

Supplemental Material for ‘Qualitative equivalence between incompatibility and Bell nonlocality’

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I. PROOFS OF TECHNICAL RESULTS IN THE MAIN TEXT

A. Every non-trivial joint measurability structure admits a resolution into N -Specker scenarios

We will use some generic properties of any partial order over a set in proving our claim. Namely,

1. Any partially ordered set, say (X, \preceq) , admits minimal elements.
2. The upward closure of any element $x \in X$ is defined by the (upward closed) set $\uparrow x := \{u | u \in X, x \preceq u\}$.¹
3. The union of the upward closures of the minimal elements of (X, \preceq) is equal to X , i.e.,

$$X = \bigcup_{x_{\min}} \uparrow x_{\min}.$$

It is easy to see this as follows: Suppose some $x \in X$ but $x \notin \bigcup_{x_{\min}} \uparrow x_{\min}$. That is, x is not above *any* minimal element x_{\min} in the partial order. Hence, for each minimal element x_{\min} , x is either below x_{\min} or incomparable to x_{\min} : the former is ruled out because nothing is below x_{\min} by definition, and in the latter case, x must be a minimal element itself and therefore cannot be outside the set $\bigcup_{x_{\min}} \uparrow x_{\min}$. Thus, our supposition is flawed and we have $X = \bigcup_{x_{\min}} \uparrow x_{\min}$.

We now apply the above properties to the partial order over incompatible subsets in any given non-trivial joint measurability structure \mathcal{J} . Given \mathcal{J} , let us define the

hypergraph, $\overline{\mathcal{J}}$, with vertices $V(\overline{\mathcal{J}}) = V(\mathcal{J})$ and hyperedges $E(\overline{\mathcal{J}}) = \{e \subseteq V(\mathcal{J}) | e \notin E(\mathcal{J})\}$.² It is now easy to see how the resolution of \mathcal{J} into N -Specker scenarios works using the joint measurability structure in Figure 1 in the main text as our working example: the incompatible subsets (i.e., hyperedges of $\overline{\mathcal{J}}$) are given by

$$\begin{aligned} & \{\{M_1, M_2, M_3, M_4\}, \{M_1, M_2, M_4\}, \{M_2, M_3, M_4\}, \\ & \{M_1, M_3, M_4\}, \{M_1, M_3, M_2\}, \{M_1, M_3\}\}. \end{aligned} \quad (1)$$

Of these incompatible subsets, the minimal ones—namely, those for which every proper subset is compatible—are given by (cf., Figure 1 in the main text)

$$\{\{M_1, M_2, M_4\}, \{M_2, M_3, M_4\}, \{M_1, M_3\}\}. \quad (2)$$

The non-minimal incompatible sets in \mathcal{J} (equivalently, hyperedges of $\overline{\mathcal{J}}$) can be generated from the minimal ones by progressively adding a new vertex to each minimal subset until the full set of vertices in \mathcal{J} is covered: e.g., the minimal incompatible (2-Specker) set $\{M_1, M_3\}$ generates the non-minimal incompatible sets $\{\{M_1, M_3, M_4\}, \{M_1, M_3, M_2\}, \{M_1, M_2, M_3, M_4\}\}$; the other two minimal incompatible sets generate the (non-minimal) incompatible set $\{M_1, M_2, M_3, M_4\}$. The same argument for resolution into N -Specker scenarios applies for any \mathcal{J} . The formal reason for this is that the incompatibility relations in \mathcal{J} form a partial order with respect to set inclusion, i.e., for any two incompatible subsets S_1, S_2 of $V(\mathcal{J})$, $S_1 \subseteq S_2$, $S_2 \subseteq S_1$, or neither is a subset of the other, i.e., $S_1 \not\subseteq S_2$ and $S_2 \not\subseteq S_1$. The minimal elements of this partial order for any non-trivial joint measurability structure are given by N -Specker scenarios since they do not contain any proper subset that is incompatible, i.e., there are no incompatible subsets below them in this partial order over incompatible subsets. Similarly, the maximal element of any non-trivial

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¹ This set is “upward closed” in the sense that it contains any element above x in the partial order (X, \preceq) .

² Note that both \mathcal{J} and $\overline{\mathcal{J}}$ are valid representations of the (in)compatibility relations, the former denoting compatibility via hyperedges and the later denoting incompatibility. Further, the partially ordered set of interest here is $(X, \preceq) := (E(\overline{\mathcal{J}}), \subseteq)$, where the set X denotes all incompatible subsets of $V(\overline{\mathcal{J}})$ and the partial order relation (\preceq) is given by set inclusion (\subseteq) .

joint measurability structure is the full set of vertices, $V(\mathcal{J})$. Denoting the minimal elements of this partial order—namely, N -Specker scenarios—by $\{\text{Sp}_s(\mathcal{J})\}_s$, we have that

$$\begin{aligned} E(\overline{\mathcal{J}}) &= \bigcup_s \uparrow E(\overline{\text{Sp}_s(\mathcal{J})}) \\ &= \bigcup_s \uparrow \{V(\text{Sp}_s(\mathcal{J}))\} \end{aligned} \quad (3)$$

where $\uparrow E(\overline{\text{Sp}_s(\mathcal{J})})$ denotes the upward-closure of N -Specker scenario $\text{Sp}_s(\mathcal{J})$ under set inclusion. Here, $\overline{\text{Sp}_s(\mathcal{J})}$ is the hypergraph obtained from $\text{Sp}_s(\mathcal{J})$ via $V(\overline{\text{Sp}_s(\mathcal{J})}) = V(\text{Sp}_s(\mathcal{J}))$ and $E(\overline{\text{Sp}_s(\mathcal{J})}) = \{V(\text{Sp}_s(\mathcal{J}))\}$.

Hence, $\uparrow \{V(\text{Sp}_s(\mathcal{J}))\}$ is obtained by constructing incompatible subsets starting from the minimal incompatible subset $V(\text{Sp}_s(\mathcal{J}))$ and progressively adding other vertices from $V(\mathcal{J})$ to it, e.g., in Figure 1 in the main text, the minimal incompatible subset $V(\text{Sp}_s(\mathcal{J})) = \{M_1, M_3\}$ has the upward-closure

$$\begin{aligned} &\uparrow \{\{M_1, M_3\}\} \\ &= \{\{M_1, M_3, M_4\}, \{M_1, M_3, M_2\}, \{M_1, M_2, M_3, M_4\}\} \end{aligned} \quad (4)$$

under set inclusion.

B. Proof that our quantum realization is Bell-violating for any non-trivial \mathcal{J}

The statistics relevant for I_{vv22} is determined by the following probabilities:

$$\begin{aligned} p_A(0|x) &= \text{Tr}(\rho M_{0|x} \otimes \mathbb{1}) \\ &= \text{Tr} \left(\rho \left(\bigoplus_s M_{0|x}^{(s)} \right) \otimes \left(\bigoplus_{s'} \mathbb{1}_{s'} \right) \right) \\ &= \text{Tr} \left(\rho \left(\bigoplus_s (M_{0|x}^{(s)} \otimes \mathbb{1}_s) \oplus \bigoplus_{s \neq s'} (M_{0|x}^{(s)} \otimes \mathbb{1}_{s'}) \right) \right) \\ &= \text{Tr} \left(\left(\bigoplus_{s_*} r_{s_*} \rho_{s_*} \right) \left(\bigoplus_s (M_{0|x}^{(s)} \otimes \mathbb{1}_s) \right) \right) \\ &= \text{Tr} \left(\bigoplus_s r_s \rho_s (M_{0|x}^{(s)} \otimes \mathbb{1}_s) \right) \\ &= \sum_s r_s \text{Tr} \left(\rho_s M_{0|x}^{(s)} \otimes \mathbb{1}_s \right) \end{aligned} \quad (5)$$

$$= \sum_s r_s p_A^{(s)}(0|x). \quad (6)$$

$$\begin{aligned} p_B(0|y) &= \text{Tr}(\rho \mathbb{1} \otimes M_{0|y}) \\ &= \text{Tr} \left(\rho \left(\bigoplus_{s'} \mathbb{1}_{s'} \right) \otimes \left(\bigoplus_s M_{0|y}^{(s)} \right) \right) \\ &= \text{Tr} \left(\rho \left(\bigoplus_s (\mathbb{1}_s \otimes M_{0|y}^{(s)}) \oplus \bigoplus_{s \neq s'} (\mathbb{1}_s \otimes M_{0|y}^{(s')}) \right) \right) \\ &= \text{Tr} \left(\left(\bigoplus_{s_*} r_{s_*} \rho_{s_*} \right) \left(\bigoplus_s (\mathbb{1}_s \otimes M_{0|y}^{(s)}) \right) \right) \\ &= \text{Tr} \left(\bigoplus_s r_s \rho_s (\mathbb{1}_s \otimes M_{0|y}^{(s)}) \right) \\ &= \sum_s r_s \text{Tr} \left(\rho_s \mathbb{1}_s \otimes M_{0|y}^{(s)} \right) \end{aligned} \quad (7)$$

$$= \sum_s r_s p_B^{(s)}(0|y). \quad (8)$$

$$\begin{aligned} p(00|xy) &= \text{Tr}(\rho M_{0|x} \otimes M_{0|y}) \\ &= \text{Tr} \left(\rho \left(\bigoplus_s M_{0|x}^{(s)} \right) \otimes \left(\bigoplus_{s'} M_{0|y}^{(s')} \right) \right) \\ &= \text{Tr} \left(\rho \left(\bigoplus_s (M_{0|x}^{(s)} \otimes M_{0|y}^{(s)}) \oplus \bigoplus_{s \neq s'} (M_{0|x}^{(s)} \otimes M_{0|y}^{(s')}) \right) \right) \\ &= \text{Tr} \left(\left(\bigoplus_{s_*} r_{s_*} \rho_{s_*} \right) \left(\bigoplus_s (M_{0|x}^{(s)} \otimes M_{0|y}^{(s)}) \right) \right) \\ &= \text{Tr} \left(\bigoplus_s r_s \rho_s (M_{0|x}^{(s)} \otimes M_{0|y}^{(s)}) \right) \\ &= \sum_s r_s \text{Tr} \left(\rho_s (M_{0|x}^{(s)} \otimes M_{0|y}^{(s)}) \right) = \sum_s r_s p^{(s)}(00|xy). \end{aligned} \quad (9)$$

We then have:

$$\begin{aligned} I_{vv22} &\equiv -p_B(0|0) - \sum_{x=1}^{v-1} p_A(0|x) + \sum_{x=0}^{v-1} p(00|x, y=0) \\ &\quad + \sum_{x=1}^{v-1} p(00|x, x) - \sum_{0 \leq x < y \leq v-1} p(00|x, y) \quad (10) \\ &= \sum_s r_s I_{vv22}^{(s)} > 0. \end{aligned} \quad (11)$$

The last inequality follows from the fact that $I_{vv22}^{(s)} > 0$ and $r_s > 0$ for all $\text{Sp}_s(\mathcal{J})$ in the resolution of \mathcal{J} . Hence, given any non-trivial joint measurability structure required of Alice's measurement settings, our construction provides a Bell-violating quantum realization of it.

C. Proof that our quantum realization is Bell-violating for every non-trivial joint measurability structure contained in \mathcal{J}

To see this for a non-trivial joint measurability structure, say \mathcal{J}_{sub} , contained in \mathcal{J} , one simply has to note that a subset of N -Specker scenarios in the decomposition of \mathcal{J} are sufficient to reconstruct \mathcal{J}_{sub} , *i.e.*, to obtain

the full set of incompatibility relations in \mathcal{J}_{sub} . Further, since $I_{vv22}^{(s)} > 0$ and $r_s > 0$ for all $\text{Sp}_s(\mathcal{J})$ in the resolution of \mathcal{J} , this is also true for the $\text{Sp}_s(\mathcal{J})$ in the resolution of \mathcal{J}_{sub} . Denoting the number of vertices in \mathcal{J}_{sub} as $w (\leq v)$, we have that

$$I_{ww22} := -p_B(0|0) - \sum_{x=1}^{w-1} p_A(0|x) + \sum_{x=0}^{w-1} p(00|x, y=0) + \sum_{x=1}^{w-1} p(00|x, x) - \sum_{0 \leq x < y \leq w-1} p(00|x, y) \quad (12)$$

$$= \sum_{s: V(\text{Sp}_s(\mathcal{J})) \subseteq V(\mathcal{J}_{\text{sub}})} r_s I_{ww22}^{(s)} > 0, \quad (13)$$

where the sum is over those N -Specker scenarios $\text{Sp}_s(\mathcal{J})$ that are contained in \mathcal{J}_{sub} .

II. BELL-VIOLATING QUANTUM REALIZATION OF $\text{Sp}_s(\mathcal{J})$

We use the Bell-violating quantum realization of N -Specker scenarios in Ref. [1] to construct a quantum realization of each such scenario $\text{Sp}_s(\mathcal{J})$ in the resolution $\{\text{Sp}_s(\mathcal{J})\}_s$ of \mathcal{J} . That is, given a $\text{Sp}_s(\mathcal{J})$ in \mathcal{J} , we assign POVMs $\{M_x^{(s)}\}_{x=1}^N$ in $\mathcal{B}(\mathcal{H}_s)$ (where $\mathcal{H}_s \cong \mathbb{C}^N$) to the N vertices of $\text{Sp}_s(\mathcal{J})$ following the construction of Ref. [1]:

$$M_{a=0|x}^{(s)} = \eta |A_x\rangle \langle A_x|, |A_x\rangle = \sum_{j=1}^N A_{xj} |j\rangle \quad (14)$$

where $\{|j\rangle\}_{j=1}^N$ is an orthonormal basis of \mathbb{C}^N , $\eta = 1/(N-1)$, and the probability amplitudes A_{xj} are defined as entries of the $N \times N$ matrix

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & -q_1 & -q_0 \\ 0 & 0 & \dots & 0 & -q_2 & \frac{q_1}{N-1} & q_0 \\ 0 & 0 & \dots & -q_3 & \frac{q_2}{N-2} & \frac{q_1}{N-1} & q_0 \\ \vdots & \vdots & \ddots & & & & \\ -q_{N-1} & \frac{q_{N-2}}{2} & \dots & \frac{q_3}{N-3} & \frac{q_2}{N-2} & \frac{q_1}{N-1} & q_0 \\ q_{N-1} & \frac{q_{N-2}}{2} & \dots & \frac{q_3}{N-3} & \frac{q_2}{N-2} & \frac{q_1}{N-1} & q_0 \end{bmatrix}, \quad (15)$$

where $q_1^2 + q_0^2 = 1$ and $q_{k+1}^2 = \left(1 - \frac{1}{(N-k)^2}\right) q_k^2$ for $k \geq 1$. This means that each row of A denotes the unit state vector $|A_x\rangle$, *i.e.*, $\sqrt{\sum_{j=1}^N |A_{xj}|^2} = 1$ [1, 2]. Hence, Alice has access to the POVMs $M_x^{(s)} \equiv \{M_{a=0|x}^{(s)}, M_{a=1|x}^{(s)} = \mathbb{1}_s - M_{a=0|x}^{(s)}\}$ for all $x \in \{1, 2, \dots, N\}$. Here, the condition that $\eta = \frac{1}{N-1}$ is sufficient to ensure that the joint measurability structure is an N -Specker scenario following the construction in Section 4 of Ref. [1].

To witness a Bell inequality violation, Bob is given access to N POVMs $M_y^{(s)} \equiv \{M_{b=0|y}^{(s)}, M_{b=1|y}^{(s)} = \mathbb{1}_s -$

$M_{b=0|y}^{(s)}\}$ given by

$$M_{b=0|y}^{(s)} = |B_y\rangle \langle B_y|, |B_y\rangle = \sum_{j=1}^N B_{yj} |j\rangle \quad (16)$$

where B_{yj} are entries of the $N \times N$ matrix

$$B = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 0 & \dots & 0 & -p_2 & \frac{p_1}{N-1} & p_0 \\ 0 & 0 & \dots & -p_3 & \frac{p_2}{N-2} & \frac{p_1}{N-1} & p_0 \\ \vdots & \vdots & \ddots & & & & \\ -p_{N-1} & \frac{p_{N-2}}{2} & \dots & \frac{p_3}{N-3} & \frac{p_2}{N-2} & \frac{p_1}{N-1} & p_0 \\ p_{N-1} & \frac{p_{N-2}}{2} & \dots & \frac{p_3}{N-3} & \frac{p_2}{N-2} & \frac{p_1}{N-1} & p_0 \end{bmatrix}, \quad (17)$$

where $p_0^2 = \frac{1}{N}$, $p_1^2 = \frac{N-1}{N} = 1 - \frac{1}{N}$, $p_{k+1}^2 = \left(1 - \frac{1}{(N-k)^2}\right) p_k^2$ for $k \geq 1$. Again, each row of B denotes the unit state vector $|B_y\rangle$, *i.e.*, $\sqrt{\sum_{j=1}^N |B_{yj}|^2} = 1$ [1, 2].

The quantum state $|\psi\rangle_\epsilon \in \mathbb{C}^N \otimes \mathbb{C}^N$ shared between Alice and Bob is given by

$$|\psi_\epsilon\rangle = \sqrt{\frac{1-\epsilon^2}{N-1}} \left(\sum_{k=1}^{N-1} |k\rangle |k\rangle \right) + \epsilon |N\rangle |N\rangle, \epsilon \in [0, 1]. \quad (18)$$

The Bell scenario therefore consists of two parties, each with N dichotomic measurements. The $I_{NN22}^{(s)}$ Bell inequality [3] for this scenario is given by

$$I_{NN22}^{(s)} := -p_B^{(s)}(0|1) - \sum_{x=2}^N p_A^{(s)}(0|x) + \sum_{x=1}^N p^{(s)}(00|x, y=1) + \sum_{x=2}^N p^{(s)}(00|x, x) - \sum_{1 \leq x < y \leq N} p^{(s)}(00|x, y) \leq 0. \quad (19)$$

Using the state and measurements above, we have (following Ref. [2]) for the probabilities entering $I_{NN22}^{(s)}$:

$$p_B^{(s)}(0|1) = \epsilon^2, \quad (20)$$

$$p_A^{(s)}(0|x) = \frac{1-\epsilon^2}{N-1} (1 - q_0^2) + \epsilon^2 q_0^2 \quad \text{for } 2 \leq x \leq N, \quad (21)$$

$$p^{(s)}(00|x, y=1) = \epsilon^2 q_0^2 \quad \text{for } 1 \leq x \leq N, \quad (22)$$

$$p^{(s)}(00|x, x) = \left(\sqrt{\frac{1-\epsilon^2}{N-1}} p_1 q_1 + \epsilon p_0 q_0 \right)^2 \quad \text{for } 2 \leq x \leq N, \quad (23)$$

$$p^{(s)}(00|x, y) = \left(\sqrt{\frac{1-\epsilon^2}{N-1}} \frac{p_1 q_1}{1-N} + \epsilon p_0 q_0 \right)^2 \quad \text{for } 1 \leq x < y \leq N. \quad (24)$$

Following Ref. [2], we choose ϵ to maximize the violation of Eq. (19), i.e.,

$$\epsilon^2 = \frac{1 - q_0^2}{1 + [(N-1)^2 - 1]q_0^2}, \quad (25)$$

so that the terms $p^{(s)}(00|x, y)$ are set to zero for all x, y such that $1 \leq x < y \leq N$. We then have that

$$I_{NN22}^{(s)} = \eta \epsilon^2 \left(-\frac{1}{\eta} + q_0^2 N \right) > 0, \text{ for } \eta > \frac{1}{Nq_0^2}, \quad (26)$$

and $\eta = \frac{1}{N-1} > \sqrt{\frac{1}{Nq_0^2}}$ requires that $q_0 > (1 - \frac{1}{N})$. Hence, $q_0 > (1 - \frac{1}{N})$ implies that $\eta = \frac{1}{N-1}$ is sufficient to violate the inequality $I_{NN22}^{(s)} \leq 0$.

III. BELL VIOLATION FROM ANY 2-SPECKER SCENARIO

We consider any $\text{Sp}_s(\mathcal{J})$ in the resolution of \mathcal{J} that is just a pair of incompatible vertices. Under the no-signalling condition, the expression for $I_{NN22}^{(s)}$ Bell inequality reduces to the CHSH inequality for $N = 2$. That is, the inequality (also, referred to as the CH inequality [4])

$$I_{2222}^{(s)}$$

$$= -p_B^{(s)}(0|1) - p_A^{(s)}(0|2) + p^{(s)}(00|1, 1) + p^{(s)}(00|2, 1) \\ + p^{(s)}(00|2, 2) - p^{(s)}(00|1, 2) \quad (27)$$

$$\leq 0 \quad (28)$$

is equivalent to the Bell-CHSH inequality [5]. We refer the reader to Ref. [4] for a proof of this equivalence (see also [6] for a more modern treatment).

Now, to obtain a violation of the CHSH inequality from a 2-Specker scenario, all we need is any pair of incompatible dichotomic POVMs. Wolf *et al.* [7] showed that Alice can use any such pair POVMs to violate the CHSH inequality for some choice of entangled state shared with Bob and some choice of POVMs for Bob. For example, one could take the shared state ρ_s to be a two-qubit maximally entangled state (hence $\mathcal{H}_s \cong \mathbb{C}^2$) and the measurements $\{M_x^{(s)}\}_{x=1}^2$ and $\{M_y^{(s)}\}_{y=1}^2$ to be those that achieve Tsirelson's bound [8]. The assignment of trivial $\{0, \mathbf{1}_s\}$ POVMs to the rest of the $v-2$ vertices in \mathcal{J} can proceed as we did in the $N \geq 3$ case.

IV. BELL VIOLATIONS IN THE 3-SPECKER SCENARIO

We investigate the ability of three qubit POVMs with coplanar Bloch vectors to violate the I_{3322} Bell inequality [3] in the 3-Specker scenario. In Ref. [1] it was shown that following set of trine spin POVMs form a 3-Specker

scenario but never violate a Bell inequality (thus showing that incompatibility does not imply Bell nonlocality):

$$\forall k \in \{1, 2, 3\}, x_k \in \{-1, +1\} :$$

$$E_k(x_k) = \frac{1}{2} (I + 0.67x_k \vec{n}_k \cdot \vec{\sigma}),$$

$$\text{where } \vec{n}_k = \cos \frac{2k\pi}{3} \vec{e}_x + \sin \frac{2k\pi}{3} \vec{e}_z,$$

$$\text{and } \vec{e}_x = (1, 0, 0), \vec{e}_z = (0, 0, 1), \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z). \quad (29)$$

This inspires us to look at two families of POVMs to which the set in Eq. (29) belongs, *i.e.*, to two generalizations of Eq. (29). In both of these generalizations we keep the trine directions but we introduce a global purity parameter $\eta \in [0, 1]$ and a bias $b \in [-1, 1]$ such that $|b| \leq 1 - \eta$. Since bias introduces a preferred orientation along the Bloch line there are two ways to assign biases producing the two families of interest, *i.e.*,

$$E_k(x_k) = \frac{1}{2} ((1 + x_k b)I + \eta x_k \vec{n}_k \cdot \vec{\sigma}), \quad (30a)$$

$$E_k(x_k) = \frac{1}{2} ((1 + (-1)^k x_k b)I + \eta x_k \vec{n}_k \cdot \vec{\sigma}), \quad (30b)$$

$$\text{where } \vec{n}_k = \cos \frac{2k\pi}{3} \vec{e}_x + \sin \frac{2k\pi}{3} \vec{e}_z,$$

$$k \in \{1, 2, 3\}, x_k \in \{-1, +1\}. \quad (30c)$$

In contrast to the trine example used in Ref. [1] (namely, Eq. (29)), Ref. [9] provides an example of POVMs with trine spin directions that does form a 3-Specker scenario and violates Bell's inequalities. Generalizing the construction of Ref. [9], we define another family of POVMs as

$$E_1(+1) = \frac{1}{2} \alpha (I + \vec{n}_1 \cdot \vec{\sigma}), \quad (31a)$$

$$E_2(+1) = \frac{1}{2} \alpha (I + \vec{n}_2 \cdot \vec{\sigma}), \quad (31b)$$

$$E_3(+1) = \frac{1}{2} \beta (I - \vec{n}_3 \cdot \vec{\sigma}), \quad (31c)$$

which recovers the example from [9] when $\beta = 3\alpha/5$, where $\alpha, \beta \in [0, 1]$.

For the three families of trine POVMs we have defined (Eqs. (30a), (30b), (31)), we will investigate the dependence of the joint measurability structure they realize and their ability to violate the I_{3322} Bell inequality on the parameters we have introduced in each case. For each family, the pairwise joint measurability is determined analytically [10], while the triplewise joint measurability condition is obtained by solving a semidefinite program (SDP) that maximizes that relevant parameters in the POVMs subject to the existence of a triplewise joint POVM.

Example 1. We consider the family in Eq. (30a), *i.e.*,

$$E_k(x_k) = \frac{1}{2} ((1 + x_k b)I + \eta x_k \vec{n}_k \cdot \vec{\sigma}),$$

$$\text{where } \vec{n}_k = \cos \frac{2k\pi}{3} \vec{e}_x + \sin \frac{2k\pi}{3} \vec{e}_z, k \in \{1, 2, 3\}. \quad (32)$$

This set is pairwise jointly measurable if and only if (blue line in Fig. 1) [10]

$$\eta \leq \begin{cases} 1+b, & -1 \leq b < -\frac{1}{3}, \\ \sqrt{3-2b^2}-1, & b \in [-\frac{1}{3}, \frac{1}{3}], \\ 1-b, & \frac{1}{3} < b \leq 1. \end{cases} \quad (33)$$

To determine triplewise joint measurability, we solve an SDP that maximizes η for each b picked from a discretization of the interval $[-1, 1]$, subject to the existence of a triplewise joint POVM. We obtain numerically a plot for $\eta_1^{(3)}(b)$ such that the set is triplewise jointly measurable iff (red line in Fig. 1)

$$\eta \leq \eta_1^{(3)}(b). \quad (34)$$

Based on our numerical evaluation and visual inspection, we conjecture the exact form of $\eta_1^{(3)}(b)$ to be

$$\eta_1^{(3)}(b) = \begin{cases} 1+b, & -1 \leq b < -\frac{1}{3}, \\ \frac{2}{3}, & b \in [-\frac{1}{3}, \frac{1}{3}], \\ 1-b, & \frac{1}{3} < b \leq 1. \end{cases} \quad (35)$$

This means that the 3-Specker scenario is realized iff the following system of inequalities is satisfied:

$$|b| \leq \frac{1}{3}, \quad \frac{2}{3} < \eta \leq \sqrt{3-2b^2}-1. \quad (36)$$

As is clear in Fig. 1, there is no violation of the I_{3322} inequality using such measurements when they form a 3-Specker scenario.

Example 2. We consider the family in Eq. (30b), *i.e.*,

$$E_k(x_k) = \frac{1}{2} \left((1 + (-1)^k x_k b) I + \eta x_k \vec{n}_k \cdot \vec{\sigma} \right),$$

where $\vec{n}_k = \cos \frac{2k\pi}{3} \vec{e}_x + \sin \frac{2k\pi}{3} \vec{e}_z$, $k \in \{1, 2, 3\}$. (37)

The pair $\{E_1, E_3\}$ is jointly measurable iff (blue line in Fig. 2)

$$\eta \leq \begin{cases} 1+b, & -1 \leq b < -\frac{1}{3}, \\ \sqrt{3-2b^2}-1, & b \in [-\frac{1}{3}, \frac{1}{3}], \\ 1-b, & \frac{1}{3} < b \leq 1. \end{cases} \quad (38)$$

while, pairs $\{E_1, E_2\}$ and $\{E_2, E_3\}$ are jointly measurable iff (green line in Fig. 2)

$$\eta \leq \begin{cases} 1+b, & -1 \leq b < 3-2\sqrt{3}, \\ \sqrt{2}\sqrt{2+b^2-\sqrt{3(1+2b^2)}}, & |b| \leq 2\sqrt{3}-3 \\ 1-b, & 2\sqrt{3}-3 < b \leq 1. \end{cases} \quad (39)$$

We solve an SDP—that maximizes η for each b picked from a discretization of the interval $[-1, 1]$, subject to the existence of a triplewise joint POVM—to obtain $\eta_2^{(3)}(b)$ such that we have triplewise joint measurability iff (red line in Fig. 2)

$$\eta \leq \eta_2^{(3)}(b). \quad (40)$$

As is clear in Fig. 2, there is no violation of the I_{3322} inequality using such measurements when they form a 3-Specker scenario.

Example 3. We consider the family given in Eq. (31). Pairs $\{E_1, E_3\}$ and $\{E_2, E_3\}$ are jointly measurable iff (blue diagonal line in Fig. 3)

$$\beta \leq \frac{4-4\alpha}{4-\alpha}. \quad (41)$$

The pair $\{E_1, E_2\}$ is jointly measurable iff (blue vertical line in Fig 3)

$$\alpha \leq \frac{2}{3}. \quad (42)$$

From an SDP—maximizing β for each value of α picked from a discretization of the interval $[0, 1]$ subject to the existence of a triplewise joint POVM—we obtain the boundary for triplewise joint measurability, *i.e.*, our family is compatible iff

$$\beta \leq \beta^{(3)}(\alpha). \quad (43)$$

Based on our numerical evaluation and visual inspection, we conjecture the boundary $\beta^{(3)}(\alpha)$ (red line in Fig 3) to be

$$\beta^{(3)}(\alpha) = \begin{cases} 1 - \frac{3}{2}\alpha, & \text{for } \alpha < \frac{2}{3}, \\ 0, & \text{for } \frac{2}{3} \leq \alpha \leq 1 \end{cases}. \quad (44)$$

We refer to Fig. 3 for a depiction of the region (“BCD”) where the I_{3322} inequality is violated by a 3-Specker scenario.

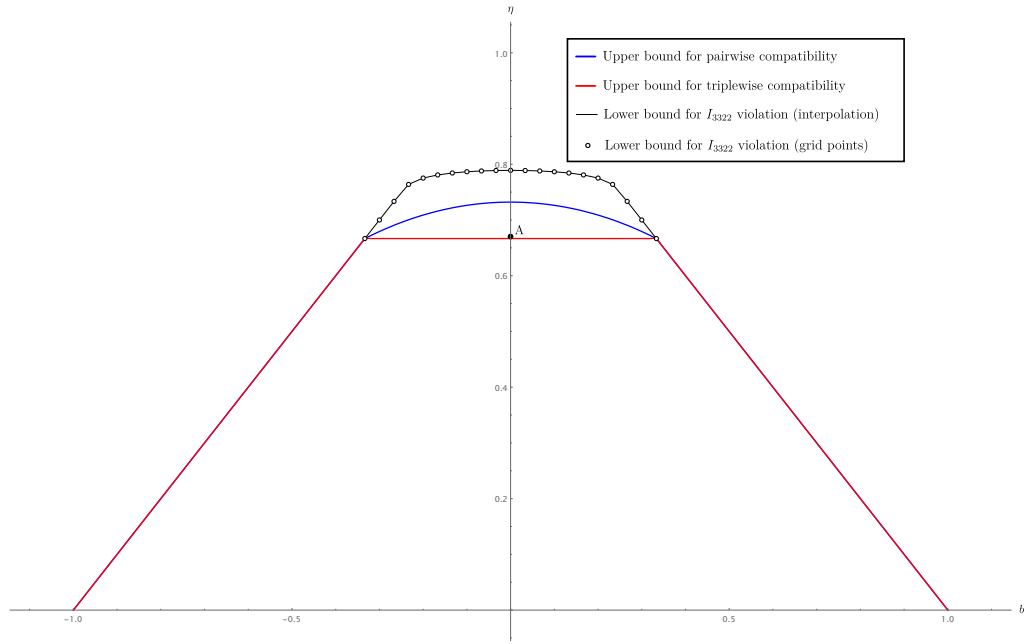


FIG. 1. Red line represents the upper bound for η given b such that the whole set is compatible. Blue line (coinciding with the red one for $|b| > 1/3$) represents the upper bound for η such that each pair is jointly measurable. Black line represents lower bound for η such that we have I_{3322} violation. The points along the black line are numerically obtained and the rest of it is an extrapolation. In the region between the blue and the red line, where we have a 3-Specker scenario, there are no I_{3322} violations. Point A represents the POVMs used for Alice (same as our Eq. (29)) in Ref. [1] to show that measurement incompatibility does not imply Bell nonlocality.

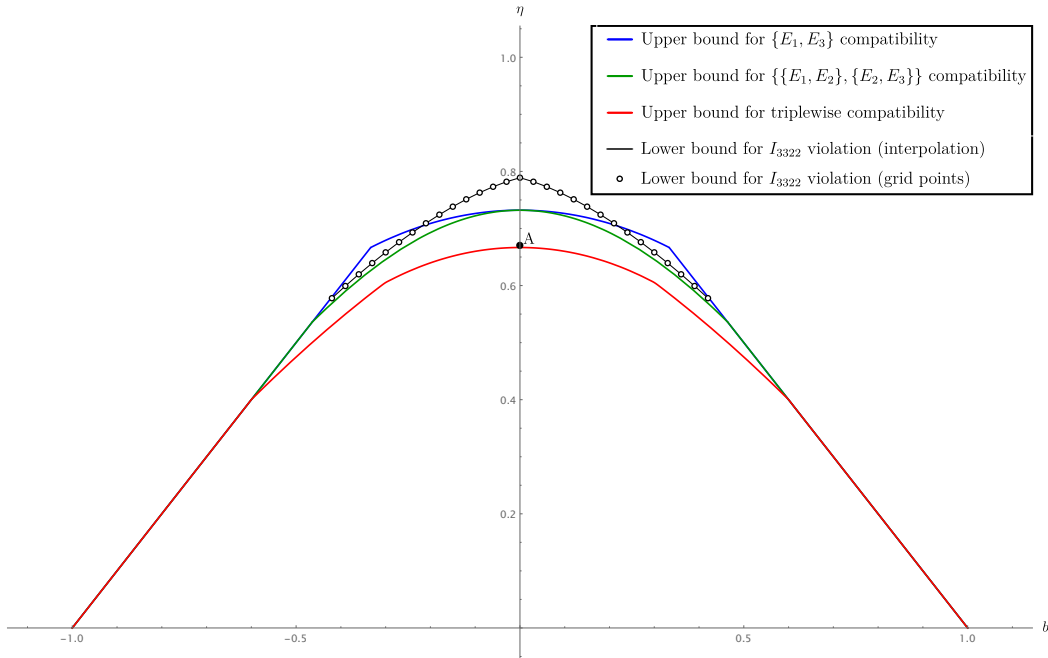


FIG. 2. Red line represents the upper bound for η given b such that the whole set is compatible. Blue line represents the upper bound for η such that $\{E_1, E_3\}$ is compatible. Green line represents the upper bound for η such that each of the pairs $\{E_1, E_2\}$ and $\{E_2, E_3\}$ is compatible. Black line represents lower bound for η such that we have I_{3322} violation. In the region between the green and the red line, where we have a 3-Specker scenario, there are no I_{3322} violations. However, there are violations in the region between the green and the blue line where only the pair $\{E_1, E_3\}$ is compatible while other two pairs are incompatible. Point A represents the POVMs used for Alice (same as our Eq. (29)) in Ref. [1] to show the that measurement incompatibility does not imply Bell nonlocality.

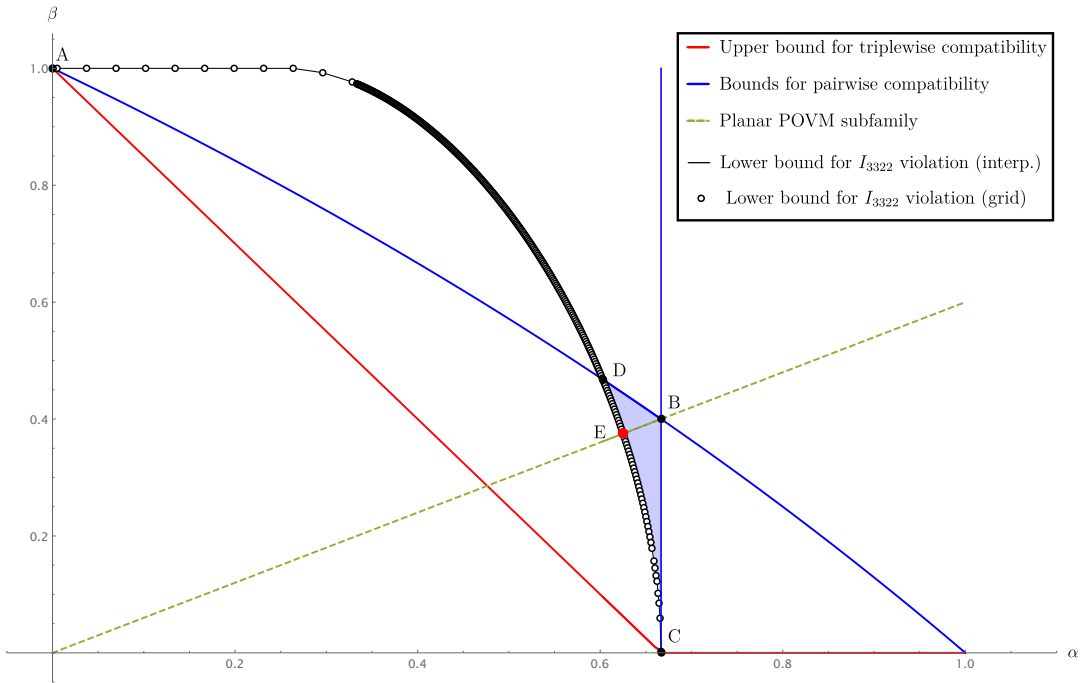


FIG. 3. Red line represents upper bound for β such that we have 3-way compatibility. Blue line passing through points A and B represents the upper bound for β such that $\{E_1, E_3\}$ and $\{E_2, E_3\}$ are compatible. Vertical blue line represents the upper bound for alpha such that the pair $\{E_1, E_2\}$ is compatible. For triangle-like region ABC, 3-Specker is realized. Black line represents the lower bound for β given α such that we have I_{3322} violation. In the region defined by the points BCD we have a 3-Specker scenario capable of violating I_{3322} Bell inequality. Dashed line represents the planar family of POVMs from Appendix B in Ref. [9]. The red dot on that line is the critical point for I_{3322} violation by the family in the same Appendix (its criticality is here confirmed).

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