_g \varphi|_g^2 + 2R_g \varphi) dv_g + 4 \int_{\partial\Sigma} k_{g'} \varphi d\lambda_{g'} \right) \right).$$

Proof. The proof is exactly the same as in the closed case (Proposition 3.9): for the part coming from the regularized determinant one has to use the conformal anomaly formula for

the boundary GFF (Proposition 4.1). For the component containing the $|Q|^2$ term we rely on Girsanov theorem for the Gaussian random variable

$$Y := -\frac{1}{4\pi} \int_{\Sigma} R_{g'} \langle Q, \mathbf{Y}^g \rangle dv_{g'} - \frac{1}{2\pi} \int_{\partial\Sigma} k_g \langle Q, \mathbf{Y}^g \rangle d\lambda_g,$$

whose variance is given by $\mathbb{E}[Y^2] = \frac{|Q|^2}{8\pi} \int_{\Sigma} |d_g \varphi|^2 dv_g$ and whose effect is to shift the law of \mathbf{Y}^g by $-\frac{Q}{2}(\varphi - m_g(\varphi))$. \square

The diffeomorphism invariance remains also valid in the boundary case:

Proposition 4.5. *If $\psi : \Sigma \rightarrow \Sigma$ is an orientation-preserving diffeomorphism that sends the boundary to itself, then for all $F : H^{-s}(\Sigma \rightarrow \mathbf{a}) \rightarrow \mathbb{R}$,*

$$\langle F \rangle_{\psi^*g}^N = \langle F(\cdot \circ \psi) \rangle_g^N.$$

4.3. Correlation functions. On the surface with boundary Σ we will consider not only N insertion points in the bulk together with their associated weights $(x_k, \alpha_k)_{k=1, \dots, N}$ but also $M \in \mathbb{N}$ insertions on the boundary $(s_l, \beta_l)_{l=1, \dots, M}$ with $s_l \in \partial\Sigma$ and $\beta_l \in \mathbf{a}$. The bulk vertex operators are defined similarly as in Equation (3.20) by

$$(4.13) \quad V_{\alpha_k, \varepsilon}(x_k) = V_{\alpha_k, \varepsilon}(x_k, \mathbf{c}) = \varepsilon^{\frac{|\alpha_k|^2}{2}} e^{\langle \alpha_k, \mathbf{X}^g(x_k) + \mathbf{c} \rangle}$$

while on the boundary they are defined by

$$(4.14) \quad V_{\beta_l, \varepsilon}(s_l) = V_{\beta_l, \varepsilon}(s_l, \mathbf{c}) = \varepsilon^{\frac{|\beta_l|^2}{4}} e^{\langle \frac{\beta_l}{2}, \mathbf{X}^g(s_l) + \mathbf{c} \rangle}.$$

In the boundary case one can slightly generalize the model by introducing different cosmological constants between the boundary insertion points. Namely, let us introduce a family of non-negative cosmological constants $(\mu_{i,l}^{\partial})$ with $i = 1, \dots, r$ and $l = 1, \dots, M$. Then we can modify the exponential potential in the action (4.9) and so redefine the latter by

$$(4.15) \quad \begin{aligned} V_g^N(\mathbf{Y}_\varepsilon^g + \mathbf{c}) &= \frac{1}{4\pi} \int_{\Sigma} R_g \langle Q, \mathbf{Y}_\varepsilon^g + \mathbf{c} \rangle dv_g + \frac{1}{2\pi} \int_{\partial\Sigma} k_g \langle Q, \mathbf{Y}_\varepsilon^g + \mathbf{c} \rangle d\lambda_g \\ &+ \sum_{i=1}^r \left(\mu_i e^{\langle \gamma e_i, \mathbf{c} \rangle} \int_{\Sigma} \varepsilon^{\gamma^2} e^{\langle \gamma e_i, \mathbf{Y}_\varepsilon^g \rangle} dv_g + e^{\langle \frac{\gamma}{2} e_i, \mathbf{c} \rangle} \int_{\partial\Sigma} \varepsilon^{\frac{\gamma^2}{2}} e^{\langle \frac{\gamma}{2} e_i, \mathbf{Y}_\varepsilon^g \rangle} d\lambda_g^{\mu_i} \right) \end{aligned}$$

where $d\lambda_g^{\mu_i} = \sum_{l=1}^M \mu_{i,l}^{\partial} \mathbf{1}_{(s_j, s_{j+1})} d\lambda_g$, (s_j, s_{j+1}) being the part of $\partial\Sigma$ between the two insertion points s_j and s_{j+1} and with the convention $s_{N+1} = s_1$. We thus define the correlation function in the boundary case by the limit

$$(4.16) \quad \left\langle \prod_{k=1}^N V_{\alpha_k}(x_k) \prod_{l=1}^M V_{\beta_l}(s_l) \right\rangle_g^N := \lim_{\varepsilon \rightarrow 0} \langle \mathbf{V}_{\alpha, \varepsilon}(\mathbf{x}) \mathbf{V}_{\beta, \varepsilon}(\mathbf{s}) \rangle_g^N$$

where we have set $\mathbf{V}_{\alpha, \varepsilon}(\mathbf{x}) = \prod_{k=1}^N V_{\alpha_k, \varepsilon}(x_k)$ and $\mathbf{V}_{\beta, \varepsilon}(\mathbf{s}) = \prod_{l=1}^M V_{\beta_l, \varepsilon}(s_l)$, and with the expectation $\langle \cdot \rangle_g^N$ taken with respect to the new action (4.15). We now establish the Seiberg bounds in the boundary case:

Theorem 4.6. *The limit (4.16) exists and belongs to $(0, \infty)$ if and only if the following conditions hold for all $1 \leq i \leq r$:*

$$(4.17) \quad \left\langle \sum_{k=1}^N \alpha_k + \sum_{l=1}^M \frac{\beta_l}{2} - Q\chi(\hat{\Sigma}), \omega_i^\vee \right\rangle > 0;$$

$$(4.18) \quad \langle \alpha_k - Q, e_i \rangle < 0 \quad \text{for all } k = 1, \dots, N \quad \text{and} \quad \langle \beta_l - Q, e_i \rangle < 0 \quad \text{for all } l = 1, \dots, M.$$

Proof. We start with the uniform type 1 metric g , the excluded cases following from the Weyl anomaly (Proposition 4.7 below). The regularized correlation function writes

$$A_\varepsilon := \int_{\mathfrak{a}} e^{-\langle Q, \mathbf{c} \rangle \chi(\Sigma)} \mathbb{E} \left[\prod \varepsilon^{\frac{|\alpha_k|^2}{2}} e^{\langle \alpha_k, \mathbf{Y}_\varepsilon^g(x_k) + \mathbf{c} \rangle} \prod \varepsilon^{\frac{|\beta_l|^2}{4}} e^{\langle \frac{\beta_l}{2}, \mathbf{Y}_\varepsilon^g(s_l) + \mathbf{c} \rangle} \right. \\ \left. \times \exp \left(- \sum_{i=1}^r \left(\mu_i e^{\langle \gamma e_i, \mathbf{c} \rangle} \int_{\Sigma} \varepsilon^{\gamma^2} e^{\langle \gamma e_i, \mathbf{Y}_\varepsilon^g \rangle} dv_g + e^{\langle \frac{\gamma}{2} e_i, \mathbf{c} \rangle} \int_{\partial \Sigma} \varepsilon^{\frac{\gamma^2}{2}} e^{\langle \frac{\gamma}{2} e_i, \mathbf{Y}_\varepsilon^g \rangle} d\lambda_g^{\mu_i} \right) \right) \right] d\mathbf{c}$$

By Girsanov theorem we obtain

$$A_\varepsilon = C_{\varepsilon, \mathbf{x}, \boldsymbol{\alpha}, \mathbf{s}, \boldsymbol{\beta}} \int_{\mathfrak{a}} e^{\langle \sum \alpha_k + \frac{1}{2} \sum \beta_l - Q\chi(\Sigma), \mathbf{c} \rangle} \mathbb{E} \left[e^{-\sum_{i=1}^r \left(\mu_i e^{\langle \gamma e_i, \mathbf{c} \rangle} Z_{i, \varepsilon}^N + e^{\langle \frac{\gamma}{2} e_i, \mathbf{c} \rangle} Z_{i, \varepsilon}^{N, \partial} \right)} \right] d\mathbf{c} (1 + o(1))$$

$$\text{where } C_{\varepsilon, \mathbf{x}, \boldsymbol{\alpha}, \mathbf{s}, \boldsymbol{\beta}} = \prod_{k \neq k'} e^{2\pi \langle \alpha_k, \alpha_{k'} \rangle G_{g, \varepsilon}^N(x_k, x_{k'})} \prod_{l \neq l'} e^{2\pi \langle \frac{\beta_l}{2}, \frac{\beta_{l'}}{2} \rangle G_{g, \varepsilon}^N(s_l, s_{l'})} \\ \times \prod_{k, l} e^{2\pi \langle \alpha_k, \frac{\beta_l}{2} \rangle G_{g, \varepsilon}(x_k, s_l)} \prod_k e^{\frac{|\alpha_k|^2}{2} W_g(x_k) + \pi |\alpha_k|^2 G_g(x_k, \sigma(x_k))} \prod_l e^{\frac{|\beta_l|^2}{4} W_g(s_l)},$$

$$Z_{i, \varepsilon}^N = \int_{\Sigma} \varepsilon^{\gamma^2} e^{\langle \gamma e_i, Y_\varepsilon(x) \rangle} H_\varepsilon^N(x) dv_g(x) \quad ; \quad Z_{i, \varepsilon}^{N, \partial} = \int_{\partial \Sigma} \varepsilon^{\frac{\gamma^2}{2}} e^{\langle \frac{\gamma}{2} e_i, Y_\varepsilon(s) \rangle} H_\varepsilon^{N, \partial}(s) d\lambda_g^{\mu_i}(s) \quad \text{and} \\ H_\varepsilon^N(x) = \prod_k e^{2\pi \langle \gamma e_i, \alpha_k \rangle G_{g, \varepsilon}^N(x_k, x)} \prod_l e^{2\pi \langle \gamma e_i, \frac{\beta_l}{2} \rangle G_{g, \varepsilon}^N(s_l, x)}, \\ H_\varepsilon^{N, \partial}(s) = \prod_k e^{2\pi \langle \frac{\gamma}{2} e_i, \alpha_k \rangle G_{g, \varepsilon}^N(x_k, s)} \prod_l e^{2\pi \langle \frac{\gamma}{2} e_i, \frac{\beta_l}{2} \rangle G_{g, \varepsilon}^N(s_l, s)}.$$

Now thanks to Lemma 4.2 we know that both $\sup_x H_\varepsilon^N(x)$ and $\sup_s H_\varepsilon^{N, \partial}(s)$ are finite, and hence there exists $A > 0$ such that

$$A_\varepsilon \geq C_{\varepsilon, \mathbf{x}, \boldsymbol{\alpha}, \mathbf{s}, \boldsymbol{\beta}}^N \int_{\mathfrak{a}} e^{\langle \sum \alpha_k + \frac{1}{2} \sum \beta_l - Q\chi(\Sigma), \mathbf{c} \rangle} e^{-\sum_{i=1}^r \left(\mu_i e^{\langle \gamma e_i, \mathbf{c} \rangle} + \mu_i^\partial e^{\langle \frac{\gamma}{2} e_i, \mathbf{c} \rangle} \right)} \times \\ \mathbb{P}(\forall i, Z_{i, \varepsilon}^N \leq A \text{ and } Z_{i, \varepsilon}^{N, \partial} \leq A) d\mathbf{c} (1 + o(1))$$

which is infinite if $\langle \sum \alpha_k + \frac{1}{2} \sum \beta_l - Q\chi(\Sigma), \omega_i^\vee \rangle \leq 0$ for some i .

We suppose now that $\langle \sum \alpha_k + \frac{1}{2} \sum \beta_l - Q\chi(\Sigma), \omega_i^\vee \rangle > 0$ for all i . The goal is to show that when both $\langle \alpha_k - Q, e_i \rangle$ and $\langle \beta_l - Q, e_i \rangle$ are negative (for all i), then the bulk GMC measures $e^{\langle \gamma e_i, Y_\varepsilon \rangle} dv_g$ integrate the singularities in the function H^N (limit of H_ε^N as $\varepsilon \rightarrow 0$), and the boundary GMC measures $e^{\langle \frac{\gamma}{2} e_i, Y_\varepsilon \rangle} d\lambda_g$ integrate the singularities in the function $H^{N, \partial}$ (limit of $H_\varepsilon^{N, \partial}$ as $\varepsilon \rightarrow 0$). The only case that differs from the closed setting is when a boundary

insertion s_l is approached by $x \in \Sigma$. This question is again a local problem and is treated in [29]. Finally by Jensen's inequality we have

$$(4.19) \quad \mathbb{E} \left[e^{-\sum_{i=1}^r (\mu_i e^{\langle \gamma e_i, c \rangle} Z_{i,\varepsilon}^N + e^{\langle \frac{\gamma}{2} e_i, c \rangle} Z_{i,\varepsilon}^{N,\partial})} \right] \leq \mathbb{E} \left[e^{-\mu_1 e^{\langle \gamma e_1, c \rangle} Z_{1,\varepsilon}^N} \right] \leq e^{-\mu_1 e^{\langle \gamma e_1, c \rangle} \mathbb{E}[Z_{1,\varepsilon}^N]}$$

which converges to 0 if $\langle \alpha_k - Q, e_1 \rangle \geq 0$ by [17]. This shows that A_ε converges to 0 if one of the $\langle \alpha_k - Q, e_i \rangle$ or $\langle \beta_l - Q, e_i \rangle$ is positive or zero. \square

As for the partition function, we obtain the Weyl anomaly formula and the diffeomorphism invariance of the correlation functions.

Proposition 4.7. *Let $g' = e^\varphi g$ be a metric in the conformal class of the uniform type 1 metric g . Then for any bounded functional $F : H^{-s}(\Sigma \rightarrow \mathfrak{a}) \rightarrow \mathbb{R}$,*

$$(4.20) \quad \langle F \mathbf{V}_\alpha(\mathbf{x}) \mathbf{V}_\beta(\mathbf{s}) \rangle_{g'}^N = \prod_{k=1}^N e^{-\Delta_{\alpha_k} \varphi(x_k)} \prod_{l=1}^M e^{-\frac{1}{2} \Delta_{\beta_l} \varphi(s_l)} \langle F(\cdot - \frac{Q}{2} \varphi) \mathbf{V}_\alpha(\mathbf{x}) \mathbf{V}_\beta(\mathbf{s}) \rangle_g^N \\ \times \exp \left(\frac{r + 6|Q|^2}{96\pi} \left(\int_\Sigma (|d_g \varphi|_g^2 + 2R_g \varphi) dv_g + 4 \int_{\partial \Sigma} k_g \varphi d\lambda_g \right) \right).$$

If $\psi : \Sigma \rightarrow \Sigma$ is an orientation-preserving diffeomorphism that sends the boundary to itself:

$$\langle F \mathbf{V}_\alpha(\mathbf{x}) \mathbf{V}_\beta(\mathbf{s}) \rangle_{\psi^* g}^N = \langle F(\cdot \circ \psi) \mathbf{V}_\alpha(\psi(\mathbf{x})) \mathbf{V}_\beta(\psi(\mathbf{s})) \rangle_g^N.$$

Proof. For the conformal anomaly we rely on the fact that, in the same fashion as in Lemma 3.12, the Wick exponentials $e^{\langle \alpha, \mathbf{Z}^{g'}(x_k) \rangle}$ and $e^{\frac{|\alpha_k|^2}{4} \varphi(x_k)} e^{\langle \alpha, \mathbf{Z}^g(x_k) \rangle}$ have the same law for a bulk insertion, while for a boundary one the prefactor is given by $e^{\frac{|\beta_l|^2}{8} \varphi(s_l)}$. The Girsanov shift by $-\frac{Q}{2} \varphi$ thus yields the factor $\prod_{k=1}^N e^{-\frac{1}{4} \langle \alpha_k, 2Q - \alpha_k \rangle \varphi(x_k)} \prod_{l=1}^M e^{-\frac{1}{8} \langle \beta_l, 2Q - \beta_l \rangle \varphi(s_l)}$ as expected. \square

5. BOUNDARY TODA CONFORMAL FIELD THEORY: MIXED BOUNDARY CONDITIONS

Up to now we have defined Toda CFTs when the model admits a path integral formulation, should it be for a Riemann surface with or without boundary. It is however argued in the physics that in the boundary case there exists an alternative way of defining Toda CFTs that takes into account the non-triviality of the outer automorphism group of \mathfrak{g} . We propose in this section a construction of a family of CFTs closely related to Toda theories and that features this group in the cases where it is non-trivial. To be more specific we define a model where a Vertex Operator $V_\alpha(z)$ gives rise to a singularity at $z \in \Sigma$ with weight α but for which the ‘‘reflected singularity’’ at $\sigma(z) \in \sigma(\Sigma)$ has weight $\tau(\alpha)$ where τ is an element of the outer automorphism group of \mathfrak{g} (see Figure 2).

Like before in this section we assume that Σ has a non-empty boundary and is of hyperbolic type. It comes equipped with a uniform type 1 metric g . We also fix an element τ of $\text{Out}(\mathfrak{g})$ and assume unless specified that it is of order two —the only remaining case, related to a phenomenon known as triality, will be discussed in Subsection 5.3. Under this assumption

the map τ is self-adjoint and allows one to decompose \mathfrak{a} as the orthogonal sum $\mathfrak{a} = \mathfrak{a}_N \oplus \mathfrak{a}_D$; we use the notation u^* for $\tau(u)$ for any $u \in \mathfrak{a}$.

5.1. The Gaussian Free Fields considered. If one wants to make sense of this heuristic about the reflected Vertex Operators, one sees that we need to prescribe the law of the underlying Toda field \mathbf{Z} in such a way that \mathbf{Z} and $\mathbf{Z}^* \circ \sigma$ have the same law. For this purpose we will rely on the same type of expression (4.2) that defines the Neumann GFF based on the doubling trick.

5.1.1. *Gaussian Free Field with “mixed” boundary conditions.* To be more specific let us consider \mathbf{X}^g to be the GFF on $\hat{\Sigma}$ with values in \mathfrak{a} defined in the previous section. Then we can define a new field via

$$(5.1) \quad \mathbf{Z}^g = \frac{\mathbf{X}^g + (\mathbf{X}^g)^* \circ \sigma}{\sqrt{2}}.$$

Some key remarks have to be made at this stage:

(1) a straightforward consequence of Equation (5.1) is that $\mathbf{Z}^g = (\mathbf{Z}^g)^* \circ \sigma$;

(2) the field \mathbf{Z}^g has covariance structure for $x, y \in \Sigma$ and $u, v \in \mathfrak{a}$:

$$(5.2) \quad \mathbb{E}[\langle u, \mathbf{Z}^g(x) \rangle \langle v, \mathbf{Z}^g(y) \rangle] = 2\pi(\langle u, v \rangle G_g(x, y) + \langle u, v^* \rangle G_g(x, \sigma(y))).$$

(3) let us decompose \mathbf{Z}^g in $\mathfrak{a} = \mathfrak{a}_N \oplus \mathfrak{a}_D$ by writing

$$(5.3) \quad \mathbf{Z}^g = \mathbf{Z}^{g,N} + \mathbf{Z}^{g,D} \quad \text{where} \quad \mathbf{Z}^{g,N} := \frac{\mathbf{Z}^g + (\mathbf{Z}^g)^*}{2} \quad \text{and} \quad \mathbf{Z}^{g,D} := \frac{\mathbf{Z}^g - (\mathbf{Z}^g)^*}{2}.$$

The fields thus introduced are such that $\mathbf{Z}^{g,N}$ has the law of a Neumann GFF from Σ to \mathfrak{a}_N while $\mathbf{Z}^{g,D}$ has the law of a Dirichlet GFF from Σ to \mathfrak{a}_D . The two fields are independent.

A key difference between this GFF and the one studied in the previous section is the inclusion of Dirichlet boundary conditions along the components in \mathfrak{a}_D . This field has covariance structure given by the Dirichlet Green function G_g^D and in particular is invariant under conformal changes of metric, in that

$$(5.4) \quad \mathbf{X}^{g',D} = \mathbf{X}^{g,D}.$$

The partition function of the (one-dimensional) Dirichlet field $\mathbf{X}^{g,D}$ is also given by a regularized determinant, $\det' \left(-\frac{\Delta_g^D}{2\pi v_g(\Sigma)} \right)^{-\frac{1}{2}}$; it satisfies a conformal anomaly under a conformal change of metric $g' = e^\varphi g$ similar to that of the Neumann GFF [1, Equation (4.38)]:

$$(5.5) \quad \log \det' \left(-\frac{\Delta_{g'}^D}{2\pi v_{g'}(\Sigma)} \right) - \log \det' \left(-\frac{\Delta_g^D}{2\pi v_g(\Sigma)} \right) \\ = -\frac{1}{96\pi} \left(\int_{\Sigma} (|d_g \varphi|_g^2 + 2R_g \varphi) dv_g + 4 \int_{\partial \Sigma} k_g \varphi d\lambda_g \right) - \frac{1}{16\pi} \int_{\partial \Sigma} \partial_{\bar{n}_g} \varphi d\lambda_g.$$

We are now in position to introduce the partition function associated to the GFFs we consider. For this purpose let $d_N := \dim(\mathfrak{a}_N)$ and $d_D := \dim(\mathfrak{a}_D)$ so that in particular

$d_N + d_D = r$ —the values of these quantities are explicit and given in Appendix A. The partition function is then defined by setting

$$(5.6) \quad \mathcal{Z}_{\mathbf{Z}}(\Sigma, g) := \det' \left(-\frac{\Delta_g^N}{2\pi v_g(\Sigma)} \right)^{-\frac{d_N}{2}} \det' \left(-\frac{\Delta_g^D}{2\pi v_g(\Sigma)} \right)^{-\frac{d_D}{2}} \exp \left(\frac{d_D - d_N}{16\pi} \int_{\partial\Sigma} k_g d\lambda_g \right).$$

The same reasoning as in the proof of Proposition 4.1 then shows the following:

Proposition 5.1. *Let $g' = e^\varphi g$ be a metric in the conformal class of g . Then*

$$(5.7) \quad \mathcal{Z}_{\mathbf{Z}}(\Sigma, g') = \mathcal{Z}_{\mathbf{Z}}(\Sigma, g) \exp \left(\frac{r}{96\pi} \left(\int_{\Sigma} (|d_g \varphi|_g^2 + 2R_g \varphi) dv_g + 4 \int_{\partial\Sigma} k_g \varphi d\lambda_g \right) \right).$$

5.1.2. *Gaussian Multiplicative Chaos.* We define the GMC measures associated to the field \mathbf{Z}^g like before by considering first the averaged GFF \mathbf{Z}_ε^g , which satisfies:

Lemma 5.2. \mathbf{Z}_ε^g has covariance kernel given by

$$(5.8) \quad \mathbb{E}[\langle u, \mathbf{Z}_\varepsilon^g(x) \rangle \langle v, \mathbf{Z}_\varepsilon^g(y) \rangle] = 2\pi \iint (|u|^2 G_g(x', y') + \langle u, u^* \rangle G_g(x', \sigma(y'))) d\mu_{x,\varepsilon}(x') d\mu_{y,\varepsilon}(y').$$

Besides as $\varepsilon \rightarrow 0$ and uniformly in x :

$$(5.9) \quad \mathbb{E}[\langle u, \mathbf{Z}_\varepsilon^g(x) \rangle^2] = \begin{cases} -|u|^2 \log \varepsilon + |u|^2 W_g(x) + 2\pi \langle u, u^* \rangle G_g(x, \sigma(x)) + o(1) & \text{if } x \notin \partial\Sigma \\ -(|u|^2 + \langle u, u^* \rangle) \log \varepsilon + (|u|^2 + \langle u, u^* \rangle) W_g(x) + o(1) & \text{if } x \in \partial\Sigma. \end{cases}$$

The GMC measures associated to \mathbf{Z}^g can therefore be defined by the limits:

$$(5.10) \quad e^{\langle u, \mathbf{Z}^g(x) \rangle} dv_g(x) := \lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{|u|^2}{2}} e^{\langle u, \mathbf{Z}_\varepsilon^g(x) \rangle} dv_g(x) \\ = \lim_{\varepsilon \rightarrow 0} e^{\langle u, \mathbf{Z}_\varepsilon^g(x) \rangle - \frac{1}{2} \mathbb{E}[\langle u, \mathbf{Z}_\varepsilon^g(x) \rangle^2]} e^{\frac{|u|^2}{2} (W_g(x) + 2\pi \langle u, u^* \rangle G_g(x, \sigma(x)))} dv_g(x)$$

for x in the bulk, while for $x \in \partial\Sigma$ the GMC measures are given by

$$(5.11) \quad e^{\langle u, \mathbf{Z}^g(x) \rangle} d\lambda_g(x) := \lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{|u|^2 + \langle u, u^* \rangle}{2}} e^{\langle u, \mathbf{Z}_\varepsilon^g(x) \rangle} d\lambda_g(x) \\ = \lim_{\varepsilon \rightarrow 0} e^{\langle u, \mathbf{Z}_\varepsilon^g(x) \rangle - \frac{1}{2} \mathbb{E}[\langle u, \mathbf{Z}_\varepsilon^g(x) \rangle^2]} e^{\frac{|u|^2 + \langle u, u^* \rangle}{2} W_g(x)} d\lambda_g(x)$$

Like before all the convergences stand in probability in the space of Radon measures equipped with the weak topology, provided $|u|^2 < 4$ for the bulk measure and $|u|^2 + \langle u, u^* \rangle < 2$ for the boundary one.

5.2. **Definition of the theory.** We are now in position to define the theory associated to this GFF.

5.2.1. *A path integral.* The potential term that appears in the Toda action (1.6) still makes sense for \mathbf{Z}^g as defined above. We can therefore consider

$$(5.12) \quad \begin{aligned} V_g^M(\mathbf{Z}_\varepsilon^g + \mathbf{c}) &:= \frac{1}{4\pi} \int_\Sigma R_g \langle Q, \mathbf{Z}_\varepsilon^g + \mathbf{c} \rangle dv_g + \frac{1}{2\pi} \int_{\partial\Sigma} k_g \langle Q, \mathbf{Z}_\varepsilon^g + \mathbf{c} \rangle d\lambda_g \\ &+ \sum_{i=1}^r \mu_i e^{\langle \gamma e_i, \mathbf{c} \rangle} \int_\Sigma \varepsilon^{\gamma^2} e^{\langle \gamma e_i, \mathbf{Z}_\varepsilon^g \rangle} dv_g + \mu_i^\partial e^{\langle \frac{\gamma}{2}, \mathbf{c} \rangle} \int_{\partial\Sigma} \varepsilon^{\frac{\gamma_i^2}{8}} e^{\langle \frac{\gamma}{2} e_i, \mathbf{Z}_\varepsilon^g \rangle} d\lambda_g \end{aligned}$$

with the $\mu_i^\partial \geq 0$ being the boundary cosmological constants and $\gamma_i^2 := \gamma^2 (|e_i|^2 + \langle e_i, e_i^* \rangle)$. As a consequence the corresponding path integral writes

$$(5.13) \quad \langle F \rangle_g^M := \lim_{\varepsilon \rightarrow 0} \langle F \rangle_{g,\varepsilon}^M, \quad \text{with} \quad \langle F \rangle_{g,\varepsilon}^M := \mathcal{Z}_{\mathbf{Z}}(\Sigma, g) \int_{\mathfrak{a}} \mathbb{E} \left[F(\mathbf{Z}_\varepsilon^g + \mathbf{c}) e^{-V_g^M(\mathbf{Z}_\varepsilon^g + \mathbf{c})} \right] d\mathbf{c}.$$

Based on the same reasoning as in the previous sections we can show the following:

Proposition 5.3. *For $\chi(\Sigma) < 0$ and $F = 1$, the limit (5.13) exists and belongs to $(0, \infty)$. The path integral satisfies a Weyl anomaly with respect to a conformal change of metric as well as diffeomorphism invariance.*

Proof. Existence and finiteness follows from the very same arguments that appear in the proof of Proposition 3.7. Likewise diffeomorphism invariance is immediate. As for the Weyl anomaly, it is based on the observation that $\langle Q, u^* \rangle = \langle Q, u \rangle$ for any $u \in \mathfrak{a}$. As a consequence and like for the Neumann case the Gaussian random variable

$$Y := -\frac{1}{4\pi} \int_\Sigma R_{g'} \langle Q, \mathbf{Z}^g \rangle dv_{g'} - \frac{1}{2\pi} \int_{\partial\Sigma} k_g \langle Q, \mathbf{Z}^g \rangle d\lambda_g$$

is seen to have variance $\mathbb{E}[Y^2] = \frac{|Q|^2}{8\pi} \int_\Sigma |d_g \varphi|_g^2 dv_g$ and has the effect of shifting the law of \mathbf{Z}^g by $-\frac{Q}{2}(\varphi - m_g(\varphi))$. \square

At this stage it is important to note the following crucial fact: since for x on the boundary $\partial\Sigma$ of the surface $\mathbf{Z}^g(x)$ belongs to the vector space \mathfrak{a}_N , the boundary potentials are associated to the simple root e_i and the other simple root e_i^* are actually the same. This means that in the defining expression for V_g^C of the r boundary terms only d_N are really different. However we will see that this is no longer the case for the correlation functions since the boundary cosmological constants will then be chosen to be piecewise constant.

5.2.2. *Correlation functions.* Indeed, let us assume that we are given $(x_k, \alpha_k)_{k=1, \dots, N} \in (\Sigma \times \mathfrak{a})^N$ and $(s_l, \beta_l)_{l=1, \dots, M} \in (\partial\Sigma \times \mathfrak{a})^M$. Regularized bulk Vertex Operators are the observables

$$(5.14) \quad V_{\alpha_k, \varepsilon}(x_k) := \varepsilon^{\frac{|\alpha_k|^2}{2}} e^{\langle \alpha_k, \mathbf{Z}^g(x_k) + \mathbf{c} \rangle}$$

with their boundary counterparts being given by

$$(5.15) \quad V_{\beta_l, \varepsilon}(s_l) = V_{\beta_l, \varepsilon}(s_l, \mathbf{c}) = \varepsilon^{\frac{|\beta_l|^2 + \langle \beta_l, \beta_l^* \rangle}{8}} e^{\langle \frac{\beta_l}{2}, \mathbf{W}^g(s_l) + \mathbf{c} \rangle}.$$

Remark 5.4. *We could a priori have chosen the boundary weights to belong to the subset \mathbf{a}_N of \mathbf{a} . Indeed if s belongs to $\partial\Sigma$ then $\mathbf{Z}^g(s) \in \mathbf{a}_N$ so that the component of β in \mathbf{a}_D only evaluates the zero mode of the field.*

We can now choose a family $(\mu_l^\partial)_{l=1,\dots,M}$ of boundary cosmological constants and redefine accordingly the potential by

$$(5.16) \quad \begin{aligned} V_g^M(\mathbf{Z}_\varepsilon^g + \mathbf{c}) &= \frac{1}{4\pi} \int_\Sigma R_g \langle Q, \mathbf{Z}_\varepsilon^g + \mathbf{c} \rangle dv_g + \frac{1}{2\pi} \int_{\partial\Sigma} k_g \langle Q, \mathbf{Z}_\varepsilon^g + \mathbf{c} \rangle d\lambda_g \\ &+ \sum_{i=1}^r \mu_i e^{\langle \gamma e_i, \mathbf{c} \rangle} \int_\Sigma \varepsilon^{\gamma^2} e^{\langle \gamma e_i, \mathbf{Z}_\varepsilon^g \rangle} dv_g + e^{\langle \frac{\gamma}{2} e_i, \mathbf{c} \rangle} \int_{\partial\Sigma} \varepsilon^{\frac{\gamma^2}{8}} e^{\langle \frac{\gamma}{2} e_i, \mathbf{Z}_\varepsilon^g \rangle} d\lambda_g^{\mu_i} \end{aligned}$$

where $d\lambda_g^{\mu_i} = \sum_{l=1}^M \mu_l^\partial \mathbf{1}_{(s_j, s_{j+1})} d\lambda_g$ like in the Neumann case.

Eventually the correlation functions are defined by

$$(5.17) \quad \left\langle \prod_{k=1}^N V_{\alpha_k}(x_k) \prod_{l=1}^M V_{\beta_l}(s_l) \right\rangle_g^M := \lim_{\varepsilon \rightarrow 0} \left\langle \prod_{k=1}^N V_{\alpha_k, \varepsilon}(x_k) \prod_{l=1}^M V_{\beta_l, \varepsilon}(s_l) \right\rangle_g^M;$$

this definition does make sense as the following statement discloses:

Theorem 5.5. *The limit (5.17) exists and belongs to $(0, \infty)$ if and only if the following conditions hold for all $1 \leq i \leq r$:*

$$(5.18) \quad \left\langle \sum_{k=1}^N \alpha_k + \sum_{l=1}^M \frac{\beta_l}{2} - Q \chi(\hat{\Sigma}), \omega_i^\vee \right\rangle > 0;$$

$$(5.19) \quad \langle \alpha_k - Q, e_i \rangle < 0 \quad \text{for all } k = 1, \dots, N \quad \text{and} \quad \langle \beta_l - Q, e_i \rangle < 0 \quad \text{for all } l = 1, \dots, M.$$

Proof. The proof parallels the one already conducted for Theorem 4.6 in the Neumann case, the main difference lying in the presence of the mixed Neumann/Dirichlet potential. The arguments are similar to those appearing in [17] and [29], but for the sake of completeness we provide a detailed proof in Appendix B. \square

This definition satisfies the same conformal anomaly and diffeomorphism invariance as in the Neumann case:

Proposition 5.6. *For $g' = e^\varphi g$ in the conformal class of the uniform type 1 metric g and F a bounded functional*

$$(5.20) \quad \begin{aligned} \langle F \mathbf{V}_\alpha(\mathbf{x}) \mathbf{V}_\beta(\mathbf{s}) \rangle_{g'}^M &= \prod_{k=1}^N e^{-\Delta_{\alpha_k} \varphi(x_k)} \prod_{l=1}^M e^{-\frac{1}{2} \Delta_{\beta_l} \varphi(s_l)} \left\langle F \left(\cdot - \frac{Q}{2} \varphi \right) \mathbf{V}_\alpha(\mathbf{x}) \mathbf{V}_\beta(\mathbf{s}) \right\rangle_g^M \\ &\times \exp \left(\frac{r + 6|Q|^2}{96\pi} \left(\int_\Sigma (|d_g \varphi|_g^2 + 2R_g \varphi) dv_g + 4 \int_{\partial\Sigma} k_g \varphi d\lambda_g \right) \right). \end{aligned}$$

In the above the conformal weights are given by $\Delta_\alpha = \langle \frac{\alpha}{2}, Q - \frac{\alpha}{2} \rangle$ for a bulk insertion and $\Delta_\beta = \langle \frac{\beta+\beta^*}{4}, Q - \frac{\beta+\beta^*}{4} \rangle$ for a boundary one. Besides, if $\psi : \Sigma \rightarrow \Sigma$ is an orientation-preserving diffeomorphism sending the boundary to itself:

$$\langle F\mathbf{V}_\alpha(\mathbf{x})\mathbf{V}_\beta(\mathbf{s}) \rangle_{\psi^*g}^M = \langle F(\cdot \circ \psi)\mathbf{V}_\alpha(\psi(\mathbf{x}))\mathbf{V}_\beta(\psi(\mathbf{s})) \rangle_g^M.$$

Proof. There is one subtlety compared to the previous cases treated. Indeed and in analogy with Lemma 3.12 we know that the Wick exponential $e^{\langle \alpha, \mathbf{Z}^{g'}(x_k) \rangle}$ in the metric $g' = e^\varphi g$ has same law as $e^{\frac{|\alpha_k|^2}{8}\varphi(x_k)} e^{\frac{|\alpha_k^*|^2}{8}\varphi(x_k)} e^{\langle \alpha, \mathbf{Z}^g(x_k) \rangle}$ where the two-terms product comes from the fact that $\mathbf{Z}^{g'} = \frac{\mathbf{X}^{g'} + (\mathbf{X}^{g'})^* \circ \sigma}{\sqrt{2}}$. The same applies for a boundary insertion, but this time the prefactor is equal to $e^{\frac{|\beta_l + \beta_l^*|^2}{16}\varphi(x_k)}$. Combining this with the shift by $-\frac{Q}{2}\varphi$ of the law of the GFF allows one to write that the factor corresponding to x_k in the conformal anomaly is given by $e^{-\frac{1}{8}\langle \alpha_k, 2Q - \alpha_k \rangle \varphi(x_k)} e^{-\frac{1}{8}\langle \alpha_k^*, 2Q - \alpha_k^* \rangle \varphi(x_k)}$ in the bulk and $e^{\left(\frac{|\beta_l + \beta_l^*|}{16} - \langle Q, \frac{\beta_l}{4} \rangle\right) \varphi(s_l)}$ on the boundary. The proof is therefore complete as soon as one recalls that $\langle u^*, Q \rangle = \langle u, Q \rangle$ for any $u \in \mathfrak{a}$ and that \cdot^* is self-adjoint. \square

It is interesting to note that the expression of the conformal weights Δ_β associated to a boundary insertion are changed compared with the Neumann case. However this weight remains unchanged if it is assumed that $\beta = \beta^*$, which is a natural assumption if we recall that the GFF \mathbf{Z}^g satisfies the property that $\mathbf{Z}^g = (\mathbf{Z}^g)^*$ on $\partial\Sigma$. Actually if one chooses to restrict the boundary weights to \mathfrak{a}_N then the only loss of information comes from the zero-mode of the Toda field.

5.3. Toda CFTs at triality. So far we have constructed a family of “twisted” models based on non-trivial outer automorphisms of order two, and to do so we have relied on Cardy’s doubling trick. Let us now discuss the special case of the Lie algebra D_4 (that is \mathfrak{so}_8) for which there exists an outer automorphism of order three. And to start with let us simply consider the free field theory on the upper-half plane \mathbb{H} (in which case σ just corresponds to the conjugacy $z \mapsto \bar{z}$ and the corresponding closed surface is the whole plane \mathbb{C}) and look at the consequences of the assumptions posed by the doubling trick.

If we want that the underlying field Φ of the theory satisfies the property that $\Phi(\bar{z}) = \tau\Phi(z)$, then formally speaking its covariance martix should be given by

$$\begin{aligned} \mathbb{E} [\langle u, \Phi(x) \rangle \langle v, \Phi(y) \rangle] = \\ \frac{1}{2} \left(\langle u, v \rangle \ln \frac{1}{x-y} + \langle u, \tau v \rangle \ln \frac{1}{\bar{x}-y} + \langle \tau u, v \rangle \ln \frac{1}{x-\bar{y}} + \langle \tau u, \tau v \rangle \ln \frac{1}{\bar{x}-\bar{y}} \right) \end{aligned}$$

for any x, y in \mathbb{H} and u, v in \mathfrak{a} , and where the branch of the logarithm is chosen to be on $(-\infty, 0)$. Put differently we should construct a field with the property that

$$\begin{aligned} \mathbb{E} [\langle u, \Phi(x) \rangle \langle v, \Phi(y) \rangle] = \frac{\langle u, v \rangle + \langle \tau u, \tau v \rangle}{2} \ln \frac{1}{|x-y|} + \frac{\langle u, \tau v \rangle + \langle \tau u, v \rangle}{2} \ln \frac{1}{|\bar{x}-y|} \\ + i \left(\frac{\langle u, v \rangle - \langle \tau u, \tau v \rangle}{2} \text{Arg} \frac{1}{x-y} + \frac{\langle u, \tau v \rangle - \langle \tau u, v \rangle}{2} \text{Arg} \frac{1}{\bar{x}-y} \right). \end{aligned}$$

In the previous cases studied, the second line was found to vanish for any value of u and v . However this is no longer the case if τ is the outer automorphism of order three. As a consequence the definition of a model with this type of “twisted” boundary conditions would not follow from the type of construction proposed in the previous sections.

We believe that the construction of the theory with this specific twist would rather involve the associated orbifold model. Put differently one should consider in the Hilbert space of the theory projection on the subspace of states invariant under this transformation.

6. PERSPECTIVES: INTEGRABILITY OF BOUNDARY TODA CONFORMAL FIELD THEORY

In this document we have proposed a definition of Toda Conformal Field Theories on a Riemann surface with boundary. To do so we have constructed two families of models based on the different types of boundary conditions that they enjoy. These models enjoy conformal symmetry but also an enhanced level of symmetry, which hints towards the fact that it might be possible to understand some of their integrability properties. We discuss this aspect in this section and provide some detailed perspectives on a future program aiming at computing certain correlation functions for these theories.

6.1. Toda Conformal Field Theory on the unit disk. Toda CFTs on the sphere have been defined and proved to be a CFT in the axiomatic sense in the article [15]. The main difference with the hyperbolic case is that the partition function is no longer convergent, and one has to add at least three insertion points in order to have a convergent correlation function: this is due to the the Seiberg bounds (3.22) (with $\chi(\Sigma) = 2$) that put constraints on the weights. The same type of obstruction exists when one constructs the Toda CFTs on the unit disk \mathbb{D} , but the correlation functions can nonetheless be defined. A way to do it is to consider Toda CFT on the sphere \mathbb{S}^2 equipped with the round metric $g = \frac{4|dz|^2}{(1+z^2)^2}$ and cut the sphere along the equator. The latter being a geodesic for the metric g , this operation induces a uniform type 1 metric on, say, the upper hemisphere, which has the topology of a disk. Then all the previous work applies, except that one has to take care that the correlation functions are well-defined, and for this one needs to ensure that the following Seiberg bounds are satisfied:

$$(6.1) \quad \begin{aligned} \left\langle \sum_{k=1}^N \alpha_k + \sum_{l=1}^M \frac{\beta_l}{2} - Q, \omega_i^\vee \right\rangle &> 0 \quad i = 1, \dots, r; \\ \langle \alpha_k - Q, e_i \rangle &< 0 \quad i = 1, 2 \text{ and } k = 1, \dots, N. \end{aligned}$$

This follows from the fact that $\chi(\mathbb{D}) = 1$. In particular we see that the Seiberg bounds impose that:

- if there are no bulk insertions then the number of boundary insertions must be greater than or equal to 3;
- correlation functions with one bulk insertion contain at least one other insertion.

This suggests that structure constants of the theory are given, like in the case of Liouville theory, by bulk-boundary two-point correlation functions together with the boundary three-point correlation functions.

Also note that using the Cayley transform $x \mapsto i\frac{1+x}{1-x}$ one maps the unit disk \mathbb{D} onto the upper half plane \mathbb{H} , and using the conformal covariance of Toda CFT one can easily deduce how to construct Toda CFTs on the upper half-plane. This setting is particularly adapted to computations, since the involution σ in that case is nothing but the complex conjugate $z \mapsto \bar{z}$. We adopt this viewpoint in the sequel.

6.2. W-symmetry of the models. A feature of Toda CFTs is that they enjoy in addition to conformal invariance a higher level of symmetry encoded by W -algebras. This enhanced symmetry manifests itself via the existence of certain observables, called *higher-spin currents*, that somehow encodes the constraints it poses. This can be seen for instance via the existence of *Ward identities*, which may be translated in differential equations satisfied by the correlation functions.

6.2.1. W -symmetry and Ward identities. The higher-spin symmetry is realized through the existence of Ward identities, which correspond to the computation of the Operator Product Expansion between a higher-spin current and a Vertex Operator. They were derived for the \mathfrak{sl}_3 Toda CFT on the sphere in the article [14], where the descendant fields associated to the Vertex Operators were constructed. It was also proved there that the global Ward identities pose 5 constraints between the W -descendants and 3 between the derivatives of a given correlation function.

The method proposed in [16] to treat the boundary case should allow us to derive such identities for the correlation functions. Besides we should be able to show the existence of degenerate fields of the theory in the same fashion as in [14].

6.2.2. Structure constants. If one considers a correlation function with N bulk Vertex Operators and M boundary ones, then by Cardy's doubling trick this amounts, as far as the Ward identities are concerned, to considering a correlation functions on the sphere with $2N + M$ Vertex Operators. The number of W -descendants that it is possible to construct is therefore given by $4N + 2M$. If we use the Ward identities we can reduce the number of unknown such descendants to $4N + 2M - 5$.

Other constraints can be added by considering degenerate fields: for instance a semi-degenerate field yields 1 more constraint, while the W -descendants of a fully degenerate field are expressible in terms of derivatives of the correlation functions. As a consequence, if we consider a correlation function containing one boundary fully degenerate field we see that the number (M, N) of remaining insertions must satisfy $4N + 2M \leq 5$ if we want to be able to find a differential equation for such a correlation function. This assumption can be relaxed if a bulk insertion is semi-degenerate in which case we must have $4N + 2M \leq 7$, while if a boundary insertion is semi-degenerate this becomes $4N + 2M \leq 6$. In theory we should therefore be able to derive BPZ differential equations that enter the computation of the following structure constants:

- (1) Bulk one-point function $\langle V_\alpha(z) \rangle$;
- (2) Boundary two-point function $\langle V_{\beta_1}(s_1)V_{\beta_2}(s_2) \rangle$;
- (3) Bulk-boundary structure constant $\langle V_\alpha(z)V_{\kappa\omega_1}(s) \rangle$ and also $\langle V_{\kappa\omega_1}(z)V_\beta(s) \rangle$;

(4) Boundary three-point function $\langle V_{\kappa\omega_1}(s_1)V_{\beta_2}(s_2)V_{\beta_3}(s_3)\rangle$.

In particular and like in the closed case we do not expect to be able to compute all the structure constants of the theory. Nevertheless there is good hope that our probabilistic framework would allow us to derive an expression for the correlation functions described above, or at least some of them.

APPENDIX A. SIMPLE LIE ALGEBRAS

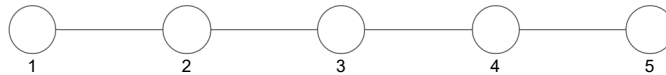
We gather here information about the Lie algebras considered throughout the document, such as their Dynkin diagrams, their Cartan matrices and their outer automorphism groups. Additional details can be found *e.g.* in the textbook [30].

Outer automorphism group: Let us denote by $\text{Aut}(\mathfrak{g})$ the group of automorphisms of the Lie algebra \mathfrak{g} and by $\text{Inn}(\mathfrak{g})$ its inner automorphism group, that is the subgroup of $\text{Aut}(\mathfrak{g})$ generated by conjugations by elements of \mathfrak{g} . The outer automorphism group of \mathfrak{g} is then the quotient $\text{Out}(\mathfrak{g}) = \text{Aut}(\mathfrak{g})/\text{Inn}(\mathfrak{g})$. A convenient way to visualize this group is to rely on the Dynkin diagram associated to the Lie algebra \mathfrak{g} . Indeed, the outer automorphism group is actually isomorphic to the group of graph automorphisms of the associated Dynkin diagram that we describe below.

The Lie algebras $(A_n)_{n \geq 1}$: This family corresponds to the simple and complex Lie algebras \mathfrak{sl}_{n+1} . The Cartan matrix of A_n is given by

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

The simple roots of A_n are pairwise separated by an angle of 120° , giving the following Dynkin diagram (here for A_5):



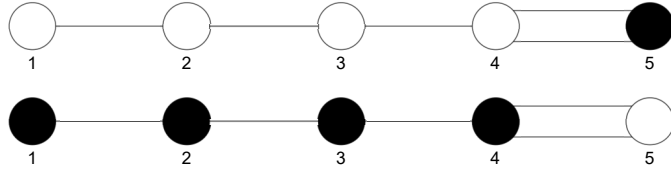
The only non-trivial outer automorphism reverses the indices in the diagram, that is in matrix form:

$$\begin{pmatrix} & & & & 1 \\ & & & & \\ & & & & \\ & & & & \\ 1 & & & & \end{pmatrix}.$$

B_n and C_n : They are associated to, respectively, \mathfrak{o}_{2n+1} and \mathfrak{sp}_n . The Cartan matrix differs from that of A_n by a -2 on the upper or lower second diagonal:

$$B_n : \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & \ddots & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & 2 & -1 & 0 \\ & & & & -1 & 2 & -2 \\ & & & & 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad C_n : \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & \ddots & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & 2 & -1 & 0 \\ & & & & -1 & 2 & -1 \\ & & & & 0 & -2 & 2 \end{pmatrix}.$$

In B_n (resp. C_n) the last simple root is shorter (resp. longer) and form an angle of 135° with its neighbor, hence the double edge in the Dynkin diagrams (here for B_5 and C_5):

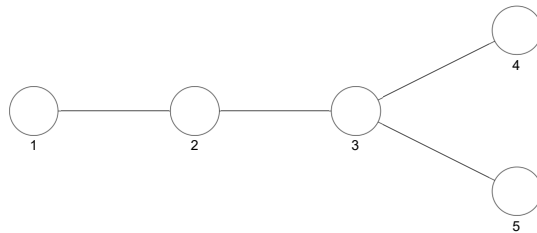


There is no non-trivial outer automorphism.

The case of D_n : The Cartan matrix of D_n (or \mathfrak{o}_{2n}) is given by:

$$\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 2 & -1 & -1 \\ & & & & -1 & 2 & 0 \\ & & & & -1 & 0 & 2 \end{pmatrix}.$$

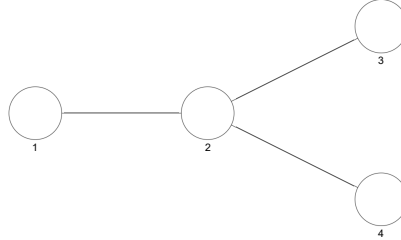
In general (that is, for $n \neq 4$), the automorphism group of D_n is isomorphic to \mathbf{Z}_2 (the cyclic group of order 2), and the only non-trivial automorphism has the effect of swapping the last two simple roots in the diagram (here for D_5):



In matrix form, this can be written as

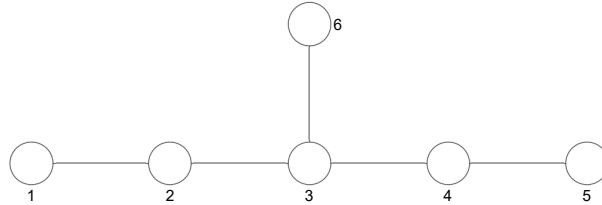
$$\begin{pmatrix} I_{n-2} & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$$

The Lie algebra D_4 is particular because its outer automorphism group is isomorphic to S_3 , the group of permutations on three elements. The elements that are permuted in D_4 are the three orthogonal simple roots, as can be seen on the following Dynkin diagram:



The five non-trivial elements of $\text{Out}(D_4)$ are the permutations $(1\ 3)$, $(1\ 4)$, $(3\ 4)$, $(1\ 3\ 4)$ and $(1\ 4\ 3)$ and can be written in matrix form as permutation matrices.

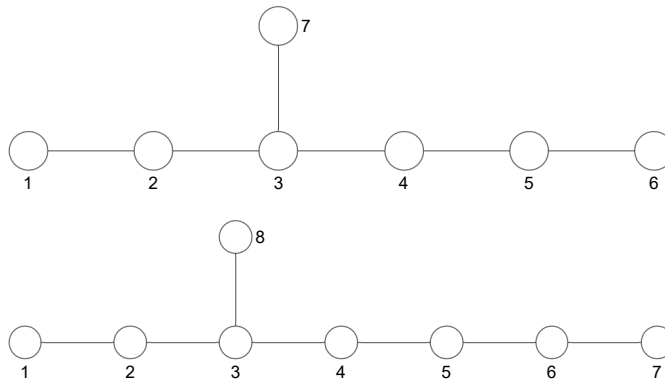
Exceptional simple Lie algebras: The only exceptional simple Lie algebra that has a non-trivial outer automorphism group is E_6 :



This group acts on the simple roots as in A_5 by reversing the indices of the first 5 roots, and leaving the last one invariant. In matrix form:

$$\begin{pmatrix} & & & & 1 \\ & & & & \\ & & \dots & & \\ 1 & & & & \\ & & & & 1 \end{pmatrix}$$

Here are the Dynkin diagrams of E_7 and E_8 :

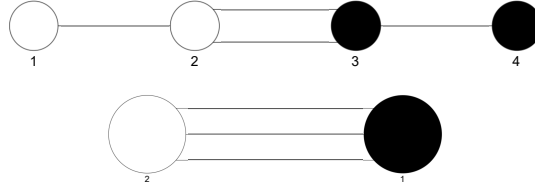


The Cartan matrix of E_k for $k = 6, 7, 8$ is the same as for A_k , except that it has the value -1 on the entries $(k, 3)$ and $(3, k)$, and 0 on the entries $(k, k - 1)$ and $(k - 1, k)$. Finally,

the Cartan matrices of F_4 and G_2 are given respectively by

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Their Dynkin diagrams are, respectively,



Lecture of the Dynkin diagrams: the vertices represent the simple roots, vertices that are not connected by one edge are orthogonal, while one edge represent an angle of 120° , two edges an angle of 135° , and three edges an angle of 150° . Further, the angles between the simple roots and their lengths being linked, the number of edges between two vertices also encodes the ratio of the lengths of two correspondent simple roots. One edge corresponds to a length ratio of 1, while two edges encode a length ratio of $\sqrt{2}$, and three edges a length ratio of $\sqrt{3}$. In the last two cases, the difference between the longest and the shortest root is encoded by the color: black for the shortest root and white for the longest one.

Invariant sub-algebras: We briefly explain here a way to compute the sub-algebra \mathfrak{a}_N appearing in the construction of the twisted models, with the help of the Dynkin diagram. This sub-algebra is the τ -invariant sub-algebra of \mathfrak{g} , with τ an outer automorphism of \mathfrak{g} , that is a graph automorphism of the Dynkin diagram. If τ is of order 2, then the (real part of the) Cartan sub-algebra \mathfrak{a} splits into two orthogonal eigenspaces of τ , \mathfrak{a}_N and \mathfrak{a}_D , associated to the eigenvalues 1 and -1 , respectively. A basis of simple roots of \mathfrak{a}_N is given by the $\{\frac{1}{2}(e_i + \tau e_i), 1 \leq i \leq r\}$ while for \mathfrak{a}_D it is given by $\{\frac{1}{2}(e_i - \tau e_i), 1 \leq i \leq r\}$. For an automorphism of order 3 (thus in D_4) the three eigenspaces are no longer orthogonal, and a basis of simple roots for the invariant sub-algebra is given by $\{e_2, \frac{1}{3}(e_1 + e_3 + e_4)\}$. In both cases studying the angles formed by the simple roots or their length ratios allows one to deduce the form of the invariant sub-algebra. However, by folding the Dynkin diagram of \mathfrak{g} on itself in order to make it invariant with respect to τ , one can easily compute the invariant sub-algebra in the different cases and in particular their dimension. It corresponds to folding the space generated by the simple roots on itself with respect to the τ -invariant hyperplanes. We gather the invariant sub-algebras in all cases in the following table.

Algebra \mathfrak{g}	Order of $\tau \in \text{Out}(\mathfrak{g})$	Invariant sub-algebra	Dimension of \mathfrak{a}_N
A_{2n}	2	B_n	n
A_{2n-1}	2	C_n	n
$D_n, n \geq 4$	2	B_{n-1}	$n - 1$
D_4	3	G_2	2
E_6	2	F_4	4

APPENDIX B. PROOF OF THEOREM 5.5

Using Girsanov theorem as before one can write the regularized partition function as

$$A_\varepsilon^M = C_{\varepsilon, \mathbf{x}, \boldsymbol{\alpha}, \mathbf{s}, \beta}^M \int_{\mathfrak{h}_3^*} e^{\langle \sum \alpha_k + \frac{1}{2} \sum \beta_l - Q\chi(\Sigma), \mathbf{c} \rangle} \mathbb{E} \left[e^{-\sum_{i=1}^2 \mu_i e^{\langle \gamma e_i, \mathbf{c} \rangle} Z_{i, \varepsilon}^C - \mu^\partial e^{\langle \frac{\gamma}{2} e_1, \mathbf{c} \rangle} Z_\varepsilon^{C, \partial}} \right] d\mathbf{c} (1 + o(1))$$

with (recall the definition of the field \mathbf{Z}^g)

$$Z_{i, \varepsilon}^M = \int_\Sigma H_\varepsilon^M(x) \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma X_\varepsilon^{N, g}(x)} \varepsilon^{\frac{\gamma^2}{2}} e^{\pm \gamma X_\varepsilon^{D, g}(x)} dv_g(x) \quad ; \quad Z_\varepsilon^{M, \partial} = \int_{\partial \Sigma} H_\varepsilon^{M, \partial}(s) \varepsilon^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2} X_\varepsilon^{N, g}(s)} d\lambda_g(s)$$

where $X^{N, g}$ and $X^{D, g}$ are independent Neumann and Dirichlet GFF, respectively. And with

$$H_\varepsilon^M(x) = \prod_k e^{2\pi(\langle \gamma e_i, \alpha_k \rangle G_{g, \varepsilon}(x_k, x) + \langle \gamma e_i, \alpha_k^* \rangle G_{g, \varepsilon}(x_k, \sigma(x)))} \prod_l e^{2\pi(\langle \gamma e_i, \frac{\beta_l + \beta_l^*}{2} \rangle G_{g, \varepsilon}(s_l, x))}$$

in the bulk, and

$$H_\varepsilon^{M, \partial}(s) = \prod_k e^{2\pi(\langle \frac{\gamma}{2} e_i, \alpha_k \rangle + \langle \frac{\gamma}{2} e_i, \alpha_k^* \rangle) G_{g, \varepsilon}(x_k, s)} \prod_l e^{2\pi(\langle \frac{\gamma}{2} e_i, \frac{\beta_l + \beta_l^*}{2} \rangle G_{g, \varepsilon}(s_l, s))}$$

on the boundary. By the same reasoning as in the Neumann case we see that A_ε^M diverges if $\langle \sum \alpha_k + \frac{1}{2} \sum \beta_l - Q\chi(\Sigma), \omega_i \rangle \leq 0$ for some i . Let us now assume the opposite and show first that the conditions (5.18) and (5.19) are necessary for the regularized partition function to converge and to be non-trivial. We start with the behavior of $Z_{i, \varepsilon}^M$ around a bulk insertion, *i.e.* we study the integral

$$\int_{B_g(x_k, r)} e^{2\pi(\langle \gamma e_i, \alpha_k \rangle G_{g, \varepsilon}(x_k, x))} \mu_\varepsilon^N(x) \mu_\varepsilon^D(x) dv_g(x)$$

where $\mu_\varepsilon^N(x)$ and $\mu_\varepsilon^D(x)$ are shortcuts for the regularized Neumann/Dirichlet GMC densities, respectively, and $r > 0$ is small. Thanks to Proposition 2.2, we can work on the Poincaré disk $(B(0, 1 - \delta), g_P)$, and since we're working on a small compact ball, it is enough to work with the Euclidean metric instead of g_P . Now if we suppose that x_k is sent to 0 without loss of generality, we are left with the integral

$$\int_{B(0, r)} e^{-\gamma\alpha \int_0^{2\pi} \log|x + \varepsilon e^{i\theta}| \frac{d\theta}{2\pi}} \mu_\varepsilon^N(x) \mu_\varepsilon^D(x) dx$$

where we recall that $\gamma\alpha < \gamma^2 + 2$ by hypothesis. Finally, the latter can be upper bounded by

$$\int_{B(0, r)} (|x| + \varepsilon)^{-\gamma\alpha} \mu_\varepsilon^N(x) \mu_\varepsilon^D(x) dx.$$

Applying Chebychev's inequality to the conclusion of Lemma B.2 (see after this proof) we see that for $\eta > 0$,

$$\sup_{\varepsilon > 0} \mathbb{P} \left(\int_{B(0, r)} (|x| + \varepsilon)^{-\gamma\alpha} \mu_\varepsilon^N(x) \mu_\varepsilon^D(x) dx > r^{2 - \gamma\alpha + \gamma^2 - \eta} \right) \leq Cr^{-\gamma^2 p^2 + \eta p}.$$

Choose $p > 0$ such that $\eta p - \gamma^2 p^2 > 0$, then by Borel-Cantelli there exists a random constant R finite almost surely such that

$$\sup_{r \in]0, 1-\delta]} r^{-(2+\gamma^2-\gamma\alpha-\eta)} \int_{B(0,r)} (|x| + \varepsilon)^{-\gamma\alpha} \mu_\varepsilon^N(x) \mu_\varepsilon^D(x) dx \leq R$$

uniformly in ε . Now divide the ball $B(0, 1 - \delta)$ into dyadic annuli $A_n = \{(1 - \delta)/2^{n+1} \leq |x| \leq (1 - \delta)/2^n\}$ and apply the latter on each annulus to obtain

$$\int_{B(0,1-\delta)} (|x| + \varepsilon)^{-\gamma\alpha} \mu_\varepsilon^N(x) \mu_\varepsilon^D(x) dx \leq \sum_{n \geq 0} R 2^{-n(2+\gamma^2-\gamma\alpha-\eta)}.$$

And the proof of this first point is complete provided we choose $0 < \eta < \gamma(q - \alpha)$ (which is compatible with the above condition on p). Let us now look at what happens around the other insertions. When $x \in \Sigma$ approaches a boundary insertion, a similar reasoning as before show that we have to study the integral

$$\int_{\mathbb{H} \cup B(0,1-\delta)} (|x| + \varepsilon)^{-\gamma\beta} \mu_\varepsilon^D(x) \mu_\varepsilon^N(x) dx$$

with $\beta < q$. The proof follows closely the previous one so we don't repeat the arguments here. Concerning the integral over $\partial\Sigma$ around a boundary insertion, the situation is exactly the same as in the Neumann case since the Dirichlet GFF vanishes almost surely on the boundary. The last thing that we have to show is that the Seiberg bounds are necessary to ensure that the integrals we considered don't blow up around the insertions, in order to ensure that the correlation function is not zero. For the integral over $\partial\Sigma$ the situation is again the same as in the Neumann case. For the integral over the bulk, however, we can restrict ourselves to the Neumann case by the use of Kahane's convexity inequality. Indeed, one has on each ball B

$$\mathbb{E}[(X_\varepsilon^N(x) \pm X_\varepsilon^D(x))(X_\varepsilon^N(y) \pm X_\varepsilon^D(y))] = 4\pi G_\varepsilon(x, y) \geq \mathbb{E}[X_\varepsilon^N(x)X_\varepsilon^N(y)] + C$$

so the integrals with mixed Dirichlet/Neumann densities can be lower-bounded by the same integral with the Neumann GMC density only, and the same reasoning as in [17] applies to end the proof.

Lemma B.1. *Let $\alpha < q$. Then for $0 < p < \frac{4}{\gamma^2} \wedge \frac{1}{\gamma}(q - \alpha)$ it holds that*

$$(B.1) \quad \sup_{\varepsilon > 0} \mathbb{E} \left[\left(\int_{B(0,1-\delta)} (|x| + \varepsilon)^{-\gamma\alpha} \mu_\varepsilon^N(x) \mu_\varepsilon^D(x) dx \right)^p \right] < \infty.$$

Proof. We use the so-called white noise cutoff approximation of the GFF. We know that there exist cutoff approximations of X^N and X^D , say \tilde{X}_ε^N and \tilde{X}_ε^D , such that the covariance kernels of $X_\varepsilon^{N/D}$ and $\tilde{X}_\varepsilon^{N/D}$ are comparable (in the sense of Kahane, that is $\mathbb{E}[X_\varepsilon^{N/D}(x)X_\varepsilon^{N/D}(y)] \leq \mathbb{E}[\tilde{X}_\varepsilon^{N/D}(x)\tilde{X}_\varepsilon^{N/D}(y)] + C$) and such that for all $\lambda < 1$,

$$(\tilde{X}_{\lambda\varepsilon}^{N/D}(\lambda x))_{|x| \leq 1} = (\tilde{X}_\varepsilon^{N/D}(x))_{|x| \leq 1} + \Omega_\lambda^{N/D}$$

in distribution, with Ω_λ^N and Ω_λ^D two $\mathcal{N}(0, \frac{1}{\lambda})$ random variables, independent of everything. For a precise construction of the white noise cutoff of the GFF, see for example [17] or [39]. If we denote by $\tilde{\mu}_\varepsilon^{N/D}$ the associated regularized GMC densities, we have

$$\tilde{\mu}_{\lambda\varepsilon}^N(\lambda x)\tilde{\mu}_{\lambda\varepsilon}^D(\lambda x)dx = e^{\gamma\Omega_\lambda^N - \frac{\gamma^2}{2}\ln\frac{1}{\lambda}}e^{\gamma\Omega_\lambda^D - \frac{\gamma^2}{2}\ln\frac{1}{\lambda}}\tilde{\mu}_\varepsilon^N(x)\tilde{\mu}_\varepsilon^D(x)dx$$

in distribution. Then, by Kahane's convexity inequality, it suffices to show the lemma with these new approximations. We start with $p \geq 1$. We have, using the above properties and classical properties of GMC,

$$\begin{aligned} & \mathbb{E} \left[\left(\int_{B(0,1-\delta)} (|x| + \varepsilon)^{-\gamma\alpha} \mu_\varepsilon^N(x) \mu_\varepsilon^D(x) dx \right)^p \right]^{1/p} \\ & \leq \mathbb{E} \left[\left(\int_{B(0, \frac{1-\delta}{2})} (|x| + \varepsilon)^{-\gamma\alpha} \mu_\varepsilon^N(x) \mu_\varepsilon^D(x) dx \right)^p \right]^{1/p} + \mathbb{E} \left[\left(\int_{\frac{1-\delta}{2} \leq |x| \leq 1-\delta} (|x| + \varepsilon)^{-\gamma\alpha} \mu_\varepsilon^N(x) \mu_\varepsilon^D(x) dx \right)^p \right]^{1/p} \\ & \leq \mathbb{E} \left[\left(\int_{B(0, \frac{1-\delta}{2})} (|x| + \varepsilon)^{-\gamma\alpha} \mu_\varepsilon^N(x) \mu_\varepsilon^D(x) dx \right)^p \right]^{1/p} + C \\ & \leq \mathbb{E} \left[2^{-2p} \left(\int_{B(0,1-\delta)} (|\frac{1}{2}y| + \frac{1}{2}\varepsilon')^{-\gamma\alpha} \mu_{\frac{1}{2}\varepsilon'}^N(\frac{1}{2}y) \mu_{\frac{1}{2}\varepsilon'}^D(\frac{1}{2}y) dy \right)^p \right]^{1/p} + C \\ & \leq \mathbb{E} \left[2^{\gamma\alpha p - 2p} \left(e^{\gamma\Omega_{1/2}^N - \frac{\gamma^2}{2}\ln 2} e^{\gamma\Omega_{1/2}^D - \frac{\gamma^2}{2}\ln 2} \int_{B(0,1-\delta)} (|y| + \varepsilon')^{-\gamma\alpha} \mu_{\varepsilon'}^N(y) \mu_{\varepsilon'}^D(y) dy \right)^p \right]^{1/p} + C \\ & \leq 2^{\gamma\alpha - \frac{\xi(p)}{p}} \mathbb{E} \left[\left(\int_{B(0,1-\delta)} (|y| + \varepsilon')^{-\gamma\alpha} \mu_{\varepsilon'}^N(y) \mu_{\varepsilon'}^D(y) dy \right)^p \right]^{1/p} + C \end{aligned}$$

where $\xi(p) = -\gamma^2 p^2 + (2 + \gamma^2)p$. Now take $\varepsilon_n = 2^{-n}$ and

$u_n = \mathbb{E} \left[\left(\int_{B(0,1-\delta)} (|x| + \varepsilon_n)^{-\gamma\alpha} \mu_{\varepsilon_n}^N(x) \mu_{\varepsilon_n}^D(x) dx \right)^p \right]^{1/p}$. By the previous argument we have $u_{n+1} \leq 2^{\gamma\alpha - \frac{\xi(p)}{p}} u_n + C$, thus (u_n) is bounded independently of n if $\frac{\xi(p)}{p} - \gamma\alpha > 0$, that is $p < \frac{1}{\gamma}(q - \alpha)$. By Kahane's convexity inequality we recover the result for general ε . If $p < 1$ then we use the sub-additivity of $x \mapsto x^p$ to conclude the same way. \square

Lemma B.2. *Under the same assumptions as in Lemma B.1, there exists $C > 0$ such that for all $0 < r \leq 1 - \delta$,*

$$(B.2) \quad \sup_{\varepsilon > 0} \mathbb{E} \left[\left(\int_{B(0,r)} (|x| + \varepsilon)^{-\gamma\alpha} \mu_\varepsilon^N(x) \mu_\varepsilon^D(x) dx \right)^p \right] < C r^{\xi(p) - \gamma\alpha p}$$

where $\xi(p) = -\gamma^2 p^2 + (2 + \gamma^2)p$.

Proof. If $\varepsilon \leq r$ then following the same reasoning as in Lemma B.1 we obtain

$$\begin{aligned} & \mathbb{E} \left[\left(\int_{B(0,r)} (|x| + \varepsilon)^{-\gamma\alpha} \mu_\varepsilon^N(x) \mu_\varepsilon^D(x) dx \right)^p \right] \\ & \leq r^{\xi(p) - \gamma\alpha p} \mathbb{E} \left[\left(\int_{B(0,1-\delta)} (|x| + \varepsilon)^{-\gamma\alpha} \mu_\varepsilon^N(x) \mu_\varepsilon^D(x) dx \right)^p \right] \\ & \leq C r^{\xi(p) - \gamma\alpha p}. \end{aligned}$$

If $\varepsilon > r$, we have

$$\begin{aligned} & \mathbb{E} \left[\left(\int_{B(0,r)} (|x| + \varepsilon)^{-\gamma\alpha} \mu_\varepsilon^N(x) \mu_\varepsilon^D(x) dx \right)^p \right] \\ & \leq r^{-\gamma\alpha p} \mathbb{E} \left[\left(\int_{B(0,r)} \mu_\varepsilon^N(x) \mu_\varepsilon^D(x) dx \right)^p \right] \\ & \leq C r^{\xi(p) - \gamma\alpha p}. \end{aligned}$$

□

REFERENCES

- [1] O. Alvarez. Theory of Strings with Boundaries: Fluctuations, Topology, and Quantum Geometry. *Nucl. Phys. B*, 216:125–184, 1983.
- [2] M. Ang. Liouville conformal field theory and the quantum zipper. *Preprint*, arXiv:2301.13200, 2023.
- [3] M. Ang, G. Remy, and X. Sun. FZZ formula of boundary Liouville CFT via conformal welding. *Journal of the European Mathematical Society*, to appear.
- [4] M. Ang, G. Remy, X. Sun, and T. Zhu. Derivation of all structure constants for boundary Liouville CFT. *Preprint*, arXiv:2305.18266, 2023.
- [5] G. Baverez and B. Wu. Higher equations of motion at level 2 in Liouville CFT. *Preprint*, arXiv:2312.13900, 2023.
- [6] A. Belavin and V. Belavin. Higher Equations of Motion in Boundary Liouville Field Theory. *JHEP*, 02:010, 2010.
- [7] A.A. Belavin, A.M. Polyakov, and A.B. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nuclear Physics B*, 241(2):333 – 380, 1984.
- [8] N. Berestycki. An elementary approach to Gaussian multiplicative chaos. *Electronic Communications in Probability*, 22:1 – 12, 2017.
- [9] J. L. Cardy. Conformal Invariance and Surface Critical Behavior. *Nucl. Phys. B*, 240:514–532, 1984.
- [10] J. L. Cardy. Effect of Boundary Conditions on the Operator Content of Two-Dimensional Conformally Invariant Theories. *Nucl. Phys. B*, 275:200–218, 1986.
- [11] J. L. Cardy. Boundary conditions, fusion rules and the verlinde formula. *Nuclear Physics B*, 324(3):581–596, 1989.
- [12] B. Cerclé. Three-point correlation functions in the \mathfrak{sl}_3 Toda theory II: the formula. *Preprint*, arXiv:2208.12085, 2022.
- [13] B. Cerclé. Three-point correlation functions in the \mathfrak{sl}_3 Toda theory I: Reflection coefficients. *Probability Theory and Related Fields*, 188:89–158, 2024.
- [14] B. Cerclé and Y. Huang. Ward identities in the \mathfrak{sl}_3 Toda conformal field theory. *Communications in Mathematical Physics*, 393:419–475, 2022.
- [15] B. Cerclé, R. Rhodes, and V. Vargas. Probabilistic construction of Toda conformal field theories. *Annales Henri Lebesgue*, 6:31–64, 2023.
- [16] B. Cerclé. Higher equations of motion for boundary Liouville conformal field theory from the Ward identities. *Preprint*, arXiv:2401.13271, 2024.

- [17] F. David, A. Kupiainen, R. Rhodes, and V. Vargas. Liouville Quantum Gravity on the Riemann Sphere. *Communications in Mathematical Physics*, 342:869–907, 2016.
- [18] H. Dorn and H.-J. Otto. Two- and three-point functions in Liouville theory. *Nuclear Physics B*, 429(2):375 – 388, 1994.
- [19] B. Duplantier and S. Sheffield. Liouville Quantum Gravity and KPZ. *Inventiones mathematicae*, 185:333, 2011.
- [20] V. Fateev, A. Zamolodchikov, and Al. Zamolodchikov. Boundary Liouville Field Theory I. Boundary State and Boundary Two-point Function. *Preprint*, arXiv:0001012, 2000.
- [21] V.A. Fateev and A.V. Litvinov. Correlation functions in conformal Toda field theory. I. *JHEP*, 11:002, 2007.
- [22] Vladimir Fateev and Sylvain Ribault. Conformal Toda theory with a boundary. *JHEP*, 12:089, 2010.
- [23] Peter Goddard and David I. Olive. Kac-Moody and Virasoro Algebras in Relation to Quantum Physics. *Int. J. Mod. Phys. A*, 1:303, 1986.
- [24] C. Guillarmou, A. Kupiainen, R. Rhodes, and V. Vargas. Conformal bootstrap in Liouville Theory. *To appear in Acta Mathematica*, 2020.
- [25] C. Guillarmou, A. Kupiainen, R. Rhodes, and V. Vargas. Segal’s axioms and bootstrap for Liouville theory. *Preprint*, arXiv:2112.14859, 2021.
- [26] C. Guillarmou, R. Rhodes, and V. Vargas. Polyakov’s formulation of $2d$ bosonic string theory. *Publications Mathématiques de l’IHÉS*, 130:111–185, 2019.
- [27] A. Honecker. Automorphisms of W-algebras and extended rational conformal field theories. *Nuclear Physics B*, 400(1-3):574–596, July 1993.
- [28] K. Hosomichi. Bulk boundary propagator in Liouville theory on a disc. *JHEP*, 11:044, 2001.
- [29] Y. Huang, R. Rhodes, and V. Vargas. Liouville quantum gravity on the unit disk. *Ann. Inst. H. Poincaré Probab. Statist.*, 54(3):1694–1730, 08 2018.
- [30] J. E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Springer, Berlin, 1972.
- [31] A. Kupiainen, R. Rhodes, and V. Vargas. Integrability of Liouville theory: proof of the DOZZ formula. *Annals of Mathematics*, 191(1):81–166, 2020.
- [32] B Osgood, R Phillips, and P Sarnak. Extremals of determinants of Laplacians. *Journal of Functional Analysis*, 80(1):148 – 211, 1988.
- [33] A. Polyakov. Quantum Geometry of bosonic strings. *Physics Letters B*, 103:207:210, 1981.
- [34] B. Ponsot and J. Teschner. Boundary Liouville field theory: boundary three-point function. *Nuclear Physics B*, 622(1):309–327, 2002.
- [35] E. Powell and W. Werner. *Lecture Notes on the Gaussian Free Field*. Cours Spécialisés. SMF, 2021.
- [36] G. Remy. Liouville quantum gravity on the annulus. *J. Math. Phys.*, 59(8):082303, 2018.
- [37] G. Remy. The Fyodorov-Bouchaud formula and Liouville conformal field theory. *Duke Math. J.*, 169(1):177–211, 2020.
- [38] G. Remy and T. Zhu. Integrability of boundary Liouville conformal field theory. *Communications in Mathematical Physics*, 395:179–268, 2022.
- [39] R. Rhodes and V. Vargas. Gaussian multiplicative chaos and applications: A review. *Probability Surveys*, 11:315 – 392, 2014.
- [40] R. Robert and V. Vargas. Gaussian multiplicative chaos revisited. *Ann. Probab.*, 38(2):605–631, 03 2010.
- [41] G. Segal. *The definition of conformal field theory*. Oxford Univ. Press 2004, 2004.
- [42] B. Wu. Liouville conformal field theory on Riemann surface with boundaries. *Preprint*, arXiv:2203.11721, 2022.
- [43] A. Zamolodchikov and Al. Zamolodchikov. Conformal bootstrap in Liouville field theory. *Nuclear Physics B*, 477(2):577–605, 1996.
- [44] A. B. Zamolodchikov. Infinite additional symmetries in two-dimensional conformal quantum field theory. *Theoretical and Mathematical Physics*, 65(3):1205–1213, December 1985.

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