Sliding mode observation for a 1D wave equation with dynamic boundary conditions

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Yacine Chitour [1], Abdelhakim Dahmani[2], Moussa Labbadi[3] and Christophe Roman[3].

***Abstract*— This study focuses on employing sliding mode observer for a wave equation subject to two dynamic boundary conditions with anti-damping coefficients, a perturbation, and a control at one of the boundaries. The exponential decay rate of the tobserver's trajectory is demonstrated using the multiplier method. The well-posedness of the error-system is proven using maximal monotone operator.**

***Index Terms*— Wave equation; Well-posedness; Observation; Sliding mode observer.**

I. Introduction

The sliding mode control (SMC) was proposed by Utkin [1], is widely recognized for its robust system performance, encompassing insensitivity to parameter fluctuations and rejection of external disturbances, it can be applied for control and observation [2]. Additionally, SMC can ensure finite-time stability [3]. It is acknowledged that finite-time stability is more significant than asymptotic stability in response to certain particular control engineering requirements [2].

In recent years, the boundary control in Partial Differential Equations (PDEs), along with distributed control concerning its spatial distribution, has become widely used and it stands out as a noteworthy approach, presenting an innovative method to reduce implementation costs. The application of the backstepping method is particularly valuable, see [4]. The wave equation serves as an infinite-dimensional harmonic oscillator, functioning as a linear system. Yet, to correctly deal with uncertainties arising from external disturbances infiltrating the system either from the interior or the boundary of the spatial domain, a robust strategy becomes essential [5].

In the domain of distributed control, several successful applications of SMC to the truncated finite-dimensional models have been demonstrated including [6], [7], [8]. For instance, control of a parabolic PDE system [9], [10], control of a heat equation [11], one-dimensional wave equation [12], [13], and Schrödinger equation [14], etc. Relevant related works will be discussed in the following part. In [15], the authors discuss the tracking and control of heat and wave equations using continuous sliding mode approaches. They employ fractional power techniques to modify the signum function. Sliding mode boundaries have been suggested for robust tracking of a diffusion equation in [16]. In [17], exponential stabilization of the wave equation is obtained with matched perturbation. In [18], a boundary control of a heat process was proposed with unbounded matched perturbation via second-order sliding-mode technique. Using sliding mode control [19], the earthquake phenomenon is simplified in this model through a cascade system consisting of a 1D wave equation to represent fault slip and wave propagation, and a 1D diffusion equation to represent the actuator dynamics as a diffusion process.

The main contribution of this paper is a sliding mode observer designed to stabilize the observation-error of an unstable wave equation together with matched disturbance rejection. The distinctiveness of this work is the use of the multiplier method to establish the exponential stability of the entire system. Additionally, we show the finite-time decay rate of a specific linear combination of state variables, based on the fact that the closed-loop system is associated with a maximal monotone operator and is therefore well-posed.

Notation

The set-valued function sign is defined on $\mathbb{R}$,

$$\operatorname{sign}(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0, \\ [-1,1] & \text{if } x = 0. \end{cases} \tag{1}$$

We denote $\operatorname{sign}(\cdot)$ as $\lceil\cdot\rfloor^0$.

II. Problem statement

Let us consider the wave equation, $\forall (t,x) \in \mathbb{R}^+ \times (0,1)$,

$$\begin{cases} u_{tt}(t,x) = u_{xx}(t,x) - q u_t(t,x), & \text{(2a)} \\ m_1 u_{tt}(t,1) = -u_x(t,1) + b_1 u_t(t,1) + Q(t) + w(t), & \text{(2b)} \\ m_0 u_{tt}(t,0) = u_x(t,0) + b_0 u_t(t,0), & \text{(2c)} \\ u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1. & \text{(2d)} \end{cases}$$

The parameter q is a positive real number representing a constant damping coefficient within the domain. Parameters b_0 and b_1 indicate damping if negative and anti-damping if positive, respectively. Parameters m_1 and m_0 denote the inertia at the dynamic boundary, and $Q(t)$ stands for the input, while $w(t)$ is a measurable function acting as a disturbance and we assume that for a.e. $t \geq 0$, $|w(t)| \leq \overline{w}$ where $\overline{w}$ is supposed to be known. Well-posedness has been established in [20] and is associated with a maximal-monotone operator. However, the well-posedness of the closed-loop system or the observer dynamics must also be studied, which is addressed subsequently for the proposed observer.

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The wave equation is associated with the following energy

$$2E(t) = \int_0^1 (u_t(t,x)^2 + u_x(t,x)^2)dx + m_1 u_t(t,1)^2 + m_0 u_t(t,0)^2. \quad (3)$$

The derivative along strong solutions of the energy yields:

$$\dot{E}(t) = -\int_0^1 q u_t(t,x)^2 dx + b_1 u_t(t,1)^2 + b_0 u_t(t,0)^2 + u_t(t,1)(Q(t)+w(t)). \quad (4)$$

The exponential decay of the energy when $b_0, b_1 < 0$ and $Q(t) = w(t) = 0$ can be deduced from [20]. The case where b_0 is positive is interesting for many applications. In the case where $m_1 = 0$, various control laws are proposed to address different output sets and objectives within this context. The equation (2b)-(2c) is commonly referred to as a dynamic (or Wentzell) boundary condition. The consideration of such boundary conditions associated with the wave equation finds several applications. Examples include crane regulation [21], [22], [23], and [24]; controlling hanging cables immersed in water [25]; piezoelectric control [26]; flexible structures [27]. Specifically, the present wave equation is directly associated with drilling torsional vibrations [28], [29].

It is crucial to emphasize that the current model lacks an associated sliding mode control, especially when the wave equation is reformulated using Riemann invariants as heterodirectional hyperbolic partial differential equations. Ongoing research delves into the backstepping transformation for coupling partial differential equations with ordinary differential equations, particularly within the realm of heterodirectional hyperbolic PDEs. Notably, the works by J. Deutscher et. al. [30] and its extension [31] are particularly comprehensive in this regard. Continuing in the same vein of research, the works presented in [32], [33], and [34] are of fairly general nature, although they are specifically applied to the case of a hanging cable immersed in water. The model (2) represents a specific instance within the broader considerations of these works. However, disturbance rejection for the case under considerations is new up to our knowledge.

Our main focus here is to investigate the sliding mode observer for the wave equation subject to disturbances. This approach serves as a preliminary step towards the SMC of the problem at hand. Since we aim to suggest control while establishing well-posedness simultaneously, we address the observation problem first and leave the control problem for future publication due to the intricacies in computation as anti-collocated control requires backstepping.

III. Sliding mode observer

Sophisticated control law may utilizes knowledge of the distributed state, which is not practically available. Therefore, we suggest designing an observer that leverages the knowledge of the boundary velocities. Let us explore the observer from [35] with a different boundary for the actuation, incorporating sliding control in a very specific manner.

$$\begin{cases} \hat{u}_{tt}(t,x) = \hat{u}_{xx}(t,x) - q\hat{u}_t(t,x), & (5a) \\ m_1\hat{u}_{tt}(t,1) + \hat{u}_x(t,1) - Q(t) - b_1\hat{u}_t(t,1) - l_1\tilde{u}_t(t,1) \\ \quad \in l_2 \lceil \tilde{u}_t(t,1) \rfloor^0, & (5b) \\ m_0\hat{u}_{tt}(t,0) = \hat{u}_x(t,0) + b_0\hat{u}_t(t,0) + l_0\tilde{u}_t(t,0). & (5c) \end{cases}$$

the observation-error system is $\tilde{u} = u - \hat{u}$. The observer needs the knowledge of the input and the boundary velocities $u_t(t,1)$ and $u_t(t,0)$.

The dynamics of the observation-error system $\tilde{u}$ are

$$\begin{cases} \tilde{u}_{tt}(t,x) = \tilde{u}_{xx}(t,x) - q\tilde{u}_t(t,x), & (6a) \\ m_1\tilde{u}_{tt}(t,1) + \tilde{u}_x(t,1) + (l_1 - b_1)\tilde{u}_t(t,1) - w(t) \\ \quad \in -l_2 \lceil \tilde{u}_t(t,1) \rfloor^0, & (6b) \\ m_0\tilde{u}_{tt}(t,0) = \tilde{u}_x(t,0) - (l_0 - b_0)\tilde{u}_t(t,0). & (6c) \end{cases}$$

The tuning parameters are l_0, l_1, and l_2. Let us consider the following Lyapunov functional

$$2\tilde{E}(t) = \int_0^1 [\tilde{u}_t^2 + \tilde{u}_x^2]dx + m_0\tilde{u}_t(t,0)^2 + m_1\tilde{u}_t(t,1)^2. \quad (7)$$

Using $\dot{\tilde{E}}(t)$ to denote the derivative along the solution, it holds

$$\dot{\tilde{E}}(t) \leq -\int_0^1 q\tilde{u}_t^2 dx - (l_1 - b_1)\tilde{u}_t(t,1)^2 - (l_0 - b_0)\tilde{u}_t(t,0)^2 - (l_2 - \overline{w})|\tilde{u}_t(t,1)|. \quad (8)$$

We have the following two theorems as the main results of this paper.

Theorem 1. *Consider system (6) with $l_0 > b_0$, $l_1 > b_1$, $l_2 > \overline{w}$. Then system (6) exhibits exponential decay rate in the sense of the energy $\tilde{E}(t)$ defined in (7), i.e., there exist $\tilde{M}$ and $\tilde{\rho}$ positive constants such that for every weak solution of (6), it holds for every $t \geq 0$*

$$\tilde{E}(t) \leq \tilde{M}\tilde{E}(0)e^{-\tilde{\rho}t}. \quad (9)$$

Let us define the sliding surface

$$s(t) := \tilde{u}_t(1,t). \quad (10)$$

Theorem 2. *Consider strong solutions of system (6) with $l_0 > b_0$, $l_1 > b_1$ and $l_2 > \overline{w}$. Then the sliding surface s defined in (10) converges in finite-time, i.e., there exists $T_0 > 0$ depending on the initial conditions such that*

$$s(t) = 0 \text{ for } t \geq T_0. \quad (11)$$

In other words, the sliding mode observer rejects the perturbation of the error dynamics $\tilde{u}$.

Ensuring that the closed-loop system retains a unique existing trajectory poses a challenge when incorporating nonlinear feedback in the PDEs setting. It is crucial to carefully discuss the well-posedness of sliding mode observer design, as errors can easily arise. Obtaining exponential stability of non-existing solutions is unsatisfactory and therefore well-posedness must be addressed diligently. This is the purpose of the following lines.

Let us express the wave partial differential equation as an abstract problem in a multi-valued operator setting.

$$\begin{cases} \dot{X}(t) + AX(t) + BX(t) \ni f(t), & \text{(12a)} \\ X(0) = X_0 \in \mathcal{D}(A) \subset \mathtt{H}. & \text{(12b)} \end{cases}$$

where $\forall z \in \mathcal{D}(A)$,

$$Az := \begin{bmatrix} -z_2 \\ -z_1'' + qz_2 \\ -z_3 \\ \frac{1}{m_1}[(l_1 - b_1)z_4 + z_1'(1)] \\ \frac{1}{m_0}[(l_0 - b_0)z_5 - z_1'(0)] \end{bmatrix}, \tag{13}$$

$$\forall z \in \mathtt{H}, \quad Bz := \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{l_2}{m_1}\lceil z_4 \rfloor^0 \\ 0 \end{bmatrix}, \quad f(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{w(t)}{m_1} \\ 0 \end{bmatrix}. \tag{14}$$

in which

$$\mathcal{D}(A) := \{z \in \mathtt{V}, z_1(1) = z_3, z_2(1) = z_4, z_2(0) = z_5\}, \tag{15}$$

$$\mathtt{V} := H^2 \times H^1 \times \mathbb{R}^3, \mathtt{H} := H^1 \times L^2 \times \mathbb{R}^3, \mathcal{D}(B) := \mathtt{H}. \tag{16}$$

The space $\mathtt{H}$ is equipped with the following semi-norm

$$\|z\|_{\mathtt{H}}^2 := \int_0^1 [z_1'(x)^2 + z_2(x)^2]dx + m_1 z_4^2 + m_0 z_5^2. \tag{17}$$

The kernel of this semi-norm is

$$S := \{z \in \mathtt{H}, \quad z_1 \overset{a.e.}{=} d, \ z_3 = d, \ d \in \mathbb{R}\}. \tag{18}$$

The reason we consider semi-norm is described in [36] and [37]. We can now state our well-posedness result.

Theorem 3. *The abstract evolution problem (12) is well-posed, i.e., the operator $-(A+B)$ is associated with a C_0 semigroup $e^{-(A+B)t}$ on $\mathtt{H}$, and it holds, for all $T > 0$,*

- $\forall X_0 \in \mathcal{D}(A)$, $f \in W^{1,1}(0,T;\mathtt{H})$,

$$X \in C^0([0,T);\mathcal{D}(A)), \ \dot{X} \in L^\infty([0,T);\mathtt{H}) \tag{19}$$

- $\forall X_0 \in \mathtt{H}$, $f \in L^1(0,T;\mathtt{H})$, *then* $X \in C^0([0,T);\mathtt{H})$.

Moreover $e^{-(A+B)t}$ is a contraction on the quotient space $\mathtt{H}/S$, and when $f = 0$, the trajectory

$$\gamma(X_0) := \bigcup_{t \geqslant 0}^{T} e^{-(A+B)t} X_0. \tag{20}$$

is precompact in $\mathtt{H}/S$ for any X_0. Moreover, it holds for the strong solution that X is differentiable from the right with

$$\frac{d^+}{dt} X(t) = (-AX(t) - BX(t) + f(t))^\circ. \tag{21}$$

The term $(AX)^\circ := \inf\{\|z\|; z \in AX\}$ is the least norm which exists due to $A + B$ maximal monotone on $\mathtt{H}/S$, see [38]. The reason we consider z_3 which is linked with $\tilde{u}(t,1)$ is to be able to insure that $\tilde{u}_t(1,t) \in L^\infty(0,T;\mathbb{R})$.

IV. Proof of Theorem 1

The proof of exponential stability is based on the multiplier method and the following result.

Theorem 4. *([39, Theorem 8.1, Page 103])*
Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and assume $\omega > 0$ such that

$$\forall S \geqslant 0, \int_S^\infty E(t)dt \leqslant \frac{1}{\omega} E(S), \tag{22}$$

then E has the following decay properties

$$E(t) \leqslant E(0) e^{1-\omega t}. \tag{23}$$

Sharper outcomes can be derived from more refined findings in [40] and [41], providing the basis for establishing distinct decay rates. We introduce a preliminary lemma before presenting the main proof.

Lemma 5. *There exists a positive constant $C > 0$ such that*

$$\int_S^T \int_0^1 \tilde{u}_x^2 dxdt \leq C\tilde{E}(S). \tag{24}$$

Proof. By a standard density argument, it is enough to establish the result for strong solutions only and we manipulate such solutions below. We multiply (6a) by $(\tilde{u} - \tilde{u}_*)$, where

$$\tilde{u}_*(t) := \tilde{u}(t,1). \tag{25}$$

and we integrate over $[S,T] \times [0,1]$. After standard computations involving several integration by parts, one gets

$$\int_S^T \int_0^1 (\tilde{u} - \tilde{u}_*)(\tilde{u}_{tt} - \tilde{u}_{xx} + q\tilde{u}_t)\, dxdt = 0. \tag{26}$$

For the first term, we have

$$\int_S^T \int_0^1 (\tilde{u} - \tilde{u}_*)\tilde{u}_{tt} dxdt = -\int_S^T \int_0^1 \tilde{u}_t^2 dxdt + \left[\int_0^1 (\tilde{u} - \tilde{u}_*)\tilde{u}_t dx\right]_S^T. \tag{27}$$

For the second term, integration by parts with respect to x yields

$$\begin{aligned} -\int_S^T \int_0^1 (\tilde{u} - \tilde{u}_*)\tilde{u}_{xx} dxdt = & \\ \int_S^T \int_0^1 ((\tilde{u} - \tilde{u}_*))_x \tilde{u}_x dxdt - \int_S^T \left[(\tilde{u} - \tilde{u}_*)\tilde{u}_x\right]_0^1 dt & \\ = \int_S^T \int_0^1 \tilde{u}_x^2 dxdt - \int_S^T \left[(\tilde{u} - \tilde{u}_*)\tilde{u}_x\right]_0^1 dt & \end{aligned} \tag{28}$$

Moreover, for the third term in (26), using integration by parts with respect to t gives

$$\begin{aligned} \int_S^T \int_0^1 (\tilde{u} - \tilde{u}_*)q\tilde{u}_t dxdt &= \frac{1}{2}\int_S^T \int_0^1 \left(q(\tilde{u} - \tilde{u}_*))^2\right)_t dxdt \\ &= \frac{1}{2}\left[\int_0^1 q(\tilde{u} - \tilde{u}_*)^2 dx\right]_S^T. \end{aligned} \tag{29}$$

It holds

$$\int_S^T \int_0^1 \tilde{u}_x^2 dxdt = \tilde{\mathbf{T}}_0 - \tilde{\mathbf{T}}_1 + \tilde{\mathbf{T}}_2 - \frac{1}{2}\tilde{\mathbf{T}}_3. \tag{30}$$

where

$$\tilde{\mathbf{T}}_0 := \int_S^T \int_0^1 \tilde{u}_t^2 dxdt, \ \tilde{\mathbf{T}}_1 := \int_0^1 [(\tilde{u}-\tilde{u}_*)\tilde{u}_t]_S^T dx, \tag{31}$$
$$\tilde{\mathbf{T}}_2 := \int_S^T [(\tilde{u}-\tilde{u}_*)\tilde{u}_x]_0^1 dt, \ \tilde{\mathbf{T}}_3 := \left[\int_0^1 q(\tilde{u}-\tilde{u}_*)^2 dx\right]_S^T.$$

We now estimate $\tilde{\mathbf{T}}_0$ to $\tilde{\mathbf{T}}_3$ defined in (31). It is direct that

$$\tilde{\mathbf{T}}_0 \leqslant \frac{2}{q}E(S). \tag{32}$$

To estimate $\tilde{\mathbf{T}}_1$, using the Young's, Cauchy-Schwartz's and Poincaré's inequalities, it holds

$$2\tilde{\mathbf{T}}_1 \leq \int_0^1 \left[((\tilde{u}-\tilde{u}_*)^2 + \tilde{u}_t^2)\right]_S^T dx \leq 4E(S). \tag{33}$$

For $\tilde{\mathbf{T}}_2$, using from the definition of u_* in (25) one observes that

$$\int_S^T [(\tilde{u}-\tilde{u}_*)\tilde{u}_x]_0^1 dt = \int_S^T (\tilde{u}(t,1)-\tilde{u}(t,0))\tilde{u}_x(t,0)dt.$$

From (6c), it holds

$$\tilde{u}_x(t,0) = m_0\tilde{u}_{tt}(0,t) + (l_0-b_0)\tilde{u}_t(t,0). \tag{34}$$

Using integration by parts and Young's inequalities it exists $c_1, c_2 > 0$ such that for every $\varepsilon > 0$

$$\tilde{\mathbf{T}}_2 \leqslant c_1(1+\varepsilon)E(S) + \frac{c_2}{2\varepsilon}\int_S^T \int_0^1 \tilde{u}_x^2 dxdt. \tag{35}$$

To estimate $\tilde{\mathbf{T}}_3$, we use the Poincaré inequality

$$\int_0^1 q(\tilde{u}-\tilde{u}_*)^2 dx \leqslant q\int_0^1 \tilde{u}_x^2 dx \leq qE(t). \tag{36}$$

Then we conclude, using that the energy is decreasing that

$$\tilde{\mathbf{T}}_3 \leq q\big(E(S)+E(T)\big) \leq 2qE(S). \tag{37}$$

Gathering (33), (35), and (37) together with (30), and taking ε big enough one concludes the proof of Lemma 5. □

Proof of Theorem 1: From (8), one gets that it exists $c_1 > 0$ such that

$$\int_S^T [\int_0^1 \tilde{u}_t(t,x)dx + m_0 u_t(t,1)^2 + m_1 u_t(t,0)^2]dt \leqslant c_1 E(S)$$

Therefore, by using Lemma 5 and applying Theorem 4, it yields (9). ■

Proof of Theorem 2: A direct consequence of the exponential decay of the energy, Theorem 1 yields

$$\lim_{t\to\infty} \int_0^1 |\tilde{u}_t(t,x)|dx + |\tilde{u}_t(t,0)| = 0. \tag{38}$$

Now, one can observe that

$$\tilde{u}_x(t,1) = \int_0^1 (\tilde{u}_{tt}(t,x) + q\tilde{u}_t(t,x))dx + m_0\tilde{u}_{tt}(t,0) + (l_0-b_0)\tilde{u}_t(t,0). \tag{39}$$

First, rewriting (21) of Theorem 3 for the second and final component on the Yosida approximation denoted A_λ, one gets

$$\left|\frac{d^+}{dt}\begin{bmatrix}\tilde{u}_{t,\lambda}(\cdot,t)\\ \tilde{u}_{t,\lambda}(0,t)\end{bmatrix}\right| \leq |A_\lambda X(t)| \leq \frac{M}{\lambda}e^{-\rho t}\|X_0\|. \tag{40}$$

Taking the limit when $t \to \infty$ with $\lambda \to 0$ and according to the Yosida approximation property, i.e., $\lim_{\lambda\to 0}(u_{tt,\lambda}(\cdot,t), u_{tt,\lambda}(0,t)) = (u_{tt}(\cdot,t), u_{tt}(0,t))$, it yields

$$\lim_{t\to\infty}|\tilde{u}_{tt}(t,0)| = 0, \quad \lim_{t\to\infty}|\tilde{u}_{tt}(t,\cdot)| \overset{a.e.}{=} 0. \tag{41}$$

Using (39), (41) and (38), one deduces that

$$\lim_{t\to\infty}\tilde{u}_x(t,1) = 0. \tag{42}$$

in order words, $\forall\varepsilon > 0$, $\exists T_0 > 0$ such that, $\forall t \leq T_0$ it holds $|\tilde{u}_x(t,0)| < \varepsilon$. The finite-time convergence is established in the following. From the definition of $s(t)$ in (10), one gets for a.e. $t \geq 0$

$$s(t)\dot{s}(t) = s(t)\left(\frac{1}{m_1}(-\tilde{u}_x(t,1) + w(t) - (l_1-b_1)\tilde{u}_t(t,1))\right) - \frac{l_2}{m_1}|s(t)|.$$

Since $l_1 > b_1$, one deduces that

$$s(t)\dot{s}(t) \leq -\frac{|s(t)|}{m_1}\big(l_2 - |w(t)| - |\tilde{u}_x(t,1)|\big). \tag{43}$$

Now, using the fact that $l_2 > \overline{w} \geq |w(t)|$ for almost every $t \geq 0$, and $\varepsilon < l_2 - \bar{w}$, we can conclude that for $t \geq T_1$, there exists $\alpha > 0$ such that

$$s(t)\dot{s}(t) \leq -\frac{\alpha}{m_1}|s(t)|, \tag{44}$$

resulting in the convergence of $s(t)$ to zero in finite-time. ■

V. Well-posedness

Let us define the following bilinear product

$$\langle z,\, w\rangle = \int_0^1 [z_1'w_1' + z_2w_2]dx + m_1z_4w_4 + m_0z_5w_5. \tag{45}$$

A. Quotient space

Lemma 6. *The quotient space $\mathtt{H}/S$ is a Hilbert space for which $\langle\cdot,\cdot\rangle$ defined in (45) is an inner product.*

The first step is to quotient the space with the kernel of the semi-norm in order to have normed space and to used monotone operator result.

The proof of this lemma follows the same steps as in [37] and is based on the following facts, as outlined in [42, Theorem 1.41].

Theorem 7. *Let S be a closed subspace of a topological vector space $\mathtt{H}$. If $\mathtt{H}$ is a Banach space so is the quotient space $\mathtt{H}/S$.*

Moreover, suppose $\|\|_{\mathtt{H}}$ is a seminorm (subadditive and absolute homogeneous) on a vector space $\mathtt{H}$. Then $\{z : \|z\|_{\mathtt{H}} = 0\}$ is a subspace of $\mathtt{H}$.

Let Λ be a linear functional on a topological vector space. Assume $\Lambda(z) \neq 0$ for some $z \in \mathcal{H}$. Then, the null space $\mathrm{Ker}(\Lambda(\cdot))$ is closed.

Proof of Lemma 6: First one directly gets that $\|\cdot\|_{\mathtt{H}}$ is a seminorm on $\mathtt{H}$, i.e.,

$$\|z+v\|_{\mathtt{H}} \leqslant \|z\|_{\mathtt{H}} + \|v\|_{\mathtt{H}}, \quad \|\alpha z\|_{\mathtt{H}} = |\alpha|\|z\|_{\mathtt{H}}. \tag{46}$$

Second one gets that $S = \mathrm{Ker}(\|z\|_{\mathtt{H}}) = \mathrm{Ker}(\langle\cdot,\cdot\rangle)$. Moreover, by employing Theorem 7, one can conclude that S is a subspace of $\mathtt{H}$.

Third, let's consider the following families of functionals:

$$\Lambda_1 s = \int_0^s z_1'(x), dx, \quad \Lambda_2 s = \int_0^s z_2(x), dx. \tag{47}$$

Both are indexed by $s \in (0,1]$. Additionally, consider the three following functionals:

$$\Lambda_3(z) = z_4, \quad \Lambda_4(z) = z_5. \tag{48}$$

It is direct that all Λ_i are linear. Note that by construction the intersection of the null spaces is

$$\bigcap_{\substack{i=1,2\\ s\in(0,1]}} \mathrm{Ker}(\Lambda_i[s](\cdot)) \bigcap_{i=3,4} \mathrm{Ker}(\Lambda_i(\cdot)) = S. \tag{49}$$

Therefore, by using Theorem 7 and the property that any intersection (finite or infinite) of closed sets is closed, one concludes that S is closed.

Fourth, using that $\mathcal{S}$ is a closed subspace of $\mathtt{H}$ and Theorem 7 it yields that $\mathtt{H}/S$ is a Banach space.

Finally, proving that (45) is a scalar product on $\mathtt{H}/S$ (this is direct) one concludes the proof. ■

B. Monotonicity of operator A and B on $\mathtt{H}/S$.

Definition 1 ([43])**.** *Let A be a nonlinear operator in H. Considering $\omega \in \mathbb{R}$, $A+\omega I$ is monotone if the following equivalent condition holds*

$$\langle z-v,\, Az-Av\rangle \geqslant -\omega\|z-v\|_{\mathtt{H}}. \tag{50}$$

Lemma 8. *The operator A and the operator B are monotone in $\mathtt{H}/S$ equipped with norm (17).*

Proof. Let us denote $\bar{z} = z - v$. This will allow us to use a compact notation for all linear parts of the operator.

$$\begin{aligned}\langle z-v,\, Az-Av\rangle = \int_0^1 &[\bar{z}_1'\bar{z}_2 + \bar{z}_2(-\bar{z}_1'' + q\bar{z}_2)]dx\\ &+ \bar{z}_4((l_1-b_1)\bar{z}_4 + \bar{z}_1'(1))\\ &+ \bar{z}_5((l_0-b_0)\bar{z}_5 - \bar{z}_1'(0)).\end{aligned} \tag{51}$$

One gets, using the fact that $z, w \in \mathcal{D}(A)$

$$\begin{aligned}\langle z-v,\, Az-Av\rangle = &\int_0^1 q\bar{z}_2^2 dx + (l_1-b_1)\bar{z}_2(1)^2\\ &+ (l_0-b_0)\bar{z}_2(0)^2 \geqslant 0.\end{aligned} \tag{52}$$

For the operator B one gets

$$\langle z-v,\, Bz-Bv\rangle = l_2(z_4-v_4)(\lceil z_4\rfloor^0 - \lceil v_4\rfloor^0) \geqslant 0. \tag{53}$$

□

C. Maximality of A and B on $\mathtt{H}/S$.

We have the following result.

Lemma 9. *The operators A and B defined in (13)-(14) are maximal monotone.*

The maximalilty of operator A is deduced from [20]. The proof for B is inspired from [44] on Page 251.

Proof. A multi-valued monotone operator B is maximal monotone on $\mathtt{H}$, if it exists $\lambda > 0$ such that

$$\mathcal{R}(B+\lambda I) = \mathtt{H}. \tag{54}$$

For $\lambda = 1$, and for $(z,y) \in \mathcal{D}(B)\times \mathtt{H}/S$, consider $Bz + z \ni y$. Then

$$z_1 \ni y_1, \quad z_2 \ni y_2, \quad z_3 \ni y_3, \quad \lambda z_5 \ni m_0 y_5, \tag{55}$$

$$l_2\lceil z_4\rfloor^0 + z_4 \ni m_1 y_4. \tag{56}$$

Then it exists a unique solution of (56).

$$z_4 = \begin{cases} \dfrac{(m_1y_4))(|m_1y_4| - l_2)}{|m_1y_4|}, & \text{if } |m_1y_4| > l_2,\\ 0, & \text{if } |m_1y_4| \leqslant l_2. \end{cases} \tag{57}$$

We conclude that the operator B is maximal monotone on $\mathtt{H}/S$. □

D. Well-posedness of the abstract problem.

There are several conditions under which the sum of two maximal monotone operators forms a maximal monotone operator.

Theorem 10 (Theorem 1.5 in [44])**.** *Let H be a reflexive Banach space, and let A and B be maximal monotone subset of $H \times H^*$. If it holds*

$$(int\,\mathcal{D}(A)) \cap \mathcal{D}(B) \neq \varnothing, \tag{58}$$

then $A+B$ is maximal monotone.

In the present case as $\mathcal{D}(B) = \mathtt{H}$ so $(\mathrm{int}\,\mathcal{D}(B)) = \mathtt{H}$ and the operator $A+B$ est maximal monotone.

Now for the regularity of the solution, we consider the following result.

Theorem 11 (Theorem 21 and following Remark in [38])**.** *Let us consider $\mathtt{H}$ being a Hilbert space, A a maximal monotone operator, and $f \in W^{1,1}(0,T;\mathtt{H})$, then for all $X_0 \in \mathcal{D}(A)$, there exists a unique $X(t) : \mathbb{R}^+ \to \mathtt{H}$ such that $\forall t \geq 0$*

- $X(t) \in \mathcal{D}(A)$.

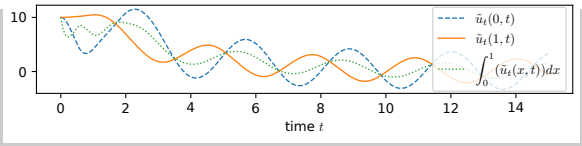

Fig. 1: Boundary and integrated velocities when $l_2 = 0$

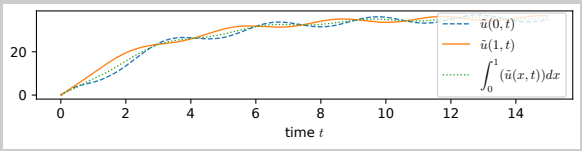

Fig. 2: Boundary and integrated positions when $l_2 = 0$

- $\frac{d^+}{dt}X(t) \in L^\infty(0,T;\mathtt{H})$.
- $\frac{d}{dt}X(t) + AX(t) \ni f$ *and* $X(0) = X_0$.
- X *is differentiable from the right* $\forall t \in [0,T)$ *and* $\frac{d^+}{dt}X(t)+(AX(t)-f(t))^\circ = 0$. *Moreover,* $|\frac{d^+}{dt}X(t)| \leq |\frac{d^+}{dt}X(0)| + \int_0^t |\frac{df}{ds}(s)|ds$.
- *the mapping* $(X_0, f) \mapsto X$ *can be extended by continuity from* $\overline{\mathcal{D}(A)} \times L^1(0,T;\mathtt{H})$ *into* $C(0,T;\overline{\mathcal{D}(A)})$. *Moreover, in the case where* $f = 0$, *the mapping* $X_0 \mapsto X(t)$ *defines a nonlinear contraction* C_0*-semigroup.*

Proof of Theorem 3: Using Lemma 6, Lemma 8, Lemma 9 and Theorem 10 the operator $A + B$ defined in (13)-(14) is maximal monotone on $\mathtt{H}/S$. We can then use Theorem 11 on the abstract problem (12) which is therefore associated with a maximal monotone operator on $\mathtt{H}/S$. To obtain the result concerning the strong solution in $\mathtt{H}$ we can either do as in [20] or using the fact that $X_1(t) = \int_0^t X_2(s)ds + X_1(0)$. The weak solutions are determined by density using the fact that A is linear maximal monotone and therefore $\overline{\mathcal{D}(A)} = \mathtt{H}$. The precompactness of the trajectory is deduced from Theorem 3 in [45], using the fact that the operator $A+I$ is maximal monotone on $\mathtt{H}/S$, or equivalently m-accretive, and that $0 \in \mathcal{R}(A)$. Applying Theorem 11, one obtains (21). This concludes the proof of Theorem 3. ■

VI. Numerical simulations

We conduct numerical simulations on the error-system (6) with the following parameters: $q = 0.01$, $l_1 - b_1 = 0.02$, $l_0 - b_0 = 0.02$, $m_1 = 10$, $m_0 = 1$. Consider the disturbance $w(t) = 40\sin(\sin(2t))$. The discrete scheme employed is a standard finite difference scheme with 40 space steps, and a time step of 0.001. No specific precautions are taken for the sign function. We aim to investigate the observation-error system (6) for $l_2 = 0$ and $l_2 = 45$. The results of both cases are plotted respectively in figures 1, 2, 3 and in figures 4, 5, 6, 7. These results illustrate the sliding manifold convergence in finite-time when $l_2 > \bar{w}$. Moreover, all objectives are achieved, unlike when $l_2 = 0$, as shown in figures 1, 2, 3.

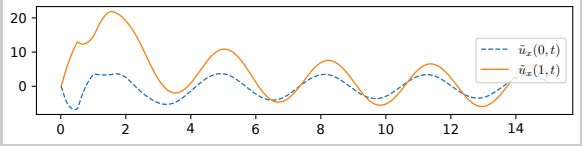

Fig. 3: Boundary torque/force $l_2 = 0$

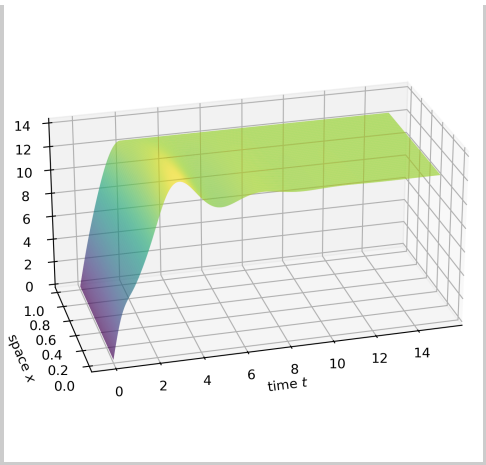

Fig. 4: Distributed position $\tilde{u}(t,\cdot)$ when $l_2 = 45$

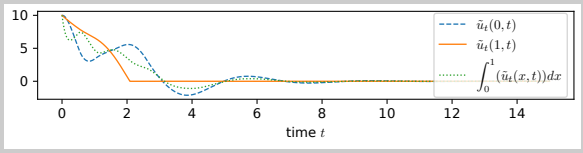

Fig. 5: Boundary and integrated velocities when $l_2 = 45$

VII. Conclusions

This paper introduces a sliding mode observer strategy tailored for a wave equation with dynamic boundary conditions together with a perturbation for one of them. The well-posedness of the observer-error system is given for the strong and the weak solutions and disturbance rejection is established. Subsequent research will concentrate on devising a sliding mode control using backstepping, and an observer relying solely on collocated velocity measurements. We also aim at performing robustness assessments.

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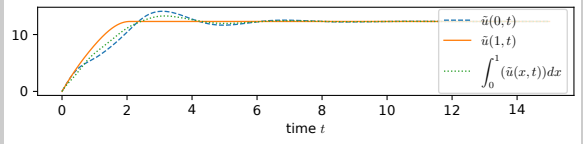

Fig. 6: Boundary and integrated positions when $l_2 = 45$

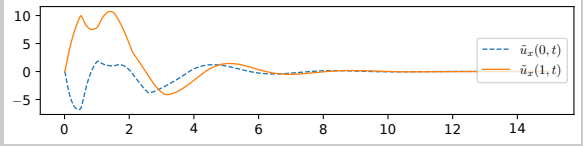

Fig. 7: Boundary torque/force $l_2 = 20$

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