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**Qualitative equivalence between incompatibility and Bell nonlocality**Shiv Akshar Yadavalli <sup>\*</sup>*Department of Physics, Duke University, Durham, North Carolina 27708, USA*Nikola Andrejic <sup>†</sup>*University of Niš, Faculty of Sciences and Mathematics, Višegradska 33, 18000 Niš, Serbia*Ravi Kunjwal <sup>‡</sup>*Université libre de Bruxelles, QuIC, 1050 Brussels, Belgium and Aix-Marseille University, CNRS, LIS, 13288 Marseille CEDEX 09, France*

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Measurements in quantum theory can fail to be jointly measurable. Like entanglement, this incompatibility of measurements is necessary but not sufficient for violating Bell inequalities. The (in)compatibility relations among a set of measurements can be represented by a joint measurability structure, i.e., a hypergraph whose vertices denote measurements and hyperedges denote all and only compatible sets of measurements. Since incompatibility is necessary for a Bell violation, the joint measurability structure on each wing of a Bell experiment must necessarily be nontrivial, i.e., it must admit a subset of incompatible vertices. Here we show that, for any nontrivial joint measurability structure with a finite set of vertices, there exists a quantum realization with a set of measurements that enables a Bell violation, i.e., given that Alice has access to this incompatible set of measurements, there exists a set of measurements for Bob and an entangled state shared between them such that they can jointly violate a Bell inequality. Hence, a nontrivial joint measurability structure is not only necessary for a Bell violation, but also sufficient.

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Unlike pre-quantum physical theories, measurements are at the heart of quantum theory. It is through measurements that the Hilbert-space picture of state vectors and unitary dynamics makes contact with experimental statistics (via the Born rule). These measurements are unlike any in classical physics because of the generic impossibility of implementing them simultaneously, i.e., they exhibit incompatibility. When combined with shared entanglement, the incompatibility of quantum measurements powers the violation of Bell inequalities. Such Bell violations are a strong form of nonclassicality, ruling out the possibility of local hidden variable (LHV) models of physics [1–7].

The experimental setup needed to violate a Bell inequality is called a Bell scenario [8]. It consists of at least two spacelike separated parties that share an entangled state and implement local measurements on their share of the state. While entanglement and incompatibility are necessary for Bell inequality violations, they are not sufficient. We know that there exist entangled states that do not violate Bell inequalities [9,10]. More recently, it has also been shown that there exist incompatible measurements that are useless for a Bell violation [11,12]. While the relationship between entanglement and Bell nonlocality has been an area of active research for over three decades [9,10,13–16], in recent years research on the

relationship between measurement incompatibility and Bell nonlocality has also picked up, often via the intermediate notion of quantum steering [11,12,17–23].

A quantum measurement is most generally represented by a positive operator-valued measure (POVM) [24]. A set of measurements is said to be jointly measurable (or compatible) if there exists a single measurement that can be coarse-grained in different ways to realize each of the measurements in the set [24]. Otherwise, it is said to be incompatible (i.e., not jointly measurable). In general, a set of measurements will exhibit various incompatibility relations among the measurements, e.g., a set of two measurements may either be compatible or incompatible while a set of three measurements can have a more complicated set of incompatibility relations such as pairwise compatibility but triplewise incompatibility [25].<sup>1</sup> Any particular pattern of such incompatibility relations can be represented by a hypergraph that we term a *joint measurability structure*:<sup>2</sup> the vertices of this hypergraph represent

<sup>1</sup>While commutativity is equivalent to joint measurability in the case of projective measurements, it is only sufficient (and not necessary) for joint measurability of general measurements [24]. A joint measurability structure that exhibits pairwise compatibility but triplewise incompatibility—often called Specker’s scenario [25–29] or hollow triangle [12,18,19,30]—is therefore *only* possible with POVMs, not projective measurements.

<sup>2</sup>This is also referred to as an *abstract simplicial complex* [27].

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measurements and its hyperedges represent all (and only) the jointly measurable subsets of vertices.

To ask whether incompatibility is sufficient for Bell nonlocality is to ask the following question: given any set of incompatible POVMs that Alice can implement on her quantum system, does there always exist a bipartite entangled state that she can share with Bob and a set of POVMs that Bob can implement on his part of the state such that their joint statistics violates some Bell inequality? This question was recently settled in the negative via explicit counterexamples [11,12]. One of the counterexamples consists of three measurements that are pairwise compatible but triplewise incompatible [11] (forming Specker's scenario [25,26,31–33]) and the other involves an uncountably infinite number of measurements [12]. Thus, measurement incompatibility does not imply Bell nonlocality, similar to how there exist entangled states that do not violate Bell inequalities [9,10].

It is interesting to note that, although Specker's scenario admits a set of POVMs that do not violate any Bell inequality, it also admits sets of POVMs that do violate a Bell inequality [11,18]. This leads us to the following line of inquiry: The necessity of incompatibility for Bell nonlocality implies the necessity of a *nontrivial joint measurability structure* on each wing of a Bell experiment, i.e., the joint measurability structure of the measurements on each wing should contain a subset of incompatible vertices. Is a nontrivial joint measurability structure, however, sufficient for a Bell inequality violation?

That is, despite the inequivalence of incompatibility and Bell nonlocality in a *quantitative* sense [11,12], we ask whether a *qualitative* equivalence between incompatibility and Bell nonlocality nevertheless holds: given any nontrivial joint measurability structure (with, say,  $v$  vertices), can Alice implement a set of  $v$  POVMs satisfying it on her part of some entangled state shared with Bob such that for some set of POVMs implemented by Bob on his part of this state their joint statistics violates a Bell inequality? Note that we want the number of settings for Alice in the Bell scenario that contribute to the Bell inequality violation—in the sense that the Bell inequality violation is a nonconstant function of the outcome probabilities associated with each such setting—to be equal to the number of vertices in the joint measurability structure. This is an important point because any joint measurability structure exhibiting some incompatibility necessarily admits a subset of incompatible POVMs and, as such, can be used to demonstrate the violation of a Bell inequality if this subset is a set of  $N$ -Specker POVMs (where  $N \geq 2$ , cf. Definition 4 and Refs. [3,11,17]) and if Alice chooses to implement only this particular subset of measurements and ignores the rest of the measurements in the joint measurability structure. Such a violation would not be a property of the intended joint measurability structure but only a substructure thereof.

In particular, arguments from input-lifting of Bell inequalities [34] do not suffice to answer the question we raise here. Briefly, input-lifting refers to the fact that a facet Bell inequality in a Bell scenario remains a facet Bell inequality even if extra inputs are added to define an extended Bell scenario. Hence, a violation of the Bell inequality in the original scenario—viewed as a restriction of the extended scenario where the extra inputs do not contribute to the Bell

expression—is a witness of Bell nonlocality for the extended scenario. A simple way to see why input-lifting does not work for our purposes is the following: we require that (1) any extra settings that are added on Alice's wing in the original Bell scenario must respect incompatibility relations dictated by Alice's particular joint measurability structure, and (2) that these settings contribute to the Bell expression involved in the Bell inequality violation. One cannot, in the absence of these two requirements, appeal merely to input-lifting to claim that a Bell inequality violation achieved by an  $N$ -Specker scenario embedded inside a joint measurability structure with  $v$  vertices ( $v > N$ ) is sufficient to conclude that the same violation is also a genuine property of the full joint measurability structure (rather than merely a substructure thereof). This would, for example, permit a situation where the  $v - N$  extra settings might as well be trivial, e.g., they always yield a fixed measurement outcome with certainty (thus being classical and compatible with all measurements) and do not, therefore, respect the  $v$ -vertex joint measurability structure whose ability to violate Bell inequalities we want to assess in the first place. To sum up, input-lifting does not on its own ensure that the full joint measurability structure will be respected on Alice's wing and, by definition, it ensures that the statistics of any extra inputs are irrelevant to the Bell inequality (violation) being lifted to the extended scenario.

With this caveat about input-lifting out of the way, we now proceed to answer our question in the affirmative. Specifically, we provide an explicit recipe for constructing a Bell-violating quantum realization of any joint measurability structure.

*Preliminaries.* We now formally define the basic notions we use in the rest of this paper.

Our definitions are equivalent to those in Refs. [27,29].

*Definition 1 (Positive operator-valued measure).* A general quantum measurement  $M$  on a Hilbert space  $\mathcal{H}$  is described by a set of positive semidefinite operators  $\{M_a | M_a : \mathcal{H} \rightarrow \mathcal{H}, M_a \geq 0\}_{a \in \mathcal{O}}$ , such that  $\sum_{a \in \mathcal{O}} M_a = \mathbb{1}$ , where  $\mathbb{1}$  denotes the identity operator on  $\mathcal{H}$  and  $\mathcal{O}$  denotes the set of measurement outcomes.

*Definition 2 (Joint measurability).* A set of POVMs  $\mathcal{M} = \{M_x\}_{x=1}^N$ , each with outcome set  $\mathcal{O}_x$ , is said to be jointly measurable or compatible if it admits a (joint) POVM  $G$ , with outcome set  $\mathcal{O} := \mathcal{O}_1 \times \mathcal{O}_2 \times \cdots \times \mathcal{O}_N$ , such that each POVM  $M_x := \{M_{a_x|x}\}_{a_x \in \mathcal{O}_x} \in \mathcal{M}$  can be obtained as coarse-graining of  $G$  over the outcomes of all other POVMs in  $\mathcal{M} \setminus \{M_x\}$ , i.e.,  $M_{a_x|x} = \sum_{\vec{a} \in \mathcal{O}}^{a_x = a} G(\vec{a})$ , for all  $a_x = a \in \mathcal{O}_x$ ,  $x \in \{1, 2, \dots, N\}$ . Here  $\vec{a} = (a_x)_{x=1}^N \in \mathcal{O}$ .

A set of POVMs that is not compatible is said to be *incompatible*. Given a set of POVMs, its different subsets may or may not be compatible and such (in)compatibility relations can be expressed via their joint measurability structure.

*Definition 3 (Joint measurability structure).* A joint measurability structure on a set of POVMs  $\mathcal{M}$  is a hypergraph  $(V_{\mathcal{M}}, E_{\mathcal{M}})$  with the set of vertices  $V_{\mathcal{M}}$ , each vertex representing a POVM in  $\mathcal{M}$ , and a set of hyperedges  $E_{\mathcal{M}} = \{e | e \subseteq V_{\mathcal{M}}\}$  denoting all and only compatible (or jointly measurable) subsets of  $\mathcal{M}$ . Since every subset of a compatible set of POVMs is also compatible, in a valid joint measurability structure we must have  $e' \subset e \in E_{\mathcal{M}} \Rightarrow e' \in E_{\mathcal{M}}$ .

A trivial joint measurability structure is one where all the vertices represent a compatible set of POVMs, i.e., it has

exactly one hyperedge containing all the vertices. Any joint measurability structure that is not trivial is said to be *non-trivial*. A joint measurability structure is said to be *quantum realizable* if and only if there exist quantum measurements that can be assigned to its vertices such that these measurements satisfy all the (in)compatibility relations specified by the joint measurability structure. In Ref. [27] it was shown that all joint measurability structures admit quantum realizations via an explicit construction. Crucial to this construction is a particular class of joint measurability structures called  $N$ -Specker scenarios.

**Definition 4** ( *$N$ -Specker scenario*). An  $N$ -Specker scenario is a joint measurability structure on a set of  $N \geq 2$  incompatible measurements where every  $(N - 1)$ -element subset of the set is compatible.

Note that a 2-Specker scenario corresponds to a pair of incompatible measurements, a 3-Specker scenario corresponds to the situation Specker originally considered [25,26,31–33], and more general  $N$ -Specker scenarios have also been studied in the literature [29,35].

**Definition 5** (*Bell-violating quantum realization of a joint measurability structure*). A quantum realization of a joint measurability structure with  $v$  vertices is said to be Bell-violating if and only if, given that a party (say, Alice) locally implements all the measurements from this realization, there exists an entangled state on a quantum system she can share with another party (say, Bob) and there exist local measurements on the other party's share of the state such that the entangled state subjected to their local measurements violates a Bell inequality in a Bell scenario with  $v$  nonredundant settings for Alice, i.e., all the  $v$  measurements from Alice's quantum realization contribute to the Bell violation.

**Resolution of any nontrivial joint measurability structure into  $N$ -Specker scenarios.** We recall now the informal argument for this resolution first presented in Ref. [27] before going on to provide a rigorous proof based on the properties of partially ordered sets in Sec. I.A of the Supplemental Material [36].<sup>3</sup>

Consider any joint measurability structure  $\mathcal{J}$  with  $v$  vertices, where  $V(\mathcal{J})$  denotes the set of vertices and  $E(\mathcal{J})$  denotes the set of hyperedges, so that the cardinality of the set  $V(\mathcal{J})$  is given by  $|V(\mathcal{J})| = v$ . We resolve  $\mathcal{J}$  into  $N$ -Specker scenarios embedded in the hypergraph, where  $N \in \{2, 3, \dots, N_{\max}\}$  and  $N_{\max} \leq v$ , following the method proposed in Ref. [27].<sup>4</sup> This resolution of  $\mathcal{J}$  into  $N$ -Specker scenarios rests on the following observation: an  $N$ -Specker scenario ( $N \geq 2$ ) is a minimal incompatible set, i.e., any

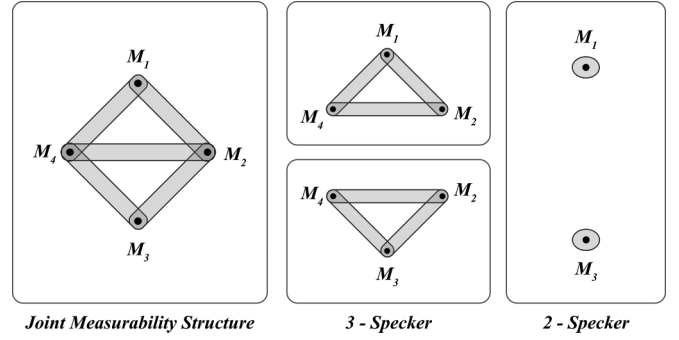


FIG. 1. A joint measurability structure with  $v = 4$  and its decomposition into 3-Specker and 2-Specker scenarios.

proper subset of it is compatible, and thus serves as an irreducible unit of incompatibility in the joint measurability structure. As such, the incompatibility relations among the  $v$  vertices of  $\mathcal{J}$  can be fully captured by identifying these irreducible units of incompatibility in  $\mathcal{J}$ . We use the joint measurability structure in Fig. 1 as our working example to illustrate how this works: the incompatible subsets in this case are given by

$$\begin{aligned} & \{\{M_1, M_2, M_3, M_4\}, \{M_1, M_2, M_4\}, \{M_2, M_3, M_4\}, \\ & \{M_1, M_3, M_4\}, \{M_1, M_3, M_2\}, \{M_1, M_3\}\}. \end{aligned} \quad (1)$$

Of these incompatible subsets, the minimal ones are

$$\{\{M_1, M_2, M_4\}, \{M_2, M_3, M_4\}, \{M_1, M_3\}\}. \quad (2)$$

The nonminimal incompatible subsets in  $\mathcal{J}$  can be generated from the minimal ones by progressively adding a new vertex to each minimal subset until the full set of vertices in  $\mathcal{J}$  is covered: e.g., the minimal set  $\{M_1, M_3\}$  generates the nonminimal sets  $\{\{M_1, M_3, M_4\}, \{M_1, M_3, M_2\}, \{M_1, M_2, M_3, M_4\}\}$ ; the other two minimal sets generate the nonminimal set  $\{M_1, M_2, M_3, M_4\}$ . Taking the union of these incompatible sets generated from the minimal ones yields the full family of incompatible subsets [Eq. (1)] in the joint measurability structure of Fig. 1. The same argument for resolution into  $N$ -Specker scenarios generalizes to any  $\mathcal{J}$ . A formal proof of this resolution into  $N$ -Specker scenarios for any  $\mathcal{J}$  can be found in Sec. I.A of the Supplemental Material [36].

We label the set of all  $N$ -Specker scenarios in the resolution of a joint measurability structure  $\mathcal{J}$  by  $\{\text{Sp}_s(\mathcal{J})\}_s$ ,  $s$  denoting a particular scenario in the resolution.

**Construction of Bell-violating quantum realization of any nontrivial joint measurability structure.** We prove our main result in two steps. First, we argue that every  $N$ -Specker scenario ( $N \geq 2$ ) admits a violation of the  $I_{NN22}$  Bell inequality [37], i.e., in a bipartite Bell scenario where each party has access to  $N$  dichotomic POVMs. Second, we build a quantum realization for any joint measurability structure with  $v$  vertices violating the  $I_{v22}$  Bell inequality.

The fact that every  $N$ -Specker scenario ( $N \geq 2$ ) admits a Bell-violating quantum realization follows from a combination of the results of Refs. [3,11]. In particular, Ref. [11] constructs a Bell-violating quantum realization for any  $N$ -Specker scenario where  $N \geq 3$ , violating the  $I_{NN22}$  Bell

<sup>3</sup>While Ref. [27] left the argument for this resolution implicit in its constructive proof of quantum realizations of arbitrary joint measurability structures, in our proof we make it explicit that the underlying argument for such hypergraphs or abstract simplicial complexes [27] follows from the properties of partially ordered sets, i.e., independently of the question of quantum realizations of a joint measurability structure in the sense of Ref. [27].

<sup>4</sup>Note that  $v = N_{\max}$  is the special case where  $\mathcal{J}$  is a  $v$ -Specker scenario, i.e., every proper subset of  $V(\mathcal{J})$  is compatible. In this case, the incompatibility of  $\mathcal{J}$  cannot be “localized” or “reduced” to that of some strict subsets of  $V(\mathcal{J})$ .

inequality. On the other hand, we know that any 2-Specker scenario can be realized by a pair of incompatible POVMs and, as such, admits a Bell-violating quantum realization, e.g., via the Clauser–Horne–Shimony–Holt (CHSH) game [3,8] with  $X$  and  $Z$  measurements for Alice. In fact, a stronger statement can be made in the 2-Specker case (which does not hold for  $N \geq 3$ ): specifically, Ref. [17] shows that every quantum realization of a 2-Specker scenario (i.e., every pair of incompatible POVMs) violates the CHSH inequality [3,8,38,39], which corresponds to the  $I_{2222}$  Bell inequality (see Sec. III of the Supplemental Material [36]). Hence, we have that every  $N$ -Specker scenario ( $N \geq 2$ ) admits a quantum realization violating the  $I_{NN22}$  Bell inequality.

We can now use these Bell-violating quantum realizations of  $N$ -Specker scenarios to construct a Bell-violating quantum realization of any nontrivial  $v$ -vertex joint measurability structure  $\mathcal{J}$  that admits the resolution  $\{\text{Sp}_s(\mathcal{J})\}_s$  into  $N$ -Specker scenarios for  $2 \leq N \leq v$ . We label the measurement settings corresponding to the Bell-violating realization of  $\mathcal{J}$  by  $x, y \in \{0, 1, 2, \dots, v-1\}$ , respectively, for Alice and Bob. However, when considering a Bell-violating quantum realization of a particular  $N$ -Specker scenario  $\text{Sp}_s(\mathcal{J})$ , for notational convenience we temporarily relabel these  $N$  measurement settings of Alice and Bob given by  $x, y \in \{0, 1, \dots, v-1\}$  (respectively) to  $x', y' \in \{1, 2, \dots, N-1\}$ .

Consider a Bell-violating quantum realization of some  $\text{Sp}_s(\mathcal{J})$ . We denote the entangled state used in the Bell-violating quantum realization of any  $\text{Sp}_s(\mathcal{J})$  as  $\rho_s \in \mathcal{B}(\mathcal{H}_s \otimes \mathcal{H}_s)$ , the POVMs of Alice as  $\{M_{0|x'}^{(s)}, M_{1|x'}^{(s)}\}$ , and the POVMs of Bob as  $\{M_{0|y'}^{(s)}, M_{1|y'}^{(s)}\}$ . For each  $N$ -Specker scenario  $\text{Sp}_s(\mathcal{J})$ , the statistics  $p(a, b|x', y') = \text{Tr}(\rho_s M_{a|x'}^{(s)} \otimes M_{b|y'}^{(s)})$  (where  $a, b \in \{0, 1\}$ ) violates the  $I_{NN22}^{(s)}$  Bell inequality, where  $N = |V(\text{Sp}_s(\mathcal{J}))|$ , the number of vertices in the scenario. This Bell inequality [11,37] is given by

$$I_{NN22}^{(s)} := -p_B^{(s)}(0|1) - \sum_{x'=2}^N p_A^{(s)}(0|x') + \sum_{x'=1}^N p^{(s)}(00|x', y' = 1) + \sum_{x'=2}^N p^{(s)}(00|x', x') - \sum_{1 \leq x' < y' \leq N} p^{(s)}(00|x', y') \leq 0. \quad (3)$$

Note that this realization does not assign POVMs to all the  $v$  vertices in  $V(\mathcal{J})$ . To the vertices that lie outside the  $N$ -Specker scenario  $\text{Sp}_s(\mathcal{J})$  under consideration—that is,  $x \in V(\mathcal{J}) \setminus V(\text{Sp}_s(\mathcal{J}))$ —we assign the trivial POVMs  $M_x^{(s)} = \{0, \mathbb{1}_s\}$ , associating the outcome labeled “0” with the impossible outcome and “1” with the certain outcome. We do the same for Bob’s measurements, adding measurements of the type  $M_y^{(s)} = \{0, \mathbb{1}_s\}$  such that Bob now has  $v$  POVMs,  $v - N$  of them trivial. Now we have  $v$  settings on each side of the Bell scenario but only  $N$  of these on each side are nontrivial POVMs, the remaining  $v - N$  settings being  $\{0, \mathbb{1}_s\}$ . We label the measurement settings by  $x, y \in \{0, 1, 2, \dots, v-1\}$ . The Hilbert space on which each  $N$ -Specker scenario  $\text{Sp}_s(\mathcal{J})$  is realized is  $\mathcal{H}_s \cong \mathbb{C}^N$  (see Sec. II of the Supplemental Material [36]) [11,40]. Since we know that the  $N$ -Specker scenario  $\text{Sp}_s(\mathcal{J})$  obtains the Bell inequality violation  $I_{NN22}^{(s)} > 0$ , we

also have that  $I_{vv22}^{(s)} > 0$ . This is because the additional terms in the expression for  $I_{vv22}^{(s)}$ , beyond those coming from  $I_{NN22}^{(s)}$ , are all zero, as  $p_A^{(s)}(0|x) = p_B^{(s)}(0|y) = 0$  for all  $x, y$  corresponding to trivial settings. Of course,  $I_{vv22}^{(s)} > 0$  is really just  $I_{NN22}^{(s)} > 0$  (since  $v - N$  settings are trivial on each side), so it is not (yet) a Bell-violating realization of  $\mathcal{J}$ .

We can now combine the Bell violations from  $N$ -Specker scenarios (i.e.,  $I_{vv22}^{(s)} > 0$ ) into a Bell-violating quantum realization of the given joint measurability structure  $\mathcal{J}$ , so that we obtain  $I_{vv22} > 0$  in a bipartite Bell scenario with  $v$  nontrivial two-outcome measurement settings per party. Iterating our argument above for all  $N$ -Specker scenarios  $\{\text{Sp}_s(\mathcal{J})\}_s$  contained in  $\mathcal{J}$ , we get a set of states  $\{\rho_s\}_s$  with measurements  $\{M_x^{(s)}\}_{x=0}^{v-1}$  and  $\{M_y^{(s)}\}_{y=0}^{v-1}$  associated with Alice and Bob, respectively. For each  $s$ , these states and measurements act on the Hilbert space  $\mathcal{H}_s \otimes \mathcal{H}_s$  ( $s$  runs through all the  $N$ -Specker scenarios in the decomposition of  $\mathcal{J}$ , cf. Fig. 1). We now combine these constructions via a direct sum so that the Hilbert space on each side becomes  $\mathcal{H} = \bigoplus_s \mathcal{H}_s$  and we have states and measurements defined on the tensor product space  $\mathcal{H} \otimes \mathcal{H} \equiv (\bigoplus_s \mathcal{H}_s) \otimes (\bigoplus_s \mathcal{H}_s)$  in such a way that they violate the  $I_{vv22}$  Bell inequality, i.e.,  $I_{vv22} > 0$ . The joint measurability structure on Alice’s side obtained via this construction is exactly the one given by  $\mathcal{J}$ : for every compatible subset  $e \in E(\mathcal{J})$ , its vertices are assigned POVMs that are compatible on every  $\mathcal{H}_s$ , and for every incompatible subset  $e' \subseteq V(\mathcal{J})$ , its vertices are assigned POVMs that are incompatible on some  $\mathcal{H}_s$ .

The Bell-violating state and measurements on  $\mathcal{H} \otimes \mathcal{H}$  are given as follows: we define  $\rho := [\bigoplus_{s_*} r_{s_*} \rho_{s_*}] \oplus [\bigoplus_{s \neq s'} 0_{s, s'}] \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) \cong \bigoplus_{s_*} \mathcal{B}(\mathcal{H}_{s_*} \otimes \mathcal{H}_{s_*}) \oplus \bigoplus_{s \neq s'} \mathcal{B}(\mathcal{H}_s \otimes \mathcal{H}_{s'})$ , where  $r_{s_*} = \dim(\mathcal{H}_{s_*})/\dim(\mathcal{H})$ ,  $0_{s, s'}$  being the null operator on  $\mathcal{H}_s \otimes \mathcal{H}_{s'}$ , and measurements  $M_{0|x} := \bigoplus_s M_{0|x}^{(s)}$ ,  $M_{0|y} := \bigoplus_s M_{0|y}^{(s)}$ . Hence,  $\rho$  is restricted to acting nontrivially only on the subspace  $\bigoplus_{s_*} (\mathcal{H}_{s_*} \otimes \mathcal{H}_{s_*})$  of  $\mathcal{H} \otimes \mathcal{H}$  and its components outside of this subspace are zero. All the indices  $s_*, s, s'$  run over the  $N$ -Specker scenarios  $\{\text{Sp}_s(\mathcal{J})\}_s$  contained in  $\mathcal{J}$ . It is easy to verify that this setup defined on  $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$  violates a Bell inequality: specifically, noting that  $x, y \in \{0, 1, \dots, v-1\}$ , we show  $I_{vv22} > 0$  for this choice of shared state and local measurements in Sec. I.B of the Supplemental Material [36]. In fact, our construction does something more: namely, it also yields a Bell-violating quantum realization for *each* nontrivial joint measurability structure contained in the given joint measurability structure, i.e., *each* joint measurability structure formed by some incompatible subset of vertices (see Sec. I.C of the Supplemental Material [36]). We therefore have the following theorem:

**Theorem 1.** Every nontrivial joint measurability structure over a finite set of vertices admits a Bell-violating quantum realization that is also Bell-violating for every nontrivial joint measurability structure contained in it.

Theorem 1 answers, in particular, the following question posed in Ref. [18] in the affirmative:

“Considering a set of arbitrarily many POVMs, it is known that any partial compatibility configuration can be realized [33]. Is it then possible to violate a Bell inequality for any possible configuration?”

The “partial compatibility configurations” of Ref. [18] are the same as the nontrivial joint measurability structures we consider here [27].

Having thus shown that every nontrivial joint measurability structure admits a Bell-violating quantum realization, the following natural question arises: Given a nontrivial joint measurability structure, which quantum realizations of it are Bell-violating and which ones are not?

Although we do not have a complete answer to this question for any given joint measurability structure, we studied it in the simplest case of interest, namely, the 3-Specker scenario. We refer to the Supplemental Material [36] (Sec. IV) for further details and simply note here that our study led us to a new family of qubit POVMs that constitute Bell-violating quantum realizations of the 3-Specker scenario, substantially generalizing a planar family of POVMs proposed in Ref. [18].

*Discussion.* We have demonstrated a qualitative equivalence between incompatibility and Bell nonlocality, i.e., a nontrivial joint measurability structure is not only necessary but also *sufficient* for Bell nonlocality. This sheds new light on the conceptual relationship between these two nonclassical features of quantum theory and raises interesting questions for future work. We already know that the mere fact of incompatibility of a set of measurements is not enough to imply a Bell inequality violation [11,12]. However, given the joint measurability structure of an incompatible set of measurements that cannot violate a Bell inequality, our result shows that we can always find another set of measurements with the same structure that do violate a Bell inequality. That is, there are no nontrivial joint measurability structures that are “useless” for a Bell inequality violation.

A fundamental question this raises is the following: Given a quantum realization of a nontrivial joint measurability structure, which features of this realization are responsible for a Bell inequality violation? A characterization of this type would help us obtain a finer handle on the relationship between incompatibility and Bell nonlocality by allowing us to target, when required, those measurements that are useful for Bell inequality violations. While the general characterization problem may be difficult to solve for arbitrary joint measurability structures, a lot of insight can be gained by studying Bell-violating vs Bell-nonviolating realizations of

the simplest joint measurability structure beyond a pair of incompatible measurements, i.e., Specker’s scenario with three binary outcome measurements.

Although we have carried out some numerical investigations of the 3-Specker scenario (see Supplemental Material [36], Sec. IV), further analytical investigations will be essential for a deeper understanding of the interplay between Bell nonlocality and measurement incompatibility. For example, the family of qubit POVMs that constitute Bell-violating quantum realizations of the 3-Specker scenario identified in the Supplemental Material [36] generalizes the planar family of POVMs proposed in Ref. [18]. Can we generalize this further, perhaps to nonplanar measurements? Similarly, do the two families of qubit POVMs that do not show  $I_{3322}$  violations (see Supplemental Material [36], Sec. IV) also fail to show Bell nonlocality more generally, going beyond known examples [11,12]? Finally, do the realizations of nontrivial joint measurability structures with qubit POVMs (including  $N$ -Specker scenarios for all finite  $N$ ) obtained in Ref. [29] enable Bell inequality violations, perhaps on two-qubit systems? Such investigations will open up avenues for better leveraging the incompatibility of measurements (in particular, qubit measurements [41]) in quantum protocols based on Bell nonlocality [8] and will be taken up in future work.

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