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# Indépendance statistique et lois limites pour quelques objets arithmétiques

Sary Drappeau

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ED 184 Mathématiques et informatique de Marseille  
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Mémoire présenté pour obtenir  
l'habilitation à diriger des recherches

Discipline : Mathématiques

Sary DRAPPEAU

Indépendance statistique et lois limites  
pour quelques objets arithmétiques

Statistical independence and limit laws  
for some arithmetical objects

Soutenu le 28 septembre 2023 devant le jury composé de :

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# Résumé

Ce mémoire présente les thèmes sur lesquels ont porté mes travaux de recherche depuis mon arrivée à l'université d'Aix-Marseille en 2015. Leur problématique commune est de mettre en évidence des comportements statistiques réguliers dans des familles d'objets arithmétiques naturels : les fonctions multiplicatives ou additives, et les valeurs centrales de certaines familles de fonctions  $L$ .

La première partie concerne une question centrale en théorie multiplicative des nombres : celle d'estimer la corrélation des valeurs  $f(n)$  et  $g(n+1)$ , où  $f$  et  $g$  sont deux fonctions multiplicatives, notamment lorsque l'une des deux fonctions est la fonction « nombre de diviseurs ». Ce problème est naturellement lié à la répartition de certaines suites dans les progressions arithmétiques, et trouvent des applications à d'autres questions arithmétiques, par exemple les zéros de petite hauteur des fonctions  $L$  de Dirichlet. Les majorations de sommes d'exponentielles algébriques sont un outil crucial dans cette partie du mémoire.

La seconde partie concerne certaines fonctions  $f : \mathbf{Q} \rightarrow \mathbf{C}$  nommées par Zagier « formes modulaires quantiques », caractérisées par certaines symétries analogues à celles des formes modulaires. Mes collaborations sur ce sujet ont consisté d'une part à établir ces relations de modularité quantiques dans certains cas : celui de tordues additives de fonctions  $L$  de Dirichlet, et celui de sommes de symboles de Pochhammer ; et d'autre part à les utiliser pour en déduire, par des méthodes de systèmes dynamiques, l'existence de lois limites pour les valeurs de  $f$  aux nombres rationnels ordonnés par dénominateurs croissants.

Mots clés : Fonctions multiplicatives, nombres premiers, progressions arithmétiques, sommes de Kloosterman, formes modulaires, formule de Kuznetsov, tordue additive, forme modulaire quantique, loi limite, invariant de Kashaev.

# Abstract

This manuscript presents the themes of my research works in Aix-Marseille university since 2015. Their common theme is the search for simple statistical behaviour among families of natural arithmetical objects: multiplicative or additive functions, functions defined in terms of numeration systems (decimal, continued fractions...) and central values of  $L$ -functions.

The first part concerns a key question in multiplicative number theory: to estimate the correlation of values of  $f(n)$  and  $g(n + 1)$ , where  $f$  and  $g$  are two multiplicative functions, with an emphasis on the case of the divisor function. This naturally involves bounds on algebraic exponential sums, and leads to applications in various problems, all linked in some way to the distribution of certain sequences in arithmetic progressions.

The second part concerns maps  $f : \mathbb{Q} \rightarrow \mathbb{C}$  called by Zagier “quantum modular forms”, which satisfy certain symmetries analogous to those satisfied by modular forms. In several collaborations, we established the modular quantum behaviour in some cases related to additive twists of central  $L$  values, or to Pochhammer symbols, and we deduced through methods from dynamical systems the existence of limit laws for values of  $f$  along rationals ordered by denominators.

Keywords: multiplicative functions, prime numbers, arithmetic progressions, Kloosterman sums, modular form, Kuznetsov formula, additive twist, quantum modular form, limit law, Kashaev invariant

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# Introduction

Ce mémoire présente les résultats obtenus depuis 2015 dans le cadre de mes recherches en théorie analytique des nombres. Leur socle commun est la mise en évidence de lois limites dans des familles d'objets arithmétiques naturels : les fonctions multiplicatives ou additives, les fonctions définies en termes de systèmes de numération (décimal, fractions continues...), les valeurs centrales de familles de fonctions  $L$ .

La première partie du mémoire est centrée sur la question d'estimer asymptotiquement la corrélation

$$C(f, g; x) := \sum_{n \leq x} f(n)g(n+1)$$

lorsque  $f$  et  $g$  sont des fonctions multiplicatives. Heuristiquement, l'on s'attend à ce que les factorisations de  $n$  et de  $n+1$  soient statistiquement indépendantes en moyenne sur  $n$ . On s'attend donc à ce que  $C(f, g; x)$  soit comparable au produit des moyennes de  $f$  et de  $g$  jusqu'à  $x$ , c'est-à-dire

$$C(f, g; x) \asymp x^{-1}C(f, 1; x)C(1, g; x).$$

Nous nous concentrons sur le cas particulier où  $f(n) = \tau(n) = \sum_{d|n} 1$ , la fonction nombre de diviseurs, et où les valeurs  $g(p)$  ont une description simple. Ce cas est d'une importance particulière puisque qu'il délimite le seuil au-delà duquel les méthodes analytiques connues actuellement sont essentiellement inefficaces. Les progrès les plus significatifs sur cette question ont été réalisés dans les années 1980, avec le développement des méthodes provenant des formes automorphes pour étudier certaines sommes d'exponentielles algébriques qui interviennent naturellement.

Dans la section 1.1, nous présentons une généralisation des résultats de Deshouillers et Iwaniec sur les coefficients de Fourier de formes modulaires, et des majorations de sommes de Kloosterman qui en découlent par la formule de Kuznetsov. Ces estimations seront ensuite employées pour en déduire un résultat arithmétique sur la majoration en moyenne des sommes de Kloosterman sous l'hypothèse de zéros de Siegel, obtenu avec J. Maynard.

Dans la section 1.2, nous revenons à l'estimation de  $C(\tau, g; x)$ , et nous présentons des travaux qui répondent à deux questions naturelles : l'influence de la localisation des zéros des fonctions  $L$  de Dirichlet sur le terme d'erreur dans



l'estimation de  $C(\tau, g; x)$ ; et une caractérisation simple des fonctions  $g$  pour lesquelles nous sommes actuellement en mesure d'estimer précisément  $C(\tau, g; x)$ . Le premier travail repose sur les majorations de sommes de Kloosterman évoquées ci-dessus. Le second, obtenu avec B. Topalogullari, dépend de nouvelles décompositions combinatoires pour certaines fonctions multiplicatives.

Dans la section 1.3, nous présentons un travail avec K. Pratt et M. Radziwiłł qui porte sur la densité des zéros de fonctions  $L$  de Dirichlet au voisinage de l'axe réel, en moyenne sur le caractère et le module. Les majorations de sommes de Kloosterman permettront d'estimer cette densité pour des fonctions test dont le support de la transformée de Fourier est élargi au-delà du domaine permis par les inégalités de grand crible.

Enfin, dans la section 1.4, nous présentons un travail avec R. de la Bretèche sur la répartition de valeurs de polynômes quadratiques dans des progressions arithmétiques, qui repose également sur les majorations de sommes de Kloosterman évoquées ci-dessus. En corollaire de ce travail, nous avons précisé un résultat de Deshouillers et Iwaniec sur le plus grand facteur premier de  $n^2 + 1$ .

La seconde partie du mémoire est centrée sur l'existence de lois limites pour certaines fonctions sur les rationnels.

Le premier résultat de cette seconde partie concerne la répartition des valeurs des fonctions  $f : \mathbf{Q} \rightarrow \mathbf{C}$  telles que pour toute homographie  $\gamma \in \mathrm{PSL}_2(\mathbf{Z})$  les fonctions  $h_\gamma(x) := f(x) - f(\gamma x)$  soient régulières, en un sens précis. Ces fonctions ont été nommées par Zagier "formes modulaires quantiques" (de poids 0 et niveau 1) car certains exemples proviennent d'invariants quantiques en théorie des nœuds. En utilisant l'algorithme d'Euclide et les propriétés dynamiques de l'application de Gauss, nous montrons sous des hypothèses assez générales sur les fonctions  $h_\gamma$ , que les valeurs de  $f(x)$  pour  $x$  variant parmi les nombres rationnels de dénominateurs au plus  $Q$  se répartissent suivant une loi stable lorsque  $Q \rightarrow \infty$ . Ce formalisme nous permet d'étudier la fonction d'Esterman, qui est la valeur au point central  $s = 1/2$  de la "tordue additive" de la fonction  $\zeta(s)^2$ ,

$$D(s, x) := \sum_{n \geq 1} \frac{\tau(n)}{n^s} e^{2\pi i n x}.$$

Cette série ne possède pas de produit eulérien; par contre, la fonction  $x \mapsto D(1/2, x)$  est une forme modulaire quantique à laquelle nos résultats s'appliquent. La méthode est essentiellement dynamique, l'arithmétique intervenant uniquement pour prouver la modularité quantique de la fonction  $x \mapsto D(\frac{1}{2}, x)$ .

Notre second résultat concerne le  $q$ -analogue du symbole de Pochhammer,

$$(q)_n = \prod_{r=1}^n (1 - q^r),$$

lorsque  $q$  est une racine de l'unité. Nous montrons que les fonctions qui à  $x \in \mathbf{Q}$

associent respectivement  $(e(x))_n$  et  $(e(\bar{x}))_n$  (où  $\overline{h/k} = (h^{-1} \pmod k)/k \pmod 1$ ) satisfont chacune, pour tout  $\gamma \in PSL_2(\mathbf{Z})$ , une formule de modularité qui relie leurs valeurs en  $x$  et en  $\gamma x$ . Ces deux formules sont cousines de l'équation de modularité de la fonction  $\eta$  de Dedekind et en partagent certains aspects. Ces formules sont cohérentes avec la conjecture de modularité de Zagier sur les invariants de Kashaev de nœuds hyperbolique. Nous montrons, en utilisant ces formules, que les invariants de Kashaev du nœud de huit

$$J_{41,0}(x) = \sum_{n \geq 0} |(e(x))_n|^2$$

ont, pour presque tout nombre rationnel  $x$  (en un sens précis), un comportement asymptotique qui s'exprime en termes très simples du développement en fraction continue de  $x$ .

# 1 Sommes d'exponentielles et indépendance statistique de fonctions arithmétiques

## 1.1 Sommes de Kloosterman

### 1.1.1 Majoration en moyenne

Pour tout  $q \in \mathbf{N}_{>0}$  et  $a, b \in \mathbf{Z}/q\mathbf{Z}$ , on définit la *somme de Kloosterman* (Kloosterman 1927)

$$S(a, b; q) := \sum_{n \in (\mathbf{Z}/q\mathbf{Z})^\times} e\left(\frac{an + bn^{-1}}{q}\right)$$

où  $e(x) := e^{2\pi ix}$ , et  $n^{-1}$  désigne un inverse de  $n$  modulo  $q$ . Nous avons bien entendu la majoration triviale

$$|S(a, b; q)| \leq \varphi(q).$$

Tout amélioration de cette inégalité est susceptible de nous renseigner sur le caractère pseudo-aléatoire de la suite  $(n^{-1})$  lorsque  $n$  parcourt  $(\mathbf{Z}/q\mathbf{Z})^\times$ . Nous avons en toute généralité la borne de Weil (Weil 1948) : pour tout  $\varepsilon > 0$ ,

$$|S(a, b; q)| \ll_\varepsilon q^{1/2+\varepsilon}, \quad ((a, b, q) = 1)$$

où  $(a, b, q)$  désigne le pgcd. L'exposant  $1/2$  est optimal. Lorsque  $a, b$  sont non nuls et fixés, et que l'on somme sur  $q$ , nous pouvons nous attendre à des compensations supplémentaires. Le premier résultat de ce type est du à Kuznetsov (Kuznetsov 1981), et affirme que pour tout  $\varepsilon > 0$  et  $(a, b) \in \mathbf{Z}^2 \setminus 0$ ,

$$\left| \sum_{q \leq x} \frac{S(a, b; q)}{q} \right| \ll_{\varepsilon, a, b} x^{1/6+\varepsilon}.$$

Cette piste a été développée de manière systématique par Deshouillers-Iwaniec (Deshouillers; Iwaniec 1982a) (voir aussi (Iwaniec 1982)). Ces idées ont prospéré et ont contribué à certains des résultats les plus précis de la théorie analytique des nombres moderne (Bombieri; Friedlander; Iwaniec 1986; Young 2011; Pitt

2013 ; Bettin ; Bui ; Li et al. 2020 ; Pratt ; Robles ; Zaharescu et al. 2020 ; Assing ; Blomer ; Li 2021 ; Maynard 2020).

Les résultats de Deshouillers-Iwaniec (Deshouillers ; Iwaniec 1982a) concernent des sommes d'exponentielles de la forme suivante :

$$\sum_n \sum_c \sum_d \sum_r \sum_s \beta_{n,r,s} g(c,d) e\left(n \frac{(rd)^{-1}}{sc}\right),$$

où  $(\beta_{n,r,s})$  est une suite à support fini, et  $g$  une fonction lisse à support compact. Le cas où  $\beta_{n,r,s} = 1$  si  $n = r = s = 1$  et 0 sinon, c'est-à-dire lorsque que la somme porte uniquement sur  $c$  et  $d$ , correspond à celui des sommes de Kloosterman considérées par Kuznetsov (Kuznetsov 1981), après une sommation de Poisson sur la variable  $d$ . La somme quintuple ci-dessus est une généralisation naturelle à un cadre réaliste, où les variables "lisses"  $c$  et  $d$  sont perturbées par convolution multiplicative.

Le résultat suivant est une généralisation du Theorem 12 de (Deshouillers ; Iwaniec 1982a). Il a été prouvé dans (Drappeau 2017), puis précisé dans (Drappeau ; Pratt ; Radziwiłł 2022).

**Théorème 1** (Drappeau 2017). *Soient  $C, D, N, R, S \geq 1$  et  $q, c_0, d_0 \in \mathbf{N}$  avec  $(q, c_0 d_0) = 1$ . Soit  $(b_{n,r,s})$  une suite à support dans  $[1, N] \times [R, 2R] \times [S, 2S] \cap \mathbf{N}^3$ . Soit  $g : \mathbf{R}_+^5 \rightarrow \mathbf{C}$  une fonction lisse à support compact dans  $[C, 2C] \times [D, 2D] \times (\mathbf{R}_+^*)^3$  qui satisfait la majoration*

$$\frac{\partial^{\nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_5} g}{\partial c^{\nu_1} \partial d^{\nu_2} \partial n^{\nu_3} \partial r^{\nu_4} \partial s^{\nu_5}}(c, d, n, r, s) \ll_{\nu_1, \nu_2, \nu_3, \nu_4, \nu_5} \{c^{-\nu_1} d^{-\nu_2} n^{-\nu_3} r^{-\nu_4} s^{-\nu_5}\}^{1-\varepsilon_0} \quad (1.1)$$

pour un certain  $\varepsilon_0 > 0$  et tous  $\nu_j \geq 0$ . Alors

$$\begin{aligned} & \sum_{\substack{c \\ c \equiv c_0}} \sum_{\substack{d \\ d \equiv d_0 \pmod{q} \\ (qrd, sc)=1}} \sum_n \sum_r \sum_s b_{n,r,s} g(c, d, n, r, s) e\left(n \frac{(rd)^{-1}}{sc}\right) \\ & \ll_{\varepsilon, \varepsilon_0} (qCDNRS)^{\varepsilon + O(\varepsilon_0)} q^{3/2} K(C, D, N, R, S) \|b_{N,R,S}\|_2, \end{aligned}$$

où  $\|b_{N,R,S}\|_2^2 = \sum_{n,r,s} |b_{n,r,s}|^2$ , et

$$\begin{aligned} K(C, D, N, R, S)^2 &= qCS(RS + N)(C + RD) \\ &+ C^{1+4\theta} DS((RS + N)R)^{1-2\theta} \left(1 + \frac{qC}{RD}\right)^{1-4\theta} + D^2 NR. \end{aligned} \quad (1.2)$$

Signalons que dans un article récent, Assing, Blomer et Li (Assing ; Blomer ; Li 2021) prouvent une estimation qui généralise le Theorem 12 de (Deshouillers ; Iwaniec 1982a) dans une autre direction, liée à l'uniformité en  $n$ .

La principale différence par rapport à (Deshouillers ; Iwaniec 1982a, Theorem 12) tient en le fait que les variables  $c, d$ , attachées à un poids lisse, peuvent

être restreintes à des progressions arithmétiques. L'idée principale pour ce faire, introduite par Blomer et Milićević (Blomer ; Milićević 2015), est d'utiliser la possibilité de faire varier le caractère central des formes automorphes pour  $\Gamma_0(q)$  considérées dans (Deshouillers ; Iwaniec 1982a). La réussite de l'argument dépend ensuite d'un choix convenable de séries de Poincaré, dont le développement en série de Fourier autour de pointes bien choisies fera apparaître les sommes de Kloosterman voulues. L'utilité de ce cadre d'étude plus étendu s'est confirmé dans d'autres travaux récents sur les fonctions  $L$  modulaires, où la formule de Kuznetsov joue un rôle : citons (Kiral ; Young 2019), (Petrow ; Young 2020) et (Zacharias 2019).

Une différence plus accessoire est la dépendance en  $\theta$ , qui est explicitée, là où la borne  $\theta \leq 1/4$  de Selberg était utilisé dans (Deshouillers ; Iwaniec 1982a). Cela n'affecte que le second terme dans (1.2). Dans la plupart des applications, ce terme n'est pas le terme limitant, mais ce n'est pas systématiquement le cas, comme nous le verrons au théorème 6 ci-dessous.

Les majorations en moyenne du type du théorème 1 confèrent aux sommes de Kloosterman une place de choix dans la théorie analytique des nombres. Il serait extrêmement souhaitable de disposer d'estimations de la même qualité pour d'autres sommes d'exponentielles, comme les sommes de Birch, les sommes de Kloosterman de rang 3, ou par exemple les sommes d'exponentielles dont la phase paramétrise les solutions de  $n^3 \equiv 2 \pmod{p}$ . Des travaux récents sur ce sujet ont notamment été menés par Buttcane (Buttcane 2022).

### 1.1.2 Valeur moyenne le long des nombres premiers et zéros de Siegel

Les travaux de Blomer et Milićević (Blomer ; Milićević 2015), auxquels il a été fait référence ci-dessus, permettent de majorer des sommes du type

$$\sum_{q \leq x} \chi(q) S(a, b; q), \quad (1.3)$$

dans laquelle  $\chi$  est un caractère de Dirichlet. Il est naturel de se demander si la flexibilité permise par la présence du caractère  $\chi$  permet d'obtenir de nouvelles applications arithmétiques. Un énoncé qu'il serait particulièrement souhaitable de démontrer est

$$\sum_{p \leq x} \frac{S(a, b; p)}{\sqrt{p}} \stackrel{?}{=} o(x/\log x), \quad (1.4)$$

où  $p$  désigne un nombre premier.

La question (1.4) est toujours ouverte. Elle découlerait de la conjecture de Satō-Tate "horizontale" énoncée par Katz (Katz 1980). En 2007, Fouvry et Michel (Fouvry ; Michel 2007) ont obtenu une majoration non-triviale lorsque la condition de primalité est relâchée à la condition que  $n$  appartient à l'ensemble  $P_k$  des entiers

ayant au plus  $k$  facteurs premiers, pour tout nombre  $k$  suffisamment grand. Leur méthode a fait l'objet de plusieurs améliorations, et la meilleure valeur connue d'un  $k$  admissible est  $k = 7$  par Xi (Xi 2018).

La possibilité d'exploiter la dépendance en le caractère  $\chi$  dans (1.3) trouve une application dans la circonstance exceptionnelle où il existerait un caractère de Dirichlet prenant la valeur  $-1$  anormalement souvent sur les nombres premiers. Plus précisément, on sait qu'il existe  $c > 0$  tel que pour chaque  $Q \geq 2$  et chaque caractère quadratique  $\chi$  primitif et non trivial de module  $q \leq Q$ , l'inégalité

$$L(1, \chi) \geq c/\log q$$

a lieu à l'exception d'au plus un caractère  $\chi$ , qui est alors quadratique. On conjecture que ces exceptions n'existent pas, et de fait l'hypothèse de Riemann généralisée pour les fonctions  $L$  de Dirichlet implique l'inégalité  $L(1, \chi) \gg 1/\log \log q$ ; mais aucun progrès substantiel n'a été fait sur cette question depuis qu'elle a été posée. En toute généralité, la borne de Siegel  $L(1, \chi) \gg_\varepsilon q^{-\varepsilon}$  pour tout  $\varepsilon > 0$ , qui est non effective dès que  $\varepsilon < 1/2$ , est la meilleure connue.

Supposons que l'on ait une suite de caractères quadratiques primitifs  $\chi_n \pmod{D_n}$  tels que  $L(1, \chi_n) \log D_n \rightarrow 0$  lorsque  $n \rightarrow \infty$ . Ces caractères fourniraient alors une famille de fonctions multiplicatives qui "imitent" la fonction de Möbius sur les entiers de taille  $D_n^{O(1)}$ , tout en étant d'une complexité analytique contrôlée puisque ce sont des fonctions périodiques de période  $D_n$ .

Cette idée a été poursuivie dans un travail fondateur de Heath-Brown en 1983 (Heath-Brown 1983), et plus récemment dans plusieurs travaux de Friedlander et Iwaniec (Friedlander ; Iwaniec 2004 ; Friedlander ; Iwaniec 2005 ; Friedlander ; Iwaniec 2013). Il résulte de ces travaux que l'existence de caractères de Siegel permet de résoudre plusieurs problèmes qui sont situés au-delà des méthodes d'analyse moderne : le problème des nombres premiers jumeaux (Heath-Brown 1983), ou en progressions arithmétiques de grands modules (Friedlander ; Iwaniec 2003), ou dans les très petits intervalles (Friedlander ; Iwaniec 2004).

Avec James Maynard, nous avons montré dans (Drappeau ; Maynard 2019) que l'existence de caractères exceptionnels entraînerait une majoration non-triviale des sommes de Kloosterman de modules premiers.

**Théorème 2** (Drappeau ; Maynard 2019). *Pour tout  $\varepsilon > 0$ , il existe  $A, B > 0$  tels que pour tout caractère quadratique primitif  $\chi \pmod{D}$  et tout  $x \geq D^A$ , l'on ait*

$$\left| \sum_{p \leq x} \frac{S(1, 1; p)}{\sqrt{p}} \right| \leq \pi(x) (\varepsilon + BL(1, \chi) \log x).$$

Puisque  $|S(1, 1; p)| \leq 2\sqrt{p}$  par la majoration de Weil, l'énoncé précédent n'est non trivial qu'à condition qu'il existe une suite  $\chi_n \pmod{D_n}$  de caractères quadratiques primitifs qui satisfasse  $L(1, \chi_n) \log D_n \rightarrow 0$ , autrement, sous l'existence

de caractères de Siegel-Landau. L'estimation du théorème 2 est alors pertinente dans un intervalle de la forme  $[D^A, D^C]$  avec  $C$  arbitraire mais fixé.

La stratégie de base est celle suivie par Heath-Brown (Heath-Brown 1983) et Friedlander-Iwaniec (Friedlander; Iwaniec 2005; Friedlander; Iwaniec 2004), qui consiste à approcher la fonction de van Mangoldt  $\Lambda(n) = \log * \mu(n)$  par la convolution  $\log * \chi(n)$ . Du point de vue de la complexité analytique, cette dernière est comparable à la fonction diviseur  $1 * 1(n)$ . Le problème se ramène donc essentiellement à l'étude des sommes

$$\sum_{N < n \leq 2N} \sum_{M < m \leq 2M} S(1, 1, mn).$$

Lorsque  $M < N^{1-\varepsilon}$  ou  $N < M^{1-\varepsilon}$ , les estimations de Deshouillers-Iwaniec, du même type que le théorème 1, fournissent une majoration acceptable. Le principal obstacle est constitué par le cas  $M = N$ . Dans ce cas, nous ne parvenons pas à tirer parti des compensations dans le signe de la somme de Kloosterman. L'idée cruciale est de remplacer  $\log(n/d)$  dans la convolution  $\Lambda(n) = \sum_{d|n} \mu(d) \log(n/d)$ , par  $\log(\sqrt{n}/d)$ . Ceci n'a aucun effet sur le reste des arguments, mais a l'avantage d'atténuer la contribution des  $d$  de l'ordre de  $\sqrt{n}$ , qui correspondent au cas problématique ci-dessus. Le reste de l'argument reprend une méthode de Fouvry-Michel (Fouvry; Michel 2003), basée sur une idée de Hooley (Hooley 1964) et la coïncidence numérique  $8 < 3\pi$ , qui prouve que  $|S(1, 1, n)|$  devient négligeable, en moyenne sur  $n$ , à mesure que  $n$  possède de plus en plus de facteurs premiers. Cette idée se retrouve dans d'autres travaux dans le domaine, notamment un travail récent de Xi (Xi 2020) sur la non coïncidence entre sommes de Kloosterman et valeurs propres de Hecke-Maass.

## 1.2 Convolution décalée avec la fonction diviseur

### 1.2.1 Terme d'erreur dans le problème de Titchmarsh

Les majoration de sommes d'exponentielles du type du théorème 1 trouvent naturellement des applications à l'estimation de "convolutions décalées"

$$\sum_{n \leq x} f(n)g(n+1), \tag{1.5}$$

où  $f$  et  $g$  sont supposées multiplicatives, ou ayant un lien fort avec une fonction multiplicative, comme la fonction indicatrice des nombres premiers.

Une approche dans ce problème consiste à approcher  $f$  et  $g$  par des convolutions de Dirichlet non triviales, ce qui revient à factoriser  $n = ab$  et  $n+1 = cd$ , puis à essayer de compter les solutions de l'équation  $cd - ab = 1$  affectées de certains coefficients (Duke; Friedlander; Iwaniec 1994). Dans les cas les plus

favorables, les coefficients varient de manière lisse avec  $a, b, c, d$ , et par analyse de Fourier, on se ramène à des sommes d'exponentielles qui sont du même type que les sommes de Kloosterman. Ceci explique que les estimations basées sur les sommes de Kloosterman trouvent un intérêt particulier en théorie analytique des nombres (Kowalski 2003).

La fonction diviseur  $\tau = 1 * 1$  peut être vue comme la fonction arithmétique la plus simple qui ne soit pas une perturbation d'une fonction constante. Les fonctions diviseurs d'ordre supérieur,  $\tau_k = 1 * \dots * 1$  ( $k$  fois) en sont des généralisations naturelles. La fonction indicatrice des nombres premiers  $n \mapsto 1_{\mathcal{P}}(n)$ , quant à elle, est vue comme une forme de limite des fonctions  $\tau_k$ , au sens où si une question concernant une fonction  $f$  peut être résolue pour  $f = \tau_k$ ,  $k$  arbitraire, avec une certaine uniformité en  $k$ , alors on peut espérer la résoudre pour  $f = 1_{\mathcal{P}}$ . Cette forme de vague classification des fonctions multiplicatives se matérialise dans des « identités combinatoires », les plus connues étant celles de Linnik, Vaughan et Heath-Brown, et dont il sera question dans cette section.

Le problème des diviseurs de Titchmarsh (Titchmarsh 1930) consiste à estimer la somme

$$T(x) := \sum_{p \leq x} \tau(p-1).$$

C'est un cas particulier du problème (1.5). La meilleure estimation connue

$$T(x) = c_1 x + c_2 \operatorname{li}(x) + O\left(\frac{x}{(\log x)^A}\right) \quad (1.6)$$

est due indépendamment à Fouvry (Fouvry 1985) et Bombieri-Friedlander-Iwaniec (Bombieri; Friedlander; Iwaniec 1986). La forme du terme d'erreur découle de l'utilisation du théorème des nombres premiers en progressions arithmétiques, et de l'utilisation du théorème de Siegel-Walfisz. La question se pose naturellement de savoir si une hypothèse comme celle de Riemann sur la fonction  $\zeta$  permet d'obtenir un terme d'erreur en puissance de  $x$ , comme c'est le cas dans le théorème des nombres premiers. Le résultat suivant, obtenu dans (Drappeau 2017), montre que la réponse est oui.

**Théorème 3** (Drappeau 2017). *Supposons l'hypothèse de Riemann généralisée aux fonctions  $L$  de Dirichlet. Alors il existe  $\delta > 0$  tel que*

$$T(x) = c_1 x + c_2 \operatorname{li}(x) + O(x^{1-\delta}) \quad (x \geq 2).$$

Ce résultat est celui qui avait motivé originellement le théorème 1. Une valeur effective de  $\delta$  a été obtenue par Tang (Tang 2020). Il serait naturel de s'attendre à ce que le terme d'erreur optimal soit  $x^{1/2}$ , mais la meilleure valeur de  $\delta$  que la méthode permettrait d'obtenir en est loin.

La stratégie est similaire à celle de Fouvry (Fouvry 1985) et Bombieri-Friedlander-Iwaniec (Bombieri; Friedlander; Iwaniec 1986), mais la contrainte d'obtenir un



terme d'erreur en puissance de  $x$  nous impose d'éviter le recours à certaines techniques, de crible notamment. En tenant compte de ces contraintes, il apparaît que la somme d'exponentielles que l'on doit estimer *in fine* a une forme semblable à

$$\sum_{c,d,n,r,s} b_{n,r,s} g(c,d) e^{\left( \frac{n(sc)^{-1}}{rd} + \frac{(cd)^{-1}}{q} \right)} \quad (1.7)$$

où  $q \geq 1$  est un entier relativement petit, de l'ordre de  $x^\varepsilon$ . Les travaux précédents (Fouvry 1985 ; Bombieri ; Friedlander ; Iwaniec 1986) utilisaient diverses réductions pour se réduire au cas  $q = 1$ , ce qui amenait la somme dans le giron des estimations originelles de Deshouillers-Iwaniec (Deshouillers ; Iwaniec 1982a). Mais cette réduction se fait au coût d'un terme d'erreur en  $O(x e^{-c\sqrt{\log x}})$  dans le meilleur cas. En revanche, munis à présent du théorème 1, nous pouvons travailler directement avec la somme originelle (1.7) en séparant les sommes sur  $c$  et  $d$  suivant les classes de congruence modulo  $q$ , avec des termes d'erreurs en puissance de  $x$  tout au long de l'argument. C'est ce qui mène à la preuve du théorème 3.

### 1.2.2 Problème de Titchmarsh pour d'autres fonctions multiplicatives

Considérons la somme (1.5) avec  $f = \tau$ . Nous savons en obtenir un développement asymptotique pour  $g = \mathbf{1}_{\mathcal{P}}$ , comme on l'a vu dans le paragraphe précédent. Cependant, en 2017, rien ne semblait connu pour des fonctions considérées comme étant de difficulté équivalente ou moindre, comme  $g = \mathbf{1}_{\mathcal{B}}$ , la fonction indicatrice de l'ensemble  $\mathcal{B}$  des entiers qui sont sommes de deux carrés.

Plus généralement, on s'attendait à pouvoir traiter le cas de la fonction  $g = \tau_z$  pour  $z \in \mathbf{C}$ , car on considère celle-ci comme étant de la même "difficulté analytique" que la fonction indicatrice des nombres premiers : par exemple, le prolongement holomorphe de sa série génératrice  $s \mapsto \zeta(s)^z$  possède des singularités aux zéros de la fonction  $\zeta$ , qui sont de nature logarithmique.

Nous avons résolu ce cas dans un travail commun avec Berke Topalogullari (Drappeau ; Topalogullari 2019). La "difficulté analytique" à laquelle il a été fait allusion se matérialise, dans notre résultat, par l'intermédiaire d'une hypothèse de périodicité sur le comportement de la fonction  $g$  le long de la suite des nombres premiers.

**Théorème 4** (Drappeau ; Topalogullari 2019). *Supposons que la fonction multiplicative  $g : \mathbf{N} \rightarrow \mathbf{C}$  satisfait les hypothèses suivantes :*

- *Il existe  $D \in \mathbf{N}$  tel que si  $p, q$  sont deux nombres premiers avec  $p \equiv q \pmod{D}$ , alors  $g(p) = g(q)$ .*
- *Il existe  $A > 0$  tel que  $|g(n)| \leq \tau(n)^A$  pour tout  $n \in \mathbf{N}$ .*

Alors pour tout  $B > 0$ , l'on a

$$\sum_{1 < n \leq x} g(n)\tau(n-1) = 2 \sum_{\substack{\chi \text{ primitif} \\ \text{cond}(\chi)|D}} \sum_{\substack{q \leq \sqrt{x} \\ \text{cond}(\chi)|q}} \frac{1}{\varphi(q)} \sum_{\substack{q^2 \leq n \leq x \\ (n,q)=1}} g(n)\chi(n) + O_{D,A,B}(x/(\log x)^B). \quad (1.8)$$

La somme sur  $n$  dans le membre de droite est facile à estimer en utilisant des résultats classiques sur les régions sans zéros de fonctions  $L$  de Dirichlet. Fouvry et Tenenbaum (Fouvry ; Tenenbaum 2022) ont récemment obtenu un résultat qui contient le théorème (1.8) et fournit des applications nouvelles sur le problème de convolution décalée (1.5) pour certaines fonctions additives  $f, g$ .

Par rapport aux arguments qui sous-tendent le problème de Titchmarsh (1.6), le point que nous améliorons est la partie communément appelée “identité combinatoire”. C’est un argument qui a beaucoup été étudié dans le cas des nombres premiers (Vinogradov, Linnik, Gallagher, Vaughan, Heath-Brown) ; nous renvoyons à l’article de survol (Ramaré 2013) pour une discussion plus détaillée de l’histoire de ces techniques. L’élément nouveau dans le théorème 4 est une identité combinatoire valable pour une fonction multiplicative  $g$  satisfaisant les conditions du théorème 4. Nous obtenons en fait deux identités.

La première est une identité de type Heath-Brown (Heath-Brown 1982) pour  $\tau_\alpha$  avec  $\alpha \in \mathbf{Q}$ . La possibilité d’obtenir une telle formule était suggérée par une identité due à Vaughan pour  $\tau_{1/2}$  qui nous a été aimablement communiquée par Hugh Montgomery par l’intermédiaire d’Olivier Ramaré. Cette identité pour  $\tau_{1/2}$ , que nous ne reporterons pas ici, apparaîtra dans le volume II de *Multiplicative Number Theory*, mais nous n’avons pas réussi à en tirer parti pour l’appliquer au problème de Titchmarsh. Après quelques tâtonnements, nous avons trouvé une généralisation convenable de l’identité de Heath-Brown. Pour reprendre le cas particulier simple de la fonction  $\tau_{1/2}$ , elle prend la forme

$$\sum_{m \geq 4} a_m (\zeta(s)^{1/2} M_x(s) - 1)^m \zeta(s)^{1/2} = \zeta(s)^{1/2} + \sum_{\ell \geq 1} b_\ell \zeta(s)^\ell M_x(s)^{2\ell-1}, \quad (1.9)$$

où  $(a_m), (b_\ell)$  sont des nombres complexes,  $\zeta$  est la fonction de Riemann et

$$M_x(s) = \sum_{n \leq x^{1/4}} \tau_{-1/2}(n) n^{-s}$$

est une approximation de  $\zeta(s)^{-1/2}$ . La série de Dirichlet formée par le membre de gauche de (1.9) ne comporte aucun terme d’indice  $\leq x$ , tandis que le membre de droite fait apparaître  $\zeta^{1/2}$  d’une part, et des puissances entières de  $\zeta$  et de la série tronquée  $M_x$  d’autre part. Les coefficients  $a_m$  et  $b_\ell$  sont obtenus en trouvant d’abord un polynôme de la forme  $P(X) = 1 + XQ(X^2)$ ,  $Q \in \mathbf{C}[X]$ , qui s’annule en 1 à l’ordre 4, par exemple  $P(X) = 1 - \frac{35}{16}X + \frac{35}{16}X^3 - \frac{21}{16}X^5 + \frac{5}{16}X^7$ . On substitue ensuite formellement  $X = \zeta^{1/2}M$  dans l’égalité  $P(X) = 1 + XQ(X^2)$ , puis on

multiplie par  $\zeta^{1/2}$ .

Il est moins évident de voir que cette méthode se généralise à la fonction  $\tau_\alpha$  pour  $\alpha = u/v$  rationnel, avec un bon contrôle en termes de  $v$ , sur la taille des coefficients  $b_\ell$ . Ce dernier point est crucial. Nous prouvons que c'est effectivement le cas ; la taille de  $b_\ell$  est alors essentiellement contrôlée par la taille (archimédienne) de  $|u/v|$ . Cela nous a permis, dans un premier temps, d'estimer la somme

$$\sum_{n \leq x} \tau_\alpha(n) \tau(n+1)$$

uniformément en  $\alpha \in \mathbf{Q}$ ,  $|\alpha| \ll 1$ ,  $\text{denom}(\alpha) \ll (\log x)^A$ , avec un terme d'erreur en  $O(x/(\log x)^A)$ .

Pour aborder le cas général  $\tau_z$ ,  $z \in \mathbf{C}$ ,  $|z| \ll 1$ , la seconde idée que nous utilisons est tout simplement une interpolation de Lagrange sur la fonction polynomiale

$$z \mapsto \sum_{n \leq x} \tau_z(n) \tau(n+1).$$

La simplicité de cette approche cache, bien entendu, le problème de maîtriser la dépendance en  $x$  dans l'étape d'interpolation. On estime d'abord la contribution des termes de degrés  $\geq C \log \log x$  par  $O_C(x/(\log x)^{C+O_z(1)})$  pour  $C \geq 7$ .<sup>1</sup> L'ordre de grandeur  $\log \log x$  provient de la valeur typique du nombre de facteurs premiers des entiers inférieurs à  $x$  (théorème de Hardy-Ramanujan). Ensuite, un choix naturel de points d'échantillonnage rationnels de dénominateurs  $O(\log \log x)$ , dans l'interpolation de Lagrange, mène à un terme d'erreur de la forme

$$[\text{terme d'erreur aux points d'échantillonnage}] \times \exp([\text{degré}] \log \log x), \quad (1.10)$$

autrement dit ici  $\frac{x}{(\log x)^A} \times \exp(C \log \log x)$ , avec  $A$  arbitrairement grand. Finalement, nous obtenons pour  $z$  quelconque une estimation avec terme d'erreur

$$O_{A,C} \left( \frac{x}{(\log x)^{C+O_z(1)}} + \frac{x}{(\log x)^{A-C}} \right),$$

et un choix convenable de  $A$  et  $C$  conclut l'argument. La réussite de cette approche dépend de façon essentielle de la dépendance en le degré dans (1.10).

La seconde preuve que nous avons obtenue présente beaucoup de similarités avec les travaux originels de Vinogradov (Vinogradov 1937). Elle ne repose pas sur une étape d'interpolation, et est donc un peu plus simple dans sa mise en œuvre. Le point de départ reprend l'identité de Linnik (Linnik 1963, Cha-

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1. L'exposant n'est pas optimal asymptotiquement mais il nous suffit qu'il tende vers l'infini avec  $C$ .

pitre VIII), et consiste à écrire

$$\zeta^z = (1 + (\zeta - 1))^z = \sum_{j \geq 0} \binom{z}{j} (\zeta - 1)^j.$$

Lorsqu'on développe le membre de droite en série de Dirichlet, le coefficient d'un entier  $n$  fait intervenir des indices  $j$  de taille  $O(\log n)$ , ce qui est trop élevé en pratique. L'idée est de factoriser la partie  $y$ -friable des entiers au niveau des séries de Dirichlet, c'est-à-dire d'écrire

$$\zeta = \zeta_y M_y, \quad \zeta_y(s) = \prod_{p \leq y} (1 - p^{-s})^{-1}, \quad M_y(s) = \prod_{p > y} (1 - p^{-s})^{-1}.$$

On choisit alors  $y = x^\varepsilon$  pour une petite valeur de  $\varepsilon > 0$  à déterminer. En appliquant la stratégie de Linnik à la partie qui concerne  $M_y$ , on obtient

$$M_y^z = \sum_{j \geq 0} \binom{z}{j} (M_y - 1)^j,$$

et si l'on souhaite détecter le coefficient d'un entier  $n \leq x$ , la somme sur  $j$  peut être tronquée à  $j < 1/\varepsilon$ . Puis pour conclure nous réintroduisons les divers facteurs  $\zeta_y$ , ce qui n'a pas d'impact sur nos arguments dès que  $\varepsilon$  est suffisamment petit. En pratique  $\varepsilon < 1/4$  convient. Ceci est dû au fait que la fonction indicatrice des entiers friables bénéficie de bonnes propriétés de factorisation au sens de la convolution de Dirichlet. Dans le cas de la fonction diviseurs, nous obtenons par exemple la décomposition

$$\tau_z(n) = \sum_{0 \leq \ell \leq 3} c_{\ell, z} \sum_{\substack{n = n_1 n_2 \\ p | n_1 \implies p \leq x^{1/4}}} \tau_{z-\ell}(n_1) \tau_\ell(n_2), \quad (n \leq x), \quad (1.11)$$

où

$$\begin{aligned} c_{0, z} &= 1 - \frac{11}{6}z + z^2 - \frac{1}{6}z^3, & c_{1, z} &= 3z - \frac{5}{2}z^2 + \frac{1}{2}z^3, \\ c_{2, z} &= -\frac{3}{2}z + 2z^2 - \frac{1}{2}z^3, & c_{3, z} &= \frac{1}{3}z - \frac{1}{2}z^2 + \frac{1}{6}z^3. \end{aligned}$$

Partant de la formule (1.11), il est relativement simple de poursuivre les arguments de Bombieri-Fouvry-Friedlander-Iwaniec. On vérifie que  $\sum_\ell c_{\ell, z} = 1$  (ce qui est cohérent avec (1.11) en  $n = 1$ ) et que  $\sum_\ell \ell c_{\ell, z} = z$  (ce qui est cohérent avec (1.11) lorsque  $n = p > x^{1/4}$ ).

L'uniformité par rapport au paramètre  $z$  permet d'exploiter la formule de Cauchy pour en déduire des formules asymptotiques pour les entiers ayant exactement  $k$  facteurs premiers.

**Corollaire 1** (Drappeau ; Topacogullari 2019). *Pour  $1 \leq k \ll \log \log x$ , l'on a*

$$\sum_{\substack{n \leq x \\ \omega(n)=k}} \tau(n-1) = \frac{x}{\log x} \sum_{j=1}^J \frac{P_{j,k}(\log \log x)}{(\log x)^j} + O_{J,\varepsilon} \left( x \frac{(\log \log x)^k}{k!(\log)^{J-1+\varepsilon}} \right),$$

où pour tout  $j \geq 1$ ,  $P_{j,k}$  est un polynôme de degré au plus  $k-1$ .

Pour  $k=1$ , l'on retrouve le théorème de Bombieri-Fouvry-Friedlander-Iwaniec avec le même terme d'erreur. On note que le nombre des entiers  $n \leq x$  tels que  $\omega(n) > C \log \log x$  est  $\ll_C x(\log x)^{C-1-C \log C}$ . La condition sur  $k$  dans le corollaire 1 est donc naturelle puisqu'elle correspond à la région dans laquelle le terme d'erreur, qui provient d'une utilisation du théorème de Siegel-Walfisz, est pertinent.

Le cas  $z=1/2$  dans (1.11) nous a permis de répondre à la question qui nous motivait initialement.

**Corollaire 2** (Drappeau ; Topacogullari 2019). *Si  $\mathcal{B}$  désigne l'ensemble des nombres qui s'écrivent comme somme de deux carrés, alors pour un certaine suite de nombres  $(\beta_j)_{j \geq 0}$  et tout  $J \geq 0$ , l'on a*

$$\sum_{n \in \mathcal{B} \cap [2,x]} \tau(n-1) = x(\log x)^{1/2} \sum_{j=0}^{J-1} \frac{\beta_j}{(\log x)^j} + O_J(x(\log x)^{1/2-J}).$$

Ces résultats sont encore valables lorsque  $n-1$  est remplacé par  $n-h$  avec  $h \in \mathbf{Z}$ ,  $h \neq 0$  et  $|h| \leq x^\delta$  pour un certain  $\delta > 0$ .

Dans la continuation de ce sujet, on souhaiterait pouvoir estimer la somme au membre de gauche de (1.8) même dans des situations où la valeur moyenne de  $f$  est négligeable devant toute puissance négative de  $\log x$ . Le terme d'erreur donné par le théorème 4 n'est alors plus négligeable devant le terme principal. Un cas particulier important, celui des entiers friables, a pu être traité dans (Fouvry ; Tenenbaum 1990 ; Drappeau 2015), en particulier grâce à un travail de Harper (Harper 2012). Il est naturel de se poser la même question pour l'ensemble des entiers ayant  $k$  facteurs premiers lorsque  $k/\log \log x \rightarrow \infty$ , autrement dit, un analogue du corollaire 1 pour ces grandes valeurs de  $k$ . La fonction caractéristique de ces entiers bénéficie de bonnes propriétés de factorisation, et il est plausible que certaines des techniques utilisées dans (Drappeau 2015) puissent être adaptées pour traiter ce cas. Le problème de remplacer la fonction  $\tau$  dans le membre de gauche de (1.8) par une fonction plus générale semble, en revanche, actuellement hors de portée. Le cas le plus simple après la fonction  $\tau$  est celui de la fonction diviseurs  $\tau_3$  ; l'estimation asymptotique de  $\sum_{n \leq x} \tau_3(n)\tau_3(n+1)$  est une question ouverte majeure en théorie analytique des nombres, sur laquelle essentiellement aucune approche n'est effective pour le moment.

## 1.3 Zéros de fonctions $L$ de Dirichlet

Dans cette section on suppose que l'hypothèse de Riemann généralisée aux fonctions  $L$  de Dirichlet est vraie.

Le problème dont il est question ici concerne la répartition verticale des zéros de fonctions  $L$  de Dirichlet. Étant donné un caractère de Dirichlet primitif  $\chi$ , on s'intéresse au multi-ensemble des zéros de sa fonction  $L$  dans la bande critique,

$$Z_\chi := \{\gamma \in \mathbf{R}, L(\frac{1}{2} + i\gamma, \chi) = 0, \},$$

puis aux multi-ensembles

$$Z_q := \bigcup_{\substack{\chi \pmod{q} \\ \text{primitif}}} Z_\chi,$$

$$Z_{\leq Q} := \bigcup_{q \leq Q} Z_q.$$

Par des méthodes dues notamment à Riemann et Weyl, il est connu que l'on a

$$\text{card}\{\gamma \in Z_\chi \cap [0, T]\} \sim \frac{T}{2\pi} \log\left(\frac{q}{2\pi}\right), \quad (T \rightarrow \infty, T \leq q^{o(1)}).$$

Au vu de cette estimation, il est donc naturel de s'attendre à trouver, à distance  $\asymp 1$  de l'axe réel, un nombre de l'ordre de  $\log q$  éléments de  $Z_\chi$ . Peu de choses sont connues pour un caractère  $\chi$  donné, lorsque  $q \rightarrow \infty$ . En revanche, on dispose de conjectures précises lorsque l'on fait varier  $\chi$  parmi tous les caractères de Dirichlet de module  $q$ , autrement dit, lorsqu'on étudie l'ensemble  $Z_q$ .

Étant donnée une fonction  $\phi : \mathbf{R} \rightarrow \mathbf{C}$  et un entier  $q \geq 1$ , on s'intéresse à la quantité

$$W(\phi, q) := \sum_{\substack{\chi \pmod{q} \\ \text{primitif}}} \sum_{\gamma \in Z_\chi} \phi\left(\frac{\gamma}{2\pi} \log q\right).$$

Notons  $\psi(q)$  le nombre de caractères primitifs modulo  $q$ .

**Conjecture 1.** *Supposons que la transformée de Fourier  $\hat{\phi}$  est à support compact. Alors l'on a*

$$W(\phi, q) \sim \psi(q) \int \phi(\xi) d\xi \tag{1.12}$$

dès que le membre de droite tend vers l'infini.

Cette conjecture est un cas particulier d'une conjecture plus vaste énoncée par Katz-Sarnak (Katz ; Sarnak 1999), qui fait un lien conjecturel avec la répartition des valeurs propres de matrices aléatoires. Ces conjectures trouvent leur source dans un article de Montgomery (Montgomery 1973) sur la corrélation de paires de zéros de la fonction  $\zeta$  de Riemann.

La conjecture 1 peut être établie sous des conditions relatives à la taille du support de la transformée de Fourier de  $\phi$ . Le résultat suivant, qui est établi par exemple dans (Sica 1998), découle relativement rapidement de l'hypothèse de Riemann généralisée.

**Théorème 5.** *Sous l'hypothèse de Riemann généralisée aux fonctions  $L$  de Dirichlet, l'équivalence (1.12) est vraie lorsque  $\hat{\phi}$  est à support dans  $]-2, 2[$ .*

Dans un travail avec Kyle Pratt et Maksym Radziwiłł (Drappeau ; Pratt ; Radziwiłł 2022), nous étendons la condition sur le support de  $\hat{\phi}$ , au prix d'effectuer une moyenne supplémentaire sur le module  $q$ .

**Théorème 6** (Drappeau ; Pratt ; Radziwiłł 2022). *Sous l'hypothèse de Riemann généralisée aux fonctions  $L$  de Dirichlet, l'on a*

$$\sum_{q \leq Q} W(\phi, q) \sim \left( \sum_{q \leq Q} \psi(q) \right) \int \phi(\xi) d\xi$$

lorsque  $\hat{\phi}$  est à support dans  $]-2 - \delta, 2 + \delta[$  avec  $\delta = 50/1093$ .

L'hypothèse de Riemann peut être retirée dans cet énoncé, sous réserve de remplacer  $\gamma$  par  $-i(\rho - 1/2)$  dans la définition de  $W(\phi, q)$ .

Le principal intérêt du théorème 6 est que l'on dépasse le seuil 2 sur la taille du support de  $\hat{\phi}$ , ce qui ne pouvait être déduit directement d'une application de l'hypothèse de Riemann comme au théorème 5. La situation est analogue à celle de compter des nombres premiers en progressions arithmétiques de modules  $q$ , où une utilisation triviale de l'hypothèse de Riemann généralisée mène à une condition  $q \leq x^{1/2-\varepsilon}$ , que l'on peut dépasser inconditionnellement dans certains cas (Fouvry 1985 ; Bombieri ; Friedlander ; Iwaniec 1986) à condition d'effectuer une moyenne supplémentaire sur  $q$ . Cette analogie est plus que formelle, puisque les mêmes outils sont au cœur de la preuve dans les deux cas.

La méthode commence par une application de la formule explicite de Riemann-Weyl, qui prend essentiellement la forme

$$\sum_{q \leq Q} W(\phi, q) \approx \left( \sum_{q \leq Q} \psi(q) \right) \int \phi(\xi) d\xi + \sum_{q \leq Q} \sum_{\substack{\chi \pmod{q} \\ \text{primitif}}} \sum_p \chi(p) \hat{\phi} \left( \frac{\log p}{\log q} \right).$$

Par l'orthogonalité des caractères de Dirichlet, la dernière somme peut être bien évaluée à condition de savoir estimer la somme

$$\sum_{q \leq Q} \psi(q) \sum_{p \equiv 1 \pmod{q}} \hat{\phi} \left( \frac{\log p}{\log q} \right).$$

Nous sommes donc réduit à compter des nombres premiers en progressions arithmétiques. Le problème revient fondamentalement au suivant : prouver qu'il

existe des réels  $\kappa > 2$  et  $\eta > 0$ , tels que l'on ait

$$\sum_{q \leq Q} \left( \sum_{\substack{p \leq X \\ p \equiv 1 \pmod{q}}} 1 - \frac{\text{li}(X)}{\varphi(q)} \right) \ll Q^{1-\eta} \sqrt{X}, \quad \text{lorsque } X = Q^\kappa. \quad (1.13)$$

Cette question peut sembler paradoxale. En effet, dans les estimations de type Bombieri-Vinogradov, le domaine  $X > Q^{2+\varepsilon}$  est le plus facile d'accès, et tout le problème est de diminuer la valeur de  $X$ , tandis qu'ici l'on souhaite prendre  $X$  légèrement plus grand que  $Q^2$ . Le principal enjeu réside en réalité dans le terme d'erreur, que nous souhaitons rendre négligeable par rapport à  $Q\sqrt{X}$ , tandis que dans l'étude des nombres premiers en progressions arithmétiques, la borne triviale est plutôt de l'ordre de  $X$ . Ainsi, lorsque  $X$  est légèrement plus grand que  $Q^2$ , le terme d'erreur que nous souhaitons atteindre est de l'ordre de  $X^{1-\eta}$ , tandis que le théorème de Bombieri-Vinogradov fournit un majorant de l'ordre de  $X/(\log X)^A$ .

La "preuve de concept" du théorème 6 est fournie par l'estimation du théorème 3. Ce théorème prouve que si l'hypothèse de Riemann généralisée est vraie, alors l'on a

$$\sum_{q \leq Q} \left( \sum_{\substack{p \leq X \\ p \equiv 1 \pmod{q}}} 1 - \frac{\text{li}(X)}{\phi(q)} \right) \ll X^{1-\eta} \quad (X = Q^2)$$

pour un certain  $\eta > 0$ , ce qui constitue une majoration de type (1.13) pour  $\kappa = 2$ . Tout le travail consiste à montrer que l'on peut augmenter légèrement la valeur de  $\kappa$  tout en gardant un terme d'erreur de la même qualité.

Le fait que nous pouvons nous passer de l'hypothèse de Riemann généralisée dans le théorème 6 est lié au fait que les caractères  $\chi$  qui interviennent dans la définition de  $W(\phi, q)$  sont tous primitifs. Nous pouvons alors retirer, dans l'ensemble des arguments, la contribution des caractères de conducteurs au plus  $x^\delta$  pour  $\delta > 0$  suffisamment petit, qui constituent la seule obstruction à obtenir un gain en puissance de  $X$  en l'absence d'une hypothèse de Riemann.

## 1.4 Valeurs friables de polynômes quadratiques

Une autre application emblématique des estimations de Deshouillers et Iwaniec (Deshouillers; Iwaniec 1982a) concerne le niveau de répartition des polynômes quadratiques. Cela tient à ce que les racines d'un polynôme quadratique donnent naturellement lieu à une somme d'exponentielle du même type que les sommes de Kloosterman.



Le problème est celui d'estimer la somme

$$\Delta(x, Q, \lambda) = \sum_{q \leq Q} \lambda_q \left( \sum_{\substack{n \leq x \\ n^2 \equiv -1 \pmod{q}}} 1 - \text{TP}(x, q) \right),$$

où  $(\lambda_q)$  est une suite de nombres complexes de modules au plus 1, et  $\text{TP}(x, q)$  est le terme principal attendu, c'est-à-dire

$$\text{TP}(x, q) := \frac{x\rho(q)}{q}, \quad \rho(q) = \text{card}\{m \pmod{q}, m^2 \equiv -1 \pmod{q}\}.$$

Il est permis de conjecturer que pour chaque  $\varepsilon > 0$ , il existe  $\delta > 0$  tel que

$$\Delta(x, Q, \lambda) = O(x^{1-\delta}), \quad (Q \leq x^{2-\varepsilon}),$$

uniformément en  $\lambda$ . En pratique, l'on a  $\sum_{q \leq Q} \text{TP}(x, q) \ll x \log(2Q)$ , et toute majoration de  $\Delta(x, q, \lambda)$  négligeable devant  $x \log Q$  est intéressante. Ce problème est de plus en plus difficile à mesure que la valeur de  $Q$  est de plus en plus grande.

Le résultat suivant a été obtenu dans travail en commun avec Régis de la Bretèche (La Bretèche ; Drappeau 2020).

**Théorème 7** (La Bretèche ; Drappeau 2020). *Supposons que la suite  $(\lambda_q)$  soit bien factorisable au sens d'Iwaniec. Pour tout  $\varepsilon > 0$ , il existe  $\delta > 0$  tel que*

$$|\Delta(x, Q, \lambda)| \ll x^{1-\delta} \quad \text{pour } Q \leq x^{1+25/178-\varepsilon}.$$

Ceci améliore quantitativement un résultat obtenu par Iwaniec en 1978 (Iwaniec 1978), c'est-à-dire avant la "kloostermanie". Une telle majoration a des conséquences sur les problèmes qui concernent la factorisation du polynôme  $n^2 + 1$  ; par exemple, Iwaniec utilise son estimation de  $|\Delta(x, Q, \lambda)|$  pour démontrer, à l'issue d'un travail conséquent, que le polynôme  $n^2 + 1$  prend infiniment souvent des valeurs ayant au plus deux facteurs premiers. C'est une approximation de la conjecture de Landau que  $n^2 + 1$  est premier pour une infinité de valeurs de  $n$ .

Nous avons appliqué le théorème 7 à un autre problème, issu de travaux de Tchébychev. La question est de trouver des valeurs de  $n$  telles que  $n^2 + 1$  a un grand facteur premier. Le meilleur résultat avant 2017 était dû à Deshouillers et Iwaniec (Deshouillers ; Iwaniec 1982b), qui établissent la minoration

$$P^+ \left( \prod_{x \leq n < 2x} (n^2 + 1) \right) \gg x^{1.2024},$$

où  $P^+(m)$  désigne le plus grand facteur premier de  $m$ , avec la convention  $P^+(1) = 1$ . Dans le cadre des travaux qui nous ont mené au théorème 7, nous avons été en mesure de mettre à jour ce qui est connu sur ce sujet, à la lumière des progrès

réalisés sur la conjecture de Selberg sur les petites valeurs propres du laplacien hyperbolique sur les surfaces de congruence. La dépendance explicite en  $\theta$  dans l'estimation (1.2), en particulier, nous a permis de déduire le résultat suivant.

**Théorème 8** (La Bretèche ; Drappeau 2020). *Nous avons*

$$P^+\left(\prod_{x \leq n < 2x} (n^2 + 1)\right) \gg x^{1.2182}.$$

Ce résultat a été ultérieurement amélioré de façon significative par Merikoski (Merikoski 2022) : l'on sait maintenant que

$$P^+\left(\prod_{x \leq n < 2x} (n^2 + 1)\right) \gg x^{1.279}.$$

Une des nouveautés du travail de Merikoski est l'utilisation d'un crible minorant du à Harman dans l'argument de Deshouillers-Iwaniec, qui a eu pour effet d'amplifier considérablement l'efficacité des majorations de sommes d'exponentielles sur ce problème.

Une seconde partie du travail avec Régis de la Bretèche (La Bretèche ; Drappeau 2020), qui en constituait en fait la principale motivation, consiste en la construction d'un crible majorant pour l'ensemble des entiers friables. Un entier  $n$  est dit  $y$ -friable si  $P^+(n) \leq y$ . C'est un ensemble d'entiers qui intervient de façon récurrente en théorie multiplicative des nombres (Granville 2008 ; Hildebrand ; Tenenbaum 1993 ; Moree 2014), et sur lequel mes travaux de thèse étaient centrés. On note

$$\begin{aligned} S(x, y) &:= \{n \leq x, p \mid n \implies p \leq y\}, \\ \Psi(x, y) &:= \text{card } S(x, y). \end{aligned}$$

Pour chaque réel  $u \geq 1$ , il est connu que

$$\Psi(x, x^{1/u}) \sim x\rho(u) \quad (x \rightarrow \infty),$$

où  $\rho : \mathbf{R}_{\geq 1} \rightarrow [0, 1]$  est la fonction de Dickman. Lorsque  $u \rightarrow \infty$ , l'on a asymptotiquement  $\rho(u) = u^{-(1+o(1))u}$ . De façon analogue aux nombres premiers, nous aimerions être en mesure de savoir compter asymptotiquement le cardinal des entiers  $n \leq x$  tel que  $n^2 + 1$  soit  $y$ -friable. Ceci peut être vu comme une instance du problème (1.5) lorsque  $f$  est la fonction caractéristique des entiers  $y$ -friables, et  $g$  est la fonction caractéristique des entiers carrés ; ou bien encore comme une version "duale" de la conjecture de Landau. L'on conjecture que

$$C(x, x^{1/u}) := \text{card}\{n \leq x : n^2 + 1 \text{ est } x^{1/u}\text{-friable}\} \sim x\rho(2u),$$

sur la base du fait que l'entier  $n^2 + 1$  est typiquement d'ordre de grandeur  $x^2$ ,

et de l'heuristique que cet entier, qui n'a pas de diviseur fixé, prend des valeurs friables à une fréquence similaire à celle des entiers de même taille.

Le problème de majorer asymptotiquement la quantité  $C(x, x^{1/u})$  est plus simple. La meilleure estimation connue, due à Khmyrova (Hmyrova 1964), fournit

$$C(x, x^{1/u}) \ll x\rho(u)$$

pour tout  $\varepsilon > 0$  et  $u \ll \log x / \log \log x$ , en se basant sur estimation élémentaire de  $\Delta(x, Q, \lambda)$  pour  $Q \leq x^{1-\varepsilon}$ . Dans l'article (La Bretèche ; Drappeau 2020), nous construisons un crible majorant pour les entiers friables, qui fournit, en association avec le théorème 7, l'estimation suivante.

**Théorème 9** (La Bretèche ; Drappeau 2020). *Pour tout  $\varepsilon > 0$  fixé, il existe  $c > 0$  telle que pour  $1 \leq u \leq c \log x / \log \log x$ , l'on ait*

$$C(x, x^{1/u}) \ll_{\varepsilon} x\rho\left(\left(1 + \frac{25}{178} - \varepsilon\right)u\right).$$

Lorsque  $u \asymp \log x / \log \log x$ , nous gagnons donc asymptotiquement une puissance de  $x$  par rapport aux estimations précédentes.

Le théorème 7 apporte deux améliorations par rapport à la littérature existante. La première concerne la valeur numérique de l'exposant. Le précédent travail sur ce sujet était dû à Iwaniec, qui a obtenu une estimation analogue avec exposant  $1/15$ , en utilisant la majoration de Weil des sommes de Kloosterman. L'utilisation des travaux récents sur les sommes de Kloosterman en moyenne, et notamment le théorème 1, est la raison principale au gain numérique. La seconde amélioration apportée par le théorème 7 est plus substantielle mais aussi plus technique : nous n'avons nullement besoin que la suite  $(\lambda_q)$  soit à support sur des nombres premiers ou sans facteur carré. Cette flexibilité découle d'une amélioration d'un argument de Duke-Friedlander-Iwaniec (Duke ; Friedlander ; Iwaniec 1995), et utilise de façon cruciale l'uniformité en  $q$  permise dans la majoration du théorème 1.

## 2 Modularité, fractions continues et lois limites

Dans cette seconde partie, les résultats présentés concernent des objets d'origine arithmétique qui sont étudiés par des méthodes relevant plutôt de systèmes dynamiques.

On décrit d'abord le problème qui a initié les travaux présentés dans cette partie. Il concerne la série

$$D(s, x) := \sum_{n \geq 1} \frac{\tau(n)}{n^s} e(nx),$$

où l'on rappelle que  $\tau(n)$  est le nombre de diviseurs de  $n$ , et  $e(z) := e^{2\pi iz}$ . Cette série est absolument convergente pour  $\operatorname{Re}(s) > 1$  et  $x \in \mathbf{R}$  quelconque. Lorsque  $x \in \mathbf{Q}$ , Estermann (Estermann 1930) a prouvé que la fonction  $D(\cdot, x)$  possède un prolongement méromorphe à  $\mathbf{C}$ , et une équation fonctionnelle qui relie la valeur  $D(s, a/q)$  avec  $D(1-s, -d/q)$  lorsque  $ad \equiv 1 \pmod{q}$ . Cela peut se voir, par exemple, en exprimant la fonction périodique  $n \mapsto e(nx)$  en termes de caractères de Dirichlet.

La valeur  $D(1/2, a/q)$  est ainsi intimement liée aux valeurs centrales  $L(\frac{1}{2}, \chi)$  des fonctions  $L$  de Dirichlet. Bettin prouve dans (Bettin 2016) que lorsque  $q$  est un nombre premier et  $(a, q) = 1$ , le second moment tordu

$$M_2(q, a) := \frac{1}{q^{1/2}} \sum_{\chi \pmod{q}} \chi(a) |L(\frac{1}{2}, \chi)|^2$$

est proche de la valeur en  $x = a/q$  de la fonction d'Estermann au point central,

$$M_2(q, a) \simeq \operatorname{Re} D(\frac{1}{2}, a/q) + \operatorname{Im} D(\frac{1}{2}, a/q). \quad (2.1)$$

Ainsi que le remarque Bettin (Bettin 2016), l'estimation de  $M_2(q, a)$  est lié à celle du quatrième moment des fonctions  $L$  de Dirichlet,

$$M_4(q) := \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} |L(\frac{1}{2}, \chi)|^4,$$

dont l'estimation avec terme d'erreur en puissance de  $q$  par Young (Young 2011), successivement améliorée dans (Blomer ; Fouvry ; Kowalski et al. 2017 ; Wu 2020),

a été un succès remarquable des méthodes décrites dans la première partie (sommes d'exponentielles algébriques, formule de Kuznetsov). En effet, l'on a par orthogonalité des caractères

$$M_4(q) = \sum_{a \pmod{q}} |M_2(q, a)|^2.$$

Au vu de (2.1), ceci motive donc naturellement la question de comprendre la taille des valeurs prises par  $D(\frac{1}{2}, x)$  lorsque  $x$  varie dans un ensemble de nombres rationnels.

## 2.1 Répartition statistique des valeurs centrales de fonctions $L$

### 2.1.1 Équirépartition d'orbites rationnelles pour l'application de Gauss

Améliorant des résultats antérieurs de Young (Kıral; Young 2019) et Conrey (Conrey [s. d.]), Bettin (Bettin 2016) démontre que la fonction

$$\phi(x) := D(\frac{1}{2}, x) - D(\frac{1}{2}, -\frac{1}{x}), \quad (2.2)$$

initialement définie pour  $x \in \mathbf{Q} \setminus \{0\}$ , se prolonge en une fonction hölderienne sur  $\mathbf{R} \setminus \{0\}$ . En conjonction avec la 1-périodicité, ceci implique plus généralement que pour tout  $\gamma \in \mathrm{SL}(2, \mathbf{Z})$ , la fonction

$$\phi_\gamma(x) : x \mapsto D(\frac{1}{2}, x) - D(\frac{1}{2}, \gamma x)$$

se prolonge en une fonction hölderienne sur  $\mathbf{R} \setminus \{\gamma^{-1}\infty\}$ . Ceci fait de  $D(\frac{1}{2}, \cdot)$  ce que Zagier, dans un article séminal (Zagier 2010), appelle une forme modulaire “quantique” pour  $\mathrm{SL}(2, \mathbf{Z})$ .

L'égalité (2.2) ouvre la voie à une offre une expression de  $D(\frac{1}{2}, x)$  en termes de l'orbite finie du rationnel  $x$  sous l'application de Gauss

$$T : ]0, 1[ \rightarrow ]0, 1[, \quad T(x) = \{1/x\},$$

où  $\{\cdot\}$  désigne la fonction « partie fractionnaire », c'est-à-dire à l'aide de l'algorithme d'Euclide :

$$D(\frac{1}{2}, x) = D(\frac{1}{2}, 0) + \sum_{j=1}^{r(x)} \phi((-1)^{j-1} T^{j-1}(x)), \quad (2.3)$$

où  $r(x) \geq 0$  est le plus petit entier tel que  $T^{r(x)}(x) = 0$ . Cette égalité ramène entiè-

rement le problème à celui de la répartition en moyenne des orbites de nombres rationnels sous l'application de Gauss. Toute l'arithmétique du problème est encodée dans les propriétés analytiques de la fonction  $\phi$ , qui est couramment appelée « l'observable » dans la théorie des systèmes dynamiques.

Notons l'ensemble des rationnels dans  $[0, 1[$  de dénominateur  $q$  fixé par

$$\Omega_q := \{a/q, 0 \leq a < q, (a, q) = 1\}.$$

La question à laquelle nous aimerions répondre est la suivante : lorsque  $x$  est un nombre rationnel pris uniformément au hasard dans  $\Omega_q$ , que peut-on dire de la variable aléatoire  $D(\frac{1}{2}, x)$  lorsque  $q \rightarrow \infty$  ?

Les méthodes dont nous disposons ne permettent pas d'obtenir des résultats précis sur  $\Omega_q$ <sup>1</sup>. Le problème devient abordable dès lors que l'on effectue une moyenne supplémentaire sur  $q$ , c'est-à-dire si l'on travaille sur l'ensemble plus grand

$$\Omega_{\leq Q} := \bigcup_{q \leq Q} \Omega_q$$

des nombres rationnels de dénominateurs au plus  $Q$ . La répartition des sommes de Birkhoff de la forme (2.3), le long des orbites de nombres rationnels de  $\Omega_{\leq Q}$  est un sujet qui est entré récemment dans le giron des méthodes de la théorie des systèmes dynamiques mesurés, par les travaux de Vallée (Vallée 2000) et Baladi et Vallée (Baladi ; Vallée 2005).

Munissant  $\Omega_{\leq Q}$  de la mesure de comptage uniforme, on note  $\mathbb{P}_Q$  et  $\mathbb{E}_Q$  la probabilité et la variance associée. Un des théorèmes principaux de Baladi et Vallée dans (Baladi ; Vallée 2005) est le suivant.

**Théorème 10.** *Soit  $(c(n))_{n \geq 1}$  une suite de nombres réels vérifiant  $c(n) = O(\log n)$  pour  $n \geq 2$ . Pour tout  $x \in \mathbf{Q} \cap (0, 1)$ , l'on pose*

$$f(x) = \sum_{j=1}^{r(x)} c(a_j(x)),$$

où  $a_j(x) = \lfloor 1/T^{j-1}(x) \rfloor$  désigne le  $j$ -ième coefficient dans le développement en fraction continue

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots}}}.$$

Il existe  $\delta > 0$  et deux fonctions  $U, V$  définies et holomorphes sur un voisinage  $W \subset \mathbf{C}$  de l'origine, telles que lorsque  $Q \rightarrow \infty$ , l'on ait

$$\mathbb{E}_Q \left( \exp(wf(x)) \right) = \exp \left( U(w) \log Q + V(w) + O(Q^{-\delta}) \right) \quad (w \in W). \quad (2.4)$$

---

1. On réfère néanmoins à (Aistleitner ; Borda ; Hauke 2022) pour quelques résultats généraux récents.

La qualité du terme d'erreur au membre de droite de (2.4) repose notamment sur l'adaptation d'une méthode de Dolgopyat (Dolgopyat 1998) permettant d'obtenir un trou spectral uniforme pour la famille d'opérateurs de transfert associée à  $T$ ,

$$\mathbb{H}_\tau : f \mapsto \sum_{n=1}^{\infty} \frac{1}{(n+x)^{2+i\tau}} f\left(\frac{1}{n+x}\right) \quad (\tau \in \mathbf{R}).$$

Ainsi qu'il est expliqué dans la section 1.1 de (Baladi ; Vallée 2005), des analogues du théorème 10 sont connus et classiques dans la situation où  $x \in [0, 1]$  est pris uniformément au hasard selon la mesure de Lebesgue, et lorsque que la fonction étudiée est  $f_N(x) = \sum_{j=1}^N c(a_j(x))$  avec  $N \rightarrow \infty$ . Le paramètre  $N$  joue formellement le rôle de  $\log Q$ , à une constante près, qui est la longueur typique du développement en fraction continue d'une fraction dans  $\Omega_{\leq Q}$ . L'analogie continue du problème est plus simple à de nombreux égards : par exemple, l'analogie continue de l'estimation (2.4) s'obtient à l'aide d'informations sur les perturbations de l'opérateur  $\mathbb{H}_\tau$  pour  $\tau = 0$  uniquement, ce qui relève de méthodes classiques de la théorie des perturbations des opérateurs, sans qu'il y ait lieu de faire appel aux techniques de Dolgopyat.

Dans le théorème 10, la fonction  $f(x)$  peut être interprétée comme une somme de Birkhoff de même nature que (2.3), où l'observable est  $\phi(x) := c(\lfloor 1/x \rfloor)$ . Pour la question qui nous intéresse, celle de la fonction  $D(\frac{1}{2}, \cdot)$ , les deux principales différences par rapport à l'énoncé du théorème 10 sont les suivantes<sup>2</sup> :

1. notre observable  $\phi$  est hölderienne, et n'est pas constante par morceaux sur les intervalles  $[\frac{1}{n+1}, \frac{1}{n}]$ ,
2. notre observable  $\phi$  n'a pas de moment exponentiel, et plus précisément, l'on a  $\phi(x) \sim cx^{-1/2} \log(1/x)$  lorsque  $x \rightarrow 0$ .

Dans (Bettin ; Drappeau 2022b), nous obtenons une généralisation du résultat de (Baladi ; Vallée 2005) qui résout ces deux points. Étant donnée une application  $\phi : [0, 1] \rightarrow \mathbf{R}$ , l'on note

$$S_\phi(x) = \sum_{j=1}^{r(x)} \phi(T^{j-1}(x)), \quad (x \in \mathbf{Q} \cap (0, 1)).$$

Dans le cas du théorème 10, nous avons donc  $f(x) = S_\phi(x)$  pour  $\phi(x) = c(\lfloor 1/x \rfloor)$ . Dans le cas de la fonction d'Estermann, nous avons  $D(\frac{1}{2}, x) = D(\frac{1}{2}, 0) + S_\phi(x)$  où  $\phi$  est une certaine fonction höldérienne.

L'expression de notre résultat fait intervenir l'intégrale

$$I_\phi(t) := \int_0^1 (e^{it\phi(x)} - 1) \frac{dx}{(1+x) \log 2} \quad (t \in \mathbf{R}). \quad (2.5)$$

---

2. Nous négligeons dans la suite l'effet du changement de signe  $(-1)^{j-1}$  dans (2.3).

**Théorème 11** (Bettin ; Drappeau 2022b). Soient  $\kappa_0, \lambda_0, \alpha_0 > 0$ . Supposons que la fonction  $\phi : (0, 1) \rightarrow \mathbf{R}$  vérifie les propriétés suivantes :

- Pour chaque  $n \in \mathbf{N}$ , la fonction  $\phi|_{]1/(n+1), 1/n[}$  se prolonge en une fonction  $\kappa_0$ -hölderienne.
- L'on a

$$\sum_{n \geq 1} \frac{1}{n^2} \left( \sup_{x \in [\frac{1}{n+1}, \frac{1}{n}]} |\phi(x)|^{\alpha_0} + \sup_{x, y \in [\frac{1}{n+1}, \frac{1}{n}]} \frac{|\phi(x) - \phi(y)|^{\lambda_0}}{|x - y|^{\lambda_0 \kappa_0}} \right) < \infty. \quad (2.6)$$

Alors il existe  $t_0, \delta > 0$  et deux fonctions  $U, V : [-t_0, t_0] \rightarrow \mathbf{C}$  telles que

$$\mathbb{E}_Q \left( \exp \left( it S_\phi(x) \right) \right) = \exp \left( U(t) \log Q + V(t) + O(Q^{-\delta}) \right), \quad (t \in [-t_0, t_0]), \quad (2.7)$$

ainsi que

$$\begin{aligned} U(t) &= \frac{12 \log 2}{\pi^2} I_\phi(t) + O_\varepsilon(t^2 + |t|^{2\alpha_0 - \varepsilon}), \\ V(t) &= O_\varepsilon(|t| + |t|^{\alpha_0 - \varepsilon}). \end{aligned}$$

Si, de plus, l'on a  $\alpha_0 > 1$ , alors il existe  $C \in \mathbf{R}$  tel que

$$U(t) = \frac{12 \log 2}{\pi^2} I_\phi(t) + Ct^2 + O_\varepsilon(t^3 + |t|^{1 + \alpha_0 - \varepsilon}).$$

La conclusion du théorème 11 correspond au cas  $t = iw$  du théorème 10 de Baladi-Vallée. Une estimation pour  $t$  complexe au voisinage de l'origine est valable, avec une preuve identique, à condition que  $\phi$  admette des moments exponentiels, au sens où

$$\sum_{n \geq 1} \frac{1}{n^2} \sup_{x \in [\frac{1}{n+1}, \frac{1}{n}]} \exp \left( \sigma_0 |\phi(x)| \right) < \infty$$

pour un certain  $\sigma_0 > 0$ . Ceci est équivalent à demander que  $\phi(x) = O(|\log(2/x)|)$ , de façon similaire à l'hypothèse de croissance dans le théorème 10. Cette hypothèse n'est pas vérifiée dans le cas de la fonction d'Estermann.

La preuve que nous proposons avec Bettin du théorème 11 repose sur une adaptation des arguments de Baladi et Vallée (Baladi ; Vallée 2005), et notamment de la partie qui utilise les travaux de Dolgopyat (Dolgopyat 1998). Pour résoudre l'objection 1., nous travaillons dans un espace de fonctions hölderiennes, ce qui relève de méthodes d'analyse standard. Cette modification s'est avérée très utile pour permettre des valeurs arbitraires de  $\lambda_0 > 0$  dans la condition (2.6). Pour résoudre l'objection 2., nous utilisons des résultats récents de Kloeckner (Kloeckner 2019) sur le spectre d'opérateurs sujets à une perturbation, qui sont valables sans condition d'analyticité de ladite perturbation. Le prix à payer pour



obtenir notre énoncé général est de faire intervenir l'intégrale  $I_\phi(t)$  sans l'évaluer *a priori*.

Dans un article compagnon (Bettin ; Drappeau 2022a), nous fournissons des énoncés généraux permettant d'évaluer l'intégrale  $I_\phi(t)$  dans la plupart des cas pratiques. Dans les cas les plus favorables,  $\phi$  est de carré intégrable, et nous pouvons estimer l'intégrale (2.5) en insérant un développement de Taylor à l'ordre 2. L'on obtient alors le corollaire suivant.

**Corollaire 3** (Bettin ; Drappeau 2022b). *L'on reprend les hypothèses et notations du théorème 11. Supposons de plus que  $\alpha_0 \geq 2$ , et qu'il n'existe pas de constante  $c \in \mathbf{R}$  et de fonction  $f$  sur  $[0, 1]$  telles que  $\phi = c + f - f \circ T$ . Alors il existe  $\mu \in \mathbf{R}$  et  $\sigma > 0$  telles que pour tout  $t \in \mathbf{R}$ , lorsque  $Q \rightarrow \infty$ , l'on ait*

$$\mathbb{P}_Q \left( \frac{S_\phi(x) - \mu \log Q}{\sigma \sqrt{\log Q}} \leq t \right) = \int_{-\infty}^t \frac{e^{-v^2/2} dv}{\sqrt{2\pi}} + o(1).$$

L'expression de  $\mu$  est

$$\mu = \frac{12}{\pi^2} \int_0^1 \phi(x) \frac{dx}{1+x}.$$

La quantité  $\sigma$ , quant à elle, ne semble pas admettre en général d'expression explicite en termes de  $\phi$ .

Nous illustrons enfin le théorème 11 par un théorème limite sur la somme des coefficients  $a_j(x)$  dans le développement en fraction continue,

$$\Sigma(x) := \sum_{j=1}^{r(x)} a_j(x).$$

L'on a manifestement  $\Sigma(x) = S_\phi(x)$  pour le choix  $\phi(x) = \lfloor 1/x \rfloor$ . Cette fonction  $\phi$  n'est pas intégrable au voisinage de 0. Pour énoncer la loi limite, l'on définit  $G_1$  comme étant la fonction de répartition de la loi stable  $S_1(\frac{6}{\pi}, 1, 0)$ , c'est-à-dire

$$G_1(v) := \int_{-\infty}^v g_1(x) dx, \quad g_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} e^{-\frac{6}{\pi}|t| - \frac{12}{\pi^2} it \log|t|} dt. \quad (2.8)$$

Nous notons aussi  $\gamma_0$  la constante d'Euler-Mascheroni.

**Corollaire 4** (Bettin ; Drappeau 2022b). *Pour chaque  $v \in \mathbf{R}$ , nous avons*

$$\mathbb{P}_Q \left( \frac{\Sigma(x)}{\log Q} - \frac{\log \log Q - \gamma_0}{\pi^2/12} \leq v \right) \rightarrow G_1(v)$$

*lorsque  $Q \rightarrow \infty$ .*

La loi  $G_1$  n'admet pas de moment d'ordre 1, ce qui correspond au fait que  $\phi(x) = \lfloor 1/x \rfloor$  n'est pas intégrable sur  $[0, 1]$ .

L'analogie continue du corollaire 4, c'est-à-dire lorsque  $x$  est pris au hasard dans  $[0, 1]$  (la somme  $\Sigma(x)$  étant tronquée à  $j \leq N$  pour un paramètre  $N \in \mathbb{N}$  qui tend vers l'infini), avait été établi par Heinrich (Heinrich 1987) en 1987.

### 2.1.2 Application à la fonction d'Estermann

Nous pouvons maintenant appliquer le théorème 11 à la fonction  $D(\frac{1}{2}, \cdot)$ . Pour simplifier l'exposition, l'on considère sa partie réelle, qui a l'avantage d'être paire, et l'on pose  $\psi = \operatorname{Re} \phi$  où  $\phi$  est définie par (2.2). Ainsi que l'a montré Bettin (Bettin 2016), la fonction  $\psi$  prend essentiellement la forme

$$\psi(x) = x^{-1/2}(c_1 \log x + c_2) + E(x),$$

où  $E$  est hölderienne sur  $\mathbb{R}$  d'exposant  $\alpha$  pour tout  $\alpha < 1/2$ , et  $c_1, c_2$  sont deux constantes. Dans (Bettin ; Drappeau 2022a), l'intégrale  $I_\psi(t)$  est estimée par

$$I_\psi(t) = 1 + i\mu t - \sigma^2 t^2 |\log t|^3 + o(t^2 |\log t|^3)$$

lorsque  $t \rightarrow 0^+$ , où  $\mu \in \mathbb{R}$  est une constante et  $\sigma = 1/\pi$ . En évaluant l'espérance de  $D(1/2, x)$  sur  $\Omega_{\leq Q}$ , on montre que  $\mu = 0$ . On en déduit alors le résultat suivant, qui répond à notre question initiale.

**Corollaire 5** (Bettin ; Drappeau 2022b). *Pour  $\sigma = 1/\pi$ , l'on a*

$$\mathbb{P}_Q \left( \frac{\operatorname{Re} D(x)}{\sigma \sqrt{(\log Q)(\log \log Q)^3}} \leq t \right) = \int_{-\infty}^t \frac{e^{-v^2/2} dv}{\sqrt{2\pi}} + o(1)$$

lorsque  $Q \rightarrow \infty$ .

L'apparition du facteur supplémentaire  $(\log \log Q)^3$ , que nous n'avions pas anticipé, est intimement lié au fait que la fonction  $\psi$  n'est pas de carré intégrable. Cela découle *in fine* de la présence d'un pôle double en  $s = 1$  pour la série de Dirichlet  $\zeta(s)^2$ , dont  $D(s, x)$  est la tordue additive.

La simplicité avec laquelle le corollaire 5 est déduit du théorème 11 occulte nos nombreuses tentatives d'obtenir le corollaire 5 par d'autres méthodes, qui ont toutes échoué. À titre d'exemple, la stratégie la plus naturelle était de passer par l'estimation des moments, afin de tirer parti de l'orthogonalité des caractères additifs. Ce calcul fut réalisé par Bettin dans (Bettin 2019) avec la conclusion que l'estimation n'est plus pertinente à partir du deuxième moment, du fait de la contribution prépondérante d'un ensemble asymptotiquement négligeable de nombres rationnels. Mener à bien cette stratégie aurait nécessité une compréhension plus fine des nombres rationnels en cause et une façon analytiquement viable de retirer leur contribution aux moments. *A contrario*, les méthodes dynamiques qui sous-tendent le théorème 11 analysent cet obstacle de façon naturelle et transparente.

En ce qui concerne le lien avec les moments des fonctions  $L$  de Dirichlet, dont nous avons parlé au début de cette partie, il est décevant que la relation exacte (2.1), entre le second moment tordu des fonctions  $L$  de Dirichlet et la fonction d'Estermann, n'ait lieu que lorsque  $q$  est premier ; lorsque  $q$  n'est pas premier, cette relation est perturbée par une convolution multiplicative supplémentaire. Cela ne nous permet pas de transférer directement les résultats ci-dessus, qui sont valables en moyenne sur  $q$  entier sans contrainte. Nous pouvons néanmoins émettre une conjecture plausible sur la répartition des valeurs des moments tordus  $M_2(q, a)$ .

**Conjecture 2.** *La répartition du multi-ensemble*

$$\left\{ \frac{M_2(a, q)}{(\log q)^{1/2} (\log \log q)^{3/2}}, 0 < a < q, (a, q) = 1 \right\}$$

*suit asymptotiquement, lorsque  $q \rightarrow \infty$ , une loi gaussienne centrée.*

### 2.1.3 Application aux tordues additives de valeurs centrales de formes modulaires

Le théorème 11 a trouvé une application à un autre type de fonctions sur  $\mathbf{Q}$ , qui sont construites à l'aide de formes modulaires. Soit  $f$  une forme modulaire holomorphe cuspidale de poids  $k$  et de niveau  $N$ , ce que l'on notera  $f \in S(k, N)$ . Notant

$$f(z) = \sum_{n \geq 1} a_n n^{(k-1)/2} e(nz), \quad (\text{Im}(z) > 0)$$

le développement de Fourier de  $f$  à l'infini, on définit la fonction  $L$

$$L(f, s) = \sum_{n \geq 1} \frac{a_n}{n^s},$$

initialement sur  $\text{Re}(s) > 1$ . Il est connu depuis Hecke que cette fonction se prolonge en une fonction entière de  $s$ , qui satisfait une équation fonctionnelle de la forme

$$L(f, s) = \gamma_f(s) L(\bar{f}, 1 - s)$$

où  $\gamma_f$  est essentiellement un quotient de fonctions  $\Gamma$  d'Euler. Lorsque  $f$  est une forme de Hecke, la fonction  $n \mapsto a_n$  est multiplicative, et  $L(f, s)$  possède alors un produit eulérien. La méthode de Hecke consiste à exprimer  $L(f, s)$  comme une intégrale, ou période, de la forme  $f$  :

$$L(f, s) = \frac{(2\pi)^s}{i\Gamma(s)} \int_0^{i\infty} f(z) (\text{Im } z)^{s + \frac{k-1}{2} - 1} dz, \quad (s \in \mathbf{C}), \quad (2.9)$$

où le chemin d'intégration est une droite verticale.

De façon analogue avec la construction de la fonction d'Estermann, l'on peut former la tordue additive :

$$L(f, s, x) := \sum_{n \geq 1} \frac{a_n}{n^s} e(nx)$$

qui converge pour  $\operatorname{Re}(s) > 1$  et  $x \in \mathbf{R}$ . Nous avons une expression analogue à celle de Hecke (2.9) :

$$L(f, s, x) = \frac{(2\pi)^s}{i\Gamma(s)} \int_x^{i\infty} f(z) (\operatorname{Im} z)^{s + \frac{k-1}{2} - 1} dz, \quad (\operatorname{Re}(s) > 1),$$

le chemin d'intégration étant encore une droite verticale. Lorsque  $x \in \mathbf{Q}$ , la décroissance de  $f$  au voisinage de  $x$  le long de la demi-droite  $\{x + iy, y > 0\}$  permet de prolonger analytiquement l'intégrale au membre de droite, et donc aussi la fonction  $L(f, \cdot, x)$ , à tout le plan complexe. La fonction qui en résulte satisfait encore une équation fonctionnelle de la forme  $L(f, s, a/q) \leftrightarrow L(f, 1 - s, -d/q)$  lorsque  $ad \equiv 1 \pmod{q}$ . Nous pouvons donc définir la valeur centrale

$$L_f(x) := L(f, 1/2, x) \quad (x \in \mathbf{Q}).$$

Lorsque  $k = 2$ , nous retrouvons, à un facteur multiplicatif près, une quantité connue sous le nom de symbole modulaire :

$$L_f(x) = c_f \langle x \rangle_f, \quad \langle x \rangle_f := \int_x^{i\infty} f(z) dz \quad (k = 2).$$

Les symboles modulaires sont notamment utilisés pour calcul explicitement des formes modulaires (Cremona 1997).

Motivés par des applications liées au rang du groupe des points rationnels d'une courbe elliptique le long d'une famille d'extensions, Mazur et Rubin ont formulé une série de conjectures sur la répartition des valeurs des symboles modulaires  $\langle x \rangle_f$  en moyenne sur  $x$  (Mazur ; Rubin 2021). Leurs conjectures originelles portaient sur le comportement en moyenne pour  $x$  de dénominateur donné, c'est-à-dire pour  $x \in \Omega_q$  avec les notations de la section qui précède. Ces conjectures sont encore ouvertes et présumées très difficiles. Le calcul du deuxième moment de ces quantités se situe à la frontière des techniques connues actuellement (Blomer ; Fouvry ; Kowalski et al. 2023).

Toujours lorsque  $k = 2$ , Petridis et Risager (Petridis ; Risager 2018) ont montré que la question peut être résolue en effectuant une moyenne supplémentaire sur les dénominateurs, ce qui revient à choisir  $x \in \Omega_{\leq Q}$  avec les notations de la section précédente. Leur théorème principal s'applique à toute forme modulaire de poids 2 sur un groupe Fuchsien de première espèce, et implique en particulier,

pour une forme  $f \in S_k(N)$ , l'existence de  $\sigma_f > 0$  telle que pour tout  $t \in \mathbf{R}$ ,

$$\mathbb{P}_Q\left(\frac{L_f(x)}{\sigma_f \sqrt{\log Q}} \leq t\right) \rightarrow \int_{-\infty}^t \frac{e^{-v^2/2} dv}{\sqrt{2\pi}}, \quad (Q \rightarrow \infty). \quad (2.10)$$

La méthode de Petridis et Risager (Petridis ; Risager 2018) passe par l'estimation de la fonction génératrice des moments, et est capable de fournir un énoncé de type “quasi-puissances” analogue à (2.4).

Lorsque  $k = 2$ , l'une des propriétés connues du symbole modulaire se traduit dans le fait que l'application

$$x \mapsto L_f(x) - L_f(\gamma x)$$

est constante pour  $\gamma \in \Gamma_0(N)$  (cf. (Mazur ; Rubin 2021)). Cette propriété est cruciale dans le formalisme de (Petridis ; Risager 2018). Cela en fait certainement une forme modulaire quantique au sens de Zagier.

Considérons maintenant le cas où le poids  $k \in \mathbf{Z}$  est supérieur à 2. Pour tout  $\gamma \in \Gamma_0(N)$ , posons

$$\phi_\gamma(x) : x \mapsto L_f(x) - L_f(\gamma x) \quad (2.11)$$

Lorsque  $k > 2$ , la fonction  $x \mapsto L_f(x) - L_f(\gamma x)$  pour  $\gamma \in \Gamma_0(N)$  n'est plus nécessairement constante. En revanche, ainsi qu'il est montré dans (Bettin ; Drappeau 2022b ; Nordentoft 2021a), les fonctions  $\phi_\gamma$  se prolongent toutes en des fonctions  $(1 - \varepsilon)$ -höldériennes et bornées de  $x \in \mathbf{R}$ . Lorsque  $N = 1$ , c'est-à-dire si  $f$  est modulaire pour  $\mathrm{SL}(2, \mathbf{Z})$ , le corollaire 3 nous permet d'en déduire le théorème de la limite centrale suivant sur les valeurs  $L_f(x)$  pour  $x \in \Omega_{\leq Q}$ , c'est-à-dire un analogue du théorème de Petridis-Risager où l'hypothèse “ $f \in S_2(N)$ ” est remplacé par “ $f \in S_k(1)$ ”.

**Théorème 12** (Bettin ; Drappeau 2022b). *Supposons que  $f \in S_k(1)$ . Il existe  $\sigma_f > 0$  tel que l'estimation (2.10) ait lieu pour tout  $t \in \mathbf{R}$ , lorsque  $Q \rightarrow \infty$ .*

L'aspect véritablement nouveau de notre travail réside en le fait que nous n'utilisons aucune information sur  $f$  au-delà de l'existence de la formule de réciprocity (2.11) et d'une faible hypothèse de régularité sur  $\phi_\gamma$ . C'est la stratégie de la méthode, basée sur l'expression en termes d'itérées de l'application de Gauss, qui permet cela.

Lee et Sun (Lee ; Sun 2019), indépendamment de nous et à peu près à la même période, ont eu la même idée d'utiliser les techniques de Baladi-Vallée sur ce sujet, et ont pu traiter le cas  $f \in S_2(N)$ . Un nouvel ingrédient particulièrement important dans leur travail est d'utiliser une extension naturelle de l'application de Gauss afin de travailler sur un groupe de congruence.

Toujours à la même période, Nordentoft (Nordentoft 2021b) a obtenu l'estimation du théorème 12 pour toute forme  $f$  modulaire holomorphe sur un

groupe Fuchsien de première espèce, ce qui inclut le théorème 12. La méthode de Nordentoft est encore différente, mais se rapproche plutôt de celle de Petridis-Risager, et passe par l'étude des moments.

Dans un travail avec Nordentoft (Drappeau ; Nordentoft 2022), nous avons obtenu un résultat de régularité sur les fonctions (2.11) dans le cas général d'une forme de Maaß sur un sous-groupe Fuchsien de  $SL(2, \mathbf{R})$  de première espèce, ce qui généralise tout à la fois le cas des formes holomorphes que nous venons d'aborder, ainsi que la situation de la fonction d'Estermann où  $a_n = \tau(n)$ , dans laquelle la forme de Maaß est une certaine série d'Eisenstein. Cela permet d'obtenir un théorème limite semblable au théorème 12 lorsque  $f$  est remplacée par une forme de Maaß pour  $SL(2, \mathbf{Z})$ .

Dans un travail en cours avec Bettin et Lee, nous nous efforçons d'obtenir un analogue du théorème 11 pour l'action d'un groupe Fuchsien de première espèce sur ses pointes. Lorsque ce travail aura convergé, nous aurons alors une loi limite du type (2.10) valable pour les tordues additives de fonctions  $L$  de rang 2, au sens de (Iwaniec ; Kowalski 2004, Chapitre V).

## 2.2 Répartition des valeurs d'invariants quantiques de nœuds

Dans l'article de Zagier (Zagier 2010) où les formes modulaires quantiques sont introduites, un exemple singulier provient des invariants de Kashaev en théorie des nœuds. La définition de cet invariant fait intervenir une construction combinatoire délicate à partir d'un diagramme du nœud. Dans le cas non-trivial le plus simple, celui du nœud "de huit" noté  $4_1$ , cet invariant prend la forme d'une fonction  $J : \mathbf{Q} \rightarrow \mathbf{R}$  donnée par

$$J(x) = \sum_{n=0}^{\infty} \prod_{r=1}^n |1 - e(rx)|^2. \quad (2.12)$$

La somme est finie lorsque  $x \in \mathbf{Q}$ . Zagier a remarqué numériquement que la fonction  $h$  définie par

$$\log J(x) - \log J(1/x) = h(x) \quad (2.13)$$

semble se prolonger à  $\mathbf{R} \setminus \{0\}$  en une fonction régulière, par exemple continue presque partout, là où la fonction  $J$  elle-même est très irrégulière. Ces deux fonctions sont tracées aux figures 3 et 4 de (Zagier 2010).

**Conjecture 3** (Zagier). — *La fonction  $h$  se prolonge en une fonction sur  $\mathbf{R} \setminus \{0\}$  qui est continue aux nombres irrationnels, et continue à gauche et à droite aux nombres rationnels.*

— L'on a  $h(x) = \frac{C}{x} - \frac{3}{2} \log x + O(1)$  pour  $x \in ]0, 1]$ , où  $C = \text{Vol}(4_1)/2\pi \approx 0,323$  avec  $\text{Vol}(4_1)$  le volume hyperbolique du complément du nœud de huit.

Cette fonction nous donne l'opportunité d'un cas d'application intéressant du théorème 11. En effet, nous avons par définition de  $h$  et par la valeur  $J(0) = 1$ ,

$$\log J(x) = \sum_{j=1}^{r(x)} h(T^{j-1}(x)). \quad (2.14)$$

La fonction  $h$  ne satisfait pas tout à fait les conditions de régularité énoncées au théorème 11, mais elle n'en est pas loin. Si la conjecture 3 était vraie, nous pourrions déduire un résultat de convergence vers une loi stable de paramètre 1 similaire à (2.8), pour les valeurs prises par  $\log J(x)$ .

Nous nous sommes donc mis avec Sandro Bettin à la recherche d'une preuve de la conjecture de Zagier. Cela nous a mené à étudier plus particulièrement le  $q$ -symbole de Pochhammer

$$(q)_N := (1 - q) \dots (1 - q^N), \quad (N \in \mathbf{N}), \quad (2.15)$$

dont le module au carré intervient dans la définition (2.12). Ce produit est aussi appelé, dans d'autres contextes, produit de Sudler (Sudler 1964), et nous reviendrons sur cela un peu plus loin.

Le symbole  $(q)_n$  peut être vu comme une version tronquée de la fonction  $\eta$  de Dedekind

$$\eta(z) = \prod_{n \geq 1} (1 - e(nz)), \quad (\text{Im}(z) > 0),$$

qui est une forme modulaire de poids  $1/2$ .

L'un des deux résultats principaux dans le travail (Bettin; Drappeau 2022c) avec Sandro Bettin est une forme de modularité pour le  $q$ -symbole de Pochhammer. On dénote  $\text{den}(x)$  le dénominateur de  $x \in \mathbf{Q}$ . Informellement énoncé, notre résultat s'exprime ainsi.

**Théorème 13** (Bettin; Drappeau 2022c). *Soit  $\gamma \in \text{SL}(2, \mathbf{Z})$ . Pour  $1 \leq r < \text{den}(\gamma x)$ , l'on a*

$$e\left(\frac{\gamma x}{24}\right)(e(\gamma x))_r = \chi(\gamma) e\left(\frac{x}{24}\right)(e(x))_{r'} \psi_\gamma(x, r), \quad (2.16)$$

pour un certain  $1 \leq r' < \text{den}(x)$ , où  $\chi$  est le caractère central associé à la fonction de Dedekind (Iwaniec 1997, p. 45), et  $\psi_\gamma$  est une fonction satisfaisant certaines conditions d'holomorphicité.

La variable en laquelle  $\psi_\gamma$  se prolonge en une fonction holomorphe est  $z = \{r \text{den}(x) / \text{den}(\gamma x)\}$ . Le point de départ de la preuve est la formule sommatoire

d'Abel-Plana appliquée à la somme

$$\sum_{n=1}^N \log(1 - e(nz)).$$

La formule d'Abel-Plana est une version de la formule classique d'Euler-Maclaurin, où le terme d'erreur est exprimé de manière exacte par une intégrale de la fonction sommée, ici  $\log(1 - e(z))$ . Notre argument exploite, de façon cachée, le fait que la fonction  $\log(1 - e(z))$  est proche d'une primitive de la fonction  $\pi \cot(\pi z)$ , laquelle est le noyau de sommation associé aux entiers relatifs (elle possède un pôle simple de résidu 1 en chaque entier relatif).

Le fait que la formule (2.16) est exacte, et la propriété d'holomorphie que l'on a évoqué sur  $\psi_r$ , sont deux aspects cruciaux lorsque l'on souhaite sommer cette formule sur  $r$ . Nous nous appuyons grandement sur les travaux d'Ohtsuki (Ohtsuki 2016) dans cette partie. Cela nous a permis d'obtenir une preuve de la conjecture de Zagier pour certains cas particuliers de nœuds hyperboliques<sup>3</sup>. Dans le cas du nœud de huit, c'est-à-dire pour la fonction  $J(x)$ , nous déduisons en particulier la propriété suivante.

**Théorème 14** (Bettin; Drappeau 2022c). *Il existe une suite  $(c_n)$  de nombres réels telle que pour  $Q, N \geq 1$  et  $x \in ]0, 1]$ , lorsque  $\text{den}(x) \leq Q$ , l'on ait*

$$h(x) = \frac{C}{x} - \frac{3}{2} \log x + \sum_{n=0}^{N-1} c_n x^n + O_{Q,N}(x^N).$$

Muni d'une telle propriété, nous aimerions en déduire, par l'intermédiaire de la définition (2.13), un théorème limite sur les valeurs  $\log J(x)$ . Malheureusement, le fait que le terme d'erreur dépende implicitement du dénominateur de  $x$  ne nous a pas permis immédiatement de mener à bien cette stratégie. Nous y sommes cependant parvenus de façon indirecte, en complétant le théorème 13 par une autre formule de réciprocité.

Écrivons  $x = [0; a_1, \dots, a_r]$  le développement en fraction continue de  $x$ . L'application de Gauss consiste à retirer de  $x$  son premier coefficient,

$$T(x) = [0; a_2, \dots, a_r].$$

Dans certaines circonstances (Bettin; Conrey 2013), il est analytiquement plus favorable de tenter de retirer à  $x$  son dernier coefficient,

$$U(x) := [0; a_1, \dots, a_{r-1}].$$

Ceci revient, à un signe près, à conjuguer l'application de Gauss par l'involution

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3. Ce sont essentiellement ceux pour lesquels la *conjecture volume* de Kashaev (Kashaev 1997) est connue; réciproquement, la conjecture de Zagier englobe la conjecture volume.



qui à une fraction  $a/q \in ]0, 1[$  associe  $(-1)^{r-1}\bar{a}/q \pmod{1}$ , où  $a\bar{a} \equiv 1 \pmod{q}$ . Cette involution revient, grossièrement, à inverser l'ordre des coefficients dans le développement en fraction continue.

La relation de Bezout

$$\frac{\bar{a}}{q} + \frac{\bar{q}}{a} = \frac{1}{aq} \pmod{1}$$

est une version précise de l'assertion que les nombres  $x$  et  $U(x)$  sont proches l'un de l'autre, ce qui se voit par ailleurs dans la définition.

Nous obtenons la relation suivante.

**Théorème 15** (Bettin ; Drappeau 2022c). *L'on a*

$$\log J(x) = \log J(U(x)) + Ca_r(x) + E(x),$$

où  $E(x)$  est un terme d'erreur qui est borné par  $\log a_r(x)$  dans la plupart des circonstances.

Ce théorème est spécifique au nœud  $4_1$ , contrairement aux précédents théorèmes 13 et 14. Le traitement du terme d'erreur a nécessité un lemme technique spécifique portant sur des sommes de cotangentes (Bettin ; Drappeau 2020), qui fut une des parties les plus délicates de l'argument.

Les théorèmes 13 et 15 fournissent juste assez d'informations sur la fonction  $J$  pour obtenir, par l'intermédiaire de l'itération (2.14), l'approximation suivante. L'on rappelle que  $\Sigma(x) = \sum_{j=1}^{r(x)} a_j(x)$ .

**Théorème 16** (Bettin ; Drappeau 2022c). *Pour tout  $x \in \mathbf{Q} \cap ]0, 1[$ , l'on a*

$$\log J(x) \sim C\Sigma(x), \quad \frac{\Sigma(x)}{r(x)} \rightarrow \infty.$$

D'un côté, il est classiquement connu que  $r(x) \ll \log Q$  lorsque  $\text{den}(x) \leq Q$ . D'un autre côté, le corollaire 4 nous assure que l'on a

$$\Sigma(x) \sim \frac{12}{\pi^2} (\log Q) \log \log Q$$

pour une proportion  $1 + o(1)$  des rationnels  $x \in \Omega_Q$ <sup>4</sup>. On en déduit immédiatement le corollaire suivant, qui peut être interprété comme une loi des grands nombres pour les valeurs prises par  $\log J(x)$  pour  $x \in \Omega_Q$ .

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4. Ceci est en fait vrai pour une proportion  $1 + o(1)$  des nombres  $x \in \Omega_q$ , c'est-à-dire sans la moyenne sur le dénominateur, par des travaux de Rukavishnikova (Rukavishnikova 2011) ; on réfère aussi à l'article récent (Aistleitner ; Borda ; Hauke 2022).

**Corollaire 6** (Bettin; Drappeau 2022c). *Il existe une fonction  $\varepsilon : \mathbf{N} \rightarrow \mathbf{R}_+$  avec  $\varepsilon(Q) \rightarrow 0$  lorsque  $Q \rightarrow \infty$ , telle que*

$$\log J(x) = \left( \frac{12}{\pi^2} C + O(\varepsilon(Q)) \right) (\log Q) \log \log Q$$

*pour chaque  $x \in \Omega_Q$ , sauf au plus  $\varepsilon(Q)Q^2$  d'entre eux.*

Très récemment, Aistleitner et Borda ont obtenu plusieurs nouveaux résultats sur  $J(x)$  à l'aide de leurs travaux sur les perturbations de produits de sinus (Borda; Aistleitner 2022c; Borda; Aistleitner 2022b; Borda; Aistleitner 2022a). En particulier, il est prouvé au corollaire 2 de (Borda; Aistleitner 2022a) que la fonction  $h$  définie en (2.13) se prolonge par densité en une fonction presque partout continue sur  $\mathbf{R} \setminus \{0\}$ . Plus précisément, Aistleitner et Borda montrent que ce prolongement de la fonction  $h$  est continu aux irrationnels  $x$  qui sont “bien approchables” au sens où la suite  $(a_j(x))_j$  est non bornée. Cela leur a permis d'obtenir, dans le Theorem 4 de (Borda; Aistleitner 2022a), un théorème de la limite centrale pour les valeurs  $\log J(x)$ , de la même précision que le corollaire 4,

L'une des principales questions en suspens est la continuité de  $h$  aux irrationnels qui sont mal approchables, et en particulier aux irrationnels quadratiques comme  $1/\phi = [0; 1, 1, \dots]$ . C'est l'objet d'une réflexion en cours avec Sandro Bettin et Bence Borda. Nous avons bon espoir en ce qui concerne les irrationnels quadratiques.

Une autre piste à explorer est la validité de la loi des grands nombres du même type que le corollaire 6, pour des nœuds plus généraux. Notre démonstration pour le nœud  $4_1$  utilise de façon cruciale la positivité des quantités sommées dans (2.12), et une nouvelle idée est nécessaire pour s'en passer. C'est plus précisément notre preuve du théorème 15 qui ne semble pas s'adapter. Les travaux d'Aistleitner-Borda (Borda; Aistleitner 2022a), ainsi que nos travaux en cours avec Sandro Bettin et Bence Borda sur la fonction  $h$ , fourniront peut-être un autre point de départ.

## 3 Mémoires encadrés

### G. de Murcia : Équirépartition de racines de congruences quadratiques modulo $p$

Guillaume de Murcia était étudiant de M2 Mathématiques fondamentales à AMU en 2019-2020. Le sujet de mémoire proposait la lecture de l'article de Duke, Friedlander et Iwaniec "Equidistribution of Roots of a Quadratic Congruence to Prime Moduli", qui mélange un panel de techniques très variées, qui vont des méthodes de cribles à l'analyse de formes automorphes. Le but du stage était de l'initier aux techniques plutôt analytiques. Les questions d'ouverture préparées pour ce sujet, qui n'ont pas été abordées par manque de temps, auraient porté sur des variantes plus simple (dimension  $1/2$ ) de la partie sur le crible. Le sujet proposé était à mi-chemin des thématiques vues par Guillaume en cours (arithmétique et géométrie hyperbolique), et du sujet de thèse proposé à son issue, qui portait sur l'estimation de convolution additive de fonctions multiplicatives. Sa demande de thèse n'ayant pas été retenue pour financement, Guillaume s'est orienté vers la préparation aux concours d'enseignement.

### H. Noireault : Biais de Tchébychev dans des corps de nombres

Húng Noireault était étudiant de M2 Mathématiques fondamentales à AMU en 2021-2022. Je suis intervenu cette année-là dans le M2 en y donnant la moitié d'un cours de théorie algébrique des nombres. Le mémoire de Húng portait sur un article récent de Fiorilli-Jouve à propos d'une généralisation du biais de Tchébychev aux corps de nombres. Le sujet était à la frontière entre la théorie algébrique des nombres (fonctions  $L$  d'Artin), et la théorie analytique des nombres avec laquelle je suis plus familier. Le but du stage était également d'initier Húng aux méthodes analytiques en théorie des nombres. Une petite question d'ouverture lui a été proposée, à laquelle il a eu le temps de répondre convenablement. Ce sujet n'a pas donné suite à une thèse, Húng s'étant orienté vers la préparation à l'agrégation.

## N. Sarkis : Spin d'idéaux premiers

Nicolas Sarkis était étudiant de M2 Mathématiques fondamentales à Paris-Sud (Orsay) en 2021-2022. Ses cours étant de nature plutôt algébriques, je lui ai proposé également un stage à la frontière entre théories analytique et algébrique des nombres. Le mémoire a porté sur l'article "The spin of prime ideals" de Friedlander, Iwaniec, Mazur et Rubin, et ses généralisations ultérieures par Chan, Koymans, McMeekin et Milović. Une question d'ouverture proposait d'étudier la faisabilité d'une caractérisation complète des fonctions spin, à laquelle Nicolas a répondu (par la négative). Le sujet ne visait pas à donner suite à une thèse sur le même sujet, puisque Nicolas avait déjà un projet de thèse en arithmétique et cryptographie à Bordeaux.

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## Annexes : articles

# Sums of Kloosterman sums in arithmetic progressions, and the error term in the dispersion method

Sary Drappeau

## ABSTRACT

We prove a bound for quintilinear sums of Kloosterman sums, with congruence conditions on the ‘smooth’ summation variables. This generalizes classical work of Deshouillers and Iwaniec, and is key to obtaining power-saving error terms in applications, notably the dispersion method. As a consequence, assuming the Riemann hypothesis for Dirichlet  $L$ -functions, we prove power-saving error term in the Titchmarsh divisor problem of estimating  $\sum_{p \leq x} \tau(p-1)$ . Unconditionally, we isolate the possible contribution of Siegel zeroes, showing it is always negative. Extending work of Fouvry and Tenenbaum, we obtain power-saving in the asymptotic formula for  $\sum_{n \leq x} \tau_k(n)\tau(n+1)$ , reproving a result announced by Bykovskiĭ and Vinogradov by a different method. The gain in the exponent is shown to be independent of  $k$  if a generalized Lindelöf hypothesis is assumed.

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## 1. Introduction

Understanding the joint multiplicative structure of pairs of neighboring integers such as  $(n, n+1)$  is an outstanding problem in multiplicative number theory. A quantitative way to look at this question is to try to estimate sums of the type

$$\sum_{n \leq x} f(n)g(n+1), \tag{1.1}$$

when  $f, g : \mathbf{N} \rightarrow \mathbf{C}$  are two functions that are of multiplicative nature—multiplicative functions for instance, or the characteristic function of primes. In this paper, we are motivated by two instances of the question (1.1): the Titchmarsh divisor problem, and correlation of divisor functions.

In what follows,  $\tau(n)$  denotes the number of divisors of the integer  $n$ , and more generally,  $\tau_k(n)$  denotes the number of ways one can write  $n$  as a product of  $k$  positive integers. Studying the function  $\tau_k$  gives some insight into the factorization of numbers<sup>†</sup>, which is deeper but more difficult to obtain as  $k$  grows.

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<sup>†</sup>There are a number of formulas relating the characteristic function of primes to linear combination of divisor-like functions, for instance Heath-Brown’s identity [32].

1.1. *The Titchmarsh divisor problem*

One would like to be able to evaluate, for  $k \geq 2$ , the sum

$$\sum_{p \leq x} \tau_k(p-1), \tag{1.2}$$

where  $p$  denotes primes. *A priori*, this would require understanding primes up to  $x$  in arithmetic progressions of moduli up to  $x^{1-1/k}$ . The case  $k \geq 3$  seems far from reach of current methods, so we consider  $k = 2$ .

In place of (1.2), one may consider

$$T(x) := \sum_{1 < n \leq x} \Lambda(n)\tau(n-1),$$

where  $\Lambda$  is the von Mangoldt function [39, formula (1.39)]. In 1930, Titchmarsh [53] first considered the problem, and proved  $T(x) \sim C_1 x \log x$  for some constant  $C_1 > 1$  under the assumption that the Riemann hypothesis holds for all Dirichlet  $L$ -functions. This asymptotic was proved unconditionally by Linnik [42] using his so-called dispersion method. Simpler proofs were later given by Rodriquez [49] and Halberstam [30] using the theorems of Bombieri–Vinogradov and Brun–Titchmarsh. Finally, the most precise known estimate was proved independently by Bombieri–Friedlander–Iwaniec [7] and Fouvry [23]. To state their result, let us denote

$$C_1 := \prod_p \left(1 + \frac{1}{p(p-1)}\right), \quad C_2 := \sum_p \frac{\log p}{1 + p(p-1)}.$$

**THEOREM A** (Fouvry [23], Bombieri–Friedlander–Iwaniec [7]). *For all  $A > 0$  and all  $x \geq 3$ ,*

$$T(x) = C_1 x \{ \log x + 2\gamma - 1 - 2C_2 \} + O_A(x/(\log x)^A).$$

In this statement,  $\gamma$  denotes Euler’s constant. See also [19, 20] for generalizations in arithmetic progressions and [1] for an analog in function fields.

The error term in Theorem A is due to an application of the Siegel–Walfisz theorem [39, Corollary 5.29]. One could wonder whether assuming the Riemann Hypothesis generalized to Dirichlet  $L$ -functions (GRH) allows for power-saving error term to be obtained (as is the case for the prime number theorem in arithmetic progressions [44, Corollary 13.8]). The purpose of this paper is to prove that such is indeed the case.

**THEOREM 1.1.** *Assume GRH. Then for some  $\delta > 0$  and all  $x \geq 2$ ,*

$$T(x) = C_1 x \{ \log x + 2\gamma - 1 - 2C_2 \} + O(x^{1-\delta}).$$

Unconditionally, we quantify the influence of hypothetical Siegel zeroes. Define, for  $q \geq 1$ ,

$$C_1(q) := \frac{1}{\varphi(q)} \prod_{p \nmid q} \left(1 + \frac{1}{p(p-1)}\right), \quad C_2(q) := \sum_{p \nmid q} \frac{\log p}{1 + p(p-1)},$$

where  $\varphi$  is Euler’s totient function. Note that  $C_1 = C_1(1)$  and  $C_2 = C_2(1)$ .

**THEOREM 1.2.** *There exist  $b > 0$  and  $\delta > 0$  such that*

$$T(x) = C_1 x \{ \log x + 2\gamma - 1 - 2C_2 \} - C_1(q) \frac{x^\beta}{\beta} \left\{ \log \left( \frac{x}{q^2} \right) + 2\gamma - \frac{1}{\beta} - 2C_2(q) \right\} + O(x e^{-\delta \sqrt{\log x}}).$$

*The second term is only to be taken into account if there is a primitive character  $\chi \pmod{q}$  with  $q \leq e^{\sqrt{\log x}}$  whose Dirichlet  $L$ -function has a real zero  $\beta$  with  $\beta \geq 1 - b/\sqrt{\log x}$ .*

By partial summation, one deduces

COROLLARY 1.3. *In the same notation as Theorem 1.2,*

$$\sum_{p \leq x} \tau(p-1) = C_1 \{x + 2 \operatorname{li}(x)(\gamma - C_2)\} - C_1(q) \left\{ \frac{x^\beta}{\beta} + 2 \operatorname{li}(x^\beta)(\gamma - \log q - C_2(q)) \right\} + O(x e^{-\delta \sqrt{\log x}}).$$

The method readily allows for more general shifts  $\tau(p-a)$ ,  $0 < |a| \leq x^\delta$  (cf. [21, Corollary 3.4] for results on the uniformity in  $a$ ). The contribution of the exceptional character in Corollary 1.3 would then have a twist by  $\chi(a)$ . Since  $\chi$ , if it exists, is a real character, then  $\chi(a) = 1$  whenever  $a$  is a perfect square (for instance,  $a = 1$ ), in which case we have an unconditional inequality.

COROLLARY 1.4. *With an effective implicit constant, we have*

$$\sum_{p \leq x} \tau(p-1) \leq C_1 \{x + 2 \operatorname{li}(x)(\gamma - C_2)\} + O(x e^{-\delta \sqrt{\log x}}).$$

We conclude our discussion of the Titchmarsh divisor problem by mentioning the important work of Pitt [46], who proves  $\sum_{p \leq x} a(p-1) \ll x^{1-\delta}$  for the sequence  $(a(n))$  of Fourier coefficients of an integral weight holomorphic cusp form (which is a special case of (1.1) when the  $(a(n))$  are Hecke eigenvalues). It is a striking feature that power-saving can be proved *unconditionally* in this situation.

### 1.2. Correlation of divisor functions

Another instance of the problem (1.1) is the estimation, for integers  $k, \ell \geq 2$ , of the quantity

$$\mathcal{T}_{k,\ell}(x) := \sum_{n \leq x} \tau_k(n) \tau_\ell(n+1).$$

The conjectured estimate is of the shape

$$\mathcal{T}_{k,\ell}(x) \sim C_{k,\ell} x (\log x)^{k+\ell-2}$$

for some constants  $C_{k,\ell} > 0$ . The case  $k = \ell$  is of particular interest when one looks at the  $2k$ th moment of the Riemann  $\zeta$  function [54, Section 7.21] (see also [11]): in that context, the size of the error term is a non-trivial issue, as well as the uniformity with which one can replace  $n+1$  above by  $n+a$ ,  $a \neq 0$ . Current methods are ineffective when  $k, \ell \geq 3$ , so we focus on the case  $\ell = 2$ . Let us denote

$$\mathcal{T}_k(x) := \sum_{n \leq x} \tau_k(n) \tau(n+1).$$

There has been several works on the estimation of  $\mathcal{T}_k(x)$ . There are nice expositions of the history of the problem in the papers of Heath-Brown [33] and Fouvry–Tenenbaum [25]. The latest published results may be summarized as follows.

THEOREM B. *There holds:*

$$\begin{aligned} \mathcal{T}_2(x) &= xP_2(\log x) + O_\varepsilon(x^{2/3+\varepsilon}), & [15], \\ \mathcal{T}_3(x) &= xP_3(\log x) + O(x^{1-\delta}), & [13, 55], \\ \mathcal{T}_k(x) &= xP_k(\log x) + O_k(x e^{-\delta \sqrt{\log x}}) \quad \text{for fixed } k \geq 4, & [28]. \end{aligned} \tag{1.3}$$

Here  $\varepsilon > 0$  is arbitrary,  $\delta > 0$  is some constant depending on  $k$ , and  $P_k$  is an explicit degree  $k$  polynomial.

The error term of (1.3) resembles that in the distribution of primes in arithmetic progressions, where it is linked to the outstanding problem of zero-free regions of  $L$ -functions. However there is no such process at work in (1.3), leaving one to wonder if power-saving can be achieved. In [10], Bykovskiĭ and Vinogradov announce results implying

$$\mathcal{T}_k(x) = xP_k(\log x) + O_k(x^{1-\delta/k}) \quad (k \geq 4, x \geq 2) \tag{1.4}$$

for some absolute  $\delta > 0$ , and sketch ideas of a proof. The proposed argument, in a way, is dual to the method adopted in [25]<sup>†</sup> (which is related to earlier work of Motohashi [45]). Here we take up the method of [25] and prove an error term of the same shape.

**THEOREM 1.5.** *For some absolute  $\delta > 0$ , the estimate (1.4) holds.*

In view of [10], Theorem 1.5 is not new. However the method is somewhat different. In the course of our arguments, the analytic obstacle to obtaining an error term  $O_k(x^{1-\delta})$  ( $\delta$  independent of  $k$ ) in the estimate (1.4) will appear clearly: it lies in the estimation of sums of the shape  $\sum_{n \leq x} \tau_k(n)\chi(n)$  for Dirichlet characters  $\chi$  of small conductors. This issue is known to be closely related to the growth of Dirichlet  $L$ -functions inside the critical strip [28].

**THEOREM 1.6.** *Assume that Dirichlet  $L$ -functions satisfy the Lindelöf hypothesis, meaning  $L(\frac{1}{2} + it, \chi) \ll_\varepsilon (q(|t| + 1))^\varepsilon$  for  $t \in \mathbf{R}$  and  $\chi \pmod q$ . Then for some absolute  $\delta > 0$ ,*

$$\mathcal{T}_k(x) = xP_k(\log x) + O_k(x^{1-\delta}) \quad (k \geq 4, x \geq 2). \tag{1.5}$$

The standard conjecture for the error term in the previous formula is  $O_{k,\varepsilon}(x^{1/2+\varepsilon})$ . We have not sought optimal values for  $\delta$  in Theorems 1.5 and 1.6. In the case of (1.4), the method of [10] seems to yield much better numerical results.

Our method readily allows to replace the shift  $n + 1$  in Theorem 1.5 by  $n + a$ ,  $0 < |a| \leq x^\delta$  with  $\delta$  independent of  $k$ . We give some explanations in Section 7.2 below regarding this point.

## 2. Overview

The method at work in Theorems 1.1, 1.2, and 1.5 is the dispersion method, which was pioneered by Linnik [42] and studied intensively in groundbreaking work of Bombieri, Fouvry, Friedlander and Iwaniec [7, 22, 24] on primes in arithmetic progressions. It has received a large publicity recently with the breakthrough of Zhang [57] (see also [47]), giving the first proof of the existence of infinitely many bounded gaps between primes (which was shown later by Maynard [43] and Tao (unpublished) not to require such strong results).

In our case, by writing  $\tau(n)$  as a convolution of the constant function 1 with itself, the problem is reduced to estimating the mean value of  $\Lambda(n)$  or  $\tau_k(n)$  when  $n \leq x$  runs over arithmetic progressions  $\pmod q$ , with an average over  $q$ . It is crucial that the uniformity be good enough to average over  $q \leq \sqrt{x}$ . In the case of  $\Lambda(n)$ , that is beyond what can currently be done for individual moduli  $q$ , even assuming the GRH. The celebrated theorem of Bombieri–Vinogradov [39, Theorem 17.1] allows to exploit the averaging over  $q$ , but if one wants error terms at least as good as  $O(x/(\log x)^2)$  for instance, it barely fails to be useful.

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<sup>†</sup>In [25, 45], the authors study the distribution of  $\tau_k(n)$  in progressions of moduli up to  $x^{1/2}$ , while in [10] the authors address the distribution of  $\tau(n)$  in progressions of moduli up to  $x^{1-1/k}$ .

Linnik’s dispersion method [42], which corresponds at a technical level to an acute use of the Cauchy–Schwarz inequality, offers the possibility for such results, on the condition that one has good bounds on some types of exponential sums related to Kloosterman sums. One then appeals to Weil’s bound [56], or to the more specific but stronger bounds of Deshouillers–Iwaniec [15] which originate from the theory of modular forms through Kuznetsov’s formula.

The Deshouillers–Iwaniec bounds apply to exponential sums of the following kind:

$$\sum_{\substack{c,d,n,r,s \\ (rd,sc)=1}} b_{n,r,s} g(c,d) e\left(n \frac{\overline{rd}}{sc}\right),$$

where  $c, d, n, r, s$  are integers in specific intervals,  $(b_{n,r,s})$  is a generic sequence, and  $g(c, d)$  depends in a smooth way on  $c$  and  $d$ . Here and in what follows,  $e(x)$  stands for  $e^{2\pi i x}$ , and  $\overline{rd}$  denotes the multiplicative inverse of  $rd \pmod{sc}$  (since  $e(x)$  is of period 1, the above is well-defined). It is crucial that the variables  $c$  and  $d$  are attached to a smooth weight  $g(c, d)$ : for the variable  $d$ , in order to reduce to complete Kloosterman sums  $\pmod{sc}$ ; and for the variable  $c$ , because the object that arises naturally in the context of modular forms is the average of Kloosterman sums over moduli (with smooth weight).

In the dispersion method, dealing with largest common divisors (appearing through the Cauchy–Schwarz inequality) causes some issues. The most important of these is that the phase function that arises in the course of the argument takes a form similar to

$$e\left(n \frac{\overline{rd}}{sc} + \frac{\overline{cd}}{q}\right) \tag{2.1}$$

rather than the above. Here  $q$  can be considered small and fixed, but even then, the second term oscillates chaotically.

Previous works avoided the issue altogether by using a sieve beforehand in order to reduce to the favorable case  $q = 1$  (see [23, Lemma 4 and Section 3] and [7, Lemma 4 and Theorem 5\*]). Two error terms are then produced, which take the form

$$e^{-\delta(\log x)/\log z} + z^{-1},$$

where  $z \leq x$  is a parameter. Roughly speaking, the first term corresponds to sieving out prime factors smaller than  $z$ , with the consequence that the ‘bad’ variable  $q$  above is either 1 or larger than  $z$ . The second term corresponds to a trivial bound on the contribution of  $q > z$ . The best error term one can achieve in this way is  $e^{-\delta\sqrt{\log x}}$ , whence the estimate (1.3).

By contrast, in the present paper, we transpose the work of Deshouillers–Iwaniec in a slightly more general context, which allows to encode phases of the kind (2.1). More specifically, whereas Deshouillers and Iwaniec worked with modular forms with trivial multiplier system, we find that working with multiplier systems defined by Dirichlet characters allows one to encode congruence conditions  $\pmod{q}$  on the ‘smooth’ variables  $c$  and  $d$ . This is partly inspired by recent work of Blomer and Milićević [5]. The main result, which extends [15, Theorem 12] and has potential for applications beyond the scope of the present paper, is the following.

**THEOREM 2.1.** *Let  $C, D, N, R, S \geq 1$ , and  $q, c_0, d_0 \in \mathbf{N}$  be given with  $(c_0 d_0, q) = 1$ . Let  $(b_{n,r,s})$  be a sequence supported inside  $(0, N] \times (R, 2R] \times (S, 2S] \cap \mathbf{N}^3$ . Let  $g : \mathbf{R}_+^5 \rightarrow \mathbf{C}$  be a smooth function compactly supported in  $]C, 2C[ \times ]D, 2D[ \times (\mathbf{R}_+^*)^3$ , satisfying the bound*

$$\frac{\partial^{\nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_5} g}{\partial c^{\nu_1} \partial d^{\nu_2} \partial n^{\nu_3} \partial r^{\nu_4} \partial s^{\nu_5}}(c, d, n, r, s) \ll_{\nu_1, \nu_2, \nu_3, \nu_4, \nu_5} \{c^{-\nu_1} d^{-\nu_2} n^{-\nu_3} r^{-\nu_4} s^{-\nu_5}\}^{1-\varepsilon_0} \tag{2.2}$$



for some small  $\varepsilon_0 > 0$  and all fixed  $\nu_j \geq 0$ . Then

$$\sum_{\substack{c \\ c \equiv c_0}} \sum_{\substack{d \\ d \equiv d_0 \pmod{q}}} \sum_{\substack{n \\ (qr d, sc)=1}} \sum_r \sum_s b_{n,r,s} g(c, d, n, r, s) e\left(n \frac{\overline{rd}}{sc}\right) \ll_{\varepsilon, \varepsilon_0} (qCDNRS)^{\varepsilon + O(\varepsilon_0)} q^{3/2} K(C, D, N, R, S) \|b_{N,R,S}\|_2, \tag{2.3}$$

where  $\|b_{N,R,S}\|_2 = (\sum_{n,r,s} |b_{n,r,s}|^2)^{1/2}$  and

$$K(C, D, N, R, S)^2 = qCS(RS + N)(C + RD) + C^2DS\sqrt{(RS + N)R} + D^2NRS^{-1}.$$

We have made no attempt to optimize the dependence in  $q$ . In all of the applications considered here, we only apply the estimate (2.3) for small values of  $q$ , say  $q = O((CDNRS)^{\varepsilon_1})$  for some small  $\varepsilon_1 > 0$ . Such being the case, the reader might still wonder why the bound tends to grow with  $q$ . The main reason is that upon completing the sum over  $d$ , we obtain a Kloosterman sum to modulus  $scq$ , which grows with  $q$ .

In the footsteps of previous work [16], for the proof of our equidistribution results, we separate from the outset of the argument the contribution of characters of small conductors (which is typically well-handled by complex-analytic methods). We only apply the dispersion method to the contribution of characters of large conductors. There is considerable simplification coming from the fact that no ‘Siegel–Walfisz’-type hypothesis is involved in the latter, which allows us to focus on the combinatorial aspect of the method<sup>†</sup>.

In Section 3, we state a few useful lemmas. In Section 4, we adapt the arguments of [15] to prove Theorem 2.1. In Section 5, we employ a variant of the dispersion method to obtain equidistribution for binary convolutions in arithmetic progressions. In Sections 6 and 7, we derive Theorems 1.1, 1.2, 1.5, and 1.6.

NOTATIONS. We use the convention that the letter  $\varepsilon$  denotes a positive number that can be chosen arbitrarily small and whose value may change at each occurrence. The letter  $\delta > 0$  will denote a positive number whose value may change from line to line, and whose dependence on various parameters will be made clear by the context.

We define the Fourier transform  $\widehat{f}$  of a function  $f$  as

$$\widehat{f}(\xi) = \int_{\mathbf{R}} f(t) e(-\xi t) dt.$$

If  $f$  is smooth and compactly supported, the above is well-defined and there holds

$$f(t) = \int_{\mathbf{R}} \widehat{f}(\xi) e(\xi t) d\xi.$$

If moreover  $f$  is supported inside  $[-M, M]$  for some  $M \geq 1$  and  $\|f^{(j)}\|_{\infty} \ll M^{-j}$  for  $j \in \{0, 2\}$ , then we have

$$\widehat{f}(\xi) \ll \frac{M}{1 + (M\xi)^2}.$$

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<sup>†</sup>It is more straightforward to study the mean value of  $\tau_k(n)$  in arithmetic progressions of small moduli, than a  $k$ -fold convolution of slowly oscillating sequences, each supported on a dyadic interval.

### 3. Lemmas

In this section we group a few useful lemmas. The first is the Poisson summation formula, which is very effective at estimating the mean value of a smooth function along arithmetic progressions.

LEMMA 3.1 [7, Lemma 2]. *Let  $M \geq 1$  and  $f : \mathbf{R} \rightarrow \mathbf{C}$  be a smooth function supported on an interval  $[-M, M]$  satisfying  $\|f^{(j)}\|_\infty \ll_j M^{-j}$  for all  $j \geq 0$ . For all  $q \geq 1$  and  $(a, q) = 1$ , with  $H := q^{1+\varepsilon}/M$ , we have*

$$\sum_{m \equiv a \pmod{q}} f(m) = \frac{1}{q} \sum_{|h| \leq H} \hat{f}\left(\frac{h}{q}\right) e\left(\frac{ah}{q}\right) + O_\varepsilon\left(\frac{1}{q}\right).$$

The next lemma is quoted from the work of Shiu [51, Theorem 2], and gives an upper bound of the right order of magnitude for sums of  $\tau_k(n)$  in short intervals and arithmetic progressions of large moduli. It is an analog of the celebrated Brun–Titchmarsh inequality [39, Theorem 6.6].

LEMMA 3.2 [51, Theorem 2]. *For  $k \geq 2$ ,  $x \geq 2$ ,  $x^{1/2} \leq y \leq x$ ,  $(q, a) \in \mathbf{N}$  with  $(a, q) = 1$  and  $q \leq y^{3/4}$ ,*

$$\sum_{\substack{x-y < n \leq x \\ n \equiv a \pmod{q}}} \tau_k(n) \ll_k \frac{y}{q} \left(\frac{\varphi(q)}{q} \log x\right)^{k-1}.$$

Note that such a result could also be deduced from earlier work of Barban and Vekhov [2]; see also [34] for the most recent results on this topic.

The next lemma is the classical form of the multiplicative large sieve inequality [39, Theorem 7.13].

LEMMA 3.3. *Let  $(a_n)$  be a sequence of numbers, and  $N, M, Q \geq 1$ . Then*

$$\sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\substack{\chi \pmod{q} \\ \chi \text{ primitive}}} \left| \sum_{M < n \leq M+N} a_n \chi(n) \right|^2 \leq (Q^2 + N - 1) \sum_{N < n \leq N+M} |a_n|^2.$$

We quote from [31, Number Theory Result 1] the following version of the Pólya–Vinogradov inequality with an explicit dependence on the conductor.

LEMMA 3.4. *Let  $\chi \pmod{q}$  be a character of conductor  $1 \neq r|q$ , and  $M, N \geq 1$ . Then*

$$\sum_{M < n \leq M+N} \chi(n) \ll \tau(q/r) \sqrt{r} \log r.$$

### 4. Sums of Kloosterman sums in arithmetic progressions

Theorem 2.1 is proved by a systematic use of the Kuznetsov formula, which establishes a link between sums of Kloosterman sums and Fourier coefficients of holomorphic and Maaß cusp forms. There is numerous bibliography about this theory; we refer the reader to the books by Iwaniec [36, 38] and to Chapters 14–16 of [39] for references.

Most of the arguments in [15] generalize without the need for substantial new ideas. We will introduce the main notations, and of course provide the required new arguments; but we will refer to [15] for the parts of the proofs that can be transposed *verbatim*.

4.1. *Setting*

4.1.1. *Kloosterman sums.* Let  $q \geq 1$ . The setting is the congruence subgroup

$$\Gamma = \Gamma_0(q) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}), c \equiv 0 \pmod{q} \right\}.$$

Let  $\chi$  be a character modulo  $q_0|q$ , and  $\kappa \in \{0, 1\}$  such that  $\chi(-1) = (-1)^\kappa$ . We warn the reader that the variable  $q$  has a different meaning in Sections 4.1 and 4.2, than in the statement of Theorem 2.1 (where it corresponds to  $qrs$ ). The character  $\chi$  induces a multiplier (that is, here, a multiplicative function) on  $\Gamma$  by

$$\chi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \chi(d).$$

The *cusps* of  $\Gamma$  are  $\Gamma$ -equivalence classes of elements  $\mathbf{R} \cup \{\infty\}$  that are parabolic, that is, each of them is the unique fixed point of some element of  $\Gamma$ . They correspond to cusps on a fundamental domain. A set of representatives is given by rational numbers  $u/w$  where  $1 \leq w, w|q, (u, w) = 1$  and  $u$  is determined (mod  $(w, q/w)$ ).

For each cusp  $\mathbf{a}$ , let  $\Gamma_{\mathbf{a}}$  denote the stabilizer of  $\mathbf{a}$  for the action of  $\Gamma$ . A *scaling matrix* is an element  $\sigma_{\mathbf{a}} \in SL_2(\mathbf{R})$  such that  $\sigma_{\mathbf{a}}\infty = \mathbf{a}$  and

$$\left\{ \sigma_{\mathbf{a}} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \sigma_{\mathbf{a}}^{-1}, b \in \mathbf{Z} \right\} = \Gamma_{\mathbf{a}}.$$

Whenever  $\mathbf{a} = u/w$  with  $u \neq 0, (u, w) = 1$  and  $w|q$ , one can choose

$$\sigma_{\mathbf{a}} = \begin{pmatrix} \mathbf{a}\sqrt{[q, w^2]} & 0 \\ \sqrt{[q, w^2]} & (\mathbf{a}\sqrt{[q, w^2]})^{-1} \end{pmatrix}. \tag{4.1}$$

A cusp  $\mathbf{a}$  is said to be *singular* if  $\chi(\gamma) = 1$  for any  $\gamma \in \Gamma_{\mathbf{a}}$ . When  $\mathbf{a} = u/w$  with  $u$  and  $w$  as above, then this merely means that  $\chi$  has conductor dividing  $q/(w, q/w)$ . The point at infinity is always a singular cusp, with stabilizer

$$\Gamma_{\infty} = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}.$$

For any pair of singular cusps  $\mathbf{a}, \mathbf{b}$  and any associated scaling matrices  $\sigma_{\mathbf{a}}, \sigma_{\mathbf{b}}$ , define the set of moduli

$$\mathcal{C}(\mathbf{a}, \mathbf{b}) := \left\{ c \in \mathbf{R}_+^* : \exists a, b, d \in \mathbf{R}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \sigma_{\mathbf{a}}^{-1}\Gamma\sigma_{\mathbf{b}} \right\}.$$

This set actually only depends on  $\mathbf{a}$  and  $\mathbf{b}$ . For all  $c \in \mathcal{C}(\mathbf{a}, \mathbf{b})$ , let  $\mathcal{D}_{\mathbf{ab}}(c)$  be the set of real numbers  $d$  with  $0 < d \leq c$ , such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \sigma_{\mathbf{a}}^{-1}\Gamma\sigma_{\mathbf{b}}$$

for some  $a, b \in \mathbf{R}$ . For each such  $d$ ,  $a$  is uniquely determined (mod  $c$ ).

For any integers  $m, n \geq 0$ , and any  $c \in \mathcal{C}(\mathbf{a}, \mathbf{b})$ , the Kloosterman sum is defined as (see formula (3.13) and [37, Chapter 4])

$$S_{\sigma_{\mathbf{a}}\sigma_{\mathbf{b}}}(m, n; c) = \sum_{d \in \mathcal{D}_{\mathbf{ab}}(c)} \bar{\chi} \left( \sigma_{\mathbf{a}} \begin{pmatrix} a & * \\ c & d \end{pmatrix} \sigma_{\mathbf{b}}^{-1} \right) e \left( \frac{am + dn}{c} \right),$$

where  $\begin{pmatrix} a & * \\ c & d \end{pmatrix}$  denotes any matrix  $\gamma$  having lower row  $(c, d)$  such that  $\sigma_{\mathbf{a}}\gamma\sigma_{\mathbf{b}}^{-1} \in \Gamma$ . This is well-defined by our hypotheses that  $\mathbf{a}$  and  $\mathbf{b}$  are singular. This definition allows for a great deal of generality. We quote from [15, Section 2.1] the remark that the Kloosterman sums essentially

depend only on the cusps  $\mathfrak{a}$ ,  $\mathfrak{b}$ , and only mildly on the scaling matrices  $\sigma_{\mathfrak{a}}$  and  $\sigma_{\mathfrak{b}}$ , in the following sense. If  $\widetilde{\mathfrak{a}}$  and  $\widetilde{\mathfrak{b}}$  are two cusps, respectively,  $\Gamma$ -equivalent to  $\mathfrak{a}$  and  $\mathfrak{b}$ , with respective scaling matrices  $\widetilde{\sigma}_{\mathfrak{a}}$  and  $\widetilde{\sigma}_{\mathfrak{b}}$ , then there exist real numbers  $t_1$  and  $t_2$ , independent of  $m$  or  $n$ , such that

$$S_{\sigma_{\mathfrak{a}}\sigma_{\mathfrak{b}}}(m, n; c) = e(mt_1 + nt_2)S_{\widetilde{\sigma}_{\mathfrak{a}}\widetilde{\sigma}_{\mathfrak{b}}}(m, n; c).$$

Moreover, the converse fact holds that for any reals  $t_1, t_2$ , any cusps  $\mathfrak{a}$  and  $\mathfrak{b}$ , and any scaling matrices  $\sigma_{\mathfrak{a}}$  and  $\sigma_{\mathfrak{b}}$ , there exist scaling matrices  $\widetilde{\sigma}_{\mathfrak{a}}$  and  $\widetilde{\sigma}_{\mathfrak{b}}$  associated to  $\mathfrak{a}$  and  $\mathfrak{b}$  such that the equality above holds. This rather simple fact is of tremendous help because all of the results obtained through the Kuznetsov formula are uniform with respect to the scaling matrices, so that one can encode oscillating factors depending on  $m$  and  $n$  at no cost (it is crucial for separation of variables). Whenever the context is clear enough, we write

$$S_{\mathfrak{a}\mathfrak{b}}(m, n; c)$$

without reference to the scaling matrices.

The first example is  $\mathfrak{a} = \mathfrak{b} = \infty$  and  $\sigma_{\mathfrak{a}} = \sigma_{\mathfrak{b}} = 1$ . Then  $\mathcal{C}(\infty, \infty) = q\mathbf{N}$  and

$$S_{\infty\infty}(m, n; c) = S_{\chi}(m, n; c) = \sum_{d \pmod{c}^{\times}} \bar{\chi}(d) e\left(\frac{\bar{d}m + dn}{c}\right) \quad (c \in q\mathbf{N}) \tag{4.2}$$

is the usual (twisted) Kloosterman sum. Here and in the rest of the paper, we write  $(\text{mod } c)^{\times}$  to mean a primitive residue class  $(\text{mod } c)$ .

The next example that we need is the case  $\mathfrak{a} = \mathfrak{b}$ . The following is an extension of [15, Lemma 2.5]. It is proven in an identical way, so we omit the details.

LEMMA 4.1. *Assume  $\mathfrak{a} = u/w$  is a cusp with  $(u, w) = 1$ ,  $w|q$  and  $u \neq 0$ . Assume that  $\mathfrak{a}$  is singular. Choose the scaling matrix as in (4.1). Then  $\mathcal{C}(\mathfrak{a}, \mathfrak{a}) = q/(w, q/w)\mathbf{N}$ , and if  $c = \gamma q/(w, q/w)$  for some  $\gamma \in \mathbf{N}$ ,*

$$S_{\mathfrak{a}\mathfrak{a}}(m, n; c) = e\left((w, q/w)\frac{m-n}{uq}\right) \sum_{\delta \pmod{c}^*} \bar{\chi}\left(\alpha + u\frac{\alpha\delta - 1}{\gamma}\right) e\left(\frac{m\alpha + n\delta}{c}\right), \tag{4.3}$$

where, in the sum  $\sum^*$ ,  $\delta$  runs over the solutions  $(\text{mod } c)$  of

$$(\delta, \gamma q/w) = 1, \quad (\gamma + u\delta, w) = 1, \quad \delta(\gamma + u\delta) \equiv u \pmod{(w, q/w)}, \tag{4.4}$$

and  $\alpha$  is determined  $(\text{mod } c)$  by the equations

$$\alpha\delta \equiv 1 \pmod{\gamma q/w}, \quad \alpha \equiv \gamma' \overline{u'} + u' \overline{(\gamma' + u'\delta)} \pmod{w\gamma'}, \tag{4.5}$$

where  $\gamma' = \gamma/(\gamma, u)$  and  $u' = u/(\gamma, u)$ .

The sums  $S_{\mathfrak{a}\mathfrak{a}}(m, n; c)$  are expressed by means of the Chinese remainder theorem (twisted multiplicativity) as a product of similar sums for moduli  $c$  that are prime powers. When  $c = p^{\nu}$  and  $\nu \geq 2$ , a bound is obtained by means of elementary methods as in [39, Section 12.3]. When  $c$  is prime, the Weil bound (cf. [41, Theorem 9.3]) from algebraic geometry can be used. In the general case, one obtains

LEMMA 4.2. *For all  $c \in \mathcal{C}(\mathfrak{a}, \mathfrak{a})$ ,  $m, n \in \mathbf{Z}$ , we have*

$$S_{\mathfrak{a}\mathfrak{a}}(m, n; c) \ll (m, n, c)^{1/2} \tau(c)^{O(1)} (cq_0)^{1/2},$$

where  $q_0$  is the modulus of  $\chi$ .

Finally, we consider as in [15] the following family of Kloosterman sums, which will be of particular interest to us.

LEMMA 4.3. *Assume that the level  $q$  is of the shape  $rs$ , with  $q_0|r$ , where  $q_0$  is the modulus of  $\chi$ , and  $(r, s) = 1$ . The two cusps  $\infty$  and  $1/s$  are singular. Choose the scaling matrices*

$$\sigma_\infty = \text{Id}, \quad \sigma_{1/s} = \begin{pmatrix} \sqrt{r} & 0 \\ s\sqrt{r} & 1/\sqrt{r} \end{pmatrix}.$$

Then  $\mathcal{C}(\infty, 1/s) = \{cs\sqrt{r}, c \in \mathbf{N}, (c, r) = 1\}$ , and for  $(c, r) = 1$ , we have

$$S_{\infty, 1/s}(m, n; cs\sqrt{r}) = \bar{\chi}(c) e\left(\frac{n\bar{s}}{r}\right) S(m\bar{r}, n; sc),$$

where  $S(\dots)$  in the right-hand side is the usual (untwisted) Kloosterman sum.

The main feature here is the presence of the character *outside* the Kloosterman sums, as opposed to (4.2). It is proven in a way identical to [15, p. 240], keeping track of an additional factor  $\bar{\chi}(D)$  in the summand.

4.1.2. *Normalization.* In order to state the Kuznetsov formula, we first fix the normalization. We largely borrow from [3]. We also refer to [17, Section 4] for useful explanations on Maaß forms, and to [48] for a discussion in the case of general multiplier systems.

For each integer  $k > 0$  with  $k \equiv \kappa \pmod{2}$ , we fix a basis  $\mathcal{B}_k(q, \chi)$  of holomorphic cusp forms. It is taken orthonormal with respect to the weight  $k$  Petersson inner product:

$$\langle f, g \rangle_k = \int_{\Gamma \backslash \mathbf{H}} y^k f(z) \overline{g(z)} \frac{dx dy}{y^2} \quad (z = x + iy).$$

We let  $\mathcal{B}(q, \chi)$  denote a basis of the space of Maaß cusp forms. In particular they are functions on  $\mathbf{H}$ , are automorphic of weight  $\kappa \in \{0, 1\}$  (meaning they satisfy [48, formula (5)]), are square-integrable on a fundamental domain and vanish at the cusps (note that when  $\kappa = 1$ , they do not induce a function on  $\Gamma \backslash \mathbf{H}$ ). They are eigenfunctions of the  $L^2$ -extension of the Laplace–Beltrami operator

$$\Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - i\kappa y \frac{\partial}{\partial x}.$$

This operator has pure point spectrum on the  $L^2$ -space of cusp forms. For  $f \in \mathcal{B}(q, \chi)$ , we write  $(\Delta + s(1 - s))f = 0$  with  $s = \frac{1}{2} + it_f$  and  $t_f \in \mathbf{R} \cup [-i/2, i/2]$ . The  $(t_f)_{f \in \mathcal{B}(q, \chi)}$  form a countable sequence with no limit point in  $\mathbf{C}$  (in particular, there are only finitely many  $t_f \in i\mathbf{R}$ ). We choose the basis  $\mathcal{B}(q, \chi)$  orthonormal with respect to the weight zero Petersson inner product. Let

$$\theta := \sup_{f \in \mathcal{B}(q, \chi)} |\Im t_f|, \tag{4.6}$$

then Selberg’s eigenvalue conjecture is that  $\theta = 0$ , that is  $t_f \in \mathbf{R}$  for all  $f \in \mathcal{B}(q, \chi)$ . Selberg proved that  $\theta \leq 1/4$  (see [15, Theorem 4]), and the current best-known result is  $\theta \leq 7/64$ , due to Kim [40] (see [50] for useful explanations on this topic).

The decomposition of the space of square-integrable, weight  $\kappa$  automorphic forms on  $\mathbf{H}$  with respect to eigenspaces of the Laplacian contains the Eisenstein spectrum  $\mathcal{E}(q, \chi)$  which turns out to be the orthogonal complement to the space of Maaß forms. It can be described explicitly by means of the Eisenstein series  $E_{\mathfrak{a}}(z; \frac{1}{2} + it)$  where  $\mathfrak{a}$  runs through singular cusps, and  $t \in \mathbf{R}$ . Care must be taken because these are not square-integrable; see [39, Section 15.4] for more explanations.

Let  $j(g, z) := cz + d$  where  $g = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in SL_2(\mathbf{R})$ . We write the Fourier expansion of  $f \in \mathcal{B}_k(q, \chi)$  around a singular cusp  $\mathfrak{a}$  with associated scaling matrix  $\sigma_{\mathfrak{a}}$  as

$$f(\sigma_{\mathfrak{a}}z)j(\sigma_{\mathfrak{a}}, z)^{-k} = \sum_{n \geq 1} \rho_{f_{\mathfrak{a}}}(n)(4\pi n)^{k/2} e(nz). \tag{4.7}$$

We write the Fourier expansion of  $f \in \mathcal{B}(q, \chi)$  around the cusp  $\mathfrak{a}$  as

$$f(\sigma_{\mathfrak{a}}z) e^{-i\kappa \arg j(\sigma_{\mathfrak{a}}, z)} = \sum_{n \neq 0} \rho_{f_{\mathfrak{a}}}(n) W_{\frac{|n|}{n}, \frac{\kappa}{2}, it_f}(4\pi|n|y) e(nx),$$

where the Whittaker function is defined as in [38, formula (1.26)]. Finally, for every singular cusp  $\mathfrak{c}$ , we write the Fourier expansion around the cusp  $\mathfrak{a}$  of the Eisenstein series associated with the cusp  $\mathfrak{c}$  as

$$\begin{aligned} E_{\mathfrak{c}} \left( \sigma_{\mathfrak{a}}z, \frac{1}{2} + it \right) e^{-i\kappa \arg j(\sigma_{\mathfrak{a}}, z)} &= c_{1,\mathfrak{c}}(t)y^{1/2+it} + c_{2,\mathfrak{c}}(t)y^{1/2-it} \\ &+ \sum_{n \neq 0} \rho_{\mathfrak{c}\mathfrak{a}}(n, t) W_{\frac{|n|}{n}, \frac{\kappa}{2}, it}(4\pi|n|y) e(nx). \end{aligned}$$

4.1.3. *The Kuznetsov formula.* Let  $\phi : \mathbf{R}_+ \rightarrow \mathbf{C}$  be of class  $\mathcal{C}^\infty$  and satisfy

$$\phi(0) = \phi'(0) = 0, \quad \phi^{(j)}(x) \ll (1+x)^{-2-\eta} \quad (0 \leq j \leq 3) \tag{4.8}$$

for some  $\eta > 0$ . In practice, the function  $\phi$  will be  $\mathcal{C}^\infty$  with compact support in  $\mathbf{R}_+^*$ . We define the integral transforms

$$\dot{\phi}(k) := 4i^k \int_0^\infty J_{k-1}(x)\phi(x) \frac{dx}{x}, \tag{4.9}$$

$$\tilde{\phi}(t) := \frac{2\pi it^\kappa}{\sinh(\pi t)} \int_0^\infty (J_{2it}(x) - (-1)^\kappa J_{-2it}(x))\phi(x) \frac{dx}{x}, \tag{4.10}$$

$$\check{\phi}(t) := 8i^{-\kappa} \cosh(\pi t) \int_0^\infty K_{2it}(x)\phi(x) \frac{dx}{x}, \tag{4.11}$$

where we refer to [38, Appendix B.4] for the definitions and estimates on the Bessel functions. We have borrowed the normalization from [4], apart from a constant factor 4 which we included in the transforms. The sizes of these transforms are controlled by the following lemma (we need only consider  $|t| \leq 1/4$  in the second estimate, by Selberg’s theorem that  $\theta \leq 1/4$ ). The bounds we state are not the best that can be obtained, but they will be sufficient for our purpose.

LEMMA 4.4. *If  $\phi$  is supported on  $x \asymp X$  with  $\|\phi^{(j)}\|_\infty \ll X^{-j}$  for  $0 \leq j \leq 4$ , then*

$$\begin{aligned} |\dot{\phi}(t)| + \frac{|\tilde{\phi}(t)|}{1+|t|^\kappa} + |\check{\phi}(t)| &\ll \frac{1+|\log X|}{1+X} \min \left\{ 1, \left( \frac{1+X^{3/2}}{1+|t|^3} \right) \right\} \quad (t \in \mathbf{R}), \\ |\tilde{\phi}(t)| + |\check{\phi}(t)| &\ll \frac{1+X^{-2|t|}}{1+X} \quad (t \in [-i/4, i/4]). \end{aligned} \tag{4.12}$$

*Proof.* These bounds are analogs of [15, Lemma 7.1; 4, Lemma 2.1]. Taking into account the factor  $t^\kappa$  in front of  $\tilde{\phi}(t)$ , the arguments there are easily adapted. The only non-trivial fact to check is that the decaying factor in (4.12) only requires the hypotheses  $\|\phi^{(j)}\|_\infty \ll X^{-j}$

for  $j \leq 4$ . This is seen by reproducing the proof of [4, Lemma 2.1] with the choices  $j = 1$  and  $i = 2$ .  $\square$

Recall that  $\kappa$  is defined by  $\chi(-1) = (-1)^\kappa$ . We are ready to state the Kuznetsov formula for Dirichlet multiplier system and general cusps.

LEMMA 4.5. *Let  $\mathfrak{a}$  and  $\mathfrak{b}$  be two singular cusps with associated scaling matrices  $\sigma_{\mathfrak{a}}$  and  $\sigma_{\mathfrak{b}}$ , and  $\phi : \mathbf{R}_+ \rightarrow \mathbf{C}$  as in (4.8). Let  $m, n \in \mathbf{N}$ . Then*

$$\sum_{c \in \mathcal{C}(\mathfrak{a}, \mathfrak{b})} \frac{1}{c} S_{\mathfrak{a}\mathfrak{b}}(m, n; c) \phi\left(\frac{4\pi\sqrt{mn}}{c}\right) = \mathcal{H} + \mathcal{E} + \mathcal{M}, \tag{4.13}$$

$$\sum_{c \in \mathcal{C}(\mathfrak{a}, \mathfrak{b})} \frac{1}{c} S_{\mathfrak{a}\mathfrak{b}}(m, -n; c) \phi\left(\frac{4\pi\sqrt{mn}}{c}\right) = \mathcal{E}' + \mathcal{M}', \tag{4.14}$$

where  $\mathcal{H}$ ,  $\mathcal{E}$ ,  $\mathcal{M}$  (‘holomorphic’, ‘Eisenstein’, ‘Maab’) are defined by

$$\mathcal{H} := \sum_{\substack{k > \kappa \\ k \equiv \kappa \pmod{2}}} \sum_{f \in \mathcal{B}_k(q, \chi)} \dot{\phi}(k) \Gamma(k) \sqrt{mn} \overline{\rho_{f\mathfrak{a}}(m)} \rho_{f\mathfrak{b}}(n), \tag{4.15}$$

$$\mathcal{E} := \sum_{c \text{ sing.}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \tilde{\phi}(t) \frac{\sqrt{mn}}{\cosh(\pi t)} \overline{\rho_{c\mathfrak{a}}(m, t)} \rho_{c\mathfrak{b}}(n, t) dt, \tag{4.16}$$

$$\mathcal{M} := \sum_{f \in \mathcal{B}(q, \chi)} \tilde{\phi}(t_f) \frac{\sqrt{mn}}{\cosh(\pi t_f)} \overline{\rho_{f\mathfrak{a}}(m)} \rho_{f\mathfrak{b}}(n), \tag{4.17}$$

$$\mathcal{E}' := \sum_{c \text{ sing.}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \check{\phi}(t) \frac{\sqrt{mn}}{\cosh(\pi t)} \overline{\rho_{c\mathfrak{a}}(m, t)} \rho_{c\mathfrak{b}}(-n, t) dt, \tag{4.18}$$

$$\mathcal{M}' := \sum_{f \in \mathcal{B}(q, \chi)} \check{\phi}(t_f) \frac{\sqrt{mn}}{\cosh(\pi t_f)} \overline{\rho_{f\mathfrak{a}}(m)} \rho_{f\mathfrak{b}}(-n). \tag{4.19}$$

*Proof.* For  $\mathfrak{a} = \mathfrak{b} = \infty$ , the formula (4.13) and the case  $\kappa = 0$  of (4.14) can be found in [3, Section 2.1.4]. The extension to general cusps  $\mathfrak{a}$ ,  $\mathfrak{b}$  is straightforward.

The case  $\kappa = 1$  of (4.14) was obtained by Topacogullari (private communication). We restrict here to mentioning that it can be proved by reproducing the computations of [15, p. 251] and [17, Section 5<sup>†</sup>].  $\square$

The right-hand side of the Kuznetsov formula (the so-called spectral side) naturally splits into two contributions. The *regular spectrum* consists in  $\mathcal{H}$ ,  $\mathcal{E}$  and the contribution to  $\mathcal{M}$  of those  $f \in \mathcal{B}(q, \chi)$  with  $t_f \in \mathbf{R}$ ; the conjecturally inexistant *exceptional spectrum* is the contribution to  $\mathcal{M}$  of those  $f$  with  $t_f \in i\mathbf{R}^*$  (similarly with  $\mathcal{E}'$  and  $\mathcal{M}'$ ). The technical reason for this distinction is the growth properties of the integral transforms. Indeed, when  $X$  is small (that is, when the average over the moduli of the Kloosterman sums is long, since  $X \asymp \sqrt{mn}/c$ ), we

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<sup>†</sup>Note that in the expression for  $h_p(t)$  given on page 518 of [17], the term  $\Gamma(1 - \frac{k}{2} - ir)$  should read  $\Gamma(1 - \frac{k}{2} + ir)$ .

see from Lemma 4.4 that while  $\dot{\phi}(t)$ ,  $\tilde{\phi}(t)$ , and  $\check{\phi}(t)$  are essentially bounded for  $t \in \mathbf{R}$ ,  $\tilde{\phi}(it)$  is roughly of size  $X^{-2|t|}$  when  $t \in [-1/2, 1/2]$ .

We remark that in contrast with other works (for example, [6]), we do not make use of Atkin–Lehner’s newform theory, nor of Hecke theory. In fact, we do not use any information about the Fourier coefficients  $\rho_{f_{\mathfrak{a}}}(n)$  and  $\rho_{c_{\mathfrak{a}}}(n, t)$  other than the fact that Kuznetsov’s formula holds, so the reader unfamiliar with the subject can go through the following sections without knowing what they are. The main feature of the Kuznetsov formula which is used is the decay properties of the integral transforms (4.9)–(4.11), and the fact that it separates the variables  $m$  and  $n$  in a way that combines very nicely with the Cauchy–Schwarz inequality.

#### 4.2. Large sieve inequalities

4.2.1. *Quadratic forms with  $S_{\mathfrak{a}\mathfrak{a}}$ .* Given  $N \in \mathbf{N}$ ,  $\vartheta \in \mathbf{R}_+^*$ ,  $\lambda \geq 0$ , a sequence  $(b_n)$  of complex numbers, a singular cusp  $\mathfrak{a}$  and  $c \in \mathcal{C}(\mathfrak{a}, \mathfrak{a})$ , let

$$B_{\mathfrak{a}}(\lambda, \vartheta; c, N) := \sum_{N < m, n \leq 2N} b_m \overline{b_n} e^{-\lambda \sqrt{mn}} S_{\mathfrak{a}\mathfrak{a}}(m, n, c) e\left(\frac{2\sqrt{mn}}{c} \vartheta\right).$$

We also define

$$\|b_N\|_2 := \left( \sum_{N < n \leq 2N} |b_n|^2 \right)^{1/2}.$$

The following extends [15, Proposition 3].

LEMMA 4.6 [15, Proposition 3]. *We have*

$$|B_{\mathfrak{a}}(\lambda, \vartheta; c, N)| \leq \tau(c)^{O(1)} (q_0 c)^{1/2} N \|b_N\|^2, \tag{4.20}$$

$$|B_{\mathfrak{a}}(\lambda, \vartheta; c, N)| \ll (c + N + \sqrt{\vartheta c N}) \|b_N\|^2,$$

$$|B_{\mathfrak{a}}(\lambda, \vartheta; c, N)| \ll_{\varepsilon} \vartheta^{-1/2} c^{1/2} N^{1/2+\varepsilon} \|b_N\|^2, \tag{4.21}$$

where the last bounds hold for  $\vartheta < 2$  and  $c < N$ .

*Proof.* Suppose  $\lambda = 0$ . The first bound is an immediate consequence of Lemma 4.2. For the second bound, the proof given in [15, p. 256] transposes without any change: after expanding out the sum  $S_{\mathfrak{a}\mathfrak{a}}(\dots)$ , one uses the triangle inequality with the effect that the factors involving  $\chi$  are trivially bounded. For the last bound, the proof is adapted with the following modification: the Cauchy–Schwarz inequality yields

$$|B_{\mathfrak{a}}(0, \vartheta; c, N)|^2 \leq \|b_N\|_2^2 \sum_{\substack{N < m_1, m_2 \leq 2N \\ \delta_1, \delta_2}} b_{m_1} \overline{b_{m_2}} \chi(r_1) \chi(r_2) e\left(\frac{m_1 \delta_1 - m_2 \delta_2}{c}\right) \sum_n f(n), \tag{4.22}$$

where  $f(n)$  is defined as in [15, p. 256],  $\delta_1$  and  $\delta_2$  run over residue classes modulo  $c$  satisfying (4.4), and  $r_j := \delta_j^{-1} + u(\alpha_j \delta_j - 1)/\gamma$  for  $j \in \{1, 2\}$ , where  $\alpha_j$  is determined by (4.5). The only difference is the presence of the  $\chi$  factors. Upon using Poisson summation on the sum  $\sum_n f(n)$ , the argument is split in two cases according to whether  $\alpha_1 \equiv \alpha_2 \pmod{c}$  or not. If  $\alpha_1 \not\equiv \alpha_2 \pmod{c}$ , then one uses the triangle inequality on (4.22) so that the  $\chi$  factors do not intervene. If on the contrary  $\alpha_1 \equiv \alpha_2 \pmod{c}$ , then we deduce from (4.5) that also  $\delta_1 \equiv \delta_2 \pmod{c}$ . The  $\chi$  factors cancel out and the rest of the argument carries through without change.



The case of arbitrary  $\lambda \geq 0$  reduces to the case  $\lambda = 0$  by Mellin inversion

$$e^{-y} = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \Gamma(s)y^{-s} ds = 1 + \frac{1}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} \Gamma(s)y^{-s} ds$$

at  $y = \lambda\sqrt{mn}$ , using the first expression when  $\lambda N \geq 1$  and the second otherwise. □

4.2.2. *Large sieve inequalities for the regular spectrum.* We proceed to state the following large sieve-type inequalities, which extend [15, Proposition 4].

PROPOSITION 4.7. *Let  $(a_n)$  be a sequence of complex numbers, and  $\mathfrak{a}$  a singular cusp for the group  $\Gamma_0(q)$  and Dirichlet multiplier  $\chi \pmod{q_0}$ . Suppose  $T \geq 1$  and  $N \geq 1/2$ . Then each of the three quantities*

$$\sum_{\substack{\kappa < k \leq T \\ k \equiv \kappa \pmod{2}}} \Gamma(k) \sum_{f \in \mathcal{B}_k(q, \chi)} \left| \sum_{N < n \leq 2N} a_n \sqrt{n} \rho_{f\mathfrak{a}}(n) \right|^2, \tag{4.23}$$

$$\sum_{\substack{f \in \mathcal{B}(q, \chi) \\ |t_f| \leq T}} \frac{(1 + |t_f|)^{\pm\kappa}}{\cosh(\pi t_f)} \left| \sum_{N < n \leq 2N} a_n \sqrt{n} \rho_{f\mathfrak{a}}(\pm n) \right|^2, \tag{4.24}$$

$$\sum_{c \text{ sing.}} \int_{-T}^T \frac{(1 + |t|)^{\pm\kappa}}{\cosh(\pi t)} \left| \sum_{N < n \leq 2N} a_n \sqrt{n} \rho_{c\mathfrak{a}}(\pm n, t) \right|^2 dt, \tag{4.25}$$

is majorized by

$$O_\varepsilon((T^2 + q_0^{1/2} \mu(\mathfrak{a}) N^{1+\varepsilon}) \|a_N\|_2^2).$$

Here, if  $\mathfrak{a}$  is equivalent to  $u/w$  with  $w|q$  and  $(u, w) = 1$ , then  $\mu(\mathfrak{a}) := (w, q/w)/q$ .

*Proof.* These formulas are deduced from two summation formulas, namely the Petersson formula [37, Theorem 3.6]

$$\mathbf{1}_{m=n} + 2\pi i^{-k} \sum_{c \in \mathcal{C}(\mathfrak{a}, \mathfrak{a})} \frac{1}{c} S_{\mathfrak{a}\mathfrak{a}}(m, n; c) J_{k-1} \left( \frac{4\pi\sqrt{mn}}{c} \right) = 4\Gamma(k-1)\sqrt{mn} \sum_{f \in \mathcal{B}_k(q, \chi)} \overline{\rho_{f\mathfrak{a}}(m)} \rho_{f\mathfrak{a}}(n), \tag{4.26}$$

valid for  $k > 1$ ,  $k \equiv \kappa \pmod{2}$ , and a ‘pre-Kuznetsov’ formula [17, Proposition 5.2] which, for general cusps, is

$$\begin{aligned} & \frac{|\Gamma(1 \mp \frac{\kappa}{2} + ir)|^2}{4\pi^2} \left\{ \mathbf{1}_{m=n} + \sum_{c \in \mathcal{C}(\mathfrak{a}, \mathfrak{a})} \frac{1}{c} S_{\mathfrak{a}\mathfrak{a}}(\pm m, \pm n; c) I_\pm \left( \frac{4\pi\sqrt{mn}}{c} \right) \right\} \\ &= \sum_{f \in \mathcal{B}(q, \chi)} \frac{\sqrt{mn}}{\cosh(\pi t_f)} H(t_f, r) \overline{\rho_{f\mathfrak{a}}(\pm m)} \rho_{f\mathfrak{a}}(\pm n) \\ &+ \frac{1}{4\pi} \sum_{c \text{ sing.}} \int_{-\infty}^{\infty} \frac{\sqrt{mn}}{\cosh(\pi t)} H(t, r) \overline{\rho_{c\mathfrak{a}}(\pm m)} \rho_{c\mathfrak{a}}(\pm n) dt \end{aligned} \tag{4.27}$$

for all real  $r$  and positive integers  $m, n$ . Here,

$$H(t, r) = \frac{\cosh(\pi t) \cosh(\pi r)}{\cosh(\pi(t - r)) \cosh(\pi(t + r))} \quad (r, t \in \mathbf{C}, r \notin \pm t + i/2 + i\mathbf{Z}),$$

$$I_{\pm}(x) = -2x \int_{-i}^i (-iv)^{\pm\kappa-1} K_{2ir}(vx) \, dv \quad (x > 0),$$

where  $v$  varies on the half-circle  $|v| = 1, \Re(v) \geq 0$  counterclockwise. Note that by the complement formula

$$\left| \Gamma\left(1 - \frac{\epsilon}{2} + ir\right) \right|^2 = \frac{\pi}{\cosh(\pi r)} \times \begin{cases} 1, & \epsilon = 1, \\ \frac{1}{4} + r^2, & \epsilon = -1. \end{cases} \tag{4.28}$$

Given formulas (4.26) and (4.27), the arguments in [15, pp. 258–261] are adapted as follows. When  $\kappa = 0$ , the details are strictly identical. Consider the case  $\kappa = 1$  of (4.23). We multiply both sides of (4.26) by  $(k - 1) e^{-(k-1)/T} \overline{a_m} a_n$  and sum over  $k, m$  and  $n$ . The analog of the function  $E_K(x)$  defined in [15, p. 258] is (up to a constant factor) the function

$$E_T(x) = \sum_{\ell \geq 1} (-1)^\ell 2\ell e^{-2\ell/T} J_{2\ell}(x) = -\frac{1}{2} \sinh\left(\frac{1}{T}\right) \int_0^1 \frac{u^2 x J_1(ux) \, du}{(\cosh(1/T)^2 - u^2)^{3/2}},$$

as can be seen by reproducing the computations in [35, p. 316]<sup>†</sup>. We then write (see [29, equation (8.411.3), p. 912])

$$J_1(y) = \frac{2}{\pi} \int_0^{\pi/2} \cos \tau \sin(y \cos \tau) \, d\tau,$$

split the integral at  $\Delta \in (0, \pi/2]$  and deduce the bound (4.23) by following the steps in [15, p. 259].

Consider next the case  $\kappa = 1$  and positive sign of (4.24) and (4.25). We multiply both sides of (4.27) by  $r^2 \cosh(\pi r) e^{-(r/T)^2} \overline{a_m} a_n$ , integrate over  $r \in \mathbf{R}$  and sum over  $m$  and  $n$ . The analog of the function  $\Phi(x)$  of [15, p. 260] is the function

$$\Phi_+(x) = \int_{-\infty}^{\infty} r^2 e^{-(r/T)^2} \int_{-i}^i K_{2ir}(xv) \, dv \, dr.$$

We use the expression  $K_{2ir}(y) = \int_0^\infty e^{-y \cosh \xi} \cos(2r\xi) \, d\xi$  ( $\Re y > 0$ ). For  $x > 0$ , we obtain by integrations by parts

$$\begin{aligned} \Phi_+(x) &= -i\sqrt{\pi} T^3 \int_0^\infty e^{-(\xi T)^2} \xi \tanh \xi \left\{ \cos(x \cosh \xi) - \frac{1}{2} \int_{-1}^1 \cos(x\vartheta \cosh \xi) \, d\vartheta \right\} \, d\xi \\ &= i\sqrt{\pi} \frac{T^3}{x} \int_0^\infty e^{-(\xi T)^2} (1 - 2(\xi T)^2) \sinh(x \cosh \xi) \frac{d\xi}{\cosh \xi}, \end{aligned}$$

and from there, the bounds (4.24) and (4.25) are obtained by reproducing the computations of [15, p. 261].

Consider finally the case of negative sign in (4.24) and (4.26). We multiply both sides of (4.27) by  $r^2 \cosh(\pi r) / (\frac{1}{4} + r^2) e^{-(r/T)^2} \overline{a_m} a_n$ . The analog of the function  $\Phi(x)$  of [15, p. 260] is now

$$\Phi_-(x) = \int_{-\infty}^{\infty} r^2 e^{-(r/T)^2} \int_{-i}^i K_{2ir}(xv) \frac{dv}{v^2} \, dr,$$

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<sup>†</sup>There is a slight convergence issue in the Fourier integral for  $yJ_1(y)$ , which is resolved by changing  $b = \cosh(1/T)$  to  $b + i\varepsilon, \varepsilon > 0$  and letting  $\varepsilon \rightarrow 0$ .

and we have by integration by parts

$$\begin{aligned} \Phi_-(x) &= i\sqrt{\pi}T^3 \int_0^\infty e^{-(\xi T)^2} \xi \tanh \xi \left\{ \cos(x \cosh \xi) - \frac{1}{2i} \int_{-i}^i \frac{e^{-vx \cosh \xi}}{v^2} dv \right\} d\xi \\ &= -i\sqrt{\pi} \frac{T^3}{x} \int_0^\infty e^{-(\xi T)^2} (1 - 2(\xi T)^2) \left\{ \sinh(x \cosh \xi) + \frac{1}{i} \int_{-i}^i \frac{e^{-xv \cosh \xi}}{v^3} dv \right\} \frac{d\xi}{\cosh \xi}. \end{aligned}$$

From there, it is straightforward to reproduce the computations of [15, p. 261] using the bounds of Lemma 4.6.  $\square$

4.2.3. *Weighted large sieve inequalities for the exceptional spectrum.* The objects we would like to bound now are of the shape

$$E_{q,\mathfrak{a}}(Y, (a_n)) := \sum_{\substack{f \in \mathcal{B}(q, \chi) \\ t_f \in i\mathbf{R}}} Y^{2|t_f|} \left| \sum_{N < n \leq 2N} a_n n^{1/2} \rho_{f\mathfrak{a}}(n) \right|^2,$$

where  $Y \geq 1$  is to be taken as large as possible while still keeping this quantity comparable to the bounds  $(1 + \mu(\mathfrak{a})N) \sum_n |a_n|^2$  coming from Proposition 4.7. The following is the analog of [15, Theorem 5].

LEMMA 4.8. *Assume that the situation is as in Proposition 4.7. Then for any  $Y \geq 1$ ,*

$$E_{q,\mathfrak{a}}(Y, (a_n)) \ll_\varepsilon (1 + (\mu(\mathfrak{a})NY)^{1/2}) (1 + (q_0\mu(\mathfrak{a})N)^{1/2+\varepsilon}) \|a_N\|_2^2.$$

The important aspect in this bound is that it is as good as those coming from the regular spectrum (that is, the upper bound in Proposition 4.7) in the situation when  $\mu(\mathfrak{a}) = 1/q$  (which will typically be the case),  $N < q$  and  $Y \leq q/N$ . Note also that the previous bound holds for any individual  $q$ .

*Proof.* The arguments in [15, Section 8.1, pp. 270–271] transpose identically<sup>†</sup>.  $\square$

The next step is to produce an analog of [15, Theorem 6], which is concerned with the situation when an average over  $q$  is done. Deshouillers and Iwaniec make use of the very nice idea that with the choice  $\mathfrak{a} = \infty$  for each  $q$ , the roles of  $q$  and  $c$  can be swapped in the Kuznetsov formula. Through an induction process, this enhances significantly the bounds obtained. This switching technique is specific to the choice  $\mathfrak{a} = \infty$  for all  $q$ , with scaling matrices independent of  $q$ .

LEMMA 4.9. *Assume the situation is as previously. Recall that  $\chi$  has modulus  $q_0 \geq 1$ . Then for all  $Y \geq 1$  and  $Q \geq q_0$ ,*

$$\sum_{\substack{q \leq Q \\ q_0 | q}} E_{q,\infty}(Y, (a_n)) \ll_\varepsilon (QN)^\varepsilon (Qq_0^{-1} + N + NY^{1/2}) \|a_N\|_2^2,$$

where the scaling matrices are chosen independently of  $q$ .

Note that now, in the situation when  $N \leq Q$ , the parameter  $Y$  is allowed to be as large as  $(Q/N)^2$  while still yielding a bound of same quality as the regular spectrum. The final situation is the special case when  $(a_n)$  is the characteristic sequence of an interval of integers.

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<sup>†</sup>Note that in the last display of the proof [15, p. 271],  $L(Y)$  should read  $L(Y^{-1})$ .

Then Deshouillers and Iwaniec are able to provide an even stronger bound [15, Theorem 7], by enhancing the initial step in the induction.

LEMMA 4.10. *Assume that the situation is as in Lemma 4.9. Assume moreover that  $(a_n)_{N < n \leq 2N}$  is the characteristic sequence of an interval of integers. Then*

$$\sum_{\substack{q \leq Q \\ q_0 | q}} E_{q, \infty}(Y, (a_n)) \ll_{\varepsilon} (QN)^{\varepsilon} (Qq_0^{-1} + N + (NY)^{1/2})N.$$

In the situation when  $N \leq Q$ , the parameter  $Y$  can then be taken as large as  $Q^2/N$  while still yielding an acceptable bound.

We now proceed to justify Lemmas 4.9 and 4.10. For the rest of this section, we rename  $q$  into  $q_0q$ , so that now  $q$  runs over intervals. The object of interest is

$$S(Q, Y, N, s) := \sum_{Q < q \leq 16Q} \sum_{\substack{f \in \mathcal{B}(q_0q, \chi) \\ t_f \in i\mathbf{R}}} Y^{2|t_f|} \left| \sum_{N < n \leq 2N} a_n n^{s+1/2} \rho_{f\infty}(n) \right|^2.$$

LEMMA 4.11. *Let  $N, Y, Q \geq 1$  and a sequence  $(a_n)$  be given. Then*

$$S(Q, Y, N, 0) \ll_{\varepsilon} \int_{-\infty}^{\infty} S\left(\frac{\pi NY}{q_0Q}, Y, N, it\right) \frac{dt}{t^4 + 1} + (QYN)^{\varepsilon} \left(Q + \frac{N}{q_0^{1/2}} + \frac{NY}{q_0^{1/2}Q}\right) \|a_N\|_2^2. \tag{4.29}$$

Moreover, if  $(a_n)$  is the characteristic sequence of an interval, then

$$S(Q, Y, N, 0) \ll_{\varepsilon} (NY)^{\varepsilon} (Q + N + Y)N \tag{4.30}$$

*Proof of (4.29).* The arguments in [15, pp. 272–273] are adapted with minimal effort; however we take the opportunity to justify more precisely one of the claims made there. Fix a smooth function  $\Phi : \mathbf{R} \rightarrow [0, 1]$  supported inside  $[1/2, 5/2]$  and majorizing  $\mathbf{1}_{[1,2]}$ . Letting  $g(q) = \Phi(q/Q)$  and  $\phi(x) = \Phi(Yx)$  (these kinds of homotheties of  $\Phi$  we refer to as *test functions*) we have

$$S(Q, Y, N, 0) \ll |\mathcal{S}_1|,$$

$$\mathcal{S}_1 := \sum_{q \geq 1} g(q) \sum_{\substack{f \in \mathcal{B}(q_0q, \chi) \\ t_f \in i\mathbf{R}}} \frac{\tilde{\phi}(t_f)}{\cosh(\pi t_f)} \left| \sum_n a_n n^{1/2} \rho_{f\infty}(n) \right|^2.$$

This is seen by approximating the Bessel function in the definition of  $\tilde{\phi}$  by its first order term, as in [15, formula (8.1)]. Opening the squares in  $\mathcal{S}_1$  and applying the Kuznetsov formula and the large sieve estimates (Lemma 4.5 and Proposition 4.7), one gets

$$\mathcal{S}_1 = \sum_{m, n} \overline{a_m} a_n \mathcal{S}_2(m, n) + O_{\varepsilon} \left( (QNY)^{\varepsilon} \left(Q + \frac{N}{q_0^{1/2}}\right) \sum_n |a_n|^2 \right),$$

$$\mathcal{S}_2(m, n) := \sum_{q, c \geq 1} \frac{g(q)}{q_0qc} \phi\left(\frac{4\pi\sqrt{mn}}{q_0qc}\right) S_{\infty\infty}(m, n; qc).$$

Letting  $h(x) = h_{m,n,c}(x) = \phi(x)g(4\pi\sqrt{mn}/q_0cx)$ , one applies the Kuznetsov formula for the group  $\Gamma_0(q_0c)$  (which requires that the scaling matrices be independent of  $q$ ) and obtains

$$\mathcal{S}_1 \ll |\mathcal{S}_3| + O_\varepsilon \left( (QNY)^\varepsilon \left( Q + \frac{N}{q_0^{1/2}} + \frac{NY}{q_0^{1/2}Q} \right) \sum_n |a_n|^2 \right),$$

$$\mathcal{S}_3 := \sum_{m,n} \overline{a_m} a_n \sum_{C < c \leq 16C} \sum_{\substack{f \in \mathcal{B}(q_0c, \chi) \\ t_f \in i\mathbf{R}}} \frac{\tilde{h}(t_f)}{\cosh(\pi t_f)} \sqrt{mn} \overline{\rho_{f\infty}(m)} \rho_{f\infty}(n).$$

Note that  $h(t_f) = h_{m,n,c}(t_f) = 0$  unless  $C < c \leq 16C$ , where  $C = \pi NY/(q_0Q)$ . Let

$$\mathcal{K}_{\kappa,t}(x) := \frac{2\pi i t^\kappa}{\sinh(\pi t)} (J_{2it}(x) - (-1)^\kappa J_{-2it}(x)),$$

and  $\check{g}(s) := \int_0^\infty g(x)x^{s-1} dx$  be the Mellin transform of  $g$ . Then

$$\tilde{h}(t) = \frac{1}{2\pi} \int_{-\infty}^\infty \check{g}(i\tau) \left( \frac{q_0c}{4\pi\sqrt{mn}} \right)^{i\tau} \int_0^\infty \mathcal{K}_{\kappa,t}(x)x^{i\tau} \phi(x) dx d\tau.$$

Inserting into the definition of  $\mathcal{S}_3$  and using the triangle inequality, we obtain

$$\mathcal{S}_3 \ll \int_{-\infty}^\infty |\check{g}(i\tau)| \sum_{C < c \leq 16C} \sum_{\substack{f \in \mathcal{B}(q_0c, \chi) \\ t_f \in i\mathbf{R}}} \left| \sum_m a_m m^{(1+i\tau)/2} \rho_{f\infty}(m) \right| \left| \sum_n a_n n^{(1-i\tau)/2} \rho_{f\infty}(m) \right|$$

$$\times \left| \int_0^\infty \mathcal{K}_{\kappa,t}(x)x^{i\tau} \phi(x) dx \right| d\tau.$$

From there, the arguments in [15, p. 273] apply and yield

$$\left| \int_0^\infty \mathcal{K}_{\kappa,t}(x)x^{i\tau} \phi(x) dx \right| \ll_\varepsilon Y^{2|t_f|} + Y^\varepsilon$$

from which the claimed bound follows in the same way as [15, p. 273]. □

*Proof of (4.30).* Assume that  $(a_n)_{N < n \leq 2N}$  is the characteristic sequence of the integers inside  $(N, N_1]$  for some  $N_1 \leq 2N$ . We proceed as in [15, p. 276]. By applying the Kuznetsov formula and the large sieve inequalities, one obtains

$$S(Q, N, Y, 0) \ll_\varepsilon \sum_{Q < q \leq 16Q} \sum_{c \geq 1} \frac{1}{q_0qc} \left| \sum_{N \leq m, n \leq N_1} \phi \left( \frac{4\pi\sqrt{mn}}{q_0qc} \right) S_{\infty\infty}(m, n; qq_0c) \right|$$

$$+ \left( Q + \frac{N^{1+\varepsilon}}{q_0^{1/2}} \right) N$$

for a test function  $\phi$  supported inside  $[1/(2Y), 5/(2Y)]$ . Here one may restrict summation to  $C/4 < c \leq 8C$  for  $C := \pi NY/(q_0Q)$ . Let  $k := q_0qc$ . The first term above is majorized by

$$T := (q_0QC)^{-1+\varepsilon} \sum_{\substack{k \asymp q_0QC \\ q_0|k}} \left| \sum_{N < m, n \leq N_1} \phi \left( \frac{4\pi\sqrt{mn}}{k} \right) S_{\infty\infty}(m, n; k) \right|.$$

Let  $\phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \check{\phi}(it)x^{-it} dt$ , where the Mellin transform  $\check{\phi}(s) = \int_0^{\infty} \phi(x)x^{s-1} dx$  satisfies  $\check{\phi}(it) \ll (1+t^4)^{-1}$ , so that (after reinterpreting  $t$  by  $2t$ )

$$T \ll (q_0QC)^{-1+\varepsilon} \int_{-\infty}^{\infty} \frac{1}{t^4+1} \sum_{\substack{k \asymp q_0QC \\ q_0|k}} \left| \sum_{N < m, n \leq N_1} (mn)^{-it} e((m-n)\vartheta) S_{\chi}(m, n; k) \right| dt$$

for some  $\vartheta \in [0, 1)$  (depending on the scaling matrix). By  $m^{-it} = N_1^{-it} + it \int_m^{N_1} u^{-it-1} du$ , we obtain

$$T \ll (q_0QC)^{-1+\varepsilon} \sup_{N \leq N', M' \leq N_1} \sum_{\substack{k \asymp q_0QC \\ q_0|k}} U_1(k, M', N'),$$

$$U_1(M', N') := \left| \sum_{\substack{m \leq M' \\ n \leq N'}} e((m-n)\vartheta) S_{\chi}(m, n; k) \right|.$$

Opening the summation in  $S_{\chi}$ , we have

$$U_1(k, M', N') \leq U_2(k, M', N') := \sum_{\delta \pmod k} \left| \sum_{m \leq M'} e\left(\frac{\delta m}{k} + m\vartheta\right) \right| \left| \sum_{n \leq N'} e\left(\frac{\delta n}{k} - n\vartheta\right) \right|.$$

It is crucial to note that the quantity on the right-hand side also exists for  $k$  not multiple of  $q_0$ , so trivially

$$T \ll (q_0QC)^{-1+\varepsilon} \sup_{N \leq M', N' \leq N_1} \sum_{k \asymp q_0QC} U_2(k, M', N'),$$

From there on, the calculations in [15, p. 276] apply and yield, in the notation of [15, Lemma 8.2],

$$U_2(k, M', N') \ll \sum_{m, n \in \mathbf{Z}} \widehat{f}_{M'}(m) e(m\vartheta) \widehat{f}_{N'}(n) e(-n\vartheta) S(m, n; k).$$

The proof of [15, Theorem 14] follows through, and yields for all  $K \geq 1$ ,

$$\sum_{k \leq K} U_2(k, M', N') \ll_{\varepsilon} (KMN)^{\varepsilon} K(K + MN).$$

Taking  $K \asymp q_0QC$ , we conclude that

$$T \ll_{\varepsilon} (q_0QC)^{\varepsilon} (q_0QC + N^2).$$

The rest of the arguments in [15, p. 277] applies and yields

$$S(Q, N, Y, 0) \ll_{\varepsilon} (NY)^{\varepsilon} (Q + N + Y)N$$

as claimed. □

*Proof of Lemmas 4.9 and 4.10.* In addition to the recurrence relation (4.29), we have the properties

$$S(Q, Y, N, 0) \leq (Y/Z)^{1/2} S(Q, Z, N, 0) \quad (1 \leq Z \leq Y),$$

$$S(Q, 1, N, 0) \ll_{\varepsilon} (QN)^{\varepsilon} \left( Q + \frac{N}{q_0^{1/2}} \right) \|a_N\|_2^2.$$

The second one follows from Proposition 4.7. Having these at hand, the induction arguments in [15, p. 274; 15, p. 277] are easily reproduced. It is useful to notice that  $q_0$  appears only with negative powers in the error terms, and that its presence in the denominator of  $\pi NY/(q_0Q)$  in (4.29) is beneficial for the induction.  $\square$

REMARK. The previous three lemmas used only Selberg’s theorem that  $\theta \leq 1/4$  (recall the definition (4.6)). One could make the bounds explicit in terms of  $\theta$  and thus benefit from recent progress toward the Ramanujan–Selberg conjecture. It is straightforward to check that Lemmas 4.8, 4.9, and 4.10 hold with the right-hand sides replaced by

$$\begin{aligned} & (1 + (\mu(\mathbf{a})NY)^{2\theta})(1 + q_0^{1/2}(\mu(\mathbf{a})N)^{1-2\theta+\varepsilon})\|a_N\|_2^2, \\ & (QN)^\varepsilon(Qq_0^{-1} + N + Y^{2\theta}N^{4\theta}Q^{1-4\theta})\|a_N\|_2^2, \\ & (QN)^\varepsilon(Qq_0^{-1} + N + Y^{2\theta}N^{2\theta}Q^{1-4\theta})N, \end{aligned}$$

respectively (cf. [39, Proposition 16.10]). We refrain from doing so because it would not impact the applications considered here.

### 4.3. Proof of Theorem 2.1

4.3.1. *Estimates for sums of generalized Kloosterman sums.* We begin by the following statement regarding the generalized Kloosterman sums  $S_{\mathbf{a},\mathbf{b}}(m, n; c)$ . For the sake of simplifying the presentation of the bound obtained, we discard powers of the modulus  $q$ . This does not have consequences on our applications.

PROPOSITION 4.12. *Let the real numbers  $M, N, R, S \geq 1$ ,  $X > 0$  and the integer  $q \geq 1$  be given, let  $\chi$  be a character modulo  $q$ , let  $\phi$  be a smooth function supported on the interval  $[X, 2X]$  such that  $\|\phi^{(j)}\|_\infty \ll X^{-j}$  for  $0 \leq j \leq 4$ , and let  $(a_m)$  and  $(b_{n,r,s})$  be sequences of complex numbers supported on  $M < m \leq 2M$ ,  $N < n \leq 2N$ ,  $R < r \leq 2R$  and  $S < s \leq 2S$ . Assume that  $(a_m)$  is the characteristic sequence of an interval of integers. Then*

$$\begin{aligned} & \sum_{\substack{m,n,r,s \\ (s,rq)=1}} a_m b_{n,r,s} \sum_{c \in \mathcal{C}(\infty, 1/s)} \frac{1}{c} \phi\left(\frac{4\pi\sqrt{mn}}{c}\right) S_{\infty, 1/s}(m, \pm n; c) \\ & \ll_\varepsilon (q(X + X^{-1})RSMN)^\varepsilon \{L_{\text{reg}} + L_{\text{exc}}\}, \end{aligned} \tag{4.31}$$

$$\begin{aligned} L_{\text{reg}} & := \left(1 + X + \sqrt{\frac{N}{RS}}\right) \left(1 + X + \sqrt{\frac{M}{RS}}\right) \frac{\sqrt{RS}}{1 + X} \sqrt{M} \|b_{N,R,S}\|_2, \\ L_{\text{exc}} & := \left(1 + \sqrt{\frac{N}{RS}}\right) \sqrt{\frac{1 + X^{-1}}{RS}} \left(\frac{MN}{RS + N}\right)^{1/4} \frac{\sqrt{RS}}{1 + X} \sqrt{M} \|b_{N,R,S}\|_2, \end{aligned}$$

where the Kloosterman sum is defined with respect to the congruence group  $\Gamma(qrs)$  with multiplier induced by  $\chi$ , with scaling matrices  $\sigma_\infty$  and  $\sigma_{1/s}$  that are both independent of  $m$  and  $n$ , with  $\sigma_\infty$  independent of  $r$  and  $s$  as well.

REMARK. If  $(a_m)$  is not the characteristic sequence of an interval, then the bound (4.31) still holds with  $L_{\text{exc}}$  replaced by  $M^{1/4}L_{\text{exc}}$  (see [15, Theorems 10 and 11]).

*Proof.* This estimate is deduced from Proposition 4.7 and Lemmas 4.8 and 4.10 by following the computations of [15, Section 9.1]. It is useful to notice that the bounds of Lemmas 4.8, 4.10, and Proposition 4.7 (for  $\mathbf{a} \in \{\infty, 1/s\}$ ) decrease with  $q_0$ .  $\square$

4.3.2. *Estimates for the complete Kloosterman sums twisted by a character.* We now justify the transition from Proposition 4.12 to an estimate for twisted sums of usual Kloosterman sums  $S(m, n; c)$ .

PROPOSITION 4.13. *Let the real numbers  $M, N, R, S, C \geq 1$ , and the integer  $q \geq 1$  be given, let  $\chi$  be a character modulo  $q$ , let  $g$  be a smooth function supported on  $[C, 2C] \times [M, 2M] \times (\mathbf{R}_+^*)^3$  such that*

$$\frac{\partial^{\nu_0+\nu_1+\nu_2+\nu_3+\nu_4} g}{\partial c^{\nu_0} \partial m^{\nu_1} \partial n^{\nu_2} \partial r^{\nu_3} \partial s^{\nu_4}}(c, m, n, r, s) \ll C^{-\nu_0} M^{-\nu_1} N^{-\nu_2} R^{-\nu_3} S^{-\nu_4} \tag{4.32}$$

for  $0 \leq \nu_j \leq 12$ . Let  $(b_{n,r,s})$  be a sequence of complex numbers supported on  $N < n \leq 2N$ ,  $R < r \leq 2R$ , and  $S < s \leq 2S$ . Then uniformly in  $t \in [0, 1)$ ,

$$\sum_{\substack{c,m,n,r,s \\ (sc,rq)=1}} b_{n,r,s} \bar{\chi}(c) g(c, m, n, r, s) e(mt) S(n\bar{r}, \pm m\bar{q}; sc) \ll_{\epsilon} (CRSMNq)^{\epsilon} q^{3/2} \{K_{\text{reg}} + K_{\text{exc}}\} \sqrt{M} \|b_{N,R,S}\|_2, \tag{4.33}$$

$$K_{\text{reg}}^2 := RS \frac{(C^2 S^2 R + MN + C^2 SN)(C^2 S^2 R + MN + C^2 SM)}{C^2 S^2 R + MN},$$

$$K_{\text{exc}}^2 := C^3 S^2 \sqrt{R(N + RS)}.$$

*Proof.* We present the proof in the case where there is a + sign in the Kloosterman sums. The complementary case is similar. The main issue is separation of variables, as explained in [15, p. 269]. The nuisance is mainly notational. We write

$$g(c, m, n, r, s) = \int_{\mathbf{R}^4} \frac{1}{sc\sqrt{rq}} G\left(\frac{4\pi\sqrt{mn}}{sc\sqrt{rq}}, \xi_1, \xi_2, \xi_3, \xi_4\right) e(-m\xi_1 - n\xi_2 - r\xi_3 - s\xi_4) \prod_{j=1}^4 d\xi_j,$$

by Fourier inversion, where for all  $(x, \xi_1, \dots, \xi_4) \in \mathbf{R}_+^* \times \mathbf{R}^4$ ,

$$G(x, \xi_1, \dots, \xi_4) := \int_{\mathbf{R}^4} g_*(x, x_1, \dots, x_4) e(x_1\xi_1 + \dots + x_4\xi_4) \prod_{j=1}^4 dx_j,$$

$$g_*(x, x_1, \dots, x_4) := \frac{4\pi\sqrt{x_1x_2}}{x} g\left(\frac{4\pi\sqrt{x_1x_2}}{xx_4\sqrt{x_3q}}, x_1, \dots, x_4\right).$$

By integration by parts, for any non-negative integers  $(\ell, \ell_1, \dots, \ell_4)$  with  $\ell \leq 4$  and  $\ell_j \leq 2$ ,

$$\begin{aligned} \frac{\partial^\ell G}{\partial x^\ell}(x, \xi_1, \dots, \xi_4) &= \prod_j (2\pi i \xi_j)^{-\ell_j} \int_{\mathbf{R}^4} \left( \frac{\partial^{\ell+\ell_1+\dots+\ell_4}}{\partial x^\ell \partial x_1^{\ell_1} \dots \partial x_4^{\ell_4}} g_*(x, x_1, \dots, x_4) \right) \\ &\quad \times e(x_1\xi_1 + \dots + x_4\xi_4) \prod_j dx_j \end{aligned}$$

assuming  $\xi_j \neq 0$  if  $\ell_j > 0$ . The derivatives are estimated using (4.32). Choose  $\ell_1 = 0$  or  $\ell_1 = 2$  according to whether  $|\xi_1|M < 1$  or not, and similarly for  $\ell_2, \ell_3, \ell_4$ . Then

$$\frac{\partial^\ell G}{\partial x^\ell}(x, \xi_1, \dots, \xi_4) \ll \frac{MNR S^2 C \sqrt{qR} (\sqrt{MN} / (CS\sqrt{qR}))^{-\ell}}{(1 + (\xi_1 M)^2)(1 + (\xi_2 N)^2)(1 + (\xi_3 R)^2)(1 + (\xi_4 S)^2)}.$$



We abbreviate further

$$\phi(x) = \phi_{\xi_1, \dots, \xi_4}(x) := \frac{(1 + (\xi_1 M)^2)(1 + (\xi_2 N)^2)(1 + (\xi_3 R)^2)(1 + (\xi_4 S)^2)}{MNRS^2 C \sqrt{qR}} G(x, \xi_1, \dots, \xi_4).$$

This function satisfies the hypotheses of Proposition 4.12, with<sup>†</sup>  $X = \sqrt{MN}/(CS\sqrt{qR})$ , uniformly in  $\xi_j$ . Define

$$\tilde{b}_{n,r,s} := b_{n,r,s} e(-n(\xi_2 + \bar{s}/(rq)) - r\xi_3 - s\xi_4).$$

Finally, by Lemma 4.3 with the scaling matrices

$$\sigma_\infty = \begin{pmatrix} 1 & \xi_1 - t \\ 0 & 1 \end{pmatrix}, \quad \sigma_{1/s} = \begin{pmatrix} \sqrt{rq} & 0 \\ s\sqrt{rq} & 1/\sqrt{rq} \end{pmatrix},$$

we have

$$\bar{\chi}(c)S(n\bar{r}, m\bar{q}; sc) e(m(t - \xi_1) + n\bar{s}/(rq)) = S_{\infty, 1/s}(m, n; sc\sqrt{rq}).$$

Proposition 4.12 can therefore be applied and yields

$$\begin{aligned} & \sum_{\substack{m,n,r,s \\ (s,rq)=1}} \tilde{b}_{n,r,s} \sum_{(c,rq)=1} \frac{1}{cs\sqrt{rq}} \phi\left(\frac{4\pi\sqrt{mn}}{sc\sqrt{rq}}\right) S_{\infty, 1/s}(m, n; sc\sqrt{rq}) \\ & \ll_\varepsilon \frac{q^{3/2}(CMNRS)^\varepsilon}{CS\sqrt{qR}} (W_{\text{reg}} + W_{\text{exc}})\sqrt{M}\|b_{N,R,S}\|_2, \end{aligned}$$

with

$$\begin{aligned} W_{\text{reg}}^2 &= RS \frac{(C^2 S^2 R + MN + C^2 SN)(C^2 S^2 R + MN + C^2 SM)}{C^2 S^2 R + MN}, \\ W_{\text{exc}}^2 &= C^3 S^2 \sqrt{R(N + RS)}. \end{aligned}$$

From the definitions of  $\phi$  and  $G$ , we deduce the claimed bound. □

4.3.3. *Bounds for incomplete Kloosterman sums.* In this section, we prove Theorem 2.1. As a first reduction, we remark that it suffices to prove the result when the sequence  $b_{n,r,s}$  is supported on  $N < n \leq 2N$ , by summing dyadically over  $N$  and by concavity of  $\sqrt{\cdot}$  (losing a factor  $(\log N)^{1/2}$  in the process). Second, we let  $s_0 \pmod{q}^\times$  be fixed and assume without loss of generality that

$$b_{n,r,s} = 0 \text{ unless } s \equiv s_0 \pmod{q}. \tag{4.34}$$

We will recover the full bound (2.3) by summing over  $s_0 \pmod{q}^\times$  (losing a factor  $q^{1/2}$  in the process by concavity). Let

$$\ddot{g}(c, m, n, r, s) := \int_{-\infty}^{\infty} g(c, \xi, n, r, s) e(\xi m) d\xi. \tag{4.35}$$

By Poisson summation, we write the left-hand side of (2.3) as

$$\sum_{\substack{c,n,r,s \\ (qr,sc)=1 \\ c \equiv c_0 \pmod{q}}} b_{n,r,s} \sum_{\substack{\delta \pmod{sc} \\ (\delta,sc)=1}} e\left(n \frac{\overline{r\delta}}{sc}\right) \sum_{\substack{d \equiv \delta \pmod{sc} \\ d \equiv d_0 \pmod{q}}} g(c, d, n, r, s)$$

---

<sup>†</sup>Note that in [15, p. 278], some occurrences of  $X$  should read  $X^{-1}$ .

$$\begin{aligned}
 &= \sum_{\substack{c,n,r,s \\ (qr,sc)=1 \\ c \equiv c_0 \pmod{q}}} \frac{b_{n,r,s}}{scq} \sum_{(\delta,sc)=1} e\left(n \frac{r\bar{\delta}}{sc}\right) \sum_m \ddot{g}(c, m/scq, n, r, s) e\left(-\frac{md_0\bar{sc}}{q} - \frac{m\bar{\delta}\bar{q}}{sc}\right) \\
 &= \sum_{\substack{c,m,n,r,s \\ (qr,sc)=1 \\ c \equiv c_0 \pmod{q}}} \frac{b_{n,r,s}}{scq} \ddot{g}(c, m/scq, n, r, s) e\left(\frac{-md_0\bar{s_0c_0}}{q}\right) S(n\bar{r}, -m\bar{q}; sc), \tag{4.36}
 \end{aligned}$$

where  $S(\dots)$  is the usual Kloosterman. Let  $M > 0$  be a parameter. We write (4.36) as  $\mathcal{A}_0 + \mathcal{A}_\infty + \mathcal{B}$ , where  $\mathcal{A}_0$  is the contribution of  $m = 0$ ,  $\mathcal{A}_\infty$  is the contribution of indices  $m$  such that  $|m| > M$ , and  $\mathcal{B}$  is the contribution of indices  $m$  with  $0 < |m| \leq M$ . By the bound for Ramanujan sums [39, formula (3.5)],

$$\mathcal{A}_0 \ll \frac{1}{q} \sum_{\substack{c,n,r,s \\ (qr,sc)=1 \\ c \equiv c_0 \pmod{q}}} \frac{|b_{n,r,s}|}{sc} |\ddot{g}(c, 0, n, r, s)|(n, sc) \ll q^{-2}(\log S)^2 D\{NR/S\}^{1/2} \|b_{N,R,S}\|_2.$$

By repeated integration by parts in the integral (4.35), for fixed  $k \geq 1$  and  $m \neq 0$  we have

$$\ddot{g}(c, m/(scq), n, r, s) \ll_k D^{1-k(1-\varepsilon_0)} \left(\frac{scq}{|m|}\right)^k.$$

Taking  $k \asymp 1/\varepsilon_0$ , we have that there is a choice of  $M \asymp (SCqD)^{\varepsilon+O(\varepsilon_0)} SCq/D$  such that the bound

$$\ddot{g}(c, m/(scq), n, r, s) \ll_\varepsilon 1/m^2 \quad (|m| > M)$$

holds. Bounding trivially the Kloosterman sum in (4.36) by  $sc$ , we obtain

$$\mathcal{A}_\infty \ll_\varepsilon (SCqD)^{\varepsilon+O(\varepsilon_0)} q^{-2} D\{NR/S\}^{1/2} \|b_{N,R,S}\|_2 \tag{4.37}$$

which is also acceptable (if  $\varepsilon_0$  is small enough, the factor  $q^{-2+\varepsilon+O(\varepsilon_0)}$  is bounded).

There remains to bound  $\mathcal{B}$ ; we may assume that  $M \geq 1$  for otherwise  $\mathcal{B}$  is void. By dyadic decomposition,

$$|\mathcal{B}| \ll \log 2M \sup_{1/2 \leq M_1 \leq M} |\mathcal{B}(M_1)|,$$

where

$$\mathcal{B}(M_1) := \sum_{\substack{c,m,n,r,s \\ (qr,sc)=1 \\ M_1 < |m| \leq 2M_1 \\ c \equiv c_0 \pmod{q}}} \frac{b_{n,r,s}}{scq} \ddot{g}(c, m/scq, n, r, s) e\left(\frac{-md_0\bar{s_0c_0}}{q}\right) S(n\bar{r}, -m\bar{q}; sc).$$

We insert the definition of  $\ddot{g}$  after having changed variables  $\xi \rightarrow \xi scq/m$ , to obtain

$$|\mathcal{B}(M_1)| \ll \frac{DM_1}{SCq} \sup_{\xi \asymp DM_1/(SCq)} |\mathcal{B}'(M_1, \xi)|,$$

where

$$\mathcal{B}'(M_1, \xi) := \sum_{\substack{c,m,n,r,s \\ (qr,sc)=1 \\ M_1 < |m| \leq 2M_1 \\ c \equiv c_0 \pmod{q}}} \frac{b_{n,r,s}}{m} g(c, \xi scq/m, n, r, s) e\left(\frac{-md_0\bar{s_0c_0}}{q}\right) S(n\bar{r}, -m\bar{q}; sc). \tag{4.38}$$

By orthogonality of multiplicative characters, we have

$$\mathcal{B}'(M_1, \xi) = \frac{1}{M_1 \varphi(q)} \sum_{\chi \pmod{q}} \chi(c_0) \mathcal{S}(M_1, \xi, \chi),$$

where

$$\begin{aligned} \mathcal{S}(M_1, \xi, \chi) &:= \sum_{\substack{r,s \\ (s,qr)=1}} \sum_{\substack{m,n \\ |m| \lesssim M_1}} b_{n,r,s} \sum_{(c,rq)=1} \overline{\chi(c)} g_1(c, m, n, r, s) e\left(\frac{-md_0 \overline{s_0 c_0}}{q}\right) S(n\bar{r}, -m\bar{q}; sc), \\ g_1(c, m, n, r, s) &:= M_1 m^{-1} g(c, \xi scq/m, n, r, s). \end{aligned}$$

Proposition 4.13 can be applied to the sums  $\mathcal{S}(M_1, \xi, \chi)$ , at the cost of enlarging the bound by a factor  $O((CDNRS)^{60\epsilon_0})$  in order for the derivative conditions (4.32) to be satisfied. We obtain

$$\begin{aligned} \mathcal{S}(M_1, \xi, \chi) &\ll_{\epsilon} q^{3/2} (CDNRSq)^{\epsilon+O(\epsilon_0)} \{L_{\text{reg}} + L_{\text{exc}}\} \sqrt{M_1} \|b_{N,R,S}\|_2, \\ L_{\text{reg}}^2 &:= RS \frac{(C^2 S^2 R + M_1 N + C^2 SN)(C^2 S^2 R + M_1 N + C^2 SM_1)}{C^2 S^2 R + M_1 N}, \\ L_{\text{exc}}^2 &:= C^3 S^2 \sqrt{R(N + RS)}. \end{aligned}$$

From there, computations identical to [15, p. 282] allow to bound

$$L_{\text{reg}}^2 \ll RS \left( C^2 S^2 R + M_1 N + \frac{C^2 M_1 N}{R} + C^2 S(M_1 + N) \right).$$

We deduce successively

$$\begin{aligned} |\mathcal{B}(M_1)| &\ll_{\epsilon} (CDNRSq)^{\epsilon+O(\epsilon_0)} \frac{qD\sqrt{M_1}}{SC} L^*(M_1) \|b_{N,R,S}\|_2, \\ L^*(M_1)^2 &:= RS(C^2 S^2 R + M_1 N + C^2 M_1 N/R + C^2 S(M_1 + N)) + C^3 S^2 \sqrt{R(N + RS)}, \end{aligned}$$

and finally

$$\begin{aligned} \mathcal{B} &\ll_{\epsilon} (CDNRSq)^{\epsilon+O(\epsilon_0)} q\mathcal{K}, \\ \mathcal{K}^2 &:= qCS(N + RS)(C + RD) + C^2 DS \sqrt{(N + RS)R}. \end{aligned} \tag{4.39}$$

Grouping our two bounds (4.37) and (4.39), and summing over  $s_0 \pmod{q}^{\times}$ , we obtain the claimed result.

### 5. Convolutions in arithmetic progressions

In this section, we proceed with an instance of the dispersion method, for convolutions of two sequences one of which is supported in  $[x^{\eta}, x^{1/3-\eta}]$  for some  $\eta > 0$ . This extends [7, Section 13; 23, Section V].

Given a parameter  $R \geq 1$ , an integer  $q \geq 1$  and a residue class  $n \pmod{q}$ , we let

$$\mathcal{X}_q(R) := \{\chi \pmod{q}, \text{cond}(\chi) \leq R\},$$

and

$$\begin{aligned} \mathbf{u}_R(n; q) &:= \mathbf{1}_{n \equiv 1 \pmod{q}} - \frac{1}{\varphi(q)} \sum_{\chi \in \mathcal{X}_q(R)} \chi(n) \\ &= \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod{q} \\ \text{cond}(\chi) > R}} \chi(n). \end{aligned} \tag{5.1}$$

Note that this vanishes when  $q \leq R$  or  $(n, q) > 1$ . We have the trivial bound

$$|\mathbf{u}_R(n; q)| \ll \mathbf{1}_{n \equiv 1 \pmod{q}} + \frac{R\tau(q)}{\varphi(q)}. \tag{5.2}$$

It will also be sometimes useful to write

$$\mathbf{u}_R(n; q) = \left( \mathbf{1}_{n \equiv 1 \pmod{q}} - \frac{\mathbf{1}_{(n, q) = 1}}{\varphi(q)} \right) - \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod{q} \\ 1 < \text{cond}(\chi) \leq R}} \chi(n). \tag{5.3}$$

**THEOREM 5.1.** *Let  $M, N, Q, R \geq 1$ , and  $\eta$  be given, with  $x := MN$  and  $x^{1/4} \leq Q$ . Then there exists  $\delta$  depending at most on  $\eta$  such that the following holds. Let two sequences  $(\alpha_m)$ ,  $(\beta_n)$  supported in  $n \in (N, 2N]$  and  $m \in (M, 2M]$  be given, which satisfy for some  $A \geq 1$ ,*

$$|\alpha_m| \leq \tau(m)^A, \quad |\beta_n| \leq \tau(n)^A. \tag{5.4}$$

Let  $a_1, a_2 \in \mathbf{Z} \setminus \{0\}$ , and assume that

$$\begin{cases} x^\eta \leq N \leq Q^{2/3-\eta}, \\ Q \leq x^{1/2+\delta}, \\ R, |a_1|, |a_2| \leq x^\delta. \end{cases} \tag{5.5}$$

Then for small enough  $\eta$ , we have

$$\sum_{\substack{Q < q \leq 2Q \\ (q, a_1 a_2) = 1}} \sum_{\substack{n, m \\ (n, a_2) = 1}} \alpha_m \beta_n \mathbf{u}_R(mn \overline{a_1} a_2; q) \ll x(\log x)^{O(1)} R^{-1}. \tag{5.6}$$

The implicit constants depend on  $\eta$  and  $A$  at most.

Introducing  $\mathbf{u}_R(n; q)$  is technically much more convenient than the usual

$$\mathbf{u}_1(n; q) = \mathbf{1}_{n \equiv 1 \pmod{q}} - \frac{\mathbf{1}_{(n, q) = 1}}{\varphi(q)}. \tag{5.7}$$

Indeed, there are no equidistribution assumptions on our sequences in Theorem 5.1.

### 5.1. Bombieri–Vinogradov range

Before we embark on the dispersion method we need an estimate which is relevant to values of the moduli less than the threshold  $x^{1/2-\varepsilon}$ .

**LEMMA 5.2.** *Let  $M, N, R \geq 1$ . Let  $x = MN$ , and suppose we are given two sequences  $(\alpha_m)$  and  $(\beta_n)$  supported on the integers of  $(M, 2M]$  and  $(N, 2N]$ , respectively, satisfying the*

bounds (5.4). Suppose that  $Q \leq x^{1/2}/R$  and  $R \leq Q$ . Then

$$\sum_{Q < q \leq 2Q} \max_{\substack{0 < a < q \\ (a,q)=1}} \left| \sum_{m,n} \alpha_m \beta_n \mathbf{u}_R(mn\bar{a}; q) \right| \ll x(\log x)^{O(1)}(R^{-1} + M^{-1/2} + N^{-1/2}).$$

*Proof.* See [39, Theorem 17.4]. Only the case  $r > R$  appears in our case. □

5.2. *First reductions*

First we apply two reductions, following [23, Section V.2] and [24, Section 3]. We replace the sharp cutoff for the sum over  $q$  by a smooth function  $\gamma(q)$ ; and we transfer the squareful part of  $n$  into the number  $a_2$ , allowing us to assume that  $n$  is squarefree. The squarefreeness assumption on  $n$  will be useful when dealing with GCDs (in particular in equation (5.24) below). Note also that the statement of Theorem 5.1 is monotonically weaker as  $\delta \rightarrow 0$ , so that whenever needed, we will take the liberty of reducing the value of  $\delta$  in a way that depends at most on  $\eta$ .

**PROPOSITION 5.3.** *Let  $x, M, N, Q, R, \eta$  and the sequences  $(\alpha_m)$  and  $(\beta_n)$  be as in Theorem 5.1. Assume that  $(\beta_n)$  is supported on squarefree integers. There exists  $\delta > 0$  such that for any smooth function  $\gamma : \mathbf{R}_+ \rightarrow [0, 1]$  with*

$$\mathbf{1}_{q \in (Q, 2Q)} \leq \gamma(q) \leq \mathbf{1}_{q \in (Q/2, 3Q/2)}, \tag{5.8}$$

and  $\|\gamma^{(j)}\|_\infty \ll_j Q^{-j+B\delta j}$  for some  $B \geq 0$  and all fixed  $j \geq 0$ , under the conditions (5.5), we have

$$\sum_{\substack{q \\ (q, a_1 a_2)=1}} \gamma(q) \sum_{\substack{n, m \\ (n, a_2)=1}} \alpha_m \beta_n \mathbf{u}_R(mn\bar{a}_1 a_2; q) \ll x(\log x)^{O(1)} R^{-1}. \tag{5.9}$$

The implicit constants depend on  $\eta, A$  (in (5.4)),  $B$  and the function  $\gamma$  at most.

*Proof that Proposition 5.3 implies Theorem 5.1.* We replace the sharp cutoff  $Q < q \leq 2Q$  by a smooth weight  $\gamma(q)$  such that

$$\mathbf{1}_{q \in (Q, 2Q]} \leq \gamma(q) \leq \mathbf{1}_{q \in (Q(1-Q^{-10\delta}), 2Q(1+Q^{-10\delta}))}.$$

We can pick  $\gamma$  such that  $\|\gamma^{(j)}\|_\infty \ll_j Q^{-j+10\delta j}$  for all fixed  $j \geq 0$ . The error term in this procedure comes from the contribution of those integers  $q$  at the transition range  $2Q < q \leq 2Q(1 + Q^{-10\delta})$  and  $Q(1 - Q^{-10\delta}) \leq q \leq Q$ . It is bounded by the triangle inequality, using our trivial bound (5.2) and following the reasoning of [7, pp. 219, 240], choosing  $Q_0 = x^{10\delta}$  there. We obtain

$$\sum_{\substack{q \\ (q, a_1 a_2)=1}} (\mathbf{1}_{Q < q \leq 2Q} - \gamma(q)) \sum_{\substack{n, m \\ (n, a_2)=1}} \alpha_m \beta_n \mathbf{u}_R(mn\bar{a}_1 a_2; q) \ll xR(\log x)^{O(1)} Q^{-10\delta}. \tag{5.10}$$

Given our hypotheses  $R \leq x^\delta$  and  $Q \geq x^{1/4}$ , this is an acceptable error term.

Let  $\mathcal{K}$  denote the set of squareful numbers:

$$\mathcal{K} = \{k \in \mathbf{N} : p|k \Rightarrow p^2|k\}.$$

Factor each integer  $n$  as  $n = n'k$  with  $\mu(n')^2 = 1$ ,  $(n', k) = 1$  and  $k \in \mathcal{K}$ , so that  $k \leq x^{1/3}$  and  $(k, a_2) = 1$ . Here  $\mu$  is the Möbius function. There are only  $O(K^{1/2})$  squareful numbers up to  $K$  [18], therefore

$$\sum_{\substack{k \geq K \\ k \in \mathcal{K}}} \frac{1}{k} \ll K^{-1/2} \quad (K \geq 1).$$

Proceeding as in [23, Section V.2] and using the trivial bound (5.2), we deduce for any  $K \geq 1$ ,

$$\begin{aligned} & \sum_{\substack{q \\ (q, a_1 a_2) = 1}} \gamma(q) \sum_{\substack{n, m \\ (n, a_2) = 1}} \alpha_m \beta_n \mathbf{u}_R(mn \bar{a}_1 a_2; q) \\ &= \sum_{\substack{k \leq K \\ k \in \mathcal{K} \\ (k, a_2) = 1}} \sum_{\substack{q \\ (q, a_1 a_2) = 1}} \gamma(q) \sum_{\substack{n, m \\ (n, ka_2) = 1}} \alpha_m \mu(n)^2 \beta_{kn} \mathbf{u}_R(mnk \bar{a}_1 a_2; q) + O(Rx(\log x)^{O(1)} K^{-1/2}). \end{aligned} \tag{5.11}$$

We are left to analyze, for  $k \in \mathcal{K}$ ,  $k \leq K$ ,  $(k, a_2) = 1$ , the sum

$$\sum_{\substack{q \\ (q, a_1 a_2) = 1}} \gamma(q) \sum_{\substack{n, m \\ (n, ka_2) = 1}} \alpha_m \beta_{kn} \mu(n)^2 \mathbf{u}_R(mn \bar{a}_1 ka_2; q).$$

Assume  $K \leq x^{4\delta}$ . For each fixed  $k$ , the sequences  $(\alpha_m)_m$  and  $(k^{-\delta} \mu(n)^2 \beta_{kn})_n$  are supported in  $m \in (M, 2M]$  and  $n \in (N/k, 2N/k]$ , respectively. We apply Proposition 5.3 with  $\eta$  replaced by  $\eta/2$ ,  $N$  replaced by  $N/k$ , and  $a_2$  replaced by  $ka_2$  (the factor  $k^{-\delta}$  ensures that condition (5.4) holds for  $(k^{-\delta} \mu(n)^2 \beta_{kn})_n$ ). If  $\delta$  is small enough in terms of  $\eta$ , we obtain, uniformly for  $k \leq K$ ,

$$\sum_{\substack{q \\ (q, a_1 a_2) = 1}} \gamma(q) \sum_{\substack{n, m \\ (n, ka_2) = 1}} \alpha_m \beta_{kn} \mu(n)^2 \mathbf{u}_R(mn \bar{a}_1 ka_2; q) \ll k^{-1+\delta} x(\log x)^{O(1)} R^{-1}.$$

Note that the sum  $\sum_{k \in \mathcal{K}} k^{-1+\delta}$  converges. Inserting in (5.11), we obtain

$$\sum_{\substack{q \\ (q, a_1 a_2) = 1}} \gamma(q) \sum_{\substack{n, m \\ (n, a_2) = 1}} \alpha_m \beta_n \mathbf{u}_R(mn \bar{a}_1 a_2; q) \ll x(\log x)^{O(1)} (R^{-1} + RK^{-1/2})$$

and so we conclude by the choice  $K = R^4$ . □

### 5.3. Applying the dispersion method

Let us prove Proposition 5.3. Recall that the sequence  $(\beta_n)$  is assumed to be supported on squarefree integers. Let  $\mathcal{D}$  denote the left-hand side of (5.9). By the triangle inequality

$$|\mathcal{D}| = \left| \sum_{\substack{q \\ (q, a_1 a_2) = 1}} \gamma(q) \sum_{\substack{m, n \\ (n, a_2) = 1}} \alpha_m \beta_n \mathbf{u}_R(mn \bar{a}_1 a_2; q) \right| \leq \sum_m \left( |\alpha_m| \left| \sum_q \sum_n \right| \right). \tag{5.12}$$

Define a smooth and non-negative function  $\alpha(m)$  (not to be confused with our sequence  $\alpha_m$ ), with  $\alpha(m) \geq 1$  for  $M < m \leq 2M$ , supported inside  $[M/2, 3M]$  and such that  $\|\alpha^{(j)}\|_\infty \ll_j M^{-j}$ .

Note that  $|\alpha_m| \leq \tau(m)^A \alpha(m)$  by the hypothesis (5.4). Therefore, the Cauchy–Schwarz inequality yields

$$\begin{aligned}
 |\mathcal{D}| &\ll \left( \sum_m \alpha(m) \tau(m)^A \right)^{1/2} \left( \sum_m \alpha(m) \left| \sum_q \sum_n \right|^2 \right)^{1/2} \\
 &\ll (\log x)^{O(1)} M^{1/2} (\mathcal{S}_1 - 2 \Re \mathcal{S}_2 + \mathcal{S}_3)^{1/2},
 \end{aligned}
 \tag{5.13}$$

where

$$\mathcal{S}_1 = \sum_{(q_1 q_2, a_1 a_2)=1} \gamma(q_1) \gamma(q_2) \sum_{\substack{n_1, n_2 \\ (n_1 n_2, a_2)=1}} \beta_{n_1} \overline{\beta_{n_2}} \sum_{\substack{mn_1 \equiv a_1 \overline{a_2} \pmod{q_1} \\ mn_2 \equiv a_1 \overline{a_2} \pmod{q_2}}} \alpha(m)$$

and  $\mathcal{S}_2$  and  $\mathcal{S}_3$  are defined similarly, replacing the sum over  $m$  by

$$\begin{aligned}
 &\frac{1}{\varphi(q_2)} \sum_{\chi_2 \in \mathcal{X}_{q_2}(R)} \chi(n_2 \overline{a_1 a_2}) \sum_{mn_1 \equiv a_1 \overline{a_2} \pmod{q_1}} \alpha(m) \chi_2(m), \\
 &\frac{1}{\varphi(q_1) \varphi(q_2)} \sum_{\chi_1 \in \mathcal{X}_{q_1}(R)} \sum_{\chi_2 \in \mathcal{X}_{q_2}(R)} \chi_1(n_1 \overline{a_1 a_2}) \overline{\chi_2(n_2 \overline{a_1 a_2})} \sum_{\substack{(mn_1, q_1)=1 \\ (mn_2, q_2)=1}} \alpha(m) \chi_1 \overline{\chi_2}(m),
 \end{aligned}$$

respectively. We will prove

$$\mathcal{S}_1 - 2 \Re \mathcal{S}_2 + \mathcal{S}_3 = O((\log x)^{O(1)} MN^2 R^{-2}).
 \tag{5.14}$$

5.3.1. *Evaluation of  $\mathcal{S}_3$ .* The term  $\mathcal{S}_3$  is defined by

$$\begin{aligned}
 \mathcal{S}_3 &= \sum_{(q_1 q_2, a_1 a_2)=1} \frac{\gamma(q_1) \gamma(q_2)}{\varphi(q_1) \varphi(q_2)} \sum_{\substack{\chi_1 \in \mathcal{X}_{q_1}(R) \\ \chi_2 \in \mathcal{X}_{q_2}(R)}} \sum_{\substack{n_1, n_2 \\ (n_j, q_j a_2)=1}} \beta_{n_1} \overline{\beta_{n_2}} \\
 &\times \sum_{(m, q_1 q_2)=1} \alpha(m) \chi_1(m n_1 \overline{a_1 a_2}) \overline{\chi_2(m n_2 \overline{a_1 a_2})}.
 \end{aligned}
 \tag{5.15}$$

Let  $W := [q_1, q_2]$  and  $H := W^{1+\varepsilon}/M$ . By Poisson summation (Lemma 3.1),

$$\begin{aligned}
 \sum_m \alpha(m) \chi_1 \overline{\chi_2}(m) &= \frac{\widehat{\alpha}(0)}{W} \sum_{b \pmod{W}^\times} \chi_1 \overline{\chi_2}(b) \\
 &\quad + \frac{1}{W} \sum_{0 < |h| \leq H} \widehat{\alpha}\left(\frac{h}{W}\right) \sum_{b \pmod{W}^\times} e\left(\frac{bh}{W}\right) \chi_1 \overline{\chi_2}(b) + O_\varepsilon(W^\varepsilon).
 \end{aligned}$$

The conductor of  $\chi_1 \overline{\chi_2}$  is at most  $R$ , so that [39, Lemma 3.2]<sup>†</sup> yields

$$\sum_{b \pmod{W}^\times} e\left(\frac{bh}{W}\right) \chi_1 \overline{\chi_2}(b) \ll R^{1/2} \sum_{d|(h, W)} d.$$

We deduce

$$\sum_m \alpha(m) \chi_1 \overline{\chi_2}(m) = \frac{\widehat{\alpha}(0)}{W} \sum_{b \pmod{W}^\times} \chi_1 \overline{\chi_2}(b) + O_\varepsilon(W^\varepsilon R^{1/2}).$$

---

<sup>†</sup>Note that in [39, Lemma 3.2],  $\tau(\chi)$  should read  $\tau(\chi^*)$  and an additional factor  $\chi^*(m/(dm^*))$  should appear in the summand.

The error term is  $O(x^\delta)$  while the trivial bound is  $M \geq x^{2/3}$ . We deduce

$$\mathcal{S}_3 = \widehat{\alpha}(0)X_3 + O(MN^2x^{-1/2}),$$

where, having changed  $b$  to  $ba_1\overline{a_2}$ ,

$$X_3 := \sum_{\substack{q_1, q_2 \\ (q_1 q_2, a_1 a_2) = 1}} \frac{\gamma(q_1)\gamma(q_2)}{[q_1, q_2]\varphi(q_1)\varphi(q_2)} \sum_{\substack{\chi_1 \in \mathcal{X}_{q_1}(R) \\ \chi_2 \in \mathcal{X}_{q_2}(R)}} \sum_{\substack{n_1, n_2 \\ (n_j, q_j a_2) = 1}} \beta_{n_1} \overline{\beta_{n_2}} \sum_{b \pmod{W}^\times} \chi_1(bn_1) \overline{\chi_2(bn_2)}.$$

By orthogonality,

$$\sum_{b \pmod{W}^\times} \chi_1 \overline{\chi_2}(b) = \varphi(W) \mathbf{1}_{\chi_1 \sim \chi_2},$$

where by  $\chi_1 \sim \chi_2$  we mean that  $\chi_1$  and  $\chi_2$  are induced by the same primitive character—which necessarily has conductor dividing  $(q_1, q_2)$ . Therefore,

$$\sum_{\substack{\chi_1 \in \mathcal{X}_{q_1}(R) \\ \chi_2 \in \mathcal{X}_{q_2}(R)}} \chi_1(n_1) \overline{\chi_2(n_2)} \mathbf{1}_{\chi_1 \sim \chi_2} = \sum_{\chi_0 \in \mathcal{X}_{(q_1, q_2)}(R)} \chi_0(n_1 \overline{n_2}).$$

Since  $\varphi([q_1, q_2]) = \varphi(q_1)\varphi(q_2)/\varphi((q_1, q_2))$ , we deduce

$$X_3 = \sum_{(q_1 q_2, a_1 a_2) = 1} \frac{\gamma(q_1)\gamma(q_2)}{[q_1, q_2]\varphi((q_1, q_2))} \sum_{\chi_0 \in \mathcal{X}_{(q_1, q_2)}(R)} \sum_{\substack{n_1, n_2 \\ (n_j, q_j a_2) = 1}} \beta_{n_1} \overline{\beta_{n_2}} \chi_0(n_1 \overline{n_2}). \tag{5.16}$$

5.3.2. *Evaluation of  $\mathcal{S}_2$ .* The term  $\mathcal{S}_2$  is defined by

$$\mathcal{S}_2 = \sum_{(q_1 q_2, a_1 a_2) = 1} \frac{\gamma(q_1)\gamma(q_2)}{\varphi(q_2)} \sum_{\substack{n_1, n_2 \\ (n_j, q_j a_2) = 1}} \beta_{n_1} \overline{\beta_{n_2}} \sum_{\chi_2 \in \mathcal{X}_{q_2}(R)} \sum_{m \equiv a_1 \overline{a_2} \overline{n_1} \pmod{q_1}} \alpha(m) \chi_2(mn_2 \overline{a_1} a_2). \tag{5.17}$$

As before, let  $W = [q_1, q_2]$  and  $H = W^{1+\varepsilon}/M$ . By Poisson summation,

$$\sum_{m \equiv a_1 \overline{a_2} \overline{n_1} \pmod{q_1}} \alpha(m) \chi_2(m) = \frac{\widehat{\alpha}(0)}{W} \sum_{\substack{b \pmod{W}^\times \\ b \equiv a_1 \overline{a_2} \overline{n_1} \pmod{q_1}}} \chi_2(b) + O_\varepsilon(\mathcal{R}_2 + W^\varepsilon), \tag{5.18}$$

where

$$\mathcal{R}_2 := \frac{M}{W} \sum_{0 < |h| \leq H} \left| \sum_{\substack{b \pmod{W}^\times \\ b \equiv a_1 \overline{a_2} \overline{n_1} \pmod{q_1}}} \chi_2(b) e\left(\frac{bh}{W}\right) \right|. \tag{5.19}$$

We wish to express the sum over  $b$  as a complete sum over residues. We write  $W = [q_1, q_2] = q'_1 q'_2$ , where  $(q'_2, q_1) = 1$  and  $q'_1 | q_1^\infty$  (meaning that  $p | q'_1 \Rightarrow p | q_1$ ). Note that then  $q_1 | q'_1$  and  $(q'_1, q'_2) = 1$ . Let

$$\psi : (\mathbf{Z}/q'_1 \mathbf{Z}) \times (\mathbf{Z}/q'_2 \mathbf{Z}) \longrightarrow (\mathbf{Z}/W \mathbf{Z})$$

denote the canonical ring isomorphism (so  $\psi^{-1}$  is the projection map). Note that

$$b_2 \mapsto \chi_2(\psi(1, b_2))$$



defines a character  $(\text{mod } q'_2)$  of conductor at most  $R$ . Finally, we have

$$\frac{1}{W} \equiv \frac{\overline{q'_1}}{q'_2} + \frac{\overline{q'_2}}{q'_1} \pmod{1}.$$

The sum over  $b$  in (5.19) is in absolute values at most

$$\sum_{\substack{b_1 \pmod{q'_1}^\times \\ b_1 \equiv a_1 \overline{a_2 n_1} \pmod{q_1}}} \left| \sum_{b_2 \pmod{q'_2}^\times} \chi_2(\psi(1, b_2)) e\left(\frac{b_2 h \overline{q'_1}}{q'_2}\right) \right| \tag{5.20}$$

since  $\psi(b_1, b_2) \equiv b_1 \pmod{q_1}$ , and by factoring

$$\chi_2(\psi(b_1, b_2)) = \chi_2(\psi(b_1, 1))\chi_2(\psi(1, b_2)).$$

The sum over  $b_2$  in (5.20) is a Gauss sum; by [39, Lemma 3.2],

$$\left| \sum_{b_2 \pmod{q'_2}^\times} \chi_2(\psi(1, b_2)) e\left(\frac{b_2 h \overline{q'_1}}{q'_2}\right) \right| \leq R^{1/2} \sum_{d|(h, q'_2)} d. \tag{5.21}$$

Note that

$$\sum_{\substack{b_1 \pmod{q'_1}^\times \\ b_1 \equiv a_1 \overline{a_2 n_1} \pmod{q_1}}} 1 = \frac{\varphi(q'_1)}{\varphi(q_1)} = (q_2, q_1^\infty) \tag{5.22}$$

which is a shorthand for  $\prod_{p^\nu || q_2, p|q_1} p^\nu$ . Multiplying (5.21) with (5.22) and summing over  $h$ , we obtain

$$\mathcal{R}_2 \ll_\varepsilon W^\varepsilon \tau(q_2)(q_2, q_1^\infty) R^{1/2}.$$

Inserting this estimate into (5.18) then (5.17), the error term contributes

$$\ll_\varepsilon R^{3/2} N^2 W^\varepsilon \sum_{q_1, q_2 \asymp Q} \frac{\tau(q_2)(q_2, q_1^\infty)}{q_2} \ll x^{3\delta/2+\varepsilon} N^2 Q.$$

In the last inequality we used standard facts about the kernel function  $k(n) = \prod_{p|n} p$ , for which we refer to [9]. The error term above is acceptable, since

$$x^{3\delta/2} Q \leq x^{1/2+3\delta} \leq x^{2/3-2\delta} \leq MR^{-2} x^{-\delta}$$

if  $\delta$  is small enough. We therefore have

$$\mathcal{S}_2 = \widehat{\alpha}(0) X_2 + O(MN^2 R^{-2})$$

with (having changed  $b$  into  $ba_1 \overline{a_2}$ )

$$X_2 = \sum_{(q_1 q_2, a_1 a_2)=1} \frac{\gamma(q_1)\gamma(q_2)}{[q_1, q_2]\varphi(q_2)} \sum_{\substack{n_1, n_2 \\ (n_j, q_j a_2)=1}} \beta_{n_1} \overline{\beta_{n_2}} \sum_{\chi_2 \in \mathcal{X}_{q_2}(R)} \chi_2(n_2) \sum_{\substack{b \pmod{W}^\times \\ b \equiv \overline{n_1} \pmod{q_1}}} \chi_2(b).$$

Fix  $\chi_2 \in \mathcal{X}_{q_2}(R)$  and let  $\tilde{\chi}_2 \pmod{\tilde{q}_2}$  be the primitive character inducing  $\chi_2$ . If  $S$  denotes the sum over  $b$  above, then  $S = \chi_2(c)S$  for any  $c \pmod{W}^\times, c \equiv 1 \pmod{q_1}$ . Thus  $S = 0$  if  $\chi_2$  is not  $q_1$ -periodic, that is, if  $\tilde{q}_2 \nmid (q_1, q_2)$ . If on the contrary  $\tilde{q}_2 | (q_1, q_2)$ , then  $S = \tilde{\chi}_2(\overline{n_1})\varphi(W)/\varphi(q_1) = \tilde{\chi}_2(\overline{n_1})\varphi(q_2)/\varphi((q_1, q_2))$ . We therefore find

$$\sum_{\substack{b \pmod{W}^\times \\ b \equiv \overline{n_1} \pmod{q_1}}} \chi_2(b) = \frac{\varphi(q_2)}{\varphi((q_1, q_2))} \mathbf{1}_{\tilde{q}_2 | (q_1, q_2)} \tilde{\chi}_2(\overline{n_1}).$$

Summing over  $\chi_2 \in \mathcal{X}_{q_2}(R)$  and since  $(n_1 n_2, (q_1, q_2)) = 1$ , we obtain

$$\sum_{\chi_2 \in \mathcal{X}_{q_2}(R)} \chi_2(n_2) \sum_{\substack{b \pmod{W}^\times \\ b \equiv \overline{n_1} \pmod{q_1}}} \chi_2(b) = \frac{\varphi(q_2)}{\varphi((q_1, q_2))} \sum_{\chi_0 \in \mathcal{X}_{(q_1, q_2)}(R)} \chi_0(\overline{n_1} n_2),$$

and so  $X_2 = X_3$ .

5.4. *Second reduction*

We now wish to evaluate

$$\mathcal{S}_1 := \sum_{(q_1 q_2, a_1 a_2) = 1} \gamma(q_1) \gamma(q_2) \sum_{\substack{n_1, n_2 \\ (n_j, q_j a_2) = 1 \\ n_1 \equiv n_2 \pmod{(q_1, q_2)}}} \beta_{n_1} \overline{\beta_{n_2}} \sum_{\substack{m \equiv a_1 \overline{a_2} \overline{n_1} \pmod{q_1} \\ m \equiv a_1 \overline{a_2} \overline{n_2} \pmod{q_2}}} \alpha(m).$$

The expected main term is  $\widehat{\alpha}(0) X_1$ , where

$$X_1 := \sum_{(q_1 q_2, a_1 a_2) = 1} \frac{\gamma(q_1) \gamma(q_2)}{[q_1, q_2]} \sum_{\substack{n_1, n_2 \\ (n_j, q_j a_2) = 1 \\ n_1 \equiv n_2 \pmod{(q_1, q_2)}}} \beta_{n_1} \overline{\beta_{n_2}}. \tag{5.23}$$

For all integers  $q_0, n_0$  with  $(n_0, q_0) = 1$ , let  $\mathcal{S}_1(q_0, n_0)$  denote the contribution to  $\mathcal{S}_1$  of those integers satisfying  $(q_1, q_2) = q_0$  and  $(n_1, n_2) = n_0$ . Then we have

$$\begin{aligned} |\mathcal{S}_1(q_0, n_0)| &\ll_\varepsilon x^\varepsilon \sum_{\substack{q_1, q_2 \asymp Q/q_0 \\ (q_0 q_2, a_2 n_0) = 1}} \sum_{\substack{n_1, n_2 \asymp N/n_0 \\ n_1 \equiv n_2 \pmod{q_0} \\ (n_2, q_0 q_2) = 1}} \sum_{\substack{a_2 n_0 n_2 m \equiv a_1 \pmod{q_0 q_2} \\ q_1 | m a_2 n_0 n_1 - a_1}} \alpha(m) \\ &\ll_\varepsilon x^\varepsilon \sum_{\substack{q_2 \asymp Q/n_0 \\ (q_0 q_2, a_2 n_0) = 1}} \sum_{\substack{n_1, n_2 \asymp N/n_0 \\ n_1 \equiv n_2 \pmod{q_0} \\ (n_2, q_0 q_2) = 1}} \sum_{m a_2 n_0 n_2 \equiv a_1 \pmod{q_0 q_2}} \alpha(m) \tau(|m a_2 n_0 n_1 - a_1|) \\ &\ll_\varepsilon x^\varepsilon \left\{ \frac{MN^2}{n_0^2 q_0^2} + \frac{MN}{n_0 q_0} \right\}, \end{aligned}$$

where we used our hypotheses on  $M$  and  $|a_1|$  to justify that  $m|a_1$  cannot be satisfied. Therefore, for some  $\delta > 0$  and all  $1 \leq K \leq x^\delta$ , we have

$$\sum_{\substack{(q_0, n_0) = 1 \\ \max\{q_0, n_0\} > K}} |\mathcal{S}_1(q_0, n_0)| \ll_\varepsilon x^\varepsilon MN^2 K^{-1}.$$

Similarly, if  $X_1(q_0, n_0)$  denotes the contribution to  $X_1$  of indices with  $(q_1, q_2) = q_0$  and  $(n_1, n_2) = n_0$ , we have

$$\sum_{\substack{(q_0, n_0) = 1 \\ \max\{q_0, n_0\} > K}} |X_1(q_0, n_0)| \ll_\varepsilon x^\varepsilon \sum_{\substack{(q_0, n_0) = 1 \\ \max\{q_0, n_0\} > K}} \frac{N}{q_0 n_0} \left( \frac{N}{q_0 n_0} + 1 \right) \ll_\varepsilon x^\varepsilon N^2 K^{-1}.$$

By choosing  $K$  appropriately, it will therefore suffice to show that

$$\mathcal{S}_1(q_0, n_0) = \widehat{\alpha}(0) X_1(q_0, n_0) + O(MN^2 x^{-\delta}) \quad (q_0, n_0 \leq x^\delta).$$

5.5. Evaluation of  $\mathcal{S}_1(q_0, n_0)$

Let the integers  $q_0, n_0$  be coprime, at most  $x^\delta$ , such that  $(q_0, a_1 a_2) = (n_0, a_2) = 1$ . Let us rename  $q_1$  into  $q_0 q_1$  and  $q_2$  into  $q_0 q_2$ , and similarly for  $n_1$  and  $n_2$ . We wish to evaluate

$$\begin{aligned} \mathcal{S}_1(q_0, n_0) = & \sum_{\substack{q_1, q_2 \\ (q_1 q_2, a_1 a_2) = (q_1, q_2) = 1}} \gamma(q_0 q_1) \gamma(q_0 q_2) \\ & \sum_{\substack{n_1, n_2 \\ (n_0 n_j, q_0 q_j a_2) = 1 \\ (n_1, n_2) = 1 \\ n_1 \equiv n_2 \pmod{q_0}}} \beta_{n_0 n_1} \overline{\beta_{n_0 n_2}} \sum_{m \equiv a_1 \overline{a_2 n_0 n_j} \pmod{q_0 q_j}} \alpha(m). \end{aligned}$$

Using Poisson summation, we have

$$\mathcal{S}_1(q_0, n_0) = \widehat{\alpha}(0) X_1(q_0, n_0) + \mathcal{R}_1 + O_\varepsilon(x^\varepsilon \mathcal{R}_2)$$

where, having put  $W = q_0 q_1 q_2$  and  $H := W^{1+\varepsilon} M^{-1}$ ,

$$\mathcal{R}_1 = \sum_{\substack{q_1, q_2 \\ (q_1 q_2, a_1 a_2) = (q_1, q_2) = 1}} \sum_{\substack{n_1, n_2 \\ (n_0 n_j, q_0 q_j a_2) = 1 \\ (n_1, n_2) = 1 \\ n_1 \equiv n_2 \pmod{q_0}}} \gamma(q_0 q_1) \gamma(q_0 q_2) \beta_{n_0 n_1} \overline{\beta_{n_0 n_2}} \sum_{0 < |h| \leq H} \frac{1}{W} \widehat{\alpha}\left(\frac{h}{W}\right) e\left(\frac{h\mu}{W}\right),$$

$$\mathcal{R}_2 = \sum_{q_1, q_2 \asymp Q/q_0} \sum_{n_1, n_2 \asymp N/n_0} \frac{1}{W} \ll Q^\varepsilon N^2,$$

and the residue class  $\mu \pmod{W}$  satisfies

$$\mu \equiv a_1 \overline{a_2 n_0 n_j} \pmod{q_0 q_j} \quad (j \in \{1, 2\}).$$

We seek an error term  $O(MN^2 x^{-\delta})$ . The contribution of  $\mathcal{R}_2$  is acceptable.

We now focus on  $\mathcal{R}_1$ . Recall that  $\beta_n$  is non-zero only when  $n$  is squarefree (so that  $(n_0, n_1) = 1$ ). We have the equality modulo 1

$$\frac{\mu}{q_0 q_1 q_2} \equiv \frac{a_1}{q_0 q_1 q_2 a_2 n_0 n_1} + a_1 \frac{n_1 - n_2}{q_0} \frac{\overline{q_1 a_2 n_0 n_2}}{n_1 q_2} - a_1 \frac{\overline{q_0 q_1 q_2 n_1}}{a_2 n_0} \pmod{1}. \tag{5.24}$$

This is found following the steps in [24, p. 208], but can also be more easily verified by multiplying each side by  $q_0 q_1 q_2 a_2 n_0 n_1$ , and checking the resulting congruence modulo  $a_2 n_0, n_1 q_0, q_0 q_1$ , and  $q_0 q_2$ , respectively. Taking the exponential, we may approximate

$$e\left(\frac{ha_1}{q_0 q_1 q_2 a_2 n_0 n_1}\right) = 1 + O\left(\frac{|ha_1|}{q_0 q_1 q_2 |a_2| n_0 n_1}\right).$$

Inserting in  $\mathcal{R}_1$ , the error term contributes a quantity

$$\ll \frac{|a_1| q_0 H}{|a_2| n_0 Q^2 N} \frac{Q^2 N^2}{q_0^2 n_0} \ll x^\varepsilon |a_1| N Q^2 M^{-1}$$

which is clearly acceptable. We therefore evaluate

$$\begin{aligned} \mathcal{R}'_1 := & \sum_{q_1, q_2, n_1, n_2} \frac{\gamma(q_0 q_1) \gamma(q_0 q_2)}{q_0 q_1 q_2} \beta_{n_0 n_1} \overline{\beta_{n_0 n_2}} \widehat{\alpha}\left(\frac{h}{q_0 q_1 q_2}\right) \\ & \times e\left(a_1 h \frac{n_1 - n_2}{q_0} \frac{\overline{q_1 a_2 n_0 n_2}}{n_1 q_2} - a_1 h \frac{\overline{q_0 q_1 q_2 n_1}}{a_2 n_0}\right). \end{aligned}$$

Now we insert the definition of  $\widehat{\alpha}$  as

$$\widehat{\alpha}\left(\frac{h}{q_0q_1q_2}\right) = q_0q_1q_2 \int_{\mathbf{R}} \alpha(q_0q_1q_2\xi) e(-h\xi) d\xi,$$

we detect the condition  $(a_1, q_1q_2) = 1$  by Möbius inversion, and we split the sums over  $q_1, q_2$  into congruence classes modulo  $n_0a_2$ . We obtain

$$|\mathcal{R}'_1| \ll_{\varepsilon} x^{\varepsilon} (n_0|a_2|)^2 \frac{Mq_0}{Q^2} \sup_{\xi \asymp Mq_0/Q^2} \sup_{\substack{\delta_1, \delta_2 | a_1 \\ (\delta_1, \delta_2) = 1 \\ (\delta_1 \delta_2, n_0 a_2) = 1}} \sup_{\lambda_1, \lambda_2 \pmod{n_0 a_2}^{\times}} \mathcal{R}''_1, \tag{5.25}$$

where

$$\begin{aligned} \mathcal{R}''_1 := & \sum_{\substack{q_1, q_2 \\ (\delta_1 q_1, \delta_2 q_2) = 1 \\ q_j \equiv \lambda_j \bar{\delta}_j \pmod{n_0 a_2}}} \gamma(q_0 \delta_1 q_1) \gamma(q_0 \delta_2 q_2) \sum_{\substack{n_1, n_2 \\ (n_0 n_j, q_0 \delta_j q_j a_2) = 1 \\ (n_1, n_2) = 1 \\ n_1 \equiv n_2 \pmod{q_0}}} \beta_{n_0 n_1} \overline{\beta_{n_0 n_2}} \\ & \times \sum_{0 < |h| \leq H} \alpha(\xi q_0 \delta_1 \delta_2 q_1 q_2) e\left(-\xi h - a_1 h \frac{q_0 \lambda_1 \lambda_2 n_1}{a_2 n_0}\right) e\left(a_1 h \frac{n_1 - n_2}{q_0} \frac{\overline{a_2 n_0 n_2 \delta_1 q_1}}{n_1 \delta_2 q_2}\right). \end{aligned}$$

We write  $\mathcal{R}''_1$  in the form (2.3), with

$$\begin{aligned} \mathbf{c} \leftarrow q_2, \quad \mathbf{d} \leftarrow q_1, \quad \mathbf{n} \leftarrow a_1 h \frac{n_1 - n_2}{q_0}, \quad \mathbf{r} \leftarrow a_2 n_0 n_2 \delta_1, \quad \mathbf{s} \leftarrow n_1 \delta_2, \quad \mathbf{q} \leftarrow n_0 a_2, \\ \mathbf{C} \leftarrow \frac{Q}{q_0 \delta_2}, \quad \mathbf{D} \leftarrow \frac{Q}{q_0 \delta_1}, \quad \mathbf{N} \leftarrow \frac{|a_1| H N}{q_0 n_0} \quad \mathbf{R} \leftarrow a_2 \delta_1 N, \quad \mathbf{S} \leftarrow \frac{N \delta_2}{n_0}. \end{aligned} \tag{5.26}$$

Here bold letters denote the ‘new’ summation variables in (2.3). The analog of the sequence  $b_{\mathbf{n}, \mathbf{r}, \mathbf{s}}$  is defined through

$$b_{\mathbf{n}, \mathbf{r}, \mathbf{s}} = \sum_{\substack{n_1 \quad n_2 \\ (n_0 n_j, q_0 \delta_j a_2) = 1 \\ (n_1, n_2) = 1 \\ n_1 \equiv n_2 \pmod{q_0} \\ \mathbf{r} = a_2 n_0 n_2 \delta_1 \\ \mathbf{s} = n_1 \delta_2}} \beta_{n_0 n_1} \overline{\beta_{n_0 n_2}} \sum_{\substack{0 < |h| \leq H \\ q_0 \mathbf{n} = a_1 h (n_1 - n_2)}} e\left(-\xi h - a_1 h \frac{q_0 \lambda_1 \lambda_2 n_1}{a_2 n_0}\right).$$

Note that this has at most one term since the case  $n_1 = n_2$  is prohibited by the conditions  $(n_1, n_2) = 1$  and  $N \geq x^n$ . Note also that it is void unless  $(\mathbf{r}, \mathbf{s}) = 1$  (here we use the fact that  $\beta$  is supported on squarefree integers). The quantity  $g(\mathbf{c}, \mathbf{d}, \mathbf{n}, \mathbf{r}, \mathbf{s})$  in (2.3) is

$$\gamma(q_0 \delta_1 \mathbf{d}) \gamma(q_0 \delta_2 \mathbf{c}) \alpha(\xi q_0 \delta_1 \delta_2 \mathbf{c} \mathbf{d}).$$

The derivative conditions (2.2) are satisfied with  $\varepsilon_0 = B\delta$ , by virtue of our hypothesis on  $\gamma$ . Note that the congruence and coprimality conditions on  $q_1$  and  $q_2$  translate exactly into

$$\mathbf{c} \equiv \lambda_2 \bar{\delta}_2 \pmod{\mathbf{q}}, \quad \mathbf{d} \equiv \lambda_1 \bar{\delta}_1 \pmod{\mathbf{q}}, \quad (\mathbf{d}, \mathbf{c} \mathbf{s}) = (\mathbf{c}, \mathbf{r}) = 1.$$

At this point, we are in a situation analogous to [7, formula (13.2)]. Applying Theorem 2.1 and estimating the resulting expression as in [7, p. 241], we obtain

$$\mathcal{R}''_1 \ll x^{O(\delta)} \mathcal{A}^{1/2} \mathcal{B}^{1/2},$$

where  $\mathcal{A} \ll HN^2$  is the contribution coming from  $\|b_{N, R, S}\|_2^2$  in (2.3), and

$$\mathcal{B} \ll Q^2 N^2 N(H + N) + Q^3 N^2 \sqrt{H + N} + Q^2 H N \ll (QN)^2 \{N(H + N) + Q\sqrt{H + N}\}.$$

We have  $H \ll x^{O(\delta)}N$ , so that  $\mathcal{B} \ll Q^2N^2x^{O(\delta)}(N^2 + Q\sqrt{N})$  (compare with [7, formula (13.4)]). Inserting in (5.25), we obtain

$$\mathcal{R}'_1 \ll x^{O(\delta)}MN^2(Q^{-1}N^{3/2} + Q^{-1/2}N^{3/4}) \ll x^{-\eta/2+O(\delta)}MN^2$$

by the hypothesis  $N \leq Q^{2/3-\eta}$ . Taking  $\delta$  sufficiently small in terms of  $\eta$ , we have the required bound  $O(MN^2x^{-\delta})$ .

5.6. *The main terms*

The main terms  $X_1$  and  $X_3$  defined in (5.23) and (5.16) are real numbers. They combine to form

$$X_1 - X_3 = \sum_{(q_1q_2, a_1a_2)=1} \frac{\gamma(q_1)\gamma(q_2)}{[q_1, q_2]} \sum_{\substack{n_1, n_2 \\ (n_j, q_j a_j)=1}} \overline{\beta_{n_1}}\beta_{n_2}\mathbf{u}_R(n_1\overline{n_2}; (q_1, q_2)).$$

Note the summands are zero unless  $(q_1, q_2) > R$ . We use Möbius inversion

$$\mathbf{1}_{(n_j, q_j)=1} = \sum_{d_j | (q_j, n_j)} \mu(d_j)$$

to detect the conditions  $(n_j, q_j) = 1$ , in order to separate the sums over  $n_1, n_2$  from those over  $q_1, q_2$ . We insert the definition of  $\mathbf{u}_R$  in the form

$$\mathbf{u}_R(n_1\overline{n_2}; (q_1, q_2)) = \frac{1}{\varphi((q_1, q_2))} \sum_{\substack{\chi \text{ primitive} \\ \text{cond}(\chi) > R \\ \text{cond}(\chi) | (q_1, q_2)}} \chi(\overline{n_1})\chi(n_2).$$

We can assume  $(d_j, \text{cond}(\chi)) = 1$  because of the factors  $\chi(n_j)$ . Quoting from [52, Theorem I.5.4] the bound  $\varphi(q) \gg q/\log \log q$ , we obtain

$$X_1 - X_3 \ll (\log \log x) \sum_{R < r \leq Q} \sum_{\substack{d_1, d_2 \\ d_j \ll Q/r}} \left( \sum_{\substack{q_1, q_2 \\ q_j \asymp Q \\ rd_j | q_j}} \frac{1}{q_1q_2} \right) \sum_{\substack{\chi \text{ primitive} \\ \chi \pmod r}} \prod_{j=1}^2 \left| \sum_{(n, a_2)=1} \beta_{d_j n} \chi(n) \right|.$$

The sum over  $q_1, q_2$  is  $O(1/(r^2d_1d_2))$ . By Cauchy–Schwarz, and the symmetry between  $n_1$  and  $n_2$ , we obtain

$$X_1 - X_3 \ll (\log x)^2 \sum_{d \ll N} \frac{1}{d} \sum_{R < r \leq Q} \frac{1}{r^2} \sum_{\substack{\chi \text{ primitive} \\ \chi \pmod r}} \left| \sum_{(n, a_2)=1} \beta_{dn} \chi(n) \right|^2.$$

For all  $t > R$ , the multiplicative large sieve inequality (Lemma 3.3) and our hypothesis (5.4) yield

$$G(t) := \sum_{R < r \leq t} \sum_{\substack{\chi \text{ primitive} \\ \chi \pmod r}} \left| \sum_{(n, a_2)=1} \beta_{dn} \chi(n) \right|^2 \ll (\log x)^{O(1)} \tau(d)^{2A} (t^2 + N)N$$

after ignoring denominators  $d$ . We obtain by partial summation

$$X_1 - X_3 \ll (\log x)^2 \sum_{d \ll N} \frac{1}{d} \left( \frac{G(Q)}{Q^2} + \int_R^Q \frac{G(t)}{t^3} dt \right) \ll (\log x)^{O(1)} (N + N^2R^{-2}).$$

By hypothesis  $R \leq x^\delta$ , so we have the desired bound  $X_1 - X_3 \ll N^2 R^{-2} (\log x)^{O(1)}$ . Given  $\widehat{\alpha}(0) \ll M$ , our claimed estimate (5.14) is proved, and therefore Proposition 5.3 as well.

### 6. Application to the Titchmarsh divisor problem

The aim of this section is to justify Theorems 1.1 and 1.2. Recall the definition

$$T(x) := \sum_{1 < n \leq x} \Lambda(n) \tau(n-1).$$

We let

$$\psi(x; q, a) := \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n), \quad \psi_q(x) := \sum_{\substack{n \leq x \\ (n, q) = 1}} \Lambda(n), \quad \psi(x, \chi) := \sum_{n \leq x} \Lambda(n) \chi(n).$$

Let us recall the following classical theorem of page [39, Theorems 5.26, 5.28].

LEMMA 6.1. *There is an absolute constant  $b$  such that for all  $Q, T \geq 2$ , the following holds. The function  $s \mapsto \prod_{q \leq Q} \prod_{\chi \pmod{q}} L(s, \chi)$  has at most one zero  $s = \beta$  satisfying  $\Re(s) > 1 - b/\log(QT)$  and  $|\Im(s)| \leq T$ . If it exists, the zero  $\beta$  is real and it is the zero of a unique function  $L(s, \tilde{\chi})$  for some primitive real character  $\tilde{\chi}$ .*

Given a large  $x$ , we shall say that  $\tilde{\chi}$  is  $x$ -exceptional if the above conditions are met with  $Q = T = e^{\sqrt{\log x}}$ . For all  $q \geq 1$  for which  $\tilde{q}|q$ , we let  $\tilde{\chi}_q$  denote the character  $(\bmod q)$  induced by  $\tilde{\chi}$ .

#### 6.1. Primes in arithmetic progressions

We deduce from the previous sections the following result about equidistribution of primes in arithmetic progressions.

THEOREM 6.2. *Assume the GRH. For some  $\delta > 0$ , all  $x \geq 1$ ,  $Q \leq x^{1/2+\delta}$  and all integers  $0 < |a_1|, |a_2| \leq x^\delta$ ,*

$$\sum_{\substack{q \leq Q \\ (q, a_1 a_2) = 1}} \left( \psi(x; q, a_1 \overline{a_2}) - \frac{1}{\varphi(q)} \psi_q(x) \right) \ll x^{1-\delta}.$$

Unconditionally, under the same assumptions,

$$\sum_{\substack{q \leq Q \\ (q, a_1 a_2) = 1}} \left( \psi(x; q, a_1 \overline{a_2}) - \frac{\psi_q(x) + \mathbf{1}_{\tilde{q}|q} \tilde{\chi}(a_2 \overline{a_1}) \psi(x, \tilde{\chi}_q)}{\varphi(q)} \right) \ll x e^{-\delta \sqrt{\log x}},$$

where the term  $\psi(x; \tilde{\chi}_q)$  is to be taken into account only if the  $x$ -exceptional character  $\tilde{\chi}$  exists.

Using the Dirichlet hyperbola method (see in particular [23, Section VII]), it follows that the same estimate holds on the condition  $q \leq x^{1-\varepsilon}$  for any fixed  $\varepsilon > 0$  (the implicit constants and  $\delta$  may then depend on  $\varepsilon$ ). Note however that the symmetry point is at  $q \approx (x|a_2|)^{1/2}$ , rather than  $x^{1/2}$  (so the flexibility of taking  $Q$  somewhat larger than  $x^{1/2}$  is not superfluous). We refer to [21] for more explanations on what happens when  $Q$  is very close to  $x$ .

As mentioned in the Introduction, the uniformity in  $a_1$  and  $a_2$  is an interesting question. At the present state of knowledge, bounds coming from the theory of automorphic forms are typically badly behaved in that aspect. By using a more refined form of the combinatorial

decomposition (6.4), Friedlander and Granville [26] prove that  $|a_1| \leq x^{1/4-\varepsilon}$  is admissible for all  $\varepsilon > 0$  (in the case  $a_2 = 1$ ), with a somewhat larger error term.

For the application to the Titchmarsh divisor problem, the following slightly weaker statement suffices.

**PROPOSITION 6.3.** *For some  $\delta > 0$ , all  $x \geq 2$  and  $0 < |a| \leq x^\delta$ , assuming the GRH, we have*

$$\sum_{\substack{q \leq \sqrt{x} \\ (q,a)=1}} \left( \psi(x; q, a) - \psi(q^2; q, a) - \frac{\psi_q(x) - \psi_q(q^2)}{\varphi(q)} \right) \ll x^{1-\delta}. \tag{6.1}$$

Unconditionally,

$$\begin{aligned} & \sum_{\substack{q \leq \sqrt{x} \\ (q,a)=1}} \left( \psi(x; q, a) - \psi(q^2; q, a) - \frac{\psi_q(x) - \psi_q(q^2)}{\varphi(q)} - \mathbf{1}_{\tilde{q}|q} \overline{\chi(a)} \frac{\psi(x; \tilde{\chi}_q) - \psi(q^2; \tilde{\chi}_q)}{\varphi(q)} \right) \\ & \ll x e^{-\delta \sqrt{\log x}}. \end{aligned} \tag{6.2}$$

We will focus here on proving Proposition 6.3 only because the presentation is slightly simpler and addresses all the essential issues.

*Proof of Proposition 6.3.* Let  $1 \leq R \leq x^{1/10}$  be a parameter. Let

$$\mathcal{S}_1 := \sum_{\substack{q \leq \sqrt{x} \\ (q,a)=1}} \sum_{\substack{q^2 < n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n).$$

By orthogonality of characters,

$$\mathcal{S}_1 = \sum_{\substack{q \leq \sqrt{x} \\ (q,a)=1}} \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \sum_{q^2 < n \leq x} \chi(n\bar{a}) \Lambda(n). \tag{6.3}$$

We decompose  $\mathcal{S}_1 = \mathcal{S}_1^- + \mathcal{S}_1^+$  where  $\mathcal{S}_1^-$  is the contribution of those characters  $\chi$  of conductor at most  $R$ , and

$$\mathcal{S}_1^+ = \sum_{\substack{q \leq \sqrt{x} \\ (q,a)=1}} \sum_{q^2 < n \leq x} \Lambda(n) \mathbf{u}_R(n\bar{a}; q).$$

We first focus of  $\mathcal{S}_1^+$ . By the Heath-Brown identity [7, Lemma 5] and a dichotomy argument similar to [25, Section 2(a)], the problem is reduced to showing

$$\begin{aligned} & \sum_{\substack{Q < q \leq 2Q \\ (q,a)=1}} \sum_{\substack{(1-\Delta)M_i < m_i \leq \min\{M_i, x^{1/4}\} \\ (1-\Delta)N_i < n_i \leq N_i \\ 1 \leq i \leq j}} \cdots \sum_{1 \leq i \leq j} \mu(m_1) \cdots \mu(m_j) (\log n_1) \mathbf{u}_R(n_1 m_1 \cdots n_j m_j \bar{a}; q) \\ & \ll x (\log x)^{O(1)} R^{-1}, \end{aligned} \tag{6.4}$$

where  $j \in \{1, 2, 3, 4\}$ ,  $x^{-1/10} < \Delta \leq 1/2$ , and  $Q, M_i, N_i \geq 1$  ( $1 \leq i \leq j$ ) are real numbers such that

$$Q^2 \leq \prod_i M_i N_i \leq x, \quad M_i \leq 2x^{1/4}.$$

Let us justify briefly this step. The Heath-Brown identity states that  $\mathcal{S}_1^+$  is a linear combination of the expression on the left-hand side of (6.4) for various values of  $j$ , with the conditions  $q \leq \sqrt{x}$ ,  $m_i \leq x^{1/4}$ , and  $q^2 < m_1 n_1 \cdots m_j n_j \leq x$ . We then localize  $q$  in dyadic intervals, and each  $n_i, m_i$  in intervals  $[(1 - \Delta)X, X]$  ( $X = N_i$  or  $M_i$ ). Having done this, the subset of  $(M_i, N_i)$  for which the condition  $q^2 < \prod_i m_i n_i \leq x$  is relevant will only concern those indices with  $\prod_i m_i n_i \in [(1 - \Delta)^8 x, (1 - \Delta)^{-8} x]$  or  $[(1 - \Delta)^8 q^2, (1 - \Delta)^{-8} q^2]$ . For those  $M_i, N_i$ , we apply Lemma 3.2 or a trivial bound (if  $q$  is very small); for the others, the bound (6.4) will apply. We deduce, respectively,

$$\mathcal{S}_1^+ \ll x \Delta (\log x)^{O(1)} + x \Delta^{-8} (\log x)^{O(1)} R^{-1} \tag{6.5}$$

and optimizing  $\Delta$  yields  $\mathcal{S}_1^+ \ll x (\log x)^{O(1)} R^{-1/9}$ .

Let  $\eta > 0$  be small. The contribution of tuples such that  $\prod_i M_i N_i \leq x^{1-\eta}$  is trivially bounded by  $O_\varepsilon(x^{1-\eta+\varepsilon})$  using Lemma 3.2. Suppose then  $\prod_i M_i N_i > x^{1-\eta}$ . For convenience we rename  $x = \prod_i M_i N_i$ . Our objective bound for (6.4) is  $O(x^{1-\delta})$  and we now have  $M_i \leq x^{1/4+\eta}$  if  $\eta$  is small enough.

Fix  $\eta \in (0, 1/100]$ . At least one of the three following cases must hold:

- (a) there exists an index  $k$  such that  $N_k > x^{1-(2j-1)\eta}$ ;
- (b) we have  $\min\{N_k, N_{k'}\} > x^{1/3-\eta}$  for two indices  $k \neq k'$ ;
- (c) there exists an index  $k$  such that  $M_k$  or  $N_k$  lies in the interval  $[x^\eta, x^{1/3-\eta}]$ .

In case (a), our sum (6.4) is at most

$$\mathcal{S}_a := x^\varepsilon \sum_{\substack{Q < q \leq 2Q \\ (q,a)=1}} \sum_{M/2 < m \leq M} \left| \sum_{(1-\Delta)N < n \leq N} \beta_n \mathbf{u}_R(mn\bar{a}; q) \right| \tag{6.6}$$

with  $\beta = \mathbf{1}$  or  $\log$ ,  $MN = x$  and  $N \geq x^{1-7\eta}$ . Choose  $\eta < 1/30$ , for the sum over  $n$ , we express  $\mathbf{u}_R$  as (5.3). Using

$$\sum_{\substack{n \leq z \\ n \equiv a \pmod{q}}} 1 = \frac{z}{q} + O(1) \quad (z \geq 1, (a, q) \in \mathbf{N}^2) \tag{6.7}$$

and partial summation in case  $\beta = \log$ , we get that the sum over  $n$  above is

$$\sum_{(1-\Delta)N < n \leq N} \beta_n \mathbf{u}_R(mn\bar{a}; q) \ll \log x + \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod{q} \\ 1 < \text{cond}(\chi) \leq R}} \left| \sum_{(1-\Delta)N < n \leq N} \beta_n \chi(n) \right|.$$

For each  $\chi$  in the above, the sum over  $n$  is estimated using Lemma 3.4 as

$$\sum_{(1-\Delta)N < n \leq N} \beta_n \chi(n) \ll R^{1/2} (\log x)^2 \tau(q).$$

Dropping the condition  $\text{cond}(\chi) \leq R$ , we obtain for (6.6) a crude bound

$$\mathcal{S}_a \ll_\varepsilon x^\varepsilon MQR^{1/2} \ll QR^{1/2} x^{8\eta} \ll x^{11/20+8\eta+\delta}$$

which is acceptable.

Consider case (b). Then the sum on the LHS of (6.4) is of the form

$$\mathcal{S}_b := \sum_{\substack{Q < q \leq 2Q \\ (q,a)=1}} \sum_{(1-\Delta)N < n \leq N} \sum_{(1-\Delta)M < m \leq M} \sum_{(1-\Delta)^{2j-2}L < \ell \leq L} \alpha(m)\beta(n)\gamma_\ell \mathbf{u}_R(mn\ell\bar{a}; q), \tag{6.8}$$



where  $M, N > x^{1/3-\eta}$ ,  $MNL = x$ ,  $\alpha$  and  $\beta$  are either  $\mathbf{1}$  or  $\log$ , and  $\gamma_\ell$  satisfies

$$|\gamma_\ell| \leq \tau_{2j-2}(\ell) \log \ell.$$

By partial summation and upon rewriting the size restrictions on  $m, n, \ell, q$  as differences of one-sided inequalities, it suffices to establish the bound

$$\mathcal{S}'_b := \sum_{\ell \leq L} \left| \sum_{\substack{q \leq Q \\ (q,a\ell)=1}} \sum_{m \leq M} \sum_{n \leq N} \mathbf{u}_R(mn\ell\bar{a}; q) \right| \ll x^{1-\delta}$$

whenever  $M, N > x^{1/3-2\eta}$  and  $Q \leq 2\sqrt{x}$ . Writing  $\mathbf{u}_R$  as in (5.3), we have by the triangle inequality

$$\mathcal{S}'_b \ll \mathcal{S}'_{b_1} + \mathcal{S}'_{b_2},$$

where

$$\begin{aligned} \mathcal{S}'_{b_1} &= \sum_{\ell \leq L} \left| \sum_{\substack{q \leq Q \\ (q,a\ell)=1}} \sum_{m \leq M} \sum_{n \leq N} \mathbf{u}_1(mn\ell\bar{a}; q) \right|, \\ \mathcal{S}'_{b_2} &= \sum_{\ell \leq L} \sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod{q} \\ 1 < \text{cond}(\chi) \leq R}} \left| \sum_{m \leq M} \chi(m) \right| \left| \sum_{n \leq N} \chi(n) \right|. \end{aligned}$$

Theorem 7 of [7] yields the acceptable bound  $\mathcal{S}'_{b_1} \ll x^{1-\delta}$  as long as  $\eta < 1/30$ . In  $\mathcal{S}'_{b_2}$ , by Lemma 3.4, the sums over  $m$  and  $n$  are majorized by  $O(\tau(q)R^{1/2+\varepsilon})$ . Dropping the condition  $\text{cond}(\chi) \leq R$ , we obtain for (6.8) a bound

$$\mathcal{S}'_{b_2} \ll_\varepsilon x^\varepsilon LRQ \ll_\varepsilon x^{14/15+5\eta}$$

which is also acceptable.

In case (c), we write our sum as

$$\mathcal{S}_c := \sum_{\substack{Q < q \leq 2Q \\ (q,a)=1}} \sum_{\substack{(1-\Delta)^{2j-1}M < m \leq M \\ (1-\Delta)N < n \leq N}} \alpha_m \beta_n \mathbf{u}_R(mn\bar{a}; q), \tag{6.9}$$

where  $x^\eta \leq N \leq x^{1/3-\eta}$ , so  $M \geq x^{2/3}$ . We may assume that  $R \leq x^{\eta/2}$ . If  $Q \leq x^{1/2-\eta/2}$ , then Lemma 5.2 is applicable. If on the contrary  $x^{1/2-\eta/2} < Q \leq \sqrt{x}$ , then Theorem 5.1 is applicable with  $\eta \leftarrow \eta/2$  (assuming  $|a| \leq x^{\delta/2}$  as we may). In both cases, we obtain that the quantity (6.9) is majorized by

$$\mathcal{S}_c \ll x(\log x)^{O(1)} R^{-1}.$$

Summarizing the above and in view of (6.5), we have obtained

$$\mathcal{S}_1^+ \ll x(\log x)^{O(1)} R^{-1/9}.$$

We consider now  $\mathcal{S}_1^-$ , which we recall is

$$\mathcal{S}_1^- = \sum_{\substack{q \leq \sqrt{x} \\ (q,a)=1}} \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod{q} \\ \text{cond}(\chi) \leq R}} \sum_{q^2 < n \leq x} \Lambda(n) \chi(n\bar{a}). \tag{6.10}$$

First let us assume the GRH. Isolating the contribution of the principal character, we write

$$\mathcal{S}_1^- = \sum_{\substack{q \leq \sqrt{x} \\ (q,a)=1}} \frac{\psi_q(x) - \psi_q(q^2)}{\varphi(q)} + \mathcal{S}_1^b,$$

say. For any non-trivial character  $\chi \pmod q$  with  $q \leq x$ , the GRH [44, formula (13.19)] yields

$$\sum_{q^2 < n \leq x} \chi(n)\Lambda(n) \ll x^{1/2}(\log x)^2.$$

We therefore have

$$\mathcal{S}_1^b \ll x^{1/2}(\log x)^2 \sum_{q \leq \sqrt{x}} \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \text{cond}(\chi) \leq R}} 1 \ll Rx^{1/2}(\log x)^3$$

which is acceptable. The choice  $R = x^\delta$  for small enough  $\delta$  concludes the proof of (6.1).

Unconditionally, for any  $q \leq e^{\sqrt{\log x}}$  and any non-principal, non  $x$ -exceptional character  $\chi \pmod q$ , we have by a straightforward adaptation of [44, Theorem 11.16] the estimate

$$\sum_{q^2 < n \leq x} \chi(n)\Lambda(n) \ll x e^{-c\sqrt{\log x}}$$

for some absolute constant  $c > 0$ . Choose  $R = e^{c\sqrt{\log x}/2}$ . We extract from  $\mathcal{S}_1^-$  the contribution from the principal character and the possible  $x$ -exceptional characters, and write accordingly

$$\mathcal{S}_1^- = \sum_{\substack{q \leq \sqrt{x} \\ (q,a)=1}} \frac{\psi_q(x) - \psi_q(q^2) + \mathbf{1}_{\tilde{q}|q} \overline{\chi(a)}(\psi(x; \tilde{\chi}_q) - \psi(q^2; \tilde{\chi}_q))}{\varphi(q)} + \mathcal{S}_1^{bb} + O(x e^{-c\sqrt{\log x}/2})$$

the error term being there to cover the trivial case when either  $\tilde{\chi}$  was inexistant, or  $\tilde{q} > R$ . By the same computation as above,

$$\mathcal{S}_1^{bb} \ll Rx(\log x) e^{-c\sqrt{\log x}} \ll x e^{-c\sqrt{\log x}/3}.$$

This concludes the proof of (6.2) hence of Proposition 6.3. □

### 6.2. Proof of Theorems 1.1 and 1.2

It is now straightforward to deduce Theorems 1.1 and 1.2. By the Dirichlet hyperbola method [25, p. 45], we have

$$T(x) = 2 \sum_{q \leq \sqrt{x}} (\psi(x; q, 1) - \psi(q^2; q, 1)) + O(\sqrt{x} \log x).$$

Assume first the GRH. Then Proposition 6.3 yields

$$T(x) = 2 \sum_{q \leq \sqrt{x}} \frac{\psi_q(x) - \psi_q(q^2)}{\varphi(q)} + O(x^{1-\delta}).$$

The GRH [44, formula (13.19)] allows us to deduce

$$T(x) = 2 \sum_{q \leq \sqrt{x}} \frac{x - q^2}{\varphi(q)} + O(x^{1-\delta}).$$

The main term is computed using [22, Lemme 6], which yields the claimed estimate.

Unconditionally, from Proposition 6.3, we merely have to add to our estimate for  $T(x)$  the additional contribution of the  $x$ -exceptional character (if it exists), which takes the form

$$2 \sum_{\substack{q \leq \sqrt{x} \\ \tilde{q}|q}} \frac{\psi(x; \tilde{\chi}_q) - \psi(q^2; \tilde{\chi}_q)}{\varphi(q)}. \tag{6.11}$$

We have from [44, Theorem 11.16]

$$\psi(x; \tilde{\chi}_q) = -\frac{x^\beta}{\beta} + O(x e^{-\delta\sqrt{\log x}})$$

and similarly

$$\psi(q^2; \tilde{\chi}_q) = -\frac{q^{2\beta}}{\beta} + O(x e^{-\delta\sqrt{\log x}})$$

at the possible cost of changing the numerical value of  $\delta$ . We obtain that (6.11) equals

$$-\frac{2}{\beta} \sum_{\substack{q \leq \sqrt{x} \\ \tilde{q}|q}} \frac{x^\beta - q^{2\beta}}{\varphi(q)} + O(x e^{-\delta\sqrt{\log x}}).$$

The sums over  $q$  are computed using [22, Lemme 6] (and partial summation in the form  $x^\beta - q^{2\beta} = \beta \int_{q^2}^x t^{\beta-1} dt$ ), which yields Theorem 1.2. Corollary 1.3 is straightforward.

There remains to justify Corollary 1.4. Note that  $C_2(\tilde{q})$  is absolutely bounded, while  $\tilde{q} \leq e^{\sqrt{\log x}}$  by definition. Therefore  $x^\beta \rightarrow \infty$ , and  $\beta \operatorname{li}(x^\beta)/x^\beta \sim (\log x)^{-1}$ . We deduce

$$\frac{\log \tilde{q} + C_2(\tilde{q}) - \gamma}{x^\beta / (\beta \operatorname{li}(x^\beta))} \xrightarrow{x \rightarrow \infty} 0$$

in an effective way. For  $x$  large enough, it is less than  $1/3$  and Corollary 1.4 follows.

REMARK. If we were to consider  $\tau(n - a)$  instead of  $\tau(n - 1)$ , for some  $a$  which is not a perfect square, then the Siegel zero contribution (if it existed) would have a twist by  $\chi(a)$ , which is *a priori* of unpredictable sign.

### 7. Application to correlation of divisor functions

In this section, we justify Theorem 1.5. The proof has the same structure as that of Theorems 1.1 and 1.2, replacing the function  $\Lambda(n)$  by  $\tau_k(n)$ .

#### 7.1. An equidistribution estimate

The analog of Theorem 6.2 is the following:

THEOREM 7.1. *There exists  $\eta > 0$  such that under the conditions  $k \geq 4$ ,  $0 < |a| \leq x^\eta$  and  $Q \leq x^{1/2+\eta}$ ,*

$$\sum_{\substack{q \leq Q \\ (q,a)=1}} \left( \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \tau_k(n) - \frac{1}{\varphi(q)} \sum_{\substack{n \leq x \\ (n,q)=1}} \tau_k(n) \right) \ll x^{1-\eta/k}. \tag{7.1}$$

*If the Lindelöf hypothesis is true for all Dirichlet  $L$ -functions, then the right-hand side can be replaced by  $x^{1-\eta}$ .*

In order to simplify the presentation, we put

$$\mathcal{E} = \begin{cases} x & \text{if the generalized Lindelöf hypothesis is assumed,} \\ x^{1/k} & \text{unconditionally.} \end{cases}$$

To handle the small conductor case, we require the following.

LEMMA 7.2. *For some  $\delta > 0$  and any non-principal character  $\chi \pmod{q}$  with  $q \leq x$ , of conductor  $r \leq \mathcal{E}^\delta$  we have*

$$\sum_{n \leq x} \tau_k(n)\chi(n) \ll_k x\mathcal{E}^{-\delta}.$$

*Proof.* Starting from the representation

$$\sum_{n \leq x} \tau_k(n)\chi(n) = \frac{1}{2\pi i} \int_{1+1/(\log x)-i\infty}^{1+1/(\log x)+i\infty} L(s, \chi)^k \frac{x^s ds}{s} \quad (x \notin \mathbf{N}),$$

one may truncate the contour at  $T = x^{\delta/k}$ , and shift it to the abscissa  $\Re(s) = 1 - \delta/k$ . The convexity bound  $|L(1 - \delta/k + it, \chi)| \ll q^\epsilon (r(|t| + 1))^{c\delta/k + \epsilon}$  (for some  $c > 0$ ) yields the desired estimate if  $\mathcal{E} = x^{1/k}$ . If the Lindelöf hypothesis  $L(\frac{1}{2} + it, \chi) \ll (q(|t| + 1))^\epsilon$  is true, then one chooses  $T = x^\delta$  and shifts the contour to  $\Re(s) = 1 - \delta$ , where the bound  $L(1 - \delta + it, \chi) \ll (q(|t| + 1))^\epsilon$  holds by convexity.  $\square$

7.1.1. *Small conductors.* Let  $\mathcal{S}_0$  denote the quantity in the left-hand side of (7.1), and let  $R \leq \mathcal{E}^\delta$ . The contribution of those characters  $\chi$  having conductors at most  $R$  is

$$\sum_{1 < r \leq R} \sum_{\substack{\chi \pmod{r} \\ \chi \text{ primitive}}} \overline{\chi(a)} \sum_{\substack{q \leq Q \\ (q,a)=1 \\ r|q}} \frac{1}{\varphi(q)} \sum_{\substack{n \leq x \\ (n,q)=1}} \tau_k(n)\chi(n).$$

By Lemma 7.2 applied to the character  $\pmod{q}$  induced by  $\chi$ , we have a bound

$$x\mathcal{E}^{-\delta} \sum_{r \leq R} \sum_{\substack{\chi \pmod{r} \\ \chi \text{ primitive}}} \sum_{\substack{q \leq Q \\ r|q}} \frac{1}{\varphi(q)} \ll x\mathcal{E}^{-\delta} R(\log x)^2.$$

Letting  $R = \mathcal{E}^{\delta/2}$ , this is an acceptable error term. There remains to bound

$$\mathcal{S}_1 := \sum_{\substack{q \leq Q \\ (q,a)=1}} \sum_{n \leq x} \tau_k(n)u_R(n\bar{a}; q).$$

7.1.2. *Dyadic decomposition.* We dyadically decompose in  $\mathcal{S}_1$  the sums over  $q$  and  $n$  in (7.1), yielding an upper bound

$$\mathcal{S}_1 \ll (\log x)^2 \sup_{\substack{Q' \leq x^{1/2+\eta} \\ N \leq x}} \left| \sum_{\substack{Q' < q \leq 2Q' \\ (q,a)=1}} \sum_{N < n \leq 2N} \tau_k(n)u_R(n\bar{a}; q) \right|. \tag{7.2}$$

Let  $\eta > 0$  and assume throughout that  $\delta$  is small with respect to  $\eta$ . When  $N \leq x^{1-\eta}$ , by the triangle inequality, our trivial bound (5.2) and Lemma 3.2, the sum over  $q$  and  $n$  above is  $O_k(x^{1-\eta/2})$ , so we may add the restriction  $N > x^{1-\eta}$  in the supremum with an acceptable

error. Then we relax the condition  $Q' \leq x^{1/2+\eta}$  into  $Q' \leq N^{1/2+2\eta}$ . Renaming  $N$  into  $x$ , and expanding out  $\tau_k(n)$ , we obtain that it will suffice to prove

$$\mathcal{S}_2 := \sum_{\substack{Q < q \leq 2Q \\ (q,a)=1}} \sum_{x < n_1 \cdots n_k \leq 2x} \mathbf{u}_R(n_1 \cdots n_k \bar{a}; q) \ll x \mathcal{E}^{-\eta} \tag{7.3}$$

under the constraints  $|a| \leq x^{2\eta}$  and  $Q \leq x^{1/2+2\eta}$ . We decompose the sums over  $n_1, \dots, n_k$  dyadically to obtain an upper bound

$$\mathcal{S}_2 \ll \mathcal{S}_3 := (\log x)^k \sup_{N_1, \dots, N_k \geq 1/2} \left| \sum_{\substack{Q < q \leq 2Q \\ (q,a)=1}} \sum_{\substack{x < n_1 \cdots n_k \leq 2x \\ N_j < n_j \leq 2N_j}} \mathbf{u}_R(n_1 \cdots n_k \bar{a}; q) \right|. \tag{7.4}$$

7.1.3. *Splitting cases.* Let the parameter  $0 < \delta_1 < 1/100$  be fixed. We separate into two cases according to whether there is a subset  $\mathcal{J} \subset \{1, \dots, k\}$  such that

$$\prod_{j \in \mathcal{J}} N_j \in (x^{\delta_1}, x^{1/3-\delta_1}],$$

or not. Suppose there is no such subset, and let

$$\mathcal{K} := \{j : 1 \leq j \leq k, N_j > x^{1/3-\delta_1}\}.$$

Necessarily  $\text{card } \mathcal{K} \leq 3$ . Since  $N_j \leq x^{\delta_1}$  for each  $j \notin \mathcal{K}$ , and by assumption there is no subset  $\mathcal{L} \subset \{1, \dots, k\} \setminus \mathcal{K}$  such that  $\prod_{j \in \mathcal{L}} N_j \in (x^{\delta_1}, x^{1/3-\delta_1}]$ , it is necessarily the case that

$$\prod_{j \notin \mathcal{K}} N_j \leq x^{\delta_1}.$$

This implies  $\text{card } \mathcal{K} \geq 1$ . Define

$$\mathcal{W} := \{(u_n) \in \mathbf{C}^{\mathbf{N}} : |u_n| \leq \tau(n)^k \quad (n \geq 1)\}.$$

Summarizing the above, we have

$$\mathcal{S}_3 \ll_{k,\varepsilon} x^\varepsilon (\mathcal{A} + \mathcal{B}_3 + \mathcal{B}_2 + \mathcal{B}_1), \tag{7.5}$$

where

$$\mathcal{A} = \sup_{\substack{x^{\delta_1} < N \leq x^{1/3-\delta_1} \\ MN=x \\ (\alpha_m), (\beta_n) \in \mathcal{W}}} \left| \sum_{\substack{Q < q \leq 2Q \\ (q,a)=1}} \sum_{\substack{N < n \leq 2^k N \\ M2^{-k} < m < 2M \\ x < mn \leq 2x}} \alpha_m \beta_n \mathbf{u}_R(nm\bar{a}; q) \right|,$$

$$\mathcal{B}_3 = \sup_{\substack{N_1, N_2, N_3 > x^{1/3-\delta_1} \\ MN_1 N_2 N_3 = x \\ (\alpha_m) \in \mathcal{W}}} \left| \sum_{\substack{Q < q \leq 2Q \\ (q,a)=1}} \sum_{\substack{N_j < n_j \leq 2N_j \\ M/8 < m \leq 2M \\ x < mn_1 n_2 n_3 < 2x}} \alpha_m \mathbf{u}_R(n_1 n_2 n_3 m \bar{a}; q) \right|,$$

$$\mathcal{B}_2 = \sup_{\substack{N_1, N_2 > x^{1/3-\delta_1} \\ N_1 N_2 > x^{1-\delta_1} \\ MN_1 N_2 = x \\ (\alpha_m) \in \mathcal{W}}} \left| \sum_{\substack{Q < q \leq 2Q \\ (q,a)=1}} \sum_{\substack{N_j < n_j \leq 2N_j \\ M/8 < m \leq 2M \\ x < mn_1 n_2 < 2x}} \alpha_m \mathbf{u}_R(n_1 n_2 m \bar{a}; q) \right|,$$

$$\mathcal{B}_1 = \sup_{\substack{N > x^{1-\delta_1} \\ MN = x \\ (\alpha_m) \in \mathcal{W}}} \left| \sum_{\substack{Q < q \leq 2Q \\ (q,a)=1}} \sum_{\substack{N < n \leq 2N \\ M/8 < m \leq 2M \\ x < mn < 2x}} \alpha_m \mathbf{u}_R(nm \bar{a}; q) \right|.$$

We will focus on  $\mathcal{A}$  and  $\mathcal{B}_3$ , since the treatment of  $\mathcal{B}_1$  and  $\mathcal{B}_2$  is analogous to  $\mathcal{B}_3$  and actually simpler.

7.1.4. *Separation of variables.* Fix another small parameter  $\delta_2 > 0$ . We smoothen the cutoff using a smooth function  $\phi : \mathbf{R} \rightarrow [0, 1]$  with  $\phi(\xi) = 1$  for  $\xi \in [1, 2]$ ,  $\phi(\xi) = 0$  for  $\xi \notin [1 - \mathcal{E}^{-\delta_2}, 2 + \mathcal{E}^{-\delta_2}]$ , whose derivatives satisfy  $\|\phi^{(j)}\|_\infty \ll_j \mathcal{E}^{j\delta_2}$ . The cost of replacing in  $\mathcal{A}$  and  $\mathcal{B}_3$  the sharp cutoff condition  $x < nm \leq 2x$  (respectively,  $x < n_1 n_2 n_3 m \leq 2x$ ) by  $\phi(nm/x)$  (respectively,  $\phi(n_1 n_2 n_3 m/x)$ ) is at most  $O(x\mathcal{E}^{-\delta_2/2})$ , by trivially bounding the contribution of the transition ranges using Lemma 3.2.

Integration by parts shows that the Mellin transform  $\check{\phi}(s) = \int_0^\infty \phi(\xi)\xi^{s-1} d\xi$  satisfies

$$\check{\phi}(it) \ll \frac{\mathcal{E}^{5\delta_2}}{1 + |t|^5} \quad (t \in \mathbf{R}).$$

We use the inversion formula  $\phi(\xi) = (2\pi)^{-1} \int_{\mathbf{R}} \check{\phi}(it)\xi^{-it} dt$  at  $\xi = nm/x$  (respectively,  $\xi = mn_1 n_2 n_3/x$ ) in the case of  $\mathcal{A}$  (respectively,  $\mathcal{B}_3$ ), to obtain the upper bounds

$$\mathcal{A} \ll_k x \mathcal{E}^{-\delta_2/2} + \mathcal{E}^{5\delta_2} \sup_{\substack{x^{\delta_1} < N \leq x^{1/3-\delta_1} \\ MN = x \\ (\alpha_m), (\beta_n) \in \mathcal{W}}} \left| \sum_{\substack{Q < q \leq 2Q \\ (q,a)=1}} \sum_{\substack{N < n \leq 2^k N \\ M2^{-k} < m \leq 2M}} \alpha_m \beta_n \mathbf{u}_R(mn \bar{a}; q) \right|, \tag{7.6}$$

$$\mathcal{B}_3 \ll_k x \mathcal{E}^{-\delta_2/2} + \mathcal{E}^{5\delta_2} \sup_{\substack{N_1, N_2, N_3 > x^{1/3-\eta} \\ (\alpha_m) \in \mathcal{W}, t \in \mathbf{R}}} \frac{1}{1 + |t|^3} \times \left| \sum_{\substack{Q < q \leq 2Q \\ (q,a)=1}} \sum_{\substack{N_j < n_j \leq 2N_j \\ M/8 < m \leq 2M}} \alpha_m (n_1 n_2 n_3)^{it} \mathbf{u}_R(n_1 n_2 n_3 m \bar{a}; q) \right|. \tag{7.7}$$

7.1.5. *The case of  $\mathcal{A}$ .* Let  $(\alpha_m)$ ,  $(\beta_n)$ , and  $N$  be given as in the supremum in (7.6). We wish to bound

$$\mathcal{S}_a := \sum_{\substack{Q < q \leq 2Q \\ (q,a)=1}} \sum_{\substack{N < n \leq 2^k N \\ M2^{-k} < m < 2M}} \alpha_m \beta_n \mathbf{u}_R(mn \bar{a}; q). \tag{7.8}$$

By dyadic decomposition, enlarging our bound by a factor of  $k^2$ , we may assume the conditions are  $N_1 < n \leq 2N_1$  and  $M_1 < m \leq 2M_1$  for  $M_1 N_1 \in [x2^{-k}, x2^{k+1}]$ . Suppose first  $Q \geq x^{1/2-\delta_1/2}$ . Then Theorem 5.1 with  $\eta \leftarrow \min\{\delta_1/2, 1/30\}$  gives the existence of  $\delta_3 > 0$  depending on  $\delta_1$  such that (7.8) is majorized by  $O(2^k x \mathcal{E}^{-\delta_3})$ , on the condition that  $|a| \leq 2^{-k} x^{\delta_3}$  and  $Q \leq 2^{-k} x^{1/2+\delta_3}$ , which are satisfied assuming  $\eta < \delta_3/4$  and taking  $x$  large enough in terms of  $k$ .

If on the contrary  $Q \leq x^{1/2-\delta_1/2}$ , we appeal to Lemma 5.2. We again obtain for (7.8) a bound

$$\mathcal{S}_a \ll_j 2^k x \mathcal{E}^{-\delta_3}$$

for some  $\delta_3$  (depending on  $\delta_1$ ).

Summarizing, we have obtained in any case

$$\mathcal{A} \ll_k x \mathcal{E}^{-\delta_2/2} + x \mathcal{E}^{5\delta_2-\delta_3} \tag{7.9}$$

for  $\delta_3 > 0$ . Choosing  $\delta_2$  appropriately, it is an acceptable error term once we can prove that  $\delta_1 > 0$  can be chosen independently of  $k$ .

7.1.6. *The case of  $\mathcal{B}_3$ .* Let  $(\alpha_m), N_1, N_2, N_3 > x^{1/3-\delta_1}$  and  $t \in \mathbf{R}$  be as in supremum in (7.7). The quantity we wish to bound is at most

$$\mathcal{S}_b := \frac{1}{1+|t|^3} \sum_{M/8 \leq m \leq 2M} \sum_{\substack{Q < q \leq 2Q \\ (q, am)=1}} \left| \sum_{\substack{n_1, n_2, n_3 \\ N_j \leq n_j \leq 2N_j}} (n_1 n_2 n_3)^{it} \mathbf{u}_R(n_1 n_2 n_3 m \bar{a}; q) \right|,$$

where  $N_1 N_2 N_3 M = x$  and  $M < x^{3\delta_1}$ . Writing  $n_j^{it} = (2N_j)^{it} - it \int_{n_j}^{2N_j} z^{it-1} dz$ , the above is bounded by

$$\mathcal{S}_b \ll_\varepsilon \sup_{\substack{N'_1, N'_2, N'_3 \\ N_j < N'_j \leq 2N_j}} \sum_{M/8 \leq m \leq M} \sum_{\substack{Q < q \leq 2Q \\ (q, am)=1}} \left| \sum_{\substack{n_1, n_2, n_3 \\ N_j \leq n_j \leq N'_j}} \mathbf{u}_R(n_1 n_2 n_3 m \bar{a}; q) \right|. \tag{7.10}$$

Fix  $N'_1, N'_2, N'_3$  as in the supremum. Using (5.3) and the triangle inequality,

$$\mathcal{S}_b \leq \mathcal{S}'_b + \mathcal{S}''_b,$$

where

$$\mathcal{S}'_b = \sum_{M/8 \leq m \leq M} \sum_{\substack{Q < q \leq 2Q \\ (q, am)=1}} \left| \sum_{\substack{n_1, n_2, n_3 \\ N_j \leq n_j \leq N'_j}} \mathbf{u}_1(n_1 n_2 n_3 m \bar{a}) \right|, \tag{7.11}$$

$$\mathcal{S}''_b = \sum_{M/8 < m \leq M} \sum_{Q < q \leq 2Q} \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod{q} \\ 1 < \text{cond}(\chi) \leq R}} \prod_{j=1}^3 \left| \sum_{N_j < n \leq N'_j} \chi(n) \right|. \tag{7.12}$$

To  $\mathcal{S}'_b$  we apply [8, Lemma 2] for each  $q$  individually (note that this is a very deep result [27, 33], relying on Deligne’s proof of the Weil conjectures [12]). For some small, absolute  $\delta_4$ , on the condition that  $Q \leq x^{1/2+\delta_4}$  (requiring  $\eta < \delta_4/2$ ), the quantity (7.11) is bounded by

$$\mathcal{S}'_b \ll M x^{1-\delta_4} \leq x^{1-\delta_4+3\delta_1}. \tag{7.13}$$

Consider then  $\mathcal{S}_b''$ . By Lemma 3.4, each sum over  $n$  is bounded by  $O_\varepsilon(x^\varepsilon R^{1/2})$ , and so we obtain a bound

$$\mathcal{S}_b'' \ll_\varepsilon x^\varepsilon R^{5/2} M$$

which is absorbed in the term (7.13). Inserting in (7.7), we have obtained for  $\mathcal{B}_3$  a bound

$$\mathcal{B}_3 \ll x\mathcal{E}^{-\delta_2} + \mathcal{E}^{5\delta_2} x^{1-\delta_4+3\delta_1}. \tag{7.14}$$

The terms  $\mathcal{B}_2$  and  $\mathcal{B}_1$  are shown in the same way to satisfy the same bound with  $\delta_4 > 0$  absolute and small enough. Choosing our parameters adequately, we can choose absolute constants  $\delta_1, \delta_2, \delta_3$  in such a way that both bounds (7.14) and (7.9) are true and  $O(x\mathcal{E}^{-\eta})$ . Inserting back into (7.5) and (7.4), we obtain the claimed bound (7.3).

7.2. Proof of Theorems 1.5 and 1.6

As a last step, we deduce from Theorem 7.1 the estimate

$$\sum_{\substack{q \leq \sqrt{x} \\ (q,a)=1}} \left( \sum_{\substack{n \leq q^2 \\ n \equiv a \pmod{q}}} \tau_k(n) - \frac{1}{\varphi(q)} \sum_{\substack{n \leq q^2 \\ (n,q)=1}} \tau_k(n) \right) \ll_k x\mathcal{E}^{-\eta} \quad (0 < |a| \leq x^\eta), \tag{7.15}$$

whereas before  $\mathcal{E} = x$  if the generalized Lindelöf is true and  $\mathcal{E} = x^{1/k}$  otherwise. Let  $\Delta \in (0, 1/10)$  be fixed and decompose the sums over  $q$  and  $n$  into intervals  $((1 + \Delta)^{-1}Q, Q]$  and  $((1 + \Delta)^{-1}N, N]$ . Calling  $\mathcal{S}'_1$  the left-hand side of (7.15), we have

$$\mathcal{S}'_1 \ll \sum_{\substack{j_0, j_1 \geq 0 \\ Q = (1+\Delta)^{-j_0} \sqrt{x} \\ N = (1+\Delta)^{-j_1} x}} \left| \sum_{(1+\Delta)^{-1}Q < q \leq Q} \sum_{\substack{(1+\Delta)^{-1}N < n \leq N \\ n \leq q^2}} \tau_k(n) \mathbf{u}_1(n\bar{a}; q) \right|,$$

where we used the notation (5.7). The inner sums are void if  $Q^2 \leq N$  and the condition  $n \leq q^2$  is automatically satisfied if  $N \leq Q^2(1 + \Delta)^{-2}$ . The contribution of  $j_0, j_1$  such that  $(1 + \Delta)^{-2}Q^2 \leq N \leq Q^2$  is at most

$$\sum_{\substack{q \leq \sqrt{x} \\ (q,a)=1}} \sum_{q^2(1+\Delta)^{-3} \leq n \leq q^2(1+\Delta)^2} \tau_k(n) |\mathbf{u}_1(n\bar{a}; q)| \ll \Delta x (\log x)^k$$

by virtue of Lemma 3.2. Therefore,

$$\mathcal{S}'_1 \ll \Delta x (\log x)^k + (\log x)^2 \Delta^{-2} \sup_{\substack{Q \leq \sqrt{x} \\ N \leq Q^2}} \left| \sum_{(1+\Delta)^{-1}Q < q \leq Q} \sum_{(1+\Delta)^{-1}N < n \leq N} \tau_k(n) \mathbf{u}_1(n\bar{a}; q) \right|.$$

Let  $Q, N$  be as in the supremum, and let  $\eta > 0$  be the real number given by Theorem 7.1. Lemma 3.2 gives the bound

$$\left| \sum_{(1+\Delta)^{-1}Q < q \leq Q} \sum_{(1+\Delta)^{-1}N < n \leq N} \tau_k(n) \mathbf{u}_1(n\bar{a}; q) \right| \ll_\varepsilon x^\varepsilon N$$



which is acceptable if  $N \leq x^{1-\eta/10}$ . Suppose  $N \geq x^{1-\eta/10}$ , then Theorem 7.1 applies with  $x \leftarrow N$  and yields a bound  $O(x\mathcal{E}^{-\eta/10})$  for  $|a| \leq x^{\eta/10}$ . Therefore,

$$\mathcal{S}'_1 \ll_{\varepsilon,k} x^{1+\varepsilon} \Delta + \Delta^{-2} x^{1+\varepsilon} \mathcal{E}^{-\eta/10}.$$

Taking for example  $\Delta = \mathcal{E}^{-\eta/30}$  and reinterpreting  $\eta$ , we have the claimed estimate (7.15).

From the Dirichlet hyperbola method, Theorem 7.1 and estimate (7.15), we deduce

$$\begin{aligned} \mathcal{T}_k(x) &= 2 \sum_{q \leq \sqrt{x}} \sum_{\substack{q^2 < n \leq x \\ n \equiv -1 \pmod{q}}} \tau_k(n) + O_\varepsilon(x^{1/2+\varepsilon}) \\ &= 2 \sum_{q \leq \sqrt{x}} \frac{1}{\varphi(q)} \sum_{\substack{q^2 < n \leq x \\ (n,q)=1}} \tau_k(n) + O(x\mathcal{E}^{-\delta}). \end{aligned}$$

The main terms are computed in [25, Théorème 2], with an error term  $O(x^{1-\delta/k})$  (unconditionally). If one assumes the generalized Lindelöf hypothesis, then the proof is adapted in the following way. Under the hypotheses and in the notations of [25, Lemma 6], there holds  $|\theta(p^\nu)| \leq Cp^{-\delta} \binom{k}{\lfloor k/2 \rfloor}$  [25, first display page 52]. Therefore the series  $F_k(s)$  in [25, Lemma 7] is bounded in terms of  $k$  only in the half-plane  $\Re(s) \geq 1 - \delta/2$ . In the proof of [25, Lemma 7], one chooses  $T = x^{\delta/2}$  and shifts the contour to  $\Re(s) = 1 - \delta/2$ , where the Lindelöf hypothesis implies  $\zeta(s) \ll t^\varepsilon$  by convexity, to produce the conclusion

$$\sum_{n \leq x} \Psi(n) \tau_k(n) = xQ_{k-1}(\log x) + O_{\varepsilon,k}(x^{1-\delta/2+\varepsilon}).$$

The rest of the argument in Corollaries 1 and 2 of Lemma 7, and Corollary of Lemma 8 of [25] are transposed *verbatim* to yield

$$2 \sum_{q \leq \sqrt{x}} \frac{1}{\varphi(q)} \sum_{\substack{q^2 < n \leq x \\ (n,q)=1}} \tau_k(n) = xP_k(\log x) + O_k(x^{1-c})$$

for some  $c > 0$ , as claimed.

### 7.3. Remark on the uniformity in $a$

If we were to replace the shift  $\tau(n + 1)$  by  $\tau(n + a)$ ,  $0 < |a| \leq x^\delta$ , then the deduction of an asymptotic formula analogous to (1.4) from Theorem 7.1 would go along similar lines. We briefly indicate how one reduces to our previous setting. From Dirichlet’s hyperbola method, the problem reduces to the evaluation of

$$\mathcal{S}_{k,a}(x) = 2 \sum_{q \leq \sqrt{x}} \sum_{\substack{q^2 \leq n \leq x \\ n \equiv -a \pmod{q}}} \tau_k(n).$$

Extracting the largest factor  $d_1|a|^\infty$  from  $n$ , we rewrite this as

$$\mathcal{S}_{k,a}(x) = 2 \sum_{d_1|a^\infty} \tau_k(d_1) \sum_{q \leq \sqrt{x}} \sum_{\substack{q^2/d_1 \leq n \leq x/d_1 \\ (n,a)=1 \\ nd_1 \equiv -a \pmod{q}}} \tau_k(n).$$

Writing  $d_2 := (q, d_1)$ , the congruence condition is equivalent to  $d_2|a$  and

$$n \equiv -(a/d_2) \overline{(d_1/d_2)} \pmod{q/d_2}.$$

We therefore have

$$\mathcal{S}_{k,a}(x) = 2 \sum_{d_1|a^\infty} \tau_k(d_1) \sum_{d_2|(d_1,a)} \sum_{\substack{q \leq \sqrt{x}/d_2 \\ (q,d_1/d_2)=(q,a/d_2)=1}} \sum_{\substack{q^2/d_1 \leq n \leq x/d_1 \\ (n,a)=1 \\ n \equiv -(a/d_2)(d_1/d_2) \pmod{q}}} \tau_k(n).$$

Summing for each  $d_j$  individually, the contribution of  $d_1 > x^\delta$  is bounded trivially using Lemma 3.2. When  $d_1 \leq x^\delta$ , the sum over  $n$  and  $q$  is handled by an adequate generalization of Theorem 7.1, involving a congruence of the type  $n \equiv b_1 \overline{b_2} \pmod{q}$ , as well as an additional coprimality condition  $(n, b_3) = 1$ , for integers  $|b_j| \leq x^\delta$ . Our arguments readily adapt to account for both these modifications. Note however that it is now important that the method is able to handle values of the modulus  $q$  up to  $x^{1/2+\delta}$ , with  $\delta$  independent of  $k$  (cf. the statement of Theorem 7.1).

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## SIGN CHANGES OF KLOOSTERMAN SUMS AND EXCEPTIONAL CHARACTERS

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ABSTRACT. We prove that the existence of exceptional real zeros of Dirichlet  $L$ -functions would lead to cancellations in the sum  $\sum_{p \leq x} \text{Kl}(1, p)$  of Kloosterman sums over primes, and also to sign changes of  $\text{Kl}(1, n)$ , where  $n$  runs over integers with exactly two prime factors. Our arguments involve a variant of Bombieri’s sieve, bounds for twisted sums of Kloosterman sums, and work of Fouvry and Michel on sums of  $|\text{Kl}(1, n)|$ .

### 1. INTRODUCTION

*Kloosterman sums.* For  $n \in \mathbb{N}_{>0}$  and a residue class  $a \pmod{n}$ , define the normalized Kloosterman sum as

$$\text{Kl}(a, n) = \frac{1}{\sqrt{n}} \sum_{\substack{\nu \pmod{n} \\ (\nu, n)=1}} e\left(\frac{\nu + a\bar{\nu}}{n}\right),$$

where we write  $e(z) = e^{2\pi iz}$  and  $\bar{\nu} \equiv 1 \pmod{n}$ . These sums have a long history [Poi11, Klo27], at the intersection of algebraic geometry and automorphic forms. The Weil bound [Wei48, Mat11] yields  $|\text{Kl}(a, n)| \leq 2^{\omega(n)}$  if  $32 \nmid n$  and  $|\text{Kl}(a, n)| \leq 2^{\omega(n)+1/2}$  in general, where  $\omega(n)$  is the number of distinct prime factors of  $n$ . In particular, for a prime  $p$

$$|\text{Kl}(1, p)| \leq 2.$$

Let  $\theta_{a,p} \in [0, \pi]$  be such that  $\text{Kl}(a, p) = 2 \cos(\theta_{a,p})$ . The “vertical” Sato-Tate law, due to Katz [Kat88], asserts that the numbers

$$\{\theta_{a,p} \mid 1 \leq a < p\}$$

become equidistributed, as  $p \rightarrow \infty$ , with respect to the Sato-Tate measure  $\frac{2}{\pi} \sin(\theta)^2 d\theta$ . The “horizontal” Sato-Tate conjecture is the claim that the numbers

$$\{\theta_{1,p} \mid p \leq x\}$$

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become equidistributed with respect to the same measure, as  $x \rightarrow \infty$ . This would of course imply that

$$\sum_{p \leq x} \text{Kl}(1, p) = o(\pi(x)) \quad (x \rightarrow \infty).$$

Unfortunately the horizontal Sato-Tate conjecture is still open, and very little is known about this sum. Fouvry and Michel [FM07] have obtained significant partial progress on replacing primes by almost-primes: they show that

$$\sum_{\substack{n < x \\ p|n \Rightarrow p > x^{1/23.9}}} (|\text{Kl}(1, n)| \pm \text{Kl}(1, n)) \gg \frac{x}{\log x}.$$

In particular, it follows that there are infinitely many sign changes in the set  $\{\text{Kl}(1, n), \omega(n) \leq 23\}$ . After further work by many authors [FM03a, SF07, SF09, Mat11, Xi15a], the best known current result is due to Xi [Xi15b] and shows that there are infinitely many sign changes in the set  $\{\text{Kl}(1, n), \omega(n) \leq 7\}$ . We refer to the recent preprint [Xi18] for more references and related questions.

*Landau-Siegel zeros.* In this work we study the implications of the existence of Landau-Siegel zeros on this question. For  $D \geq 3$  an integer and  $\chi \pmod{D}$  a real primitive character, define the value

$$\eta_\chi := L(1, \chi) \log D.$$

It would follow from GRH that  $\eta_\chi \gg (\log D)/\log \log D$  and so  $\eta_\chi \rightarrow \infty$  as  $D \rightarrow \infty$ . Unconditionally, however, Siegel's non-effective lower bound [Sie35]  $\eta_\chi \gg_\varepsilon D^{-\varepsilon}$  remains unsurpassed. It is as yet not even known if  $\eta_\chi$  is bounded away from 0; this is equivalent to the non-existence of real zeros of  $L(s, \chi)$  close to 1. On the other hand, if there were a sequence of characters  $\chi_1 \pmod{D_1}, \chi_2 \pmod{D_2}, \dots$  such that  $\eta_{\chi_i} \rightarrow 0$ , then many desirable consequences would follow: the existence of twin primes [HB83], equidistribution of primes in arithmetic progressions to large moduli [FI03], primes in very short intervals [FI04], or prime values of discriminants of elliptic curves [FI05, FI13], for example.

We are interested in obtaining cancellations in the sum

$$\sum_{p \leq x} \text{Kl}(1, p)$$

which go beyond the bound  $2\pi(x)$  implied by the Weil bound should such exceptional characters exist.

**Theorem 1.1.** *Let  $\varepsilon > 0$ . Then there are constants  $A, B > 0$ , depending only on  $\varepsilon$ , such that for  $D \geq 3$ ,  $x \geq D^A$  and any primitive real character  $\chi \pmod{D}$ , we have*

$$\left| \sum_{p < x} \text{Kl}(1, p) \right| \leq \pi(x) \left( \varepsilon + BL(1, \chi) \log x \right).$$

This statement is unconditional, but is only non-trivial if the value  $\eta_\chi$  is suitably small. We note that if there is a sequence of characters with  $\eta_{\chi_i} \rightarrow 0$ , then Theorem 1.1 shows that for a suitable sequence of values of  $x$ ,

$$\sum_{p < x} \text{Kl}(1, p) = o(\pi(x)),$$

as predicted by the horizontal Sato-Tate conjecture.

Unfortunately we do not know unconditionally the expected lower bound

$$(1.1) \quad \sum_{p < x} |\text{Kl}(1, p)| \gg \pi(x).$$

In particular, even if there was a sequence of characters with  $\eta_{\chi_i} \rightarrow 0$ , we would not be able to conclude from Theorem 1.1 that there are even infinitely many sign changes in the sequence  $\text{Kl}(1, p)$ . If instead of considering primes we consider products of exactly two primes, then the equivalent lower bound to (1.1) is known thanks to work of Fouvry-Kowalski-Michel [FKM14]. For technical reasons, when working with products of two primes we consider the variant

$$S(x) = \sum_{p, q \text{ prime}} \phi\left(\frac{pq}{x}\right) \log(pq)(\log p)(\log q) \text{Kl}(1, pq)$$

where  $\phi : \mathbb{R}_+ \rightarrow \mathbb{C}$  is a smooth function compactly supported inside  $\mathbb{R}_+^*$ . We note that the Weil bound implies unconditionally that  $S(x) \ll x \log^2 x$ , whilst a variant of the horizontal Sato-Tate conjecture would suggest that we should have  $S(x) = o_\phi(x \log^2 x)$ .

**Theorem 1.2.** *Let  $\varepsilon > 0$ . Then there exist  $A, B > 0$ , depending at most on  $\varepsilon$ , such that for any  $D \geq 3$ ,  $x \geq D^A$  and any primitive real character  $\chi \pmod{D}$ , we have*

$$(1.2) \quad |S(x)| \ll_\phi x(\log x)^2 \{ \varepsilon + BL(1, \chi) \log x \}.$$

*The implied constant depends only on the function  $\phi$ .*

As with Theorem 1.1, this is unconditional but non-trivial only if a sequence of exceptional characters exist. Thus, in the presence of Siegel zeros, we are able to establish infinitely many sign changes of  $\text{Kl}(1, pq)$ .

**Corollary 1.3.** *For some absolute constants  $A, c > 0$ , if  $\eta_\chi \leq c$ , then every interval  $[x, 2x] \subset [D^A, D^{100A}]$  contains two numbers  $(n_1, n_2)$  with  $\omega(n_1) = \omega(n_2) = 2$ , and  $\text{Kl}(1, n_1) \text{Kl}(1, n_2) < 0$ .*

*Proof.* Choose  $\phi$  to be real-valued with  $\phi \geq \mathbf{1}_{[1, 2]}$ , and consider the unsigned sum

$$A(x) = \sum_{\substack{p, q \\ x < pq \leq 2x}} (\log pq)(\log p)(\log q) |\text{Kl}(1, pq)|.$$

We have the following lower bound, due to Fouvry-Kowalski-Michel [FKM14, Proposition 5.1]:

$$(1.3) \quad A(x) \gg x(\log x)^2 \quad (x \geq 2).$$

Comparing (1.2) and (1.3) yields the claimed statement. □

*Notation.* We denote by  $P^-(n)$  (resp.,  $P^+(n)$ ) the smallest (resp., largest) prime factor of  $n$ , with the conventions  $P^-(1) = \infty$  and  $P^+(1) = 1$ . The letter  $p$  denotes a prime.

## 2. OUTLINE

If there is a character  $\chi \pmod{D}$  such that  $L(1, \chi)$  is very small, then “most” primes  $p \in [D^A, D^{100A}]$  have  $\chi(p) = -1 = \mu(p)$  for some suitable constant  $A$ . By multiplicativity, this means that the (poorly understood) Moebius function  $\mu$  can be well-approximated by the character  $\chi$  (which is periodic  $\pmod{D}$ ), and so better

understood) on integers  $n < D^{100A}$  with no small prime factors. In particular, for  $n \leq D^{100A}$

$$(2.1) \quad \Lambda(n) = (\mu * \log)(n) \approx (\chi * \log)(n),$$

and so we can approximate  $\Lambda$  by the convolution of two simpler sequences. By applying the hyperbola method, this would allow us to estimate a sum  $\sum_{n < x} \Lambda(n) a_n$  provided we could suitably estimate

$$(2.2) \quad \sum_{d \leq Y_1} \chi(d) \sum_{\substack{n \leq x \\ d|n}} a_n \log \frac{n}{d} \quad \text{and} \quad \sum_{d \leq Y_2} \log d \sum_{\substack{n < x \\ d|n}} a_n \chi\left(\frac{n}{d}\right)$$

for some choice of  $Y_1, Y_2$  with  $Y_1 Y_2 = x = D^{100A}$ . Often one can suitably estimate such sums for  $Y_1 = Y_2 = x^{1/2-\varepsilon}$ , which just falls short of this requirement. Much of the work on the distribution of primes under the assumption of a Siegel-Landau zero followed this strategy, and the key technical challenge is then to obtain a suitable estimate for one of the sums in (2.2) with  $Y_1$  or  $Y_2$  slightly beyond  $x^{1/2}$ .

In our situation,  $a_n = \text{Kl}(1, n)$ , and estimates for the two sums in (2.2) with  $Y_1 = Y_2 = x^{1/2-\varepsilon}$  are obtained in essentially the same way as Fouvry and Michel [FM07]. Unfortunately we do not know how to extend this work beyond  $x^{1/2}$ , and so this strategy fails. However, in the convolution identity  $\Lambda(n) = \sum_{d|n} \mu(d) \log(n/d)$  it is only terms with  $d \in [x^{1/2-\varepsilon}, x^{1/2+\varepsilon}]$  which we are unable to handle. We note the alternative identity

$$\Lambda(n) = \sum_{d|n} \mu(d) \left( \log \frac{\sqrt{n}}{d} \right).$$

The presence of the term  $\log(\sqrt{n}/d)$  means we expect that terms with  $d \approx \sqrt{n}$  to contribute less, and so we might hope that these central values would be negligible. This is a variant of the idea that Bombieri introduced in his asymptotic sieve [Bom76], where terms in  $\Lambda_2(n) = \sum_{d|n} \mu(d) \log^2(n/d)$  from  $d < n^{1-\varepsilon}$  could be handled by assumptions on equidistribution of congruence sums, and terms with  $d \in [n^{1-\varepsilon}, n]$  could be bounded by virtue of the fact that  $\log^2(n/d)$  was small in this range.

Unfortunately, as in Bombieri's work, this strategy fails if we wish to count primes. To maintain the feature that the support is essentially restricted to numbers with no small prime factors one multiplies by a short sieve weight, which loses a factor  $\varepsilon^2$  from the two variables  $d$  and  $n/d$ . This precisely cancels out the gains of a factor  $\varepsilon^2$  coming from the range of  $d$  and from the size of  $\log(\sqrt{n}/d)$ . Whilst this issue might appear to be a technicality, at least in Bombieri's work this is an expression of the fundamental parity problem of sieve methods. If instead we counted with a weight involving a higher power of  $\log(\sqrt{n}/d)$  (thereby counting products of a bounded number of primes), then this strategy can succeed.

In our case, we are interested in  $a_n = \text{Kl}(1, n)$ . Although in general we expect the Weil bound  $|\text{Kl}(1, n)| \leq 2^{\omega(n)}$  to be essentially sharp, for *most* integers  $n$  we expect  $|\text{Kl}(1, n)|$  is actually much smaller than this. Indeed, the horizontal Sato-Tate conjecture would predict that for any fixed  $a$ , the average size of  $|\text{Kl}(a, p)|$  is  $\frac{2}{\pi} \int_0^\pi 2 |\cos(t)| \sin^2(t) dt = 8/3\pi < 1$ . By multiplicativity, we might then expect  $|\text{Kl}(1, n)| \approx (8/3\pi)^{\omega(n)}$  on average over  $n$ . Fouvry and Michel [FM03b, FM06] combined an argument of Hooley [Hoo64] based on the identity  $\text{Kl}(1, ab) =$



$\text{Kl}(\overline{a^2}, b) \text{Kl}(\overline{b^2}, a)$  (for coprime  $a, b$ ), with the *vertical* Sato-Tate law, to show unconditionally that the factor  $2^{\omega(n)}$  can be indeed improved to  $(8/3\pi)^{\omega(n)}$  when considering suitable averages. Since  $8/3\pi < 1$ , numbers with a larger number of prime factors contribute less to the problematic sums, and so there is less of a loss from being restricted to a short sieve weight. This ultimately allows us to win an additional factor of  $(\varepsilon^{1-8/3\pi})^2$  for these sums involving middle sized  $d$ , which is enough to conclude that such terms make a negligible contribution, and so we are able to bound  $\sum_{p < x} \text{Kl}(1, p)$ .

### 3. PREPARATORY LEMMAS

**3.1. Level of distribution for twisted Kloosterman sums.** Here we make precise the claim that the sums in (2.2) can be estimated with  $Y_1 = Y_2 = x^{1/2-\varepsilon}$  by a variation of the work of Fouvry-Michel [FM07].

**Proposition 3.1.** *Let  $\varepsilon > 0$  be fixed. There exists  $\eta = \eta(\varepsilon) > 0$ , such that for any real  $x \geq 2$ , positive integer  $D \leq x^\eta$ , all characters  $\chi \pmod{D}$ , and any smooth function  $\phi : \mathbb{R}_+^* \rightarrow \mathbb{C}$  compactly supported inside  $\mathbb{R}_+^*$ , we have*

$$(3.1) \quad \sum_{q \leq x^{1/2-\varepsilon}} \left| \sum_{n \equiv 0 \pmod{q}} \phi\left(\frac{n}{x}\right) \chi(n) \text{Kl}(1, n) \right| \ll_{\varepsilon, \phi} x^{1-\eta}.$$

*Remark.* Note that the case  $D = 1$  of the previous statement is a weaker form of Proposition 2.1 of [FM07].

*Proof.* We may plainly assume that the sums are restricted to  $(q, D) = (n, D) = 1$ . The bound we claim is a variant of Proposition 2.1 of [FM07], which is based on the Kuznetsov formula [Kuz80, DI83], the Weil bound for Kloosterman sums, and a uniform bound  $\theta \leq 1/4 - \varepsilon$  towards Ramanujan-Petersson, which was first due to Luo-Rudnick-Sarnak [LRS95].

The difference in our case is the presence of the character. Recently, Blomer and Milićević [BM15] have succeeded in analyzing such sums in the context of modular forms with non-trivial nebentypus; another argument was used in [Dra17], which is simpler for our purpose here. We will rely on work of Topalogullari [Top15] to estimate the spectral sums.

In our case, we will use the notation and normalization described in section 4.1.2 of [Dra17]. Our aim is to apply the Kuznetsov formula [Dra17, Lemma 4.5] for the group  $\Gamma_0(qD)$ , nebentypus  $\bar{\chi} \pmod{D}$ , with cusps  $\infty$  and  $1/q$ , and parameters  $m \leftarrow D$ ,  $n \leftarrow 1$ . For each  $q$  in the left-hand side of (3.1), Lemma 4.3 of [Dra17], with the choice of scaling matrices (depending only on  $q$  and  $D$ ) given there, yields

$$\chi(n) \text{Kl}(1, n) = \chi(q) e(-\bar{q}/D) S_{\infty, 1/q}(D, 1; n\sqrt{D}).$$

Let  $\kappa \in \{0, 1\}$  be such that  $\chi(-1) = (-1)^\kappa$ . The Kuznetsov formula with test function  $\psi(t) = \phi(4\pi/(tx)) \sqrt{4\pi/(tx)}$  yields

$$\overline{\chi(q)} e(\bar{q}/D) \sum_{n \equiv 0 \pmod{q}} \phi\left(\frac{n}{x}\right) \chi(n) \text{Kl}(1, n) = \sqrt{x} \{ \mathcal{H} + \mathcal{E} + \mathcal{M} \},$$

where

$$\mathcal{H} = \sum_{\substack{k > \kappa \\ k \equiv \kappa \pmod{2}}} \sum_{f \in \mathcal{B}_k(qD, \bar{\chi})} \dot{\psi}(k) \Gamma(k) \overline{\rho_{f, \infty}(D)} \rho_{f, 1/q}(1),$$

$$\begin{aligned} \mathcal{E} &= \sum_{\mathfrak{c} \text{ sing.}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\tilde{\psi}(t)}{\cosh(\pi t)} \overline{\rho_{\mathfrak{c},\infty}(D,t)} \rho_{\mathfrak{c},1/q}(1,t) dt, \\ \mathcal{M} &= \sum_{f \in \mathcal{B}(qD, \bar{\chi})} \frac{\tilde{\psi}(t_f)}{\cosh(\pi t_f)} \overline{\rho_{f,\infty}(D)} \rho_{f,1/q}(1). \end{aligned}$$

Split  $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$  where  $\mathcal{M}_1$  is the contribution of those  $f$  with  $t_f \in \mathbb{R}$ , and  $\mathcal{M}_2$  is the remainder contribution, which consists of  $f$  with  $t_f \in [-i/4, i/4]$ . Consider first  $\mathcal{M}_2$ . Using the bound  $|\tilde{\psi}(t_f)| \ll x^{2|t_f|}$ , we obtain by the Cauchy-Schwarz inequality

$$\mathcal{M}_2 \ll \left( \sum_{f \in \mathcal{B}(qD, \bar{\chi})} x^{2|t_f|} |\rho_{f,\infty}(D)|^2 \right)^{1/2} \left( \sum_{f \in \mathcal{B}(qD, \bar{\chi})} x^{2|t_f|} |\rho_{f,1/q}(1)|^2 \right)^{1/2}.$$

We have  $q\sqrt{D} \leq x^{1/2-\varepsilon+\eta} \leq x^{1/2}$ , so that Lemma 2.9 of [Top15] may be applied to both sums, which yields, as  $x \rightarrow \infty$ ,

$$(3.2) \quad \mathcal{M}_2 \ll x^{o(1)} \left( \frac{\sqrt{xD}}{q} \right)^{4\theta} \left\{ 1 + \frac{\sqrt{D}}{q} \right\} \ll x^{2\eta} (x/q^2)^{2\theta}.$$

The contribution of  $\mathcal{E}$ ,  $\mathcal{H}$  and  $\mathcal{M}_1$  is handled by similar standard arguments [DI83, page 267], using instead Lemma 2.8 of [Top15]. We obtain

$$(3.3) \quad \mathcal{H} + \mathcal{E} + \mathcal{M}_1 \ll x^{o(1)} D^{1/2} \ll x^\eta$$

as  $x \rightarrow \infty$ . Grouping our bounds (3.3) and (3.2), we find

$$\sum_{q \leq x^{1/2-\varepsilon}} \left| \sum_{n \equiv 0 \pmod{q}} \phi\left(\frac{n}{x}\right) \chi(n) \text{Kl}(1, n) \right| \ll_{\varepsilon, \phi} x^{1+\eta-\varepsilon} + x^{1+2\eta-(1/2-2\theta)\varepsilon}.$$

By work of Kim-Sarnak [Kim03], we are ensured that  $\theta \leq 7/64 < 1/4$ , and therefore our claimed bound follows if  $Q \leq x^{1/2-\varepsilon}$  and  $0 < \eta \leq 3\varepsilon/16$ .  $\square$

From Proposition 3.1 and the convolution  $\mu^2(n) = \sum_{d^2|n} \mu(d)$ , we deduce the analogous bound with  $n$  restricted to square-free numbers.

**Corollary 3.2.** *In the setting and notation of Proposition 3.1, we have*

$$(3.4) \quad \sum_{q \leq x^{1/2-\varepsilon}} \left| \sum_{n \equiv 0 \pmod{q}} \mu^2(n) \phi\left(\frac{n}{x}\right) \chi(n) \text{Kl}(1, n) \right| \ll_{\varepsilon, \phi} x^{1-\eta}$$

for a possibly smaller value of  $\eta > 0$ , but depending on  $\varepsilon$  at most.

**3.2. Sieve weights.** To control the terms around the central point

$$d \in [x^{1/2-\varepsilon}, x^{1/2+\varepsilon}]$$

we will introduce a short sieve weight to maintain the feature that our sums are essentially supported on integers free of small prime factors. This is also crucial for carrying out the approximation (2.1) as effectively as possible. We recall the construction of the  $\beta$ -sieve from [FI10], and a few of its relevant properties for our application. Let

$$\theta(n) = \sum_{d|n} \xi_d$$

be an upper bound  $\beta$ -sieve of level  $y$  and dimension  $\kappa \geq 0$ , for the primes less than  $z$ . The parameters  $y, z$ , and  $\kappa$  will be chosen later to be small powers of  $x$  with the condition that  $z^2 \leq y$ . (We will ultimately choose  $z = x^{\varepsilon/(\log \varepsilon)^2}$  and

$y = x^\varepsilon$ .) In particular, we have  $\xi_1 = 1$  and  $|\xi_d| \leq 1$ ;  $(\xi_d)$  is supported on integers up to  $y$  free of prime factor  $> z$ , and we have  $\theta(n) \geq 0$  for all  $n \geq 1$ . Moreover, for any multiplicative function  $f$  with  $f(p) \in [0, \kappa]$ , we have

$$(3.5) \quad \sum_d \frac{\xi_d f(d)}{d} \ll \prod_{p \leq z} \left(1 + \frac{f(p)}{p}\right)^{-1}.$$

For some technical simplifications, we will work with the smoothed version

$$\theta'(n) := \sum_{d|n} \xi_d \log\left(\frac{n}{d}\right) = \sum_{d|n} \theta\left(\frac{n}{d}\right) \Lambda(d).$$

Note that  $\theta'(n) = \log n$  if  $P^-(n) > z$ .

The key property we use of  $\theta'$  is the following estimate on averages of multiplicative functions weighted by the sieve weight.

**Lemma 3.3.** *Let the multiplicative function  $f$  be supported on square-free numbers, with  $0 \leq f(p) \leq \kappa$ . Then*

$$(3.6) \quad \sum_{n \leq x} \theta'(n) f(n) \ll x \prod_{z < p \leq x} \left(1 + \frac{f(p)}{p}\right).$$

*Proof.* Denote  $S$  the sum on the left-hand side of (3.6). By definition of  $\theta'$ , we have

$$S = \sum_{n \leq x} \sum_{p|n} f(n) \theta\left(\frac{n}{p}\right) \log p \leq \sum_{n \leq x} f(n) \theta(n) \sum_{p \leq x/n} f(p) \log p$$

by positivity of the summand. Therefore,

$$S \ll_\kappa x \sum_{n \leq x} \frac{\theta(n) f(n)}{n} \leq x \sum_{P^+(n) \leq x} \frac{\theta(n) f(n)}{n}.$$

Opening the convolution in  $\theta(n)$ , we arrive at

$$(3.7) \quad \begin{aligned} S &\ll_\kappa x \sum_{P^+(d) \leq x} \frac{\xi_d f(d)}{d} \sum_{\substack{P^+(n) \leq x \\ (n,d)=1}} \frac{f(n)}{n} \\ &= x \prod_{p \leq x} \left(1 + \frac{f(p)}{p}\right) \sum_{P^+(d) \leq x} \frac{\xi_d f(d)}{d} \prod_{p|d} \left(1 + \frac{f(p)}{p}\right)^{-1}. \end{aligned}$$

By (3.5), the sum over  $d$  above is  $\ll_\kappa \prod_{p \leq z} \left(1 + \frac{f(p)}{p}\right)^{-1}$ , and our claimed bound follows.  $\square$

#### 4. ARGUMENT FOR THE UPPER BOUND

In what follows, we let  $\varepsilon > 0$  be fixed, and  $\eta > 0$  be a small parameter, to be chosen in terms of  $\varepsilon$ . We assume that  $D \leq x^\eta$ .

**4.1. Initial setup.** For a positive integer  $r$ , define the arithmetic functions  $\tilde{\Lambda}_r(n)$  by

$$\tilde{\Lambda}_r(n) = \sum_{d|n} \mu(d) \left(\log \frac{\sqrt{n}}{d}\right)^r.$$

We note that  $\tilde{\Lambda}_1(n) = \Lambda(n)$  and  $\tilde{\Lambda}_2(n) = \Lambda_2(n) - \Lambda(n) \log n$ . On square-free  $n$ ,  $\tilde{\Lambda}_1$  is supported on primes, and  $\tilde{\Lambda}_2$  is supported on products of exactly two primes.

Let  $\phi$  be a smooth bounded compactly supported function approximating the characteristic function  $\mathbf{1}_{[0,1]}$  of the interval  $[0, 1]$  from below, such that  $\phi$  and  $\mathbf{1}_{[0,1]}$  differ only on  $[0, \delta]$  and  $[1 - \delta, 1]$  for some small constant  $\delta > 0$ . If we can show that for any  $\varepsilon > 0$

$$(4.1) \quad \left| \sum_n \mu^2(n) \phi\left(\frac{n}{x}\right) (\log n) \tilde{\Lambda}(n) \text{Kl}(1, n) \right| \ll_{\phi} \varepsilon x \log x + O_{\varepsilon}(xL(1, \chi) \log^2 x),$$

then, using the Weil bound to control the contribution from  $n < \delta x$  or  $n > (1 - \delta)x$ , we see from partial summation that

$$\left| \sum_{p \leq x} \text{Kl}(1, p) \right| \ll \delta \pi(x) + O_{\delta}(\varepsilon \pi(x)) + O_{\delta, \varepsilon}(L(1, \chi) \log x).$$

Thus, by choosing  $\delta$  small and then  $\varepsilon$  sufficiently small in terms of  $\delta$ , we see it suffices to show (4.1) for any fixed  $\varepsilon > 0$ .

Since  $\mu^2(n)\tilde{\Lambda}(n)$  is only supported on primes, we have

$$\begin{aligned} & \sum_n \mu^2(n) \phi\left(\frac{n}{x}\right) (\log n) \tilde{\Lambda}(n) \text{Kl}(1, n) \\ &= \sum_{(n,D)=1} \mu^2(n) \phi\left(\frac{n}{x}\right) \theta'(n) \tilde{\Lambda}(n) \text{Kl}(1, n) + O_{\varepsilon}(x^{\varepsilon}). \end{aligned}$$

For products of two primes, we note that for all  $x \geq 2$ ,

$$(4.2) \quad \sum_{\substack{n \leq x \\ n \neq pq \text{ for all } p \neq q, \\ p, q \text{ prime}}} \tilde{\Lambda}_2(n) \ll x,$$

with the main contribution arising from pairs  $n = pq^2$  ( $p, q$  primes), while

$$(4.3) \quad \sum_{\substack{n \leq x \\ (n,D) > 1}} \tilde{\Lambda}_2(n) \ll x \log \log x$$

for  $2 \leq D \leq x$ . Thus

$$\begin{aligned} & \sum_{\substack{p, q \\ \text{prime}}} \phi\left(\frac{pq}{x}\right) (\log pq)(\log p)(\log q) \text{Kl}(1, pq) \\ &= \sum_{\substack{p, q \\ \text{prime}}} \phi\left(\frac{pq}{x}\right) \theta'(pq)(\log p)(\log q) \text{Kl}(1, pq) + O(x(\log x)(\log z)) \\ (4.4) \quad &= \frac{1}{2} \sum_{(n,D)=1} \mu^2(n) \phi\left(\frac{n}{x}\right) \theta'(n) \tilde{\Lambda}_2(n) \text{Kl}(1, n) + O(x(\log x)(\log z)). \end{aligned}$$

We have used both (4.2) and (4.3) in the last line.

Thus, we see that to establish Theorem 1.1 and Theorem 1.2 it is sufficient to show for every  $\varepsilon > 0$ , every smooth compactly supported  $\phi$  and each  $r \in \{1, 2\}$  that

$$(4.5) \quad \begin{aligned} S(x) = S(x, r) &= \sum_{(n,D)=1} \mu^2(n) \phi\left(\frac{n}{x}\right) \theta'(n) \tilde{\Lambda}_r(n) \text{Kl}(1, n) \\ &\ll_{\phi} \varepsilon x \log^r x + O_{\varepsilon}(xL(1, \chi) \log^{r+1} x), \end{aligned}$$

where  $D$  and  $x$  satisfy the hypotheses of Theorem 1.1 and  $\chi$  is any real primitive character (mod  $D$ ). Let

$$\nu(n) = (\mu * \mu\chi)(n) = \sum_{n=uv} \mu(u)\mu(v)\chi(v).$$

We write

$$(4.6) \quad \tilde{\Lambda}_r(n) = \frac{1}{2^r} \sum_{n=abc} \nu(b)\chi(a) \left(\log \frac{ab}{c}\right)^r,$$

so that for  $r \in \{1, 2\}$ ,

$$(4.7) \quad S(x) = \frac{1}{r2^r} \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ (abc,D)=1}} \mu^2(abc)\phi\left(\frac{abc}{x}\right)\theta'(abc)\nu(b)\chi(a) \left(\log \frac{ab}{c}\right)^r \text{Kl}(1, abc).$$

We split the sum over  $(a, b, c)$  into  $b > 1$  or  $b = 1$  and  $\min\{a, c\} \leq x^{1/2}/y^3$ , or  $b = 1$  and  $\min\{a, c\} > x^{1/2}/y^3$ . This gives

$$(4.8) \quad r2^r S(x) = S_L(x) + S_N(x) + S_C(x),$$

with

$$(4.9) \quad S_L(x) = \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ b > 1 \\ (abc,D)=1}} \mu^2(abc)\phi\left(\frac{abc}{x}\right)\theta'(abc)\nu(b)\chi(a) \left(\log \frac{ab}{c}\right)^r \text{Kl}(1, abc),$$

$$(4.10) \quad S_N(x) = \sum_{\substack{(a,c) \in \mathbb{N}^2 \\ \min\{a,c\} \leq \sqrt{x}/y^3 \\ (ac,D)=1}} \mu^2(ac)\phi\left(\frac{ac}{x}\right)\theta'(ac)\chi(a) \left(\log \frac{a}{c}\right)^r \text{Kl}(1, ac),$$

$$(4.11) \quad S_C(x) = \sum_{\substack{(a,c) \in \mathbb{N}^2 \\ \min\{a,c\} > \sqrt{x}/y^3 \\ (ac,D)=1}} \mu^2(ac)\phi\left(\frac{ac}{x}\right)\theta'(ac)\chi(a) \left(\log \frac{a}{c}\right)^r \text{Kl}(1, ac).$$

We can control  $S_L(x)$  by  $L(1, \chi)$ , since  $\mu^2\nu$  is supported on integers with all prime factors satisfying  $\chi(p) = 1$  and  $\theta'$  is concentrated on integers free of small prime factors. Thus if  $L(1, \chi)$  is small, we expect the support to be a lacunary sequence and so the sum  $S_L(x)$  to be small.

We can control  $S_N(x)$  by the level of distribution estimates of Section 3.1, since the sum over one of  $a$  or  $c$  is a long sum.

Finally,  $S_C$  is the contribution near the central point, and we will show this is small since  $|\log(a/c)| \leq 6 \log y + O_\phi(1)$ , which is small compared with  $\log x$ ,  $a$  is restricted to a short range on the logarithmic scale, and  $\text{Kl}(1, n)$  is typically small on numbers with many prime factors.

**4.2. Bounding the lacunary sum  $S_L$ .** We begin by bounding  $S_L(x)$  in terms of  $L(1, \chi)$ .

**Lemma 4.1.** *For all small enough  $\varepsilon > 0$ , and  $D^8 \leq x^{\varepsilon^2} \leq z \leq y^{1/(\log \varepsilon)^2}$ , we have*

$$S_L(x) \ll_\phi x(\log x)^r \left(\varepsilon^{-\kappa-2} L(1, \chi) \log x + \varepsilon\right).$$

*Proof.* We note that for square-free  $n$ , we have  $|\nu(n)| = (1 * \chi)(n)$ , and we recall that  $\theta'(n) = (\Lambda * \theta)(n)$ . Thus, bounding the summand of  $S_L(x)$  and letting  $d = ac$ , and then expanding the definition of  $\theta$ , we find

$$\begin{aligned} S_L(x) &\ll (\log x)^r \sum_{\substack{db \ll x \\ b > 1}} (1 * \chi)(b) \theta'(bd) \tau(bd)^2 \mu(bd)^2 \\ &= (\log x)^r \sum_{\substack{bef \ll x \\ b > 1}} (1 * \chi)(b) \theta(be) \Lambda(f) \tau(bef)^2 \mu(bef)^2. \end{aligned}$$

When  $f > z$  is prime we see  $\theta(be) = \theta(bef)$ . Thus the contribution from  $f > z$  can be bounded by

$$(4.12) \quad (\log x)^{r+1} \sum_{\substack{bg \ll x \\ b > 1}} (1 * \chi)(b) \theta(bg) \mu^2(bg) \tau(bg)^4.$$

The contribution from  $f \leq z$  can be bounded by

$$(4.13) \quad (\log x)^r \sum_{f \ll z} \Lambda(f) \sum_{\substack{be \ll x/f \\ b > 1}} (1 * \chi)(b) \theta(be) \mu^2(be) \tau(be)^2.$$

These are precisely sums of the type considered in Section 24.7 of [FI10]. Indeed, there it is established [FI10, equation (24.58)] that if  $D^8 \leq z \leq y \leq Y^{1/8}$  and  $\rho$  is a completely multiplicative function with  $0 \leq \rho(p) \leq \kappa/6$ , then

$$\sum_{\substack{ab < Y \\ b > 1}} (1 * \chi)(b) \theta(ab) \rho(ab) \ll \frac{Y}{\log Y} \left( L(1, \chi) \log Y + e^{-t} \right) \left( \frac{\log Y}{\log z} \right)^{\kappa/2+1},$$

where  $t = \log y / \log z$ . Since  $\mu^2(n) \tau(n)^k \leq 2^{k\Omega(n)}$  ( $k = 2, 4$ ), we may apply this to the bounds (4.12) and (4.13) above, and find that provided  $D^8 \leq z \leq y \leq x^{1/9}$  and  $\kappa > 6 \times 2^4$ , we have

$$\begin{aligned} &(\log x)^{r+1} \sum_{\substack{bg \ll x \\ b > 1}} (1 * \chi)(b) \theta(bg) \tau(bg)^4 \\ &\ll x (\log x)^r \left( L(1, \chi) \log x + e^{-t} \right) \left( \frac{\log x}{\log z} \right)^{\kappa/2+1}, \\ &(\log x)^r \sum_{f \ll z} \Lambda(a) \sum_{\substack{be \ll x/f \\ b > 1}} (1 * \chi)(b) \theta(be) \tau(be)^2 \\ &\ll (\log x)^{r-1} \left( L(1, \chi) \log x + e^{-t} \right) \left( \frac{\log x}{\log z} \right)^{\kappa/2+1} \sum_{f \ll z} \frac{\Lambda(f)}{f} \\ &\ll x (\log x)^r \left( L(1, \chi) \log x + e^{-t} \right) \left( \frac{\log x}{\log z} \right)^{\kappa/2}. \end{aligned}$$

Putting this together, we find

$$S_L(x) \ll x (\log x)^r \left( L(1, \chi) \log x + e^{-t} \right) \left( \frac{\log x}{\log z} \right)^{\kappa/2+1}.$$

If  $x^{\varepsilon^2} \leq z \leq y^{1/(\log \varepsilon)^2}$ , then this gives

$$(4.14) \quad S_L(x) \ll x (\log x)^r \left( \varepsilon^{-\kappa-2} L(1, \chi) \log x + \varepsilon \right),$$

as required. □

### 4.3. Bounding the non-central sum $S_N$ .

**Lemma 4.2.** *Let  $y \geq x^\varepsilon$ . Then there is a constant  $\eta > 0$  depending only on  $\varepsilon$  such that*

$$S_N(x) \ll_\varepsilon x^{1-\eta}.$$

*Proof.* Denote by  $\chi_0(n)$  the principal character (mod  $D$ ). Opening the summation in  $\theta'(ac)$ , we obtain

$$\begin{aligned} & S_N(x) \\ & \leq \sum_{d \leq y} \sum_{a < \sqrt{x}/y^3} \left| \sum_{n \equiv 0 \pmod{[a,d]}} \mu^2(n) \phi\left(\frac{n}{x}\right) \chi_0(n) \left(\log\left(\frac{a^2}{n}\right)\right)^r \left(\log\frac{n}{d}\right) \text{Kl}(1, n) \right| \\ & + \sum_{d \leq y} \sum_{c < \sqrt{x}/y^3} \left| \sum_{n \equiv 0 \pmod{[c,d]}} \mu^2(n) \phi\left(\frac{n}{x}\right) \chi(n) \left(\log\left(\frac{c^2}{n}\right)\right)^r \left(\log\frac{n}{d}\right) \text{Kl}(1, n) \right| \\ & \ll \sup_{\chi_1 \pmod{D}} \sup_{0 \leq j \leq r+1} \sum_{q \leq \sqrt{x}/y} \tau(q) (\log x)^{r+1} \\ (4.15) \quad & \times \left| \sum_{n \equiv 0 \pmod{q}} \mu^2(n) \phi\left(\frac{n}{x}\right) \chi_1(n) (\log(n/x))^j \text{Kl}(1, n) \right|. \end{aligned}$$

Recalling that we assume  $y \geq x^\varepsilon$  and applying Corollary 3.2, we obtain

$$(4.16) \quad S_N(x) \ll_\varepsilon x^{1-\eta}$$

for some  $\eta$  depending at most on  $\varepsilon$ . □

### 4.4. Bounding the central point sum $S_C$ .

**Lemma 4.3.** *Let  $x^{\varepsilon^2} \leq z \leq y^{1/(\log \varepsilon)^2}$  and  $y \leq x^{2\varepsilon}$ . Then we have*

$$S_C(x) \ll \varepsilon^{10} x (\log x)^r + \varepsilon^{3/10} x (\log x)^r \left(\frac{\log y}{\log z}\right)^2.$$

*Proof.* Recall that the sum  $S_C(x)$  is defined in (4.11). By construction, for all  $(a, c)$  in the summation range, we have  $|\log(a/c)| \ll \log y$ .

Therefore, by the triangle inequality and the support condition on  $\phi$ , our task is to bound

$$(\log y)^r \sum_{\substack{x/y^3 \leq a \ll xy^3 \\ ac \asymp x}} \mu^2(ac) \theta'(ac) |\text{Kl}(1, ac)|.$$

Following an idea of Hooley [Hoo64], we let a parameter  $u \in [y, x^{1/4}]$  be given, and decompose accordingly

$$a = a_1 a_2, \quad P^+(a_1) \leq u, \quad P^-(a_2) > u,$$

and similarly let  $c = c_1 c_2$ . Notice that

$$\theta'(ac) = \theta(a_1 c_1) \log(a_2 c_2) + \theta'(a_1 c_1) \ll \theta(a_1 c_1) \log x + \theta'(a_1 c_1)$$

since  $P^-(a_2c_2) > u > z$ . We write

$$(4.17) \quad \sum_{\substack{a_1 a_2 c_1 c_2 \ll x \\ x^{1/2}/y^3 < a_1 a_2 \ll x^{1/2} y^3 \\ P^+(a_1 c_1) \leq u < P^-(a_2 c_2)}} \mu^2(a_1 a_2 c_1 c_2) \theta'(a_1 a_2 c_1 c_2) |\text{Kl}(1, a_1 a_2 c_1 c_2)| = S_1 + S_2,$$

where  $S_1$  is the contribution of  $a_1 c_1 \leq x^{1/10}$ , and  $S_2$  is the contribution of  $a_1 c_1 > x^{1/10}$ .

Concerning  $S_1$ , we have

$$S_1 \ll \sum_{\substack{a_1 c_1 \leq x^{1/10} \\ P^+(a_1 c_1) \leq u}} \mu^2(a_1 c_1) \{ \theta(a_1 c_1) \log x + \theta'(a_1 c_1) \} T(a_1, c_1),$$

with

$$T(a_1, c_1) = \sum_{\substack{a_2 c_2 \ll x/a_1 c_1 \\ P^-(a_2 c_2) > u \\ x^{1/2}/y^3 < a_1 a_2 \ll x^{1/2} y^3}} \mu^2(a_2 c_2) |\text{Kl}(1, a_1 a_2 c_1 c_2)|.$$

By a reasoning identical to page 277 of [FM03b], for  $a_1 c_1 \leq x^{1/10}$ , we have

$$\begin{aligned} & T(a_1, c_1) \\ & \leq \sum_{\substack{b \pmod{a_1 c_1} \\ (b, a_1 c_1) = 1}} |\text{Kl}(b^2, a_1 c_1)| \sum_{\substack{a_2 c_2 \ll x/a_1 c_1 \\ x^{1/2}/a_1 y^3 < a_2 \ll x^{1/2} y^3/a_1 \\ P^-(a_2 c_2) > u \\ ba_2 c_2 \equiv 1 \pmod{a_1 c_1}}} \mu^2(a_2 c_2) 2^{\omega(a_2 c_2)} \\ & \ll \sum_{\substack{b \pmod{a_1 c_1} \\ (b, a_1 c_1) = 1}} |\text{Kl}(b^2, a_1 c_1)| \frac{x}{a_1 c_1 \varphi(a_1 c_1)} \frac{1}{\log x} \left( \frac{\log x}{\log u} \right)^2 \sum_{\substack{x^{1/2}/a_1 y^3 < a_2 \ll x^{1/2} y^3/a_1 \\ P^-(a_2) > u}} \frac{2^{\omega(a_2)}}{a_2} \\ & \ll \frac{x}{a_1 c_1 \varphi(a_1 c_1)} \frac{\log y}{\log^2 x} \left( \frac{\log x}{\log u} \right)^4 \sum_{\substack{b \pmod{a_1 c_1} \\ (b, a_1 c_1) = 1}} |\text{Kl}(b^2, a_1 c_1)| \\ & = \frac{x \log y}{\log^2 x} \left( \frac{\log x}{\log u} \right)^4 \frac{2^{\omega(a_1 c_1)} \kappa(a_1 c_1)}{a_1 c_1} \end{aligned}$$

for a multiplicative function  $\kappa$ . Fouvry and Michel [FM03b, equation (3.4)] show that

$$\kappa(p) = \frac{4}{3\pi} + O(p^{-1/4}),$$

which they deduce from Katz's Sato-Tate law for Kloosterman sums [Kat88]. We insert this back into  $S_1$ , and we relax the summation conditions  $a_1 c_1 \leq x^{1/10}$ . Letting

$$f(n) := \mu^2(n) 4^{\omega(n)} \kappa(n),$$

we obtain

$$(4.18) \quad S_1 \ll \frac{x \log y}{\log^2 x} \left( \frac{\log x}{\log u} \right)^4 \sum_{P^+(n) \leq u} \frac{(\theta(n) \log x + \theta'(n)) f(n)}{n}.$$

Note that

$$f(p) = \frac{16}{3\pi} + O(p^{-1/4}) \quad (p \leq u).$$



Taking Euler products, computations similar to (3.7) yield

$$(4.19) \quad \sum_{P^+(n) \leq u} \frac{\theta(n)f(n)}{n} \ll \prod_{z < p \leq u} \left(1 + \frac{f(p)}{p}\right) \ll \left(\frac{\log u}{\log z}\right)^{16/(3\pi)}.$$

On the other hand,

$$\sum_{P^+(n) \leq u} \frac{\theta'(n)f(n)}{n} = \sum_{\substack{q \leq u \\ q \text{ prime}}} \frac{f(q) \log q}{q} \sum_{\substack{P^+(n) \leq u \\ (n,q)=1}} \frac{\theta(n)f(n)}{n} \ll (\log u) \left(\frac{\log u}{\log z}\right)^{16/(3\pi)}$$

by dropping the condition  $(n, q) = 1$  and using (4.19). Inserting in (4.18), we find

$$(4.20) \quad S_1 \ll x \left(\frac{\log y}{\log x}\right) \left(\frac{\log x}{\log u}\right)^4 \left(\frac{\log u}{\log z}\right)^{16/(3\pi)}.$$

Consider now the contribution  $S_2$  to (4.17). Using the Weil bound  $|\text{Kl}(1, n)| \leq 2^{\omega(n)}$ , we have

$$S_2 \leq \sum_{n \ll x} \theta'(n)g_0(n),$$

where

$$g_0(n) := \begin{cases} \mu^2(n)4^{\omega(n)} & \text{if } \prod_{p|n, p \leq u} p > x^{1/4}, \\ 0 & \text{otherwise.} \end{cases}$$

By Rankin's inequality, we have  $g_0(n) \leq x^{-1/(4 \log u)} g_1(n)$ , where

$$g_1(n) := \mu^2(n)4^{\omega(n)} \prod_{p|n, p \leq u} p^{1/\log u}.$$

Note that the function  $g_1$  is multiplicative with

$$g(p) = \begin{cases} 4p^{1/\log u} & \text{if } p \leq u, \\ 4 & \text{if } p > u. \end{cases}$$

In any case we have  $g(p) \leq 4e$ . Assuming  $\kappa \geq 4e$ , we obtain

$$S_2 \leq x^{-1/(4 \log u)} \sum_{n \ll x} \theta'(n)g_1(n) \ll x \left(\frac{\log x}{\log u}\right)^4 \left(\frac{\log u}{\log z}\right)^{4e} \exp\left\{-\frac{\log x}{4 \log u}\right\}.$$

We now choose  $u = x^{1/(\log \varepsilon)^2}$ . Recalling that we have assumed  $x^{\varepsilon^2} \leq z$ , we see that this gives

$$(4.21) \quad S_2(x) \ll \varepsilon^{10} x.$$

Similarly, we substitute  $u = x^{1/(\log \varepsilon)^2}$  into (4.20). Using the assumption  $z \leq y \leq x^{2\varepsilon}$  and noting that  $2 - 16/3\pi > 3/10$ , we find

$$(4.22) \quad \begin{aligned} (\log y)S_1 &\ll x \log x \left(\log \frac{1}{\varepsilon}\right)^4 \left(\frac{\log y}{\log x}\right)^{2-16/3\pi} \left(\frac{\log y}{\log z}\right)^{16/3\pi} \\ &\ll \varepsilon^{3/10} x \log x \left(\frac{\log y}{\log z}\right)^2. \end{aligned}$$

Recalling that  $S_C(x) \ll (S_1 + S_2)(\log y)^r$ , that  $r \in \{1, 2\}$ , and inserting the bounds (4.22) and (4.21), we obtain

$$S_C(x) \ll \varepsilon^{10} x (\log x)^r + \varepsilon^{3/10} x (\log x)^r \left(\frac{\log y}{\log z}\right)^2,$$

as required. □

4.5. **Conclusion.** We choose

$$y = x^\varepsilon, \quad z = x^{\varepsilon/(\log \varepsilon)^2}.$$

With this choice, we see that  $y$  and  $z$  satisfy the assumptions of Lemmas 4.1, 4.2, and 4.3. Thus, provided  $D^8 \leq x^{\varepsilon^2}$  we may apply these lemmas, giving

$$\begin{aligned} S(x) &\ll S_L(x) + S_N(x) + S_C(x) \\ &\ll \varepsilon^{-\kappa-2} x L(1, \chi) (\log x)^{r+1} + \varepsilon x (\log x)^r + O_\varepsilon(x^{1-\eta}) + \varepsilon^{3/10} x (\log x)^r \left(\log \frac{1}{\varepsilon}\right)^4 \end{aligned}$$

for some quantity  $\eta > 0$  depending only on  $\varepsilon$ . Reinterpreting  $\varepsilon$ , we obtain the claimed statement (4.5), which then gives Theorem 1.1 and Theorem 1.2.

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# Combinatorial identities and Titchmarsh's divisor problem for multiplicative functions

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Given a multiplicative function  $f$  which is periodic over the primes, we obtain a full asymptotic expansion for the shifted convolution sum  $\sum_{|h| < n \leq x} f(n)\tau(n-h)$ , where  $\tau$  denotes the divisor function and  $h \in \mathbb{Z} \setminus \{0\}$ . We consider in particular the special cases where  $f$  is the generalized divisor function  $\tau_z$  with  $z \in \mathbb{C}$ , and the characteristic function of sums of two squares (or more generally, ideal norms of abelian extensions). As another application, we deduce a full asymptotic expansion in the generalized Titchmarsh divisor problem  $\sum_{|h| < n \leq x, \omega(n)=k} \tau(n-h)$ , where  $\omega(n)$  counts the number of distinct prime divisors of  $n$ , thus extending a result of Fouvry and Bombieri, Friedlander and Iwaniec.

We present two different proofs: The first relies on an effective combinatorial formula of Heath-Brown's type for the divisor function  $\tau_\alpha$  with  $\alpha \in \mathbb{Q}$ , and an interpolation argument in the  $z$ -variable for weighted mean values of  $\tau_z$ . The second is based on an identity of Linnik type for  $\tau_z$  and the well-factorability of friable numbers.

## 1. Introduction

Understanding correlations of arithmetic functions is a fundamental question in analytic number theory. In an explicit form, the problem can be stated as determining the asymptotic behavior of the sum

$$\sum_{1 < n \leq x} f(n)g(n-1), \quad (1-1)$$

where  $f, g : \mathbb{N} \rightarrow \mathbb{C}$  are arithmetic functions of multiplicative nature. Many important problems in number theory can be rephrased in terms of correlations of arithmetic functions, the twin prime conjecture or the Goldbach conjecture being two famous examples (see e.g., [Elliott 1994, Chapter 1]). Sums of the form (1-1) also come up prominently in the study of growth properties of  $L$ -functions in the critical strip. In this context, the problem is known as the shifted convolution problem and has a long and rich history (see [Michel 2007] for an overview).

In general, determining the precise asymptotic behavior of the unweighted correlation (1-1) is a difficult task and only very few unconditional results are known in this direction, all of them requiring at least one

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of the involved functions to be very close — in the convolution sense — to the constant function  $\mathbf{1}$ , the divisor function  $\tau(n)$  or to Fourier coefficients of  $\mathrm{GL}_2$ -automorphic forms. Note that when  $f$  and  $g$  are bounded, the *logarithmically* weighted correlation

$$\sum_{1 < n \leq x} \frac{f(n)g(n-1)}{n}$$

has been the object of a recent breakthrough of Tao [2016]. The case of higher-order correlations of bounded functions with logarithmic weight was also recently settled in [Tao and Teräväinen 2019].

In the present paper, we focus on the particularly important case  $g(n) = \tau(n)$  of the unweighted problem (1-1), which is at the edge of current techniques. If the average value of  $f$  is not too small, it was already observed by Vinogradov [1965] (in the case of primes; see also [Rodríguez 1965; Halberstam 1967]) that simple asymptotic equivalences for the sum

$$\sum_{1 < n \leq x} f(n)\tau(n-1) \tag{1-2}$$

can be obtained from analogues of the Bombieri–Vinogradov and Brun–Titchmarsh inequalities. We refer to [Green 2018; Granville and Shao 2018; Fouvry and Radziwiłł 2018] for recent works on this topic. In particular, Corollary 1.3 in [Fouvry and Radziwiłł 2018] leads to a partial asymptotic formula for (1-2) (including all terms with nonnegative exponent of  $\log x$ ) for a large set of arithmetic functions  $f$ , including the generalized divisor function  $\tau_z$  which we discuss further below.

It is a considerably more difficult problem to obtain full asymptotic expansions for (1-2), say, with an error term of the form  $\mathcal{O}(x(\log x)^{-N})$  where  $N > 0$  is fixed but can be chosen arbitrarily large. The gap in difficulty is related to the “ $x^{1/2}$ ”-barrier for primes in arithmetic progressions on average over moduli. To our knowledge full asymptotic expansions are known for only very few specific examples of functions  $f$  of arithmetic interest:

- The indicator function of primes [Fouvry 1985; Bombieri et al. 1986].
- The indicator function of integers without large prime factors [Fouvry and Tenenbaum 1990; Drappeau 2015].
- The  $k$ -fold divisor functions  $\tau_k(n)$ ,  $k \in \mathbb{N}$ ,  $k \geq 2$  [Motohashi 1980; Topacogullari 2016; 2018].

The methods from the last example can also be used to handle the case where  $f$  is given by Fourier coefficients of  $\mathrm{GL}_2$ -automorphic forms, although this does not seem to be worked out explicitly in the literature.

The purpose of the present paper is to introduce two new methods which lead to an asymptotic expansion for (1-2) for a wide class of multiplicative functions. Let  $A, D \geq 1$  be fixed integers. Define  $\mathcal{F}_D(A)$  to be the set of all multiplicative functions  $f : \mathbb{N} \rightarrow \mathbb{C}$  which are  $D$ -periodic over the primes in the sense that

$$f(p_1) = f(p_2) \quad \text{for any primes } p_1 \text{ and } p_2 \text{ with } p_1 \equiv p_2 \pmod{D},$$

and which satisfy the growth condition,

$$|f(n)| \leq \tau_A(n) \quad \text{for all } n \in \mathbb{N},$$

where  $\tau_A(n)$  denotes the generalized divisor function. Our main result is the following preliminary asymptotic formula for the sum (1-2) for  $f \in \mathcal{F}_D(A)$ .

**Theorem 1.1.** *Let  $A, D, N \geq 1$ . For all  $f \in \mathcal{F}_D(A)$  and all  $x \geq 2$ , we have*

$$\sum_{1 < n \leq x} f(n)\tau(n-1) = 2 \sum_{\substack{\chi \text{ primitive} \\ \text{cond}(\chi) | D}} \sum_{\substack{q \leq \sqrt{x} \\ \text{cond}(\chi) | q}} \frac{1}{\varphi(q)} \sum_{\substack{q^2 \leq n \leq x \\ (n,q)=1}} f(n)\chi(n) + \mathcal{O}\left(\frac{x}{(\log x)^N}\right), \quad (1-3)$$

where the implied constant depends only on  $A, D$  and  $N$ .

**Remarks.** • The main term in (1-3) can be evaluated asymptotically by classical methods, for instance the Selberg–Delange method [Tenenbaum 1995, Chapter II.5]. The ensuing expression will in general take the form

$$x \sum_{\kappa \in K_f} (\log x)^\kappa \sum_{\ell=0}^N \frac{c_{\kappa,\ell}}{(\log x)^\ell} + \mathcal{O}\left(\frac{x}{(\log x)^{N-\max(K_f)+1}}\right), \quad (1-4)$$

for some finite set  $K_f \subset \mathbb{C}$  and some sequences  $(c_{\kappa,\ell})_{\ell=0}^N$  of complex numbers. We spell this out in detail in three particular cases below.

• If  $f$  satisfies a Siegel–Walfisz estimate in the sense that

$$\sum_{n \leq x} f(n)\chi(n) = \mathcal{O}_A(x(\log x)^{-A}),$$

uniformly for all primitive characters of conductor  $1 < q \leq (\log x)^A$ , then only the trivial character contributes to the main term in (1-3), and the formula simplifies to

$$\sum_{1 < n \leq x} f(n)\tau(n-1) = 2 \sum_{q \leq \sqrt{x}} \frac{1}{\varphi(q)} \sum_{\substack{q^2 \leq n \leq x \\ (n,q)=1}} f(n) + \mathcal{O}\left(\frac{x}{(\log x)^N}\right).$$

For the main term one then has an expansion as in (1-4) with  $K_f = \{\kappa_f\}$ , where  $\kappa_f$  is the average value of  $f(p)$  over all primes  $p$ .

- We stress that the implied constant is uniform in all  $f \in \mathcal{F}_D(A)$ , and depends only on  $A, D$  and  $N$ . This feature can be useful in applications (see Section 1C).
- On the other hand, our result is badly behaved with respect to  $D$ , partly due to the use of the Siegel–Walfisz theorem. The arguments presented here do not seem sufficient to obtain an improvement in this aspect, although this does not affect our applications.
- The error term in (1-3) corresponds to an application of the Siegel–Walfisz theorem. If the Riemann hypothesis is true for all Dirichlet  $L$ -functions, then it can be improved to  $\mathcal{O}(x^{1-\delta})$  for some absolute constant  $\delta > 0$ .

**Theorem 1.1** may also be interpreted as a result of Bombieri–Vinogradov type “beyond  $\sqrt{x}$ ” for the average of  $f \in \mathcal{F}_D(A)$  in the residue classes of a fixed integer and without absolute values. By a slight modification of the method presented here, it is possible to show that for  $f \in \mathcal{F}_D(A)$ ,

$$\sum_{q \leq \sqrt{x}} \left( \sum_{\substack{1 < n \leq x \\ n \equiv 1 \pmod q}} f(n) - \frac{1}{\varphi(q)} \sum_{\substack{\chi \text{ primitive} \\ \text{cond}(\chi) | (D, q)}} \sum_{\substack{1 < n \leq x \\ (n, q) = 1}} f(n) \chi(n) \right) = \mathcal{O}_{A, D, N} \left( \frac{x}{(\log x)^N} \right).$$

We refer to [Green 2018; Granville and Shao 2018] for recent works related to this point of view.

In many applications correlation sums with more general shifts appear and it is important to have results which are uniform in large ranges of the involved parameters. Our methods are robust enough to be applied to these cases as well, and **Theorem 1.1** is in fact the special case  $a = h = 1$  of the following more general result.

**Theorem 1.2** (general shifts). *Let  $A, D, N \geq 1$ . There exists an absolute constant  $\delta > 0$ , such that, for all  $f \in \mathcal{F}_D(A)$ , all  $x \geq 2$  and all  $a, h \in \mathbb{Z}$  satisfying  $1 \leq a, |h| \leq x^\delta$ , we have*

$$\sum_{|h|/a < n \leq x} f(n) \tau(an - h) = M_f(x; h, a) + \mathcal{O} \left( \tau((a, h)) \frac{x}{(\log x)^N} \right),$$

where  $M_f(x; a, h)$  is given by

$$M_f(x; a, h) := 2 \sum_{\substack{\chi \text{ primitive} \\ \text{cond}(\chi) | D}} \sum_{\substack{q \leq \sqrt{ax} \\ \text{cond}(\chi) | \frac{q}{(q, h)}}} \frac{\bar{\chi} \left( \frac{h}{(h, q)} \right)}{\varphi \left( \frac{q}{(h, q)} \right)} \sum_{\substack{q^2/a \leq n \leq x \\ (an, q) = (h, q)}} f(n) \chi \left( \frac{an}{(an, q)} \right),$$

and where the implied constant depends only on  $A, D$  and  $N$ .

Unfortunately, the range of uniformity in  $h$  in **Theorem 1.2** is comparatively short. This is due to a known uniformity issue of arguments based on exponential sums estimates underlying our bilinear sums estimate (see [Fouvry and Iwaniec 1983, page 200]). Out of the same reason, the methods used here are not able to address the dual problem

$$\sum_{n=1}^{N-1} f(n) \tau(N - n)$$

(for which results are available for instance when  $f = \tau$  or  $f = \tau_3$ , see [Motohashi 1994; Topacogullari 2016]).

We mention that results are known for affine correlations whose linear parts are pairwise independent [Matthiesen 2012; 2016], or when there is an additional, long enough average over the shift [Mikawa 1992; Matomäki et al. 2019a; 2019b]. See also [Andrade et al. 2015; Bary-Soroker and Fehm 2019] for a function field analogue in the large  $q$  limit.

Finally, we mention the work of Pitt [2013]. He considered an analogue of the Titchmarsh divisor problem (see **Section 1C**) with the divisor function replaced by Fourier coefficients of holomorphic cusp forms. In many situations, these Fourier coefficients and the divisor function exhibit a similar

behavior, since the latter can also be viewed as the Fourier coefficients of an Eisenstein series (see e.g., [Iwaniec 2002, Chapter 3.4]). Remarkably, Pitt obtained an estimate with a power saving in the error term unconditionally, something which is not known for the original Titchmarsh divisor problem. It seems possible that his ideas can be adapted to our setting, and that one might obtain an analogue of [Theorem 1.2](#) with the divisor function replaced by Fourier coefficients of holomorphic cusp forms and with a power saving in the error term. We do not pursue this here.

We apply [Theorem 1.2](#) to three functions  $f$  of particular arithmetic interest:

- (1) The generalized divisor functions  $\tau_z(n)$  with  $z \in \mathbb{C}$ .
- (2) The indicator function of integers  $n$  which are norms of an integral ideal in an abelian extension.
- (3) The indicator function of integers  $n$  with exactly  $k$  different prime factors.

**1A. Correlations of divisor functions.** Our first application is related to the generalized additive divisor problem, which asks for an asymptotic evaluation of

$$D_{k,\ell}(x, h) := \sum_{|h| < n \leq x} \tau_k(n) \tau_\ell(n+h)$$

for integers  $k, \ell \geq 2$ . This problem has received a lot of attention, partly motivated by its connection to the  $2k$ -th moment of the Riemann zeta function (see [Ivić 1991, Chapter 4] or [Conrey and Keating 2016; Ng and Thom 2019]).

It is conjectured that for some constant  $C_{k,\ell}(h) > 0$ ,

$$D_{k,\ell}(x, h) \sim C_{k,\ell}(h)x(\log x)^{k+\ell-2},$$

and it is known [Henriot 2012] that this is the correct order of magnitude. However, this has been proven only for the cases where either  $k = 2$  or  $\ell = 2$ . In these cases, the best-known results in the literature are of the form

$$D_{k,2}(x, h) = xP_{k,h}(\log x) + \mathcal{O}(x^{\theta_k+\varepsilon}) \quad \text{for } h \ll x^{\eta_k},$$

where  $P_{k,h}$  is a degree  $k$  polynomial depending on  $h$ , with:

- $\theta_2 = \frac{2}{3}$  and  $\eta_2 = \frac{2}{3}$  [Deshouillers and Iwaniec 1982a; Motohashi 1994].
- $\theta_3 = \frac{8}{9}$  and  $\eta_3 = \frac{2}{3}$  [Friedlander and Iwaniec 1985; Topacogullari 2016].
- $\theta_k = \max\left(1 - \frac{4}{15k-9}, \frac{56}{57}\right)$  and  $\eta_k = \frac{15}{19}$  ( $k \geq 4$  fixed) [Linnik 1963; Fouvry and Tenenbaum 1985; Topacogullari 2018].

In the case  $k = \ell = 2$ , a similar asymptotic formula holds in a much larger range of uniformity for  $h$ , although with a weaker error term (see [Meurman 2001] for the currently best results in this direction). For  $k, \ell \geq 3$  the problem remains completely open.



The functions  $\tau_k$  are special cases of coefficients of the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\tau_z(n)}{n^s} := \zeta(s)^z \quad \text{for } z \in \mathbb{C} \text{ and } \Re(s) > 1.$$

On prime powers, they are given explicitly by

$$\tau_z(p^\ell) = \binom{z + \ell - 1}{\ell}. \tag{1-5}$$

The functions  $\tau_z$  for  $z \notin \mathbb{N}$  have a more complicated behavior than those for  $z \in \mathbb{N}$ . When  $z = -1$  for instance, we recover the Möbius function  $\tau_{-1}(n) = \mu(n)$ .

**Theorem 1.2** leads to an asymptotic expansion of  $D_{z,2}(x, h)$  for arbitrary  $z \in \mathbb{C}$ , uniformly in any fixed disk  $|z| \ll 1$ .

**Theorem 1.3.** *Let  $A, N \geq 1$  and  $\varepsilon > 0$ . There exist a constant  $\delta > 0$  and holomorphic functions  $\lambda_{h,\ell} : \mathbb{C} \rightarrow \mathbb{C}$ , such that, for  $|z| \leq A, x \geq 2$  and  $1 \leq |h| \leq x^\delta$ ,*

$$\sum_{|h| < n \leq x} \tau_z(n)\tau(n+h) = x(\log x)^z \sum_{\ell=0}^N \frac{\lambda_{h,\ell}(z)}{(\log x)^\ell} + \mathcal{O}\left(\frac{x(\log x)^{\Re(z)}}{(\log x)^{N+1-\varepsilon}}\right), \tag{1-6}$$

where the implicit constant only depends on  $A, N$  and  $\varepsilon$ .

The coefficients  $\lambda_{h,\ell}(z)$  can be computed explicitly; see (8-4) infra for an expression of the leading coefficient. If  $z$  is a nonpositive integer, all the coefficients  $\lambda_{h,\ell}(z)$  vanish and (1-6) effectively becomes an upper bound.

Our method leads to a power saving error term in **Theorem 1.3** when  $z = k \in \mathbb{N}$ . This is solely due to the fact that in these cases the  $k$ -th power of Dirichlet  $L$ -functions  $L(s, \chi)^k$  can be continued analytically to a strip  $\Re(s) \geq 1 - \delta$  for some  $\delta > 0$  (excluding the possible pole at  $s = 1$ ). We do not focus of the case  $z \in \mathbb{N}$  here, since the works mentioned above then give quantitatively stronger estimates.

**1B. Norms of integral ideals.** Let  $K/\mathbb{Q}$  be a Galois extension with discriminant  $\Delta_K$ . We define

$$\mathcal{N}_K := \{N(\alpha) : \alpha \text{ ideal of } \mathcal{O}_K, \alpha \neq 0\}.$$

This set has a rich multiplicative structure, described by the Artin reciprocity law. When the extension is abelian, the Dedekind function  $\zeta_K(s)$  factorizes into Dirichlet  $L$ -functions mod  $\Delta_K$ , so that the integers in  $\mathcal{N}_K$  can be detected by looking at the congruence classes of their prime factors mod  $\Delta_K$ . **Theorem 1.2** eventually applies and leads to the following result.

**Theorem 1.4.** *Let  $K/\mathbb{Q}$  be an abelian field extension. Let  $N \geq 1$  and  $\varepsilon > 0$ . There exist a constant  $\delta > 0$  and real numbers  $\kappa_{h,\ell}(K)$ , such that, for  $x \geq 2$  and  $1 \leq |h| \leq x^\delta$ ,*

$$\sum_{\substack{|h| < n \leq x \\ n \in \mathcal{N}_K}} \tau(n-h) = x(\log x)^{1-1/[K:\mathbb{Q}]} \sum_{\ell=0}^N \frac{\kappa_{h,\ell}(K)}{(\log x)^\ell} + \mathcal{O}\left(\frac{x}{(\log x)^{N+1/[K:\mathbb{Q}]-\varepsilon}}\right), \tag{1-7}$$

where the implicit constant depends only on  $K, N$  and  $\varepsilon$ .

An interesting special case is given by the extension  $\mathbb{Q}(i)/\mathbb{Q}$ . In this case,  $\mathcal{N}_{\mathbb{Q}(i)}$  is simply the set of integers which can be written as a sum of two squares, and [Theorem 1.4](#) takes the following form.

**Corollary 1.5.** *Let  $\mathcal{B}$  be the set of all integers which can be written as a sum of two squares. Let  $N \geq 1$  and  $\varepsilon > 0$ . There exist a constant  $\delta > 0$  and real numbers  $\beta_{h,\ell}$ , such that, for  $x \geq 2$  and  $1 \leq |h| \leq x^\delta$ ,*

$$\sum_{\substack{|h| < n \leq x \\ n \in \mathcal{B}}} \tau(n-h) = x(\log x)^{1/2} \sum_{\ell=0}^N \frac{\beta_{h,\ell}}{(\log x)^\ell} + \mathcal{O}\left(\frac{x}{(\log x)^{N+1/2-\varepsilon}}\right). \tag{1-8}$$

where the implicit constant depends only on  $N$  and  $\varepsilon$ .

The first term in the asymptotic formula for the left-hand side of (1-8) can also be obtained using a recent extension of the Bombieri–Vinogradov theorem due to Granville and Shao [2018], along with the Brun–Titchmarsh inequality. The coefficients  $\kappa_{h,\ell}(K)$  and  $\beta_{h,\ell}$  can be computed explicitly; see (8-5) infra for an evaluation of the leading coefficient  $\beta_{h,0}$  in (1-8). Note that, since the indicator function  $b(n)$  of the set  $\mathcal{B}$  correlates with both the principal and the nonprincipal character mod 4, there are two genuine contributions on the right-hand side in (1-3) when  $f(n) = b(n)$ . This also explains the discrepancy between the conjectures made in [Iwaniec 1976] and [Freiberg et al. 2017] on autocorrelations of  $b(n)$ .

We stress that the multiplicity of representations as ideal norms in [Corollary 1.5](#) is *not* taken into account. Thus the estimate (1-8) is more difficult to obtain than an estimate for the correlation sum

$$\sum_{|h| < n \leq x} r_2(n)\tau(n-h) \quad \text{with } r_2(n) := |\{(r, s) \in \mathbb{Z}^2 : r^2 + s^2 = n\}|,$$

for which classical methods suffice.

**1C. Integers with  $k$  prime divisors.** The Titchmarsh divisor problem [1930] asks for an asymptotic evaluation of the sum

$$\sum_{|h| < p \leq x} \tau(p-h), \tag{1-9}$$

where  $p$  runs over all primes up to  $x$ . Following the initial works by Titchmarsh [1930] and Linnik [1963], the best known result was obtained independently by Fouvry [1985] and Bombieri, Friedlander and Iwaniec [Bombieri et al. 1986]: For any fixed  $N > 0$ , we have, for  $1 \leq |h| \leq (\log x)^N$ ,

$$\sum_{|h| < p \leq x} \tau(p-h) = C_h x + C'_h \text{li}(x) + \mathcal{O}\left(\frac{x}{(\log x)^N}\right), \tag{1-10}$$

where

$$C_h = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \prod_{p|h} \left(1 - \frac{p}{p^2 - p + 1}\right), \quad C'_h = \left(\gamma - \sum_p \frac{\log p}{p^2 - p + 1} + \sum_{p|h} \frac{p^2 \log p}{(p-1)(p^2 - p + 1)}\right) C_h.$$

An interesting generalization of this problem concerns the sum

$$\sum_{\substack{|h| < n \leq x \\ \omega(n)=k}} \tau(n-h), \quad (1-11)$$

where  $\omega(n)$  denotes the number of distinct prime divisors of an integer  $n$ . An asymptotic equivalence for this sum was proven by Khripunova [1998, Theorem 3], uniformly for  $k \ll \log \log x$  and  $h \ll x$ .

Our methods allow to obtain a full asymptotic expansion for (1-11), at least for small shifts  $h$ . In order to circumvent the obstacle that the indicator function for integers  $n$  with  $\omega(n) = k$  is not multiplicative, we use a classical method due to Selberg [1954], which allows us to reduce the evaluation of (1-11) to the evaluation of the correlation sum of the divisor function with the multiplicative function  $n \mapsto z^{\omega(n)}$ . This eventually leads to the following result.

**Theorem 1.6.** *Let  $N \geq 1$  and  $\varepsilon > 0$ . There exist a constant  $\delta > 0$  and polynomials  $P_{h,\ell}^k(X)$  of degree  $k-1$  such that, for  $1 \leq k \ll \log \log x$  and  $|h| \leq x^\delta$ ,*

$$\sum_{\substack{|h| < n \leq x \\ \omega(n)=k}} \tau(n-h) = x \sum_{0 \leq \ell \leq N} \frac{P_{h,\ell}^k(\log \log x)}{(\log x)^\ell} + \mathcal{O}\left(\frac{x(\log \log x)^k}{k!(\log x)^{N+1-\varepsilon}}\right), \quad (1-12)$$

where the implicit constants depend only on  $N$  and  $\varepsilon$ .

The case  $k = 1$  recovers the best-known asymptotic formula (1-10) for the Titchmarsh divisor problem. As before, the polynomials  $P_{h,\ell}^k$  can be computed explicitly; in particular, the leading coefficient in the asymptotic expansion is given by  $C_h/(k-1)!$ .

This result is nontrivial throughout the range  $k \ll \log \log x$ . The case  $k/\log \log x \rightarrow +\infty$  is an interesting question which would require different tools, due to the sparsity of the set of integers under consideration (not unlike the situation for friable integers [Harper 2012]). We do not address this here.

**1D. Overview of the proof of Theorem 1.2.** For the sake of clear exposition, we will focus here on the case  $D = 1$ , as our arguments extend without much difficulty to the case of general moduli and the arising complications are mainly of technical nature. Note that any  $f \in \mathcal{F}_1(A)$  can be approximated (in the convolution sense) by a suitable generalized divisor function, so that it suffices to consider the case  $f = \tau_z$  with  $z \in \mathbb{C}$ .

We will give two distinct proofs of Theorem 1.2. They are based on two different kinds of combinatorial identities for the generalized divisor function  $\tau_z$ , both of which we believe are of independent interest. Our first approach relies on an effective combinatorial formula of Heath-Brown's type for the divisor function  $\tau_\alpha$  with  $\alpha \in \mathbb{Q}$ , and an interpolation argument in the  $z$ -variable for weighted mean values of  $\tau_z$ . Our second approach, which is more direct and avoids the interpolation step, is instead based on an identity of Linnik type for  $\tau_z$  and the well-factorability of friable numbers<sup>1</sup>.

<sup>1</sup>The second proof was found only after a preliminary version of the present manuscript was uploaded online.

**1D.1. Proof by Heath-Brown's identity and interpolation.** Our first proof of [Theorem 1.2](#) divides into two parts: We first prove the theorem for rational  $z$ , and then extend this result to all  $z \in \mathbb{C}$ .

For  $z \in \mathbb{Q}$ , the general structure of the proof of [Theorem 1.2](#) follows the setup of [[Fouvry 1985](#); [Bombieri et al. 1986](#)] (see also [[Fouvry 1984](#)]). The strategy naturally splits into two steps:

- (1) We decompose the function  $f$  into convolutions with either large smooth components (type I) or suitably localized components (type II).
- (2) We solve the question for both types of sums.

The bulk of the present work concerns the first step. Combinatorial decompositions for prime numbers have a long history since the works of Vinogradov [[1937](#)] (we refer to the survey [[Ramaré 2013](#)] for an account and further references). Yet, it was not until recently that analogous identities emerged for generalized divisor functions. Montgomery and Vaughan (private communication) have recently developed a combinatorial identity of Vaughan's type [[1975](#)] for  $\tau_{1/2}$ , which initially motivated largely the present work. Unfortunately, as for primes, the bilinear sums coming from a raw application of this identity are not quite localized enough to be effective for Titchmarsh's problem, and even though this can sometimes be fixed by iterating the formula [[Fouvry 1981](#)], our early attempts were unsuccessful. Instead we follow the more flexible approach of Heath-Brown [[1982](#)] (which is related to [[Gallagher 1968](#)]).

Our first result ([Theorem 3.2](#) below) is a uniform combinatorial formula of Heath-Brown's type for the divisor function  $\tau_{u/v}$  with  $u/v \in \mathbb{Q}$ . In the simplest case  $0 < u < v$ , it reads

$$\tau_{u/v}(n) = \sum_{\ell=1}^K c_{\ell, K, u/v} \sum_{\substack{m_1 \cdots m_\ell n_1 \cdots n_{\ell v-u} = n \\ n_1, \dots, n_{\ell v-u} \leq x^{1/K}}} \tau_{-1/v}(n_1) \cdots \tau_{-1/v}(n_{\ell v-u}) \quad \text{for } n \leq x, \quad (1-13)$$

where  $K \in \mathbb{N}_{>0}$  is arbitrary and where  $c_{\ell, K, u/v} \in \mathbb{Q}$ . A more general formula holds for any rational number  $u/v$  (see [Theorem 3.2](#)). A crucial property of this formula is that it is sensitive almost only to the archimedean size of  $u/v$ . Indeed, for  $|u/v| \leq A$ , the coefficients  $c_{\ell, K, u/v}$ , the length of the  $\ell$ -sum and the value at primes  $n = p$  of each  $\ell$ -summand on the right-hand side are bounded in terms of  $A$  and  $K$  only (but not of  $v$ ). Thus, the only loss due to the size of  $v$  comes from the number  $\mathcal{O}(v)$  of terms in the convolution, which has essentially no effect on what follows.

In the same way, we can express any rational convolution power  $\ast^{u/v} f$  of a multiplicative function in terms of higher convolutions  $\ast^k f$  with  $1 \leq k \leq K$  and a bilinear term with one component supported on the interval  $[x^\varepsilon, x^{1/K}]$ . However, to our knowledge asymptotic formulae for the correlation sums

$$\sum_{n \leq x} (\ast^k f)(n) \tau(n+1), \quad (1-14)$$

for  $k \geq 2$  are currently known for only very few functions  $f$  (essentially constant functions and Dirichlet characters). This is the main obstacle towards using decompositions of this form to prove [Theorem 1.2](#) for complex-fold convolutions of multiplicative functions.

Regarding the second step, we are mostly able to use the harmonic analysis arguments underlying [Fouvry 1985; Bombieri et al. 1986]. They are based on bounds on Kloosterman sums on average [Deshouillers and Iwaniec 1982b], along with Voronoi summation (for type I) and Linnik's dispersion method (for type II). We will follow the treatment made in [Drappeau 2017; Topacogullari 2018], although some work is needed in order to cast the main terms from these works in a form suitable for us.

Eventually, the arguments described above yield a proof of [Theorem 1.2](#) for  $f = \tau_{\frac{u}{v}}$  uniformly in the range  $v \leq (\log x)^N$ . As it turns out, this is already sufficient information to be able to conclude.

To see why, we return to the correlation sum

$$D(z) := \sum_{|h|/a < n \leq x} \tau_z(n) \tau(an - h)$$

with  $z \in \mathbb{C}$ ,  $|z| \ll 1$ . The main observation is that this expression is a polynomial in  $z$ , and that we know how to evaluate it on rational numbers with small denominators. Even though  $D(z)$  initially has degree of the order of  $\log x$ , we can use large deviation bounds on the function  $\omega(n)$  (and a convolution argument) to approximate it, up to an admissible error, by the polynomial

$$\tilde{D}(z) := \sum_{\substack{|h|/a < n \leq x \\ \omega(n) \ll \log \log x}} z^{\omega(n)} \tau(an - h),$$

which has degree at most  $\mathcal{O}(\log \log x)$ . This enables us to use Lagrange interpolation on a suitably chosen set of rational sample points to transfer our estimates for  $z \in \mathbb{Q}$  to estimates of the same quality for  $z \in \mathbb{C}$ . Indeed, this process introduces an error which grows exponentially in the degree of the polynomial. As our estimates for  $D(z)$  for  $z \in \mathbb{Q}$  save an arbitrarily large power of  $\log x$ , we are still able to obtain an asymptotic formula at the end.

Note that for the above arguments to work it is crucial that estimates with a saving of a large power of  $\log x$  for  $D(z)$  for  $z \in \mathbb{Q}$  are available, which we can fortunately obtain here from the Siegel–Walfisz bound (an unfortunate consequence of the last fact, however, is that most of our results are not effective).

We mention that, as in Heath-Brown's work [1982], the arguments sketched above can be used to obtain asymptotic formulae for short sums

$$\sum_{x < n \leq x+y} f(n)$$

for  $y \geq x^{7/12+\varepsilon}$  and  $f \in \mathcal{F}_D(A)$ , as well as theorems of Bombieri–Vinogradov type. However, unlike Titchmarsh's divisor problem, such results could in principle also be obtained by zero-density estimates for Dirichlet  $L$ -functions (see [Iwaniec and Kowalski 2004, Chapter 10.5; Bombieri 1965]).

**1D.2. Proof by Linnik's identity.** Our second proof uses a different decomposition for  $\tau_z$ , which has the major advantage that it holds uniformly for all  $z$  in a fixed bounded subset of  $\mathbb{C}$ . This avoids the interpolation step necessary in the first proof, although the resulting combinatorial identity is not as elegant as the identity of Heath-Brown's type described above.

A naive attempt to find a combinatorial formula for  $\tau_z$  which is uniform in  $z$  might start with Linnik's formula [Iwaniec and Kowalski 2004, Section 13.3], which relies on the Taylor series expansion

$$\zeta(s)^z = (1 + (\zeta(s) - 1))^z = \sum_{j \geq 0} \binom{z}{j} (\zeta(s) - 1)^j.$$

The main technical difficulty at this point is to truncate the sum over  $j$ . In the context of Linnik's formula, this truncation is performed by restricting to almost-primes from the outset (or inserting a sieve weight), see [Linnik 1963, page 21], but unfortunately this approach is not available in our situation.

Instead we write  $\zeta(s) = \zeta_y(s)M_y(s)$ , where

$$\zeta_y(s) := \prod_{p \leq y} \left(1 - \frac{1}{p^s}\right)^{-1} \quad \text{and} \quad M_y(s) := \frac{\zeta(s)}{\zeta_y(s)},$$

with  $y = x^{1/K}$  for some  $K \in \mathbb{N}$ , and then apply the Taylor series expansion only on the second factor  $M_y(s)$ , so that

$$\zeta(s)^z = \zeta_y(s)^z \sum_{j \geq 0} \binom{z}{j} (M_y(s) - 1)^j.$$

This expression has the advantage that the  $j$ -th summand has no coefficient for  $n \leq y^j$  in its Dirichlet series expansion. After expanding and comparing the Dirichlet coefficients on both sides, we are therefore led to the following “raw” combinatorial decomposition (see Theorem 3.3),

$$\tau_z(n) = \sum_{0 \leq \ell < K} c_\ell \sum_{\substack{n=n_1 n_2 \\ n_1 \text{ is } y\text{-friable}}} \tau_{z-\ell}(n_1) \tau_\ell(n_2) \quad \text{for } n \leq x,$$

where the  $c_\ell$  are some complex numbers which depend on  $z$ , but which can be bound uniformly for  $z \ll 1$  (we recall that an integer is said to be  $y$ -friable if all of its prime factors are bounded by  $y$ ).

In order to apply this formula, it is of course necessary to be able to control the factors  $\tau_{z-\ell}(n_1)$ . However, the characteristic function of  $y$ -friable numbers has good factorability properties (see [Vaughan 1989, page 66; Fouvry and Tenenbaum 1996, Lemme 3.1]): we can essentially replace them in the formula above by convolutions of sequences supported on  $[1, y]$  (see Lemma 3.4). This in turn enables us to apply estimates of type I and type II, leading eventually to the desired asymptotic formula.

**Plan.** In Section 2, we introduce our main notations and the subsets of functions of  $\mathcal{F}_D(A)$  we will mainly work with. In Section 3, we present the combinatorial decompositions for  $\tau_z$ , on which our proofs are based. In Section 4, we state some auxiliary computations in order to use the results of [Topacogullari 2018; Drappeau 2017]. In Sections 5 and 6, we proof Theorem 1.2 using the combinatorial identity of Heath-Brown's type, first by treating the case of rational parameters, and then by interpolating the obtained results to all functions in  $\mathcal{F}_D(A)$ . In Section 7, we sketch an alternative proof using the combinatorial identity of Linnik's type. Finally, in Section 8, we estimate the main terms and prove Theorems 1.3, 1.4 and 1.6.



### 2. First reductions

**2A. Statement of the main proposition.** For  $n, h \in \mathbb{Z}$  with  $n \geq 1$  and  $n - h \geq 1$ , let

$$\tilde{\tau}_h(n; R) := 2 \sum_{\substack{q \leq \sqrt{n-h} \\ (n,q)=(h,q)}} \frac{1}{\varphi\left(\frac{q}{(h,q)}\right)} \sum_{\substack{\chi \pmod{q/(h,q)} \\ \text{cond}(\chi) \leq R}} \chi\left(\frac{h}{(h,q)}\right) \overline{\chi\left(\frac{n}{(h,q)}\right)}. \tag{2-1}$$

Note that  $\tilde{\tau}_h(n; R) = \tau(n - h)$  if  $R > \sqrt{n - h}$  and  $n - h$  is not a perfect square. We will eventually choose  $R$  of size  $(\log n)^{O(1)}$ . We have a trivial bound

$$\tilde{\tau}_h(n; R) \ll_{\varepsilon} n^{\varepsilon} R^{1+\varepsilon}. \tag{2-2}$$

The function  $\tilde{\tau}_h(n; R)$  should be thought of as an approximation to  $\tau(n - h)$  on average. The main work in proving [Theorem 1.2](#) consists in showing that, for any  $f \in \mathcal{F}_D(A)$ , we have

$$\sum_{n \leq x} f(n) \tau(an - h) \sim \sum_{n \leq x} f(n) \tilde{\tau}_h(an; R_x) \quad \text{for } x \rightarrow \infty, \tag{2-3}$$

where  $R_x$  is some slowly growing function in  $x$  (some appropriate power of  $\log x$ ). Once this is established, we can evaluate the sum on the right by standard methods. In view of this, it is convenient to define

$$\Delta_h(n; R) := \tau(n - h) - \tilde{\tau}_h(n; R) \quad \text{and} \quad \Sigma_f(I; a, h; R) := \sum_{n \in I} f(n) \Delta_h(an; R),$$

for any interval  $I \subseteq \mathbb{R}^+$ . The main part of this article is concerned with proving the following proposition, which puts the statement (2-3) into precise terms, and from which the results described in the introduction can be deduced easily (see [Section 8](#)).

**Proposition 2.1.** *Let  $A, D \geq 1$  be fixed. Then we have, for  $x \geq 3$ ,  $I \subset [x/2, x]$  an interval and  $f \in \mathcal{F}_D(A)$ , the following estimate,*

$$|\Sigma_f(I; a, h; R)| \leq C \tau((a, h)) \frac{x (\log x)^B}{R^{1/3}} \quad \text{for } 1 \leq a, |h|, R \leq x^{\delta}, \tag{2-4}$$

where  $\delta > 0$  is some absolute constant and where  $B, C > 0$  are constants which depend only on  $A$  and  $D$ .

**2B. Restricting the set of functions.** It is known in multiplicative number theory that, to a certain degree of precision, the magnitude of the mean value of a multiplicative function  $f$  depends mostly on the values  $f(p)$ ,  $p$  prime. The following lemma quantifies the analogous phenomenon in our case.

**Lemma 2.2.** *Let  $f, g : \mathbb{N} \rightarrow \mathbb{C}$  be multiplicative functions, which satisfy the following conditions:*

- (i)  $|g(n)| \leq \tau_M(n)$  for some  $M \geq 1$  and all  $n \in \mathbb{N}$ .
- (ii)  $H := \sum_{n \geq 1} |(f * g^{-1})(n)| / n^{\sigma} < +\infty$  for some  $\sigma < 1$ , where  $g^{-1}$  denotes the Dirichlet convolution inverse of  $g$ .

Furthermore, assume there are constants  $\varrho, \delta \in (0, 1]$  and  $B, C \geq 1$  such that, for all  $x \geq 1$  and all intervals  $I \subset [x/2, x]$ ,

$$|\Sigma_g(I; a, h; R)| \leq C\tau((a, h)) \frac{x(\log x)^B}{R^\varrho} \quad \text{for } 1 \leq a, |h|, R \leq x^\delta. \tag{2-5}$$

Then there exists  $C', \delta' > 0$  depending only on  $\varrho, \delta, \sigma$  and  $M$ , such that, for all  $x \geq 1$  and all intervals  $I \subset [x/2, x]$ ,

$$|\Sigma_f(I; a, h; R)| \leq HCC'\tau((a, h)) \frac{x(\log x)^B}{R^\varrho} \quad \text{for } 1 \leq a, |h|, R \leq x^{\delta'}. \tag{2-6}$$

*Proof.* Let  $h := f * g^{-1}$ . We have

$$\begin{aligned} \Sigma_f(I; a, h; R) &= \sum_{n_1 n_2 \in I} g(n_1)h(n_2)\Delta_h(an_1 n_2; R) \\ &= \sum_{n_2 \leq T} h(n_2)\Sigma_g(I/n_2; an_2, h; R) + \sum_{n_2 > T} h(n_2)\Sigma_g(I/n_2; an_2, h; R), \end{aligned}$$

for some parameter  $T \geq 1$ . For the sum on the left we use the assumption (2-5), so that

$$\sum_{n_2 \leq T} h(n_2)\Sigma_g(I/n_2; an_2, h; R) \ll_\sigma CH\tau((a, h)) \frac{x(\log x)^B}{R^\varrho},$$

provided that the parameters  $a, h$  and  $R$  satisfy

$$1 \leq a \leq \frac{x^\delta}{T^{1+\delta}} \quad \text{and} \quad 1 \leq |h|, R \leq \frac{x^\delta}{T^\delta}.$$

For the sum on the right we use the trivial bound  $\Sigma_g(I/n_2; an_2, h; R) \ll_{\varepsilon, M} Rx^{1+\varepsilon}/n_2$ , and get

$$\sum_{n_2 > T} h(n_2)\Sigma_g(I/n_2; an_2, h; R) \ll_{\varepsilon, M} x^{1+\varepsilon} RT^{-1+\sigma} H.$$

The lemma follows on setting  $T = x^{\delta/3}$  and  $\delta' = \min(\delta/3, \delta(1 - \sigma)/(4(1 + \rho)))$ . □

In view of this, in order to prove [Proposition 2.1](#), we will restrict to the following two subsets of  $\mathcal{F}_D(A)$ . The first subset, denoted by  $\mathcal{F}_D^\tau(A)$ , consists of functions  $f : \mathbb{N} \rightarrow \mathbb{C}$ , which are the coefficients of Dirichlet series of the form

$$\sum_{n=1}^\infty \frac{f(n)}{n^s} = \prod_{\chi \bmod D} L(s, \chi)^{b_\chi}, \tag{2-7}$$

where the parameters  $b_\chi$  are complex numbers such that  $|b_\chi| \leq A$ . Note that  $\tau_z \in \mathcal{F}_D^\tau(A)$  for  $|z| \leq A$ . A particularly important role will be played by the subset  $\mathcal{F}_D^{\tau\mathbb{Q}}(A) \subset \mathcal{F}_D^\tau(A)$  formed by functions of this form where all the parameters  $b_\chi$  are rational.



The second subset  $\mathcal{F}_D^\omega(A)$  is defined to be the set of functions  $f : \mathbb{N} \rightarrow \mathbb{C}$ , which are the coefficients of Dirichlet series of the form

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{r \in (\mathbb{Z}/D\mathbb{Z})^\times} \prod_{p \equiv r \pmod{D}} \left( 1 + \frac{z_r}{p^s - 1} \right), \tag{2-8}$$

where the coefficients  $z_r$  are complex numbers such that  $|z_r| \leq A$ . This includes the functions  $n \mapsto z^{\omega(n)}$  for all  $|z| \leq A$ .

**Lemma 2.3.** *For any  $f \in \mathcal{F}_D(A)$ , there exist  $g_1 \in \mathcal{F}_D^\tau(A)$  and  $g_2 \in \mathcal{F}_D^\omega(A)$  which satisfy the conditions (i)–(ii) stated in Lemma 2.2 for  $\sigma = \frac{2}{3}$ , and  $M, H$  bounded only in terms of  $A$  and  $D$ .*

*Proof.* We first prove the lemma with respect to the set  $\mathcal{F}_D^\tau(A)$ . Let  $f \in \mathcal{F}_D(A)$  be fixed, and let  $v_f : \mathbb{Z} \rightarrow \mathbb{C}$  be the  $D$ -periodic function defined by

$$v_f(r) = \begin{cases} f(p) & \text{if there exists a prime } p \text{ such that } (p, D) = 1 \text{ and } p \equiv r \pmod{D}, \\ 0 & \text{if } (r, D) > 1. \end{cases} \tag{2-9}$$

We then set

$$b_\chi := \frac{1}{\varphi(D)} \sum_{r \pmod{D}} v_f(r) \bar{\chi}(r) \quad \text{for any character } \chi \pmod{D},$$

and define  $g_1$  as the coefficients of the following Dirichlet series,

$$\sum_{n=1}^{\infty} \frac{g_1(n)}{n^s} := \prod_{\chi \pmod{D}} L(s, \chi)^{b_\chi}. \tag{2-10}$$

We have  $(f * g_1^{-1})(p) = 0$  if  $p \nmid D$ . Moreover, since  $|b_\chi| \leq A$ , we get  $|g_1(n)| \leq \tau_{AD}(n)$  for all  $n$ . Therefore,

$$\sum_{n \geq 1} \frac{|(f * g_1^{-1})(n)|}{n^{2/3}} = \prod_{p \mid D} \left( 1 + \mathcal{O}_{A,D} \left( \frac{1}{p^{2/3}} \right) \right) \prod_{p \nmid D} \left( 1 + \mathcal{O}_{A,D} \left( \frac{1}{p^{4/3}} \right) \right) = \mathcal{O}_{A,D}(1).$$

This proves the first part of the lemma.

For the second part, we define  $g_2$  by its Dirichlet series

$$\sum_{n=1}^{\infty} \frac{g_2(n)}{n^s} := \prod_{r \in (\mathbb{Z}/D\mathbb{Z})^\times} \prod_{\substack{p \text{ prime} \\ p \equiv r \pmod{D}}} \left( 1 + \frac{v_f(r)}{p^s - 1} \right).$$

The fact that  $g_2$  satisfies the required conditions can be shown using similar computations as above.  $\square$

Let us at this point also note the following result, which is an easy consequence of the proofs of Lemmas 2.2 and 2.3, and which will become useful later on.

**Lemma 2.4.** *Let  $f \in \mathcal{F}_D(A)$  and let  $\psi \pmod q$  be a Dirichlet character. Then the Dirichlet series associated to  $\psi f$  is given by*

$$\sum_{n=1}^{\infty} \frac{\psi(n)f(n)}{n^s} = H_{\psi}(s) \prod_{\chi \pmod D} L(s, \psi\chi)^{b_{\chi}} \quad \text{for } \Re(s) > 1,$$

where  $H_{\psi}(s)$  is some holomorphic function defined in  $\Re(s) > \frac{1}{2}$  and where

$$b_{\chi} := \frac{1}{\varphi(D)} \sum_{r \pmod D} v_f(r)\bar{\chi}(r),$$

with  $v_f(n)$  as defined in (2-9). Moreover, for any fixed  $\sigma_0 > \frac{1}{2}$ , we have  $H_{\psi}(s) \ll 1$  uniformly in  $\Re(s) > \sigma_0$ , with the implicit constant depending at most on  $\sigma_0, A$  and  $D$ .

From Lemmas 2.2 and 2.3, we deduce the following statement.

**Lemma 2.5.** *To prove Proposition 2.1 in full generality, it suffices to prove it under either one of the additional hypotheses  $f \in \mathcal{F}_D^{\tau}(A)$  or  $f \in \mathcal{F}_D^{\omega}(A)$ .*

### 3. Combinatorial identities for $\tau_z(n)$

In this section we describe the two combinatorial identities for the generalized divisor function  $\tau_z$  on which the proofs of Theorem 1.2 are based.

**3A. A generalization of Heath-Brown's identity.** We first derive a combinatorial decomposition analogous to [Heath-Brown 1982] for the function  $n \mapsto \tau_{\alpha}(n)$  in the case  $\alpha \in \mathbb{Q}$ . Our argument is based on the following polynomial identity.

**Lemma 3.1.** *Let  $u$  and  $v$  be integers such that  $v > u \geq 0$ . Let  $K \geq 1$  and  $N \geq 0$ . Then there exist rational coefficients  $a_m$  and  $b_{\ell}$  such that there holds*

$$\sum_{K \leq m \leq (K+N)v-u} a_m (X-1)^m = 1 + X^{Nv} \sum_{1 \leq \ell \leq K} b_{\ell} X^{\ell v-u}. \tag{3-1}$$

The coefficients  $(b_{\ell})$  are unique and given explicitly by

$$b_{\ell} = \frac{(-1)^{\ell}}{(\ell-1)!(K-\ell)!} \prod_{\substack{1 \leq j \leq K \\ j \neq \ell}} \left( j + N - \frac{u}{v} \right). \tag{3-2}$$

*Proof.* An identity of the form (3-1) exists if and only if we can find  $b_1, \dots, b_K$  such that the first  $K-1$  derivatives of the polynomial on the right hand side of (3-1) vanish at  $X = 1$ . This is equivalent to saying

that the  $b_1, \dots, b_K$  solve the equation

$$\begin{pmatrix} 1 & \cdots & 1 \\ v + Nv - u & \cdots & Kv + Nv - u \\ \vdots & \ddots & \vdots \\ (v + Nv - u)^{K-1} & \cdots & (Kv + Nv - u)^{K-1} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_K \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3-3}$$

Let  $C$  be the matrix on the left, and  $B_\ell$  the same matrix but with the upper row and the  $\ell$ -th column removed. Note that  $C$  is a Vandermonde matrix, and  $B_\ell$  is a product of a Vandermonde matrix with a diagonal matrix. Hence, we deduce

$$\begin{aligned} \det C &= \prod_{1 \leq i < j \leq K} (jv - iv) = 2!3! \cdots (K - 1)!v^{K(K-1)/2}, \\ \det B_\ell &= \prod_{\substack{1 \leq i < j \leq K \\ i, j \neq \ell}} (jv - iv) \prod_{\substack{1 \leq j \leq K \\ j \neq \ell}} (jv + Nv - u). \end{aligned}$$

Since  $\det C \neq 0$ , we obtain by Cramer’s rule that there is a unique solution  $(b_\ell)$ , given by

$$b_\ell = (-1)^\ell \frac{\det B_\ell}{\det C}, \tag{3-4}$$

which yields (3-2). □

**Theorem 3.2.** *Let  $v > 0$  and  $r$  be integers such that  $v > u \geq 0$  and  $r \geq 0$ . Let  $K \geq 1$  and  $x \geq 1$ . Then for any  $n \leq x$ , we have*

$$\tau_{r+u/v}(n) = \sum_{\ell=1}^K c_\ell^+ \sum_{\substack{m_1 \cdots m_{\ell+r} n_1 \cdots n_{\ell v-u} = n \\ n_1, \dots, n_{\ell v-u} \leq x^{1/K}}} \tau_{-1/v}(n_1) \cdots \tau_{-1/v}(n_{\ell v-u}), \tag{3-5}$$

and, for  $r \geq 1$ ,

$$\tau_{-r+u/v}(n) = \sum_{\ell=1}^K c_\ell^- \sum_{\substack{m_1 \cdots m_{\ell-1} n_1 \cdots n_{\ell v+(r-1)v-u} = n \\ n_1, \dots, n_{\ell v+(r-1)v-u} \leq x^{1/K}}} \tau_{-1/v}(n_1) \cdots \tau_{-1/v}(n_{\ell v+(r-1)v-u}), \tag{3-6}$$

where the  $c_\ell^+$  and  $c_\ell^-$  are certain rational numbers, which can be bounded by

$$c_\ell^+, c_\ell^- \ll 1 \quad \text{for } 1 \leq \ell \leq K,$$

the implicit constant depending only on  $K$  and  $r$ .

*Proof.* Let

$$G(s) := \sum_{n=1}^\infty \frac{\tau_{-1/v}(n)g(n)}{n^s} \quad \text{with} \quad g(n) := \begin{cases} 1 & \text{if } n \leq x^{1/K}, \\ 0 & \text{otherwise.} \end{cases}$$

We first look at (3-5). Here we use Lemma 3.1 with  $N = 0$  and  $X = \zeta(s)^{\frac{1}{v}}G(s)$ , and then multiply both sides by  $\zeta(s)^{r+u/v}$ , which leads to the identity

$$\sum_{K \leq m \leq Kv-u} a_m (\zeta(s)^{1/v}G(s) - 1)^m \zeta(s)^{r+u/v} = \zeta(s)^{r+u/v} + \sum_{1 \leq \ell \leq K} b_\ell \zeta(s)^{r+\ell} G(s)^{\ell v-u}.$$

Then (3-5) follows by comparing the Dirichlet coefficients on both sides and noting that, by construction, the left-hand side has no Dirichlet coefficient for  $n \leq x$ .

In order to show (3-6), we use Lemma 3.1 with the same  $X$  as before and with  $N = r - 1$ , and then multiply both sides by  $\zeta(s)^{-r+u/v}$ . This gives

$$\sum_{K \leq m \leq (K+r-1)v-u} a_m \left( \zeta(s)^{1/v}G(s) - 1 \right)^m \zeta(s)^{-r+u/v} = \zeta(s)^{-r+u/v} + \sum_{1 \leq \ell \leq K} b_\ell \zeta(s)^{\ell-1} G(s)^{\ell v+(r-1)v-u},$$

and (3-6) follows again by comparing the Dirichlet coefficients on both sides. □

**Remark.** With  $r = v = 1$  and  $u = 0$ , the identity (3-6) leads to the decomposition of  $\mu(n)$  described in [Iwaniec and Kowalski 2004, (13.38)].

**3B. A combinatorial identity of Linnik's type.** Here we derive a combinatorial decomposition for  $\tau_z$  using an approach analogous to [Linnik 1963].

We denote by  $P^+(n)$  the largest, and by  $P^-(n)$  the smallest prime factor of an integer  $n > 1$ , with the convention that  $P^+(1) = 1$  and  $P^-(1) = \infty$ . Given an arbitrary multiplicative function  $f$  and a complex number  $z \in \mathbb{C}$ , we define the  $z$ -fold convolution of  $f$  as the multiplicative function given by

$$f^{(*z)}(p^\nu) := \sum_{1 \leq r \leq \nu} \binom{z}{r} \sum_{\substack{\lambda_1, \dots, \lambda_r \geq 1 \\ \lambda_1 + \dots + \lambda_r = \nu}} f(p^{\lambda_1}) \cdots f(p^{\lambda_r}) \quad (\nu \geq 1).$$

The notation is motivated by the fact that if  $F(s) := \sum_{n \geq 1} f(n)n^{-s}$  is the Dirichlet series associated to  $f$ , then for  $\Re(s)$  large enough the function  $\log F(s)$  is well defined and we have  $F(s)^z = \sum_n f^{(*z)}(n)n^{-s}$ . Indeed, by expressing  $F(s)$  as an Euler product, we see immediately that

$$F(s)^z = \prod_p \left( 1 + \sum_{\nu \geq 1} \frac{f(p^\nu)}{p^{\nu s}} \right)^z = \prod_p \left( 1 + \sum_{r \geq 1} \binom{z}{r} \left( \sum_{\nu \geq 1} \frac{f(p^\nu)}{p^{\nu s}} \right)^r \right) = \prod_p \left( 1 + \sum_{\nu \geq 1} \frac{f^{(*z)}(p^\nu)}{p^{\nu s}} \right).$$

Note that  $f^{(*z)}(p) = zf(p)$ , and that for  $\ell \in \mathbb{N}$  the  $\ell$ -fold convolution as defined here coincides with the  $\ell$ -fold convolution defined in the traditional sense. We will be eventually interested in the case when  $f = \chi$  is a Dirichlet character, in which case we have  $f^{(*z)} = \tau_z^\chi$ .

**Theorem 3.3.** *Let  $K \in \mathbb{N}_{>0}$  and  $A, x \geq 1$ . Then for all  $z \in \mathbb{C}$  there exist complex numbers  $(c_\ell)_{0 \leq \ell \leq K}$ , such that for all  $x \geq 1$  and all multiplicative functions  $f$ , we have the following identity for  $n \leq x$ ,*

$$f^{(*z)}(n) = \sum_{0 \leq \ell < K} c_\ell \sum_{\substack{n = n_1 n_2 \\ P^+(n_1) \leq x^{1/K}}} f^{(*z-\ell)}(n_1) f^{(*\ell)}(n_2), \tag{3-7}$$

where the coefficients  $c_\ell$  can be bound by  $c_\ell = O_{K,A}(1)$  uniformly for  $|z| \leq A$ .

*Proof.* Let  $y := x^{1/K}$ . As before we set  $F(s) := \sum_{n \geq 1} f(n)n^{-s}$ . We may certainly assume that  $f(p^k)$  vanishes if  $p > x$ . Let

$$F(s, y) = \prod_{p \leq y} \left( \sum_{k \geq 0} \frac{f(p^k)}{p^{ks}} \right), \quad G(s, y) = \prod_{p > y} \left( \sum_{k \geq 0} \frac{f(p^k)}{p^{ks}} \right).$$

For  $\Re(s) > 0$ , the decomposition  $F(s) = F(s, y)G(s, y)$  yields

$$\begin{aligned} F(s)^z &= F(s, y)^z (1 + (G(s, y) - 1))^z \\ &= F(s, y)^z \sum_{k \geq 0} \binom{z}{k} (G(s, y) - 1)^k \\ &= F(s, y)^z \sum_{0 \leq k < K} \binom{z}{k} (G(s, y) - 1)^k + R(s) \end{aligned}$$

with

$$R(s) := F(s, y)^z \sum_{k \geq K} \binom{z}{k} (G(s, y) - 1)^k.$$

Note that the series converge absolutely if  $\Re(s)$  is large enough in terms of  $f$ . By expanding, we get

$$F(s)^z = F(s, y)^z \sum_{0 \leq \ell < K} c_\ell G(s, y)^\ell + R(s),$$

with

$$c_\ell := (-1)^\ell \sum_{\ell \leq k < K} (-1)^k \binom{z}{k} \binom{k}{\ell}.$$

We read the coefficients of  $n^{-s}$ , for  $n \leq x$ , on each side. Note that for  $k \geq K$ , the series  $(G(s, y) - 1)^k$  has no corresponding Dirichlet coefficients, so there is no contribution from  $R(s)$ . The claimed equality follows on writing  $G(s, x^{1/K}) = F(s)F(s, x^{1/K})^{-1}$ . □

**Remarks.** • Compared with (3-5)–(3-6), this identity has the significant advantage that it is uniform for  $z \ll 1$  complex.

- The case  $K = 2$  only involves the exponents  $\ell \in \{0, 1\}$ . It follows, for instance, that if  $f^{(*z)}$  satisfies a Siegel–Walfisz estimate (in the sense of [Granville and Shao 2018, equation (1.2)]), and if  $f$  satisfies a Bombieri–Vinogradov theorem, then  $f^{(*z)}$  satisfies a Bombieri–Vinogradov theorem as well.
- The case  $K = 2, f = \mathbf{1}$  leads to Eratosthenes’ sieve identity: for all  $n \in (\sqrt{x}, x]$ , we have

$$\mathbf{1}_{n \text{ prime}} = \sum_{\substack{d | n \\ p | d \Rightarrow p \leq \sqrt{x}}} \mu(d).$$

For any  $\eta \in (0, 1/2)$ , either we have  $d \leq x^\eta$  (which corresponds to type I sums), or  $d > x^\eta$ , in which case we can localize a factor of  $d$  in the interval  $[x^\eta, x^{1/2+\eta}]$  (and this corresponds to type II sums).

The main property which allows [Theorem 3.3](#) to be used in our arguments is the following factorization lemma, in the spirit of Lemma 3.1 of [[Vaughan 1989](#), page 29]; see [[Hmyrova 1964](#)] for an early use of this property, and [[Fouvry and Tenenbaum 1996](#)] for an application in a context similar to ours.

**Lemma 3.4.** *For any multiplicative function  $f : \mathbb{N} \rightarrow \mathbb{R}$ , any compactly supported function  $g : \mathbb{N} \rightarrow \mathbb{C}$ , and all  $y, w \geq 2$ , we have*

$$\sum_{P^+(n) \leq y} f(n)g(n) = \Sigma_{\text{triv}} + \Sigma_I + \mathcal{O}(\Sigma_{\text{II}}), \tag{3-8}$$

where

$$\Sigma_I = \sum_{\substack{n \leq w \\ P^+(n) \leq y}} f(n)g(n), \quad \Sigma_{\text{triv}} = \sum_{\substack{n > w \\ P^+(n) \leq y \\ \exists p^v \parallel n, p^v > y}} f(n)g(n), \quad \Sigma_{\text{II}} = (\log y) \sup_{\alpha, \beta} \left| \sum_{w < m \leq yw} \sum_n \alpha_m \beta_n g(mn) \right|,$$

the supremum in  $\Sigma_{\text{II}}$  being taken over all sequences  $(\alpha_m), (\beta_n)$  of complex numbers satisfying

$$|\alpha_m| \leq |f(m)|, \quad |\beta_n| \leq (|f| * |f|)(n).$$

*Proof.* If an integer  $n$  with  $P^+(n) \leq y$  is not counted in the first two sums on the right-hand side, then  $n > w$  and all prime powers  $p^v \parallel n$  satisfy  $p^v \leq y$ . By incorporating these prime powers as  $p$  increases, we may factor  $n = n_1 n_2$  uniquely in such a way that

$$P^+(n_1) < P^-(n_2), \quad w < n_1 \leq w Q^+(n_1),$$

where  $Q^+(n_1)$  is the prime power corresponding to the largest prime of  $n_1$ :  $Q^+(n_1) = P^+(n_1)^v \parallel n_1$ . Note that this implies  $(n_1, n_2) = 1$ . Our statement follows after separating variables [[Iwaniec and Kowalski 2004](#), Lemma 13.11] in the condition  $P^+(n_1) < P^-(n_2)$ . □

### 4. Auxiliary estimates

In this section we collect some estimates on  $\Delta_h(n, R)$ , which will be needed in the following sections.

**4A. The second moment of  $\Delta_h(n; R)$ .** On several occasions, we will require the following rough upper-bound for the “main terms”.

**Lemma 4.1.** *For  $x \geq 3, R \geq 1$  and  $(a, h) \in \mathbb{Z}^2$  such that  $1 \leq a, |h|, R \leq x^{1/4}$ , the following estimate holds,*

$$\sum_{x/2 < n \leq x} |\Delta_h(an; R)|^2 \ll \tau((a, h))^2 x (\log x)^4.$$

*Proof.* We have

$$\sum_{x/2 < n \leq x} |\Delta_h(an; R)|^2 \ll \sum_{x/2 < n \leq x} \tau(an - h)^2 + \sum_{x/2 < n \leq x} |\tilde{\tau}_h(an; R)|^2 =: G_1 + G_2,$$

and we now proceed to estimate the two sums  $G_1$  and  $G_2$  separately.

We first look at  $G_1$ . For notational convenience, let

$$a' := \frac{a}{(a, h)}, \quad h' := \frac{h}{(a, h)} \quad \text{and} \quad t := (a, h).$$

We start by splitting the sum according to the size of  $t^* = (an - h, t^\infty)$  as follows,

$$G_1 = \sum_{\substack{t^* | t^\infty \\ (t^*, a')=1 \\ t^* \leq x^{1/2}}} \sum_{\substack{x/2 < n \leq x \\ a'n \equiv h' \pmod{t^*} \\ ((a'n-h')/t^*, t)=1}} \tau(an - h)^2 + \sum_{\substack{t^* | t^\infty \\ (t^*, a')=1 \\ t^* > x^{1/2}}} \sum_{\substack{x/2 < n \leq x \\ a'n \equiv h' \pmod{t^*} \\ ((a'n-h')/t^*, t)=1}} \tau(an - h)^2 =: G_{1a} + G_{1b}.$$

In order to estimate  $G_{1a}$  we choose  $b, y \in \mathbb{Z}$  such that  $a'b = 1 + yt^*$  and write

$$\begin{aligned} G_{1a} &= \sum_{\substack{t^* | t^\infty \\ (t^*, a')=1 \\ t^* \leq x^{1/2}}} \tau(t^*t)^2 \sum_{\substack{(x-2bh')/2t^* < n' \leq (x-bh')/t^* \\ (yh' + n'a', t)=1}} \tau(yh' + n'a')^2 \\ &\leq \sum_{\substack{t^* | t^\infty \\ (t^*, a')=1 \\ t^* \leq x^{1/2}}} \tau(t^*t)^2 \sum_{\substack{(a'x-2h')/2t^* < m \leq (a'x-h')/t^* \\ m \equiv yh' \pmod{a'}}} \tau(m)^2. \end{aligned}$$

The sum over  $m$  can now be estimated via [Shiu 1980, Theorem 2] or [Barban and Vehov 1969, Theorem 1], which leads to

$$G_{1a} \ll x \log^3 x \sum_{\substack{t^* | t^\infty \\ t^* \leq x^{1/2}}} \frac{\tau(t^*t)^2}{t^*} \ll \tau((a, h))^2 x \log^4 x. \tag{4-1}$$

In  $G_{1b}$  we bound all the summands trivially and get

$$G_{1b} \ll \sum_{\substack{t^* | t^\infty \\ (t^*, a')=1 \\ t^* > x^{1/2}}} \sum_{\substack{x/2 < n \leq x \\ a'n \equiv h' \pmod{t^*}}} \tau(t(a'n - h'))^2 \ll x^{1+\varepsilon} \sum_{\substack{t^* | t^\infty \\ x^{1/2} < t^* \leq 2a'x}} \frac{1}{t^*} \ll x^{3/4+\varepsilon},$$

so that together with (4-1) we deduce

$$G_1 \ll \tau((a, h))^2 x \log^4 x.$$

Next we look at  $G_2$ . Here we first rewrite  $\tilde{\tau}_h(an; R)$  as

$$\tilde{\tau}_h(an; R) := 2 \sum_{\substack{\alpha | (a, h) \\ (\delta, a/\alpha)=1}} \sum_{\substack{\delta | (h/\alpha, n) \\ (\delta, a/\alpha)=1}} \sum_{q \leq \sqrt{an-h}/(\alpha\delta)} \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod{q} \\ \text{cond}(\chi) \leq R}} \chi\left(\frac{h}{\alpha\delta}\right) \overline{\chi\left(\frac{an}{\alpha\delta}\right)},$$

so that after expanding the square we are led to

$$G_2 \leq 4 \sum_{\substack{\alpha_1, \alpha_2 | (a, h) \\ \delta_1 | h/\alpha_1, \delta_2 | h/\alpha_2}} \sum_{\substack{q_1 \leq \sqrt{ax-h}/(\alpha_1\delta_1) \\ q_2 \leq \sqrt{ax-h}/(\alpha_2\delta_2)}} \frac{1}{\varphi(q_1)\varphi(q_2)} \sum_{\substack{\chi_1 \pmod{q_1} \\ \chi_2 \pmod{q_2} \\ \text{cond}(\chi_1), \text{cond}(\chi_2) \leq R}} S(\bar{\chi}_1 \chi_2, \frac{x}{[\delta_1, \delta_2]}),$$

with

$$S(\chi, y) := \max_{y/2 \leq y_0 < y} \left| \sum_{y_0 < n \leq y} \chi(n) \right|.$$

If  $\chi_1$  and  $\chi_2$  are induced by the same primitive character, we use the trivial bound  $S(\bar{\chi}_1 \chi_2, y) \leq y$ . Otherwise, the Pólya–Vinogradov bound applies and  $S(\bar{\chi}_1 \chi_2, y) \ll \tau(q_1 q_2) R \log R$ . Inserting these bounds, we eventually obtain

$$G_2 \ll \tau((a, h))^2 x \log^4 x + x^\varepsilon R^3 \ll \tau((a, h))^2 x \log^4 x$$

by our assumption  $R \leq x^{1/4}$ . This concludes the proof. □

**4B. Comparison of main terms.** We begin by two technical lemmas related to the main terms that will appear later. Let  $X \geq 1$ , and let  $f$  and  $v$  be two smooth functions which are both compactly supported inside  $\mathbb{R}_+^*$ . We assume that  $\text{supp } f \subset [C_1 X, C_2 X]$  and  $\text{supp } v \subset [C_1, C_2]$ , where  $C_1$  and  $C_2$  are some positive constants, and that for some  $\Omega \in (0, 1]$ , we have

$$\|v^{(j)}\|_\infty \ll_j 1, \quad \|f^{(j)}\|_\infty \ll_j (\Omega X)^{-j}, \quad \int_{\mathbb{R}} |f^{(j+1)}| \ll (\Omega X)^{-j}, \tag{4-2}$$

for all  $j \geq 0$ . Furthermore, we define

$$M_{f,v}(b, h; M) := \frac{1}{b} \sum_{d|b} \frac{c_d(h)}{d} \int (\log(\xi - h) + 2\gamma - 2 \log d) f(\xi) v\left(\frac{\xi}{bM}\right) d\xi, \tag{4-3}$$

where

$$c_d(h) := \sum_{v \pmod{d}} e(vh/d) = \sum_{\delta | (h,d)} \delta \mu(d/\delta)$$

denotes the Ramanujan sum.

**Lemma 4.2.** *For  $(b, h) \in \mathbb{Z}^2$ ,  $b, M \geq 1$ , and  $R \geq 1$ , we have*

$$\sum_m f(bm) v\left(\frac{m}{M}\right) \tilde{\tau}_h(bm; R) = M_{f,v}(b, h; M) + \mathcal{O}\left(X^\varepsilon R^{3/2} + X^{1/2+\varepsilon} \frac{(h, b)}{b}\right),$$

where the implied constant depends on  $\varepsilon, C_1, C_2$  and on the implied constants in (4-2).

*Proof.* By partial summation and the Pólya–Vinogradov inequality, given a character  $\chi \pmod{q/(q, h)}$  of conductor  $1 < r \leq R$ , we have

$$\sum_n f(bm) v\left(\frac{m}{M}\right) \chi\left(\frac{bm}{(h, q)}\right) = \chi\left(\frac{b}{(b, q)}\right) \iint f'(t_1) v'(t_2) \sum_{\substack{m \geq \frac{h+q^2}{(h,q)/(b,q)} \\ m \leq \frac{t_1/b + Mt_2}{(h,q)/(b,q)}} \chi(m) dt_1 dt_2 \ll R^{1/2} q^\varepsilon.$$

By definition (2-1), we deduce

$$\sum_m f(bm) v\left(\frac{m}{M}\right) \tilde{\tau}_h(bm; R) = 2 \sum_m f(bm) v\left(\frac{m}{M}\right) \sum_{\substack{q \leq \sqrt{bm-h} \\ (bm,q)=(h,q)}} \frac{1}{\varphi(q/(q, h))} + \mathcal{O}(X^\varepsilon R^{3/2}).$$



The condition  $(bm, q) = (h, q)$  in the sum on the right-hand side is equivalent to

$$(b, q) \mid h, \quad \left. \frac{(h, q)}{(b, q)} \right| m, \quad \left( \frac{m(b, q)}{(h, q)}, \frac{q}{(h, q)} \right) = 1.$$

Using Möbius inversion and our hypotheses on  $f$  and  $v$ , we can replace the  $m$ -sum by the corresponding integral and obtain

$$\sum_m f(bm)v\left(\frac{m}{M}\right)\tilde{\tau}_h(bm; R) = \frac{2}{b} \int f(\xi)v\left(\frac{\xi}{bM}\right) \sum_{\substack{q \leq \sqrt{\xi-h} \\ (b,q) \mid h}} \frac{(b, q)}{q} d\xi + \mathcal{O}(X^\varepsilon R^{3/2}).$$

The main term on the right-hand side may be rewritten as

$$\frac{2}{b} \sum_{d \mid b} \frac{c_d(h)}{d} \int f(\xi)v\left(\frac{\xi}{bM}\right) H\left(\frac{\sqrt{\xi-h}}{d}\right) d\xi + \mathcal{O}(X^\varepsilon R^{3/2})$$

where  $H(x) = \sum_{q \leq x} 1/q = \log x + \gamma + \mathcal{O}(x^{-1})$ . This gives the claimed estimate. □

Next, we define

$$M_{f,v}^\chi(b, h; M) := \sum_{\substack{a \pmod D \\ (a,D)=1}} \chi(a) M_{f_{ab}, v_{a/M}}(bD, h - ab; M/D) \tag{4-4}$$

where  $f_{ab}(\xi) := f(\xi + ab)$  and  $v_{a/M}(\xi) := v(\xi + a/M)$ .

**Lemma 4.3.** *If  $b = b^\circ b^*$  with  $b^\circ \mid D^\infty$  and  $(b^*, D) = 1$ , then*

$$M_{f,v}^\chi(b, h; M) = \frac{1}{bD} \chi\left(\frac{h}{(h, b)}\right) \bar{\chi}\left(\frac{b}{(h, b)}\right) \sum_{d \mid b^*} \frac{c_d(h)}{d} \int \left( \log\left(\frac{\xi - h}{(Db^\circ d)^2}\right) + 2\gamma \right) f(\xi)v\left(\frac{\xi}{bM}\right) d\xi. \tag{4-5}$$

Moreover, if  $\chi \pmod D$  is primitive, we have

$$\begin{aligned} \sum_m f(bm)v\left(\frac{m}{M}\right)\chi(m)\tilde{\tau}_h(bm; R) \\ = M_{f,v}^\chi(b, h; M) + \mathcal{O}\left(\mathbf{1}_{D > R}(b, h) \frac{X(\log X)^3}{bD} + X^\varepsilon D^{1/2} R^{3/2} + X^{1/2+\varepsilon} \frac{(h, b^*)}{b^*}\right) \end{aligned}$$

where  $\mathbf{1}_{D > R} = 1$  if  $D > R$  and 0 otherwise.

*Proof.* We rewrite

$$M_{f,v}^\chi(b, h; M) = \frac{1}{bD} \sum_{\substack{a \pmod D \\ (a,D)=1}} \chi(a) \sum_{d \mid bD} \frac{c_d(h-ab)}{d} \int (\log(\xi - h) + 2\gamma - 2 \log d) f(\xi)v\left(\frac{\xi}{bM}\right) d\xi.$$

Using Gauss sums,

$$\sum_{\substack{a \pmod{D} \\ (a,D)=1}} \chi(a) c_d(h-ab) = G(\chi) \bar{\chi} \left( \frac{-bD}{d} \right) \sum_{v \pmod{d}} \bar{\chi}(v) e \left( \frac{hv}{d} \right).$$

This last expression vanishes unless  $D(b, D^\infty) \mid d$ . Denoting  $b^\circ = (b, D^\infty)$  and  $b^* = b/b^\circ$ , we obtain for  $d \mid b^*$

$$\begin{aligned} \sum_{\substack{a \pmod{D} \\ (a,D)=1}} \chi(a) c_{Db^\circ d}(h-ab) &= G(\chi) \bar{\chi}(-b^*/d) G(\bar{\chi}) \sum_{\delta \mid (b^\circ d, h)} \delta \chi(h/\delta) \mu(b^\circ d/\delta) \bar{\chi}(b^\circ d/\delta) \\ &= b^\circ D \mathbf{1}_{b^\circ \mid h} \bar{\chi}(b^*) \chi(h/b^\circ) c_d(h) \\ &= b^\circ D \chi \left( \frac{h}{(h, b)} \right) \bar{\chi} \left( \frac{b}{(h, b)} \right) c_d(h). \end{aligned}$$

This yields our first claim.

For the second, the computations are similar to the previous lemma. If  $D > R$ , we get

$$\sum_m f(bm) v \left( \frac{m}{M} \right) \chi(m) \tilde{\tau}_h(bm; R) \ll X^\varepsilon D^{1/2} R^{3/2}, \quad (4-6)$$

while on the other hand  $M_{f,v}^\chi(b, h; M) \ll (b, h)(bD)^{-1} X(\log X)^2$  by a simple computation from (4-5). If  $D \leq R$ , the bound (4-6) applies to all the characters involved in the definition of  $\tilde{\tau}_h(bm; R)$ , except all those which are induced by  $\chi$ . We obtain

$$\begin{aligned} \sum_m f(bm) v \left( \frac{m}{M} \right) \chi(m) \tilde{\tau}_h(bm; R) \\ = 2 \sum_{D \mid q/(h, h)} \frac{\chi(h/(b, q)) \bar{\chi}(b/(b, q))}{\varphi(q/(h, q))} \sum_{\substack{(bm, q) = (h, q) \\ q^2 \leq bm-h}} f(bm) v \left( \frac{m}{M} \right) + \mathcal{O}(X^\varepsilon D^{1/2} R^{3/2}). \end{aligned}$$

Similarly as above, the main term in the right-hand side can be rewritten

$$\frac{2}{b} \chi \left( \frac{h}{(h, b)} \right) \bar{\chi} \left( \frac{b}{(h, b)} \right) \int f(\xi) v \left( \frac{\xi}{bM} \right) \sum_{\substack{q \leq \sqrt{\xi-h} \\ (b, q) \mid h, D \mid q/(h, q) \\ (D, (b, h)/(b, q)) = 1}} \frac{(q, b)}{q} d\xi + \mathcal{O}(X^\varepsilon)$$

The  $\chi$ -factors impose the conditions  $b^\circ \mid h$  and  $(D, h/b^\circ) = 1$ . We rewrite the  $q$ -sum as

$$\sum_{\substack{q \leq \sqrt{\xi-h} \\ (b, q) \mid h, D \mid q/(h, q) \\ (D, (b, h)/(b, q)) = 1}} \frac{(b, q)}{q} = \frac{1}{D} \sum_{\substack{q \leq \sqrt{\xi-h}/(Db^\circ) \\ (b^*, q) \mid h}} \frac{(q, b^*)}{q} = \frac{1}{D} \sum_{d \mid b^*} \frac{c_d(h)}{d} H \left( \frac{\sqrt{\xi-h}}{Db^\circ d} \right)$$

whence the claimed expression.  $\square$

**4C. Type  $\tau_1$  estimates.** The following estimate is relevant for convolutions with one smooth component of size  $\gg x^{1/3+\varepsilon}$ . It can be viewed as a generalization of a result of Selberg [1991, page 235] on the equidistribution of  $\tau_2$  in arithmetic progressions.

**Lemma 4.4.** *Let  $\varepsilon > 0$ , let  $C_2 > C_1 > 0$ , let  $v : (0, \infty) \rightarrow \mathbb{R}$  be a smooth and compactly supported function, and let  $\chi \pmod D$  be a Dirichlet character of modulus  $D \geq 1$ . Then we have, for any  $X, M \geq 1$  and  $R \geq D$ , any  $1 \leq bD, |h| \ll X^{1-\varepsilon}$ , and any interval  $I \subset [C_1X, C_2X]$ ,*

$$\sum_{m:bm \in I} \chi(m)v\left(\frac{m}{M}\right)\Delta_h(bm; R) \ll X^\varepsilon(DX^{1/3} + (b, hD^\infty)MX^{-1/2} + D^{1/2}R^3/2). \tag{4-7}$$

The implied constants depend only on the function  $v$  and the constants  $\varepsilon, C_1$  and  $C_2$ .

*Proof.* Note that we can always assume  $bM \asymp X$ , since otherwise the sums in consideration are empty. Let  $f : (0, \infty) \rightarrow [0, \infty)$  be a smooth weight function, which is compactly supported in  $\text{supp } f \subset [C_1X/2, 2C_2X]$ , which has value  $f(\xi) = 1$  for all  $\xi \in I$ , and whose derivatives satisfy

$$f^{(v)}(\xi) \ll \frac{1}{(\Omega X)^v} \quad \text{for } v \geq 0 \quad \text{and} \quad \int |f^{(v)}(\xi)| d\xi \ll \frac{1}{(\Omega X)^{v-1}} \quad \text{for } v \geq 1,$$

for some constant  $\Omega \leq 1$ . We can then encode the condition  $bm \in I$  by using the function  $f(\xi)$  via

$$\sum_{m:bm \in I} \chi(m)v\left(\frac{m}{M}\right)\Delta_h(bm; R) = \sum_m f(bm)v\left(\frac{m}{M}\right)\chi(m)\Delta_h(bm; R) + \mathcal{O}(\Omega X^{1+\varepsilon}b^{-1}), \tag{4-8}$$

so that it suffices to consider the smoothed sum on the right-hand side.

Assume first that  $\chi$  is the trivial character. In [Topacogullari 2018, Section 3] it is shown that

$$\sum_m f(bm)v\left(\frac{m}{M}\right)\tau(bm - h) = M_{f,v}(b, h; M) + \mathcal{O}(X^\varepsilon b^{1/2}\Omega^{-1/2}), \tag{4-9}$$

where the main term  $M_{f,v}(b, h; M)$  is given by (4-3). By Lemma 4.2, we obtain

$$\sum_m f(bm)v\left(\frac{m}{M}\right)\tilde{\tau}_h(bm; R) = M_{f,v}(b, h; M) + \mathcal{O}(X^\varepsilon R^{3/2} + (b, h)b^{-1}X^{1/2+\varepsilon}). \tag{4-10}$$

The estimate (4-7), in the case  $D = 1$  and  $\chi = \mathbf{1}$ , now follows from (4-8) with the choice  $\Omega = bX^{-2/3}$ .

Now assume that  $\chi$  is a primitive character modulo  $D$ , where  $D \leq R$  and  $bD \ll X^{1-\varepsilon}$ . We write

$$\sum_m f(bm)v\left(\frac{m}{M}\right)\chi(m)\tau(bm - h) = \sum_{\substack{a \pmod D \\ (a, D)=1}} \chi(a)\left(\sum_m \tilde{f}(\tilde{b}m)\tilde{v}\left(\frac{m}{\tilde{M}}\right)\tau(\tilde{b}m - \tilde{h})\right),$$

with

$$\tilde{b} := Db, \quad \tilde{M} := M/D, \quad \tilde{h} := h - ab, \quad \tilde{f}(\xi) := f(\xi + ab) \quad \text{and} \quad \tilde{v}(\xi) := v\left(\xi + \frac{a}{M}\right),$$

so that we can use our former result (4-9) to get

$$\sum_m f(bm)v\left(\frac{m}{M}\right)\chi(m)\tau(bm-h) = M_{f,v}^\chi(b,h;M) + \mathcal{O}(X^\varepsilon D^{3/2}b^{1/2}\Omega^{-1/2}),$$

where  $M_{f,v}^\chi(b,h;M)$  is defined in (4-4). By Lemma 4.3, we obtain

$$\begin{aligned} \sum_m f(bm)v\left(\frac{m}{M}\right)\chi(m)\tau(bm-h) \\ = \sum_m f(bm)v\left(\frac{m}{M}\right)\chi(m)\tilde{\tau}_h(bm;R) + O_\varepsilon\left(X^\varepsilon D^{3/2}b^{1/2}\Omega^{-1/2} + X^\varepsilon D^{1/2}R^{3/2} + X^{1/2+\varepsilon}\frac{(h,b^*)}{b^*}\right) \end{aligned}$$

We choose  $\Omega = bDX^{-2/3}$ , and hence get (4-7) also in this case.

The case when  $\chi$  is not necessarily primitive follows at once using Möbius inversion. □

**4D. Type  $\tau_2$  estimates.** The following estimate is a uniform version of the  $\tau_2 - \tau_2$  shifted convolution problem obtained recently by the second author.

**Lemma 4.5.** *Let  $\varepsilon > 0$ , let  $C_2 > C_1 > 0$ , let  $v_1, v_2 : (0, \infty) \rightarrow \mathbb{R}$  be smooth and compactly supported weight functions, and let  $\chi_1$  and  $\chi_2$  be Dirichlet characters mod  $D$ . Then for any  $X, b \geq 1$  and  $R \geq D$ , any  $M_1 \geq M_2 \geq 1$  with  $X^{1/2} \leq M_1M_2$ , any  $h \in \mathbb{Z}$  with  $1 \leq |h|, D \leq X^{1/4}$  and any interval  $I \subset [C_1X/2, C_2X]$ , we have*

$$\begin{aligned} \sum_{\substack{m_1, m_2: \\ bm_1m_2 \in I}} v_1\left(\frac{m_1}{M_1}\right)v_2\left(\frac{m_2}{M_2}\right)\chi_1(m_1)\chi_2(m_2)\Delta_h(bm_1m_2;R) \\ \ll b^\circ D^{5/2}(XM_1M_2)^{1/3+\varepsilon}\left(1 + \left(\frac{|h|M_1M_2}{XD}\right)^{1/4}\right) + X^{-1/2+\varepsilon}R^{3/2}b^\circ(h,b)M_1M_2^2. \end{aligned} \quad (4-11)$$

The implied constant depends only on the constants  $\varepsilon, C_1$  and  $C_2$ , and the functions  $v_1$  and  $v_2$ .

*Proof.* Note that we can make the assumption  $b \asymp \frac{X}{M_1M_2}$ , as otherwise the sum in consideration is empty. Also, as in Lemma 4.4, we can exchange the original sum by its smoothed version,

$$\sum_{m_1, m_2} f(bm_1m_2)v_1\left(\frac{m_1}{M_1}\right)v_2\left(\frac{m_2}{M_2}\right)\chi_1(m_1)\chi_2(m_2)\Delta_h(bm_1m_2;R),$$

with an error of the size of  $\mathcal{O}(\Omega X^{1+\varepsilon}b^{-1})$ .

Let  $\chi_0 := \bar{\chi}_1\chi_2$ . The results of [Topacogullari 2018] cannot be quoted as a black box, however, the computations of [Topacogullari 2017] on which they are based may be adapted with little change. We write

$$\sum_{m_1, m_2} f(bm_1m_2)v_1\left(\frac{m_1}{M_1}\right)v_2\left(\frac{m_2}{M_2}\right)\chi_1(m_1)\chi_2(m_2)\tau(bm_1m_2-h) = \sum_{a \pmod{D}} \chi_0(a)D(a),$$

where  $D(a)$  is the defined as

$$D(a) := \sum_n w_1\left(\frac{r_1n+f_1}{x_1}\right)w_2\left(\frac{r_2n+f_2}{x_2}\right)\tau(r_1n+f_1) \sum_{\substack{n_1, n_2 \\ n_1n_2=r_2n+f_2}} \chi_0(n_2)h_{M_2M_1}(n_1, n_2),$$

with

$$r_1 := bD, \quad r_2 := D, \quad f_1 := ab - h, \quad f_2 := a, \quad x_1 := X, \quad x_2 := \frac{X}{b},$$

and

$$w_1(\xi) := \sqrt{f(X\xi + h)}, \quad w_2(\xi) := \sqrt{f(X\xi)}, \quad h_{M_2M_1}(n_1, n_2) := v_2\left(\frac{n_1}{M_2}\right)v_1\left(\frac{n_2}{M_1}\right).$$

The sum  $D(a)$  is now of the same shape as the sum  $D_{AB}(x_1, x_2)$  defined in [Topacogullari 2017, page 157], with the function  $\tilde{f}(a, b)$  there replaced by  $\chi_0(a)\tilde{f}(a, b)$ . The computations of Section 3 of [loc. cit.] can then be adapted with the following changes. In Section 3.1 of [loc. cit.] the expressions  $\Sigma_{AB}^0$  and  $\Sigma_{AB}^\pm$  have an additional factor  $\chi_0(au_2/u_2^*)$  in the summands. In the sums in the definition of  $R_{AB}^\pm$ , [loc. cit., page 159], the summand has to be multiplied by an additional factor  $\chi_0(c)$ , and the altered relation

$$\Sigma_{AB}^\pm = \sum_{\substack{u_2^* | u_2 \\ r_2^* | r_2}} \chi_0\left(\frac{u_2}{u_2^*}\right) \sum_{\substack{d \\ (d, r_1^* s_2 u_2^*)=1}} \chi_0(d) \frac{R_{AB}^\pm}{d}$$

holds. Consequently, the relationship between  $R_{AB}^\pm(N; \chi)$  and  $K_{AB}^\pm(N; \chi)$  becomes

$$R_{AB}^\pm(N; \chi) = \sum_{N < n \leq 2N} \tau(n) \hat{S}_v(\bar{\chi}; n) K_{AB}^\pm(\chi \chi_0; n).$$

The rest of the argument of [loc. cit.] is adapted with the only change that the Kuznetsov formula is applied with nebentypus  $\chi \chi_0$  instead of  $\chi$ . This has no effect on the error terms, since the bounds in Theorem 2.6 and Lemmas 2.7, 2.8 and 2.9 of [loc. cit.] are uniform with respect to the nebentypus.

By the bound (3.4) of [loc. cit.], with  $b^\circ = (b, D^\infty)$ ,  $r_0 \leftarrow Db^\circ$  and  $h \leftarrow hD$ , we obtain

$$\begin{aligned} D(a) = & \sum_{\substack{m_2 \\ (m_2, D)=1}} \frac{\chi_0(m_2)v_2\left(\frac{m_2}{M_2}\right)}{bDm_2} \sum_{d | bDm_2} \frac{c_d(abm_2\bar{m}_2 - h)}{d} \\ & \times \int (\log(\xi - h) + 2\gamma - 2 \log d) \cdot f(\xi)v_1\left(\frac{\xi}{bM_1m_2}\right) d\xi \\ & + \mathcal{O}\left(b^\circ D^{3/2} X^{1/2+\varepsilon} \left(\frac{1}{\Omega^{1/2}} + \left(\frac{(b, h)X}{Db^2}\right)^\theta \left\{1 + \left(\frac{|h|}{bD}\right)^{1/4}\right\}\right)\right), \end{aligned}$$

where  $\bar{m}_2$  denotes any integer such that  $\bar{m}_2 \cdot m_2 \equiv 1 \pmod{D}$ . We sum over  $a \pmod{D}$ , exchange the  $a$ - and  $m_2$ -sums, and change variables  $a \leftarrow am_2$ . We obtain

$$\begin{aligned} & \sum_{a \pmod{D}} \chi_1(a) D(a) \\ = & \sum_{m_2} v\left(\frac{m_2}{M_2}\right) \chi_2(m_2) M_{f, v_1}^{\chi_1}(m_2b, h; M_1) + \mathcal{O}\left(b^\circ D^{5/2} X^{1/2+\varepsilon} \left(\frac{1}{\Omega^{1/2}} + \left(\frac{(b, h)X}{Db^2}\right)^\theta \left\{1 + \left(\frac{|h|}{bD}\right)^{1/4}\right\}\right)\right), \end{aligned}$$

with  $M_{f, v_1}^{\chi_1}(m_2b, h; M_1)$  defined as in (4-4), for which we can use Lemma 4.3. The bound (4-11) follows after choosing  $\Omega = X^{1/3}(M_1M_2)^{-2/3}$ . □

**4E. Type II estimates.** The following estimate, the first version of which was obtained in [Fouvry 1985], concerns convolutions with one component supported inside  $[x^\varepsilon, x^{1/3-\varepsilon}]$ .

**Lemma 4.6.** *For all  $\eta, A > 0$ , there exist  $\delta, B > 0$  such that the following holds. Whenever  $X, R \geq 1$ ,  $(a, h) \in \mathbb{Z}^2$ , an interval  $I \subset [X/2, X]$ , and two sequences  $(\beta_n), (\gamma_n)$  are given, under the conditions  $1 \leq R, |a|, |h| \leq X^\delta$ , and*

$$|\beta_n| \leq \tau_A(n), \quad |\gamma_n| \leq \tau_A(n), \quad \gamma_n \neq 0 \implies n \in [X^\eta, X^{\frac{1}{3}-\eta}],$$

we have

$$\sum_{n \in I} (\beta * \gamma)(n) \Delta_h(an; R) \ll_{A, \eta} \tau((a, h)) R^{-1/2} X (\log X)^B. \tag{4-12}$$

*Proof.* Recall that  $\Delta_h(an; R) \ll RX^\varepsilon$ . In the left-hand side of (4-12), the contribution of those  $n$  such that  $(n, (ah)^\infty) > X^\delta$  is therefore at most

$$RX^\varepsilon \sum_{\substack{n \ll X \\ (n, (ah)^\infty) > X^\delta}} 1 \ll RX^{1-\delta+\varepsilon}.$$

Next, we have

$$\sum_{\substack{d | (ah)^\infty \\ d \leq X^\delta}} \sum_{\substack{n \in I \\ d | n \\ (n/d, ah) = 1}} (\beta * \gamma)(n) \Delta_h(an; R) = \sum_{\substack{\lambda_1, \lambda_2 | (ah)^\infty \\ \lambda_1 \lambda_2 \leq X^\delta}} \sum_{\substack{mn \in (\lambda_1 \lambda_2)^{-1} I \\ (mn, ah) = 1 \\ (m, \lambda_2) = 1}} \beta_{\lambda_1 m} \gamma_{\lambda_2 n} \Delta_h(a \lambda_1 \lambda_2 mn; R).$$

Finally, we note that there are at most  $\mathcal{O}(X^{1/2+\varepsilon})$  tuples  $(\lambda_1, \lambda_2, m, n)$  with  $\lambda_1 \lambda_2 mn \in I$  for which the expression  $a \lambda_1 \lambda_2 mn - h$  is a perfect square, and

$$\Delta_h(a \lambda_1 \lambda_2 mn; R) = 2 \sum_{\lambda_3 | (h, a \lambda_1 \lambda_2)} \sum_{\substack{q \leq \sqrt{a \lambda_1 \lambda_2 mn - h} / \lambda_3 \\ (q, a \lambda_1 \lambda_2 h / \lambda_3^2) = 1}} \mathfrak{u}_R\left(mn \frac{a \lambda_1 \lambda_2}{\lambda_3} \frac{\bar{h}}{\lambda_3}; q\right) + \mathcal{O}(\mathbf{1}_{a \lambda_1 \lambda_2 mn - h \text{ is a square}}),$$

where the notation  $\mathfrak{u}_R(n; q)$  is defined in formula (5.1) of [Drappeau 2017]. Now, for each  $(\lambda_1, \lambda_2, \lambda_3)$ , the sum

$$S(\lambda_1, \lambda_2, \lambda_3) = \sum_{\substack{q \leq \sqrt{a \lambda_1 \lambda_2 mn - h} / \lambda_3 \\ (q, a \lambda_1 \lambda_2 h / \lambda_3^2) = 1}} \sum_{\substack{mn \in (\lambda_1 \lambda_2)^{-1} I \\ (mn, ah) = 1 \\ (m, \lambda_2) = 1}} \mathfrak{u}_R\left(mn \frac{a \lambda_1 \lambda_2}{\lambda_3} \frac{\bar{h}}{\lambda_3}; q\right)$$

is of the same shape as in formula (5.6) of [Drappeau 2017], with three differences:

- (1) The quantity  $\tau_A(\lambda_1) \tau_A(\lambda_2)$  has to be factored out for the condition (5.4) of [loc. cit.] to hold.
- (2) The sums over  $m$  and  $n$  must be restricted to dyadic intervals, which is done at the cost of an additional factor  $(\log x)^2$ .
- (3) The sums over  $m, n$  and  $q$  are not separated.

The last point can be implemented by a standard argument (see e.g., page 720 of [loc. cit.]), cutting the  $(m, n)$  sums into intervals of type  $[M, (1 + \xi)M] \times [N, (1 + \xi)N]$  with  $\xi \asymp R^{-1/2}$ . Assuming  $\delta$  is small enough in terms of  $\eta$ , we obtain

$$\begin{aligned} S(\lambda_1, \lambda_2, \lambda_3) &\ll \tau_A(\lambda_1)\tau_A(\lambda_2)(\lambda_1\lambda_2)^{-1} X(\log X)^B (\xi + \xi^{-1}R^{-1}) \\ &\ll \tau_A(\lambda_1)\tau_A(\lambda_2)(\lambda_1\lambda_2)^{-1} R^{-1/2} X(\log X)^B. \end{aligned}$$

We sum this over  $(\lambda_1, \lambda_2, \lambda_3)$  satisfying

$$\lambda_1\lambda_2 \mid (ah)^\infty, \quad \lambda_1\lambda_2 \leq x^\delta, \quad \lambda_3 \mid (h, a\lambda_1\lambda_2).$$

Since  $\sum_{\lambda \mid (ah)^\infty} \tau_{2A}(\lambda)\tau(\lambda)\lambda^{-1} \ll_A (\log \log x)^{O_A(1)}$ , we obtain

$$\sum_{n \in I} (\beta * \gamma)(n) \Delta_h(an; R) \ll \tau((a, h)) \{RX^{1-\delta/2} + R^{-1/2} X(\log X)^{B+1}\},$$

which yields our claim by reinterpreting  $\delta$  and  $B$ . □

### 5. The case of rational parameters

Let  $\chi_1, \dots, \chi_T$  be distinct Dirichlet characters mod  $D$ , and the function  $f \in \mathcal{F}_D^{\tau\mathbb{Q}}(A)$  be defined by

$$\sum_{n=1}^\infty \frac{f(n)}{n^s} := \prod_{j=1}^T L(s, \chi_j)^{b_j}, \tag{5-1}$$

with  $b_1, \dots, b_T \in \mathbb{Q}$ , which we write in the form

$$b_j = r_j + \frac{u_j}{v_j} \quad \text{with } r_j \in \mathbb{Z} \text{ and } u_j, v_j \in \mathbb{N} \text{ such that } 0 \leq u_j < v_j.$$

For notational convenience we also define

$$\|r\|_1 := \sum_{1 \leq j \leq T} |r_j|, \quad \|v\|_1 := \sum_{1 \leq j \leq T} v_j.$$

Our goal is to prove estimate (2-4) for the function  $f$  defined in (5-1). In fact, we will prove a result which is slightly more precise in term of uniformity in  $D$  and  $T$ .

**Proposition 5.1.** *Let  $A, D, T \geq 1$  be fixed. Then we have, for  $x \geq 3, I \subset [x/2, x]$  and  $f \in \mathcal{F}_D^{\tau\mathbb{Q}}(A)$  as described above, the following estimate,*

$$|\Sigma_f(I; a, h; R)| \leq C \tau((a, h)) D^{5/2} \frac{x(\log x)^{B+\omega(D)}}{R^{1/2}} \|v\|_1 \quad \text{for } 1 \leq a, |h|, R \leq x^\delta, \tag{5-2}$$

where  $\delta > 0$  is some absolute constant, and where  $B, C > 0$  are constants which depend only on  $A$  and  $T$ .

The rest of this section is now concerned with proving [Proposition 5.1](#).

**5A. Application of the combinatorial identity.** Denote  $\tau_z^\chi(n) := \tau_z(n)\chi(n)$ , so that

$$f(n) = \tau_{b_1}^{\chi_1} * \cdots * \tau_{b_T}^{\chi_T}. \quad (5-3)$$

The expression on the left-hand side of (5-2) now reads

$$\Sigma_f(I; a, h; R) = \sum_{m_1 \cdots m_T \in I} \tau_{b_1}^{\chi_1}(m_1) \cdots \tau_{b_T}^{\chi_T}(m_T) \Delta_h(am_1 \cdots m_T; R). \quad (5-4)$$

By Theorem 3.2 with  $K = 4$  we can write  $\tau_{b_j}^{\chi_j}(m_j)$  as

$$\tau_{b_j}^{\chi_j}(m_j) = \sum_{\ell=1}^4 c_{\ell,j} \sum_{\substack{m_1 \cdots m_{k_{\ell,j}} n_1 \cdots n_{k'_{\ell,j}} = m_j \\ n_1, \dots, n_{k'_{\ell,j}} \leq x^{1/4}}} \chi_j(m_1) \cdots \chi_j(m_{k_{\ell,j}}) \tau_{-1/v_j}^{\chi_j}(n_1) \cdots \tau_{-1/v_j}^{\chi_j}(n_{k'_{\ell,j}}), \quad (5-5)$$

where  $(k_{\ell,j})_{\ell=1}^4$  and  $(k'_{\ell,j})_{\ell=1}^4$  are two sequences of integers satisfying

$$0 \leq k_{\ell,j} \leq |r_j| + 4, \quad 1 \leq k'_{\ell,j} \leq (|r_j| + 4)v_j,$$

and where  $(c_{\ell,j})_{\ell=1}^4$  is a set of complex numbers whose moduli are bounded in terms of  $A$ . We replace each factor  $\tau_{b_j}^{\chi_j}(m_j)$  in (5-4) by its decomposition, and after expanding the resulting expression, we end up with a linear combination (whose coefficients are bounded by  $\mathcal{O}_A(1)$ ) of  $\mathcal{O}_T(1)$  sums of the form

$$\Xi := \sum_{\substack{m_1 \cdots m_k n_1 \cdots n_{k'} \in I \\ n_1, \dots, n_{k'} \leq x^{1/4}}} \sigma_1(m_1) \cdots \sigma_k(m_k) \varrho_1(n_1) \cdots \varrho_{k'}(n_{k'}) \Delta(am_1 \cdots m_k n_1 \cdots n_{k'}; R), \quad (5-6)$$

where each function  $\sigma_i$  is some Dirichlet character mod  $D$ , where each function  $\varrho_i$  is equal to  $\tau_{-1/v_j}^{\chi_j}$  for some  $j$ , and where  $k$  and  $k'$  are integers bounded by

$$0 \leq k \leq 4T + \|r\|_1 \quad \text{and} \quad 1 \leq k' \leq 4\|v\|_1 + \sum_{1 \leq j \leq T} |r_j|v_j.$$

We consider each sum  $\Xi$  separately.

For technical reasons, it will be necessary to use a smooth dyadic decomposition for the variables  $m_1, \dots, m_k$ . Let  $u : (0, \infty) \rightarrow \mathbb{R}$  be a smooth and compactly supported function, which satisfies

$$\text{supp } u \subset \left[\frac{1}{4}, 2\right] \quad \text{and} \quad \sum_{\ell \in \mathbb{Z}} u\left(\frac{\xi}{2^\ell}\right) = 1 \quad \text{for all } \xi \in (0, \infty),$$

and define

$$u_0(\xi) := \sum_{\ell \leq 0} u\left(\frac{\xi}{M_\ell}\right) \quad \text{and} \quad u_\ell(\xi) := u\left(\frac{\xi}{M_\ell}\right) \quad \text{for } \ell > 0,$$

where we have set

$$M_\ell := x^{1/4 + \eta} 2^\ell,$$



with  $0 < \eta < \frac{1}{24}$  an arbitrary, but fixed constant. For a  $k$ -tuple  $\ell = (\ell_1, \dots, \ell_k) \in \mathbb{N}^k$ , we then define

$$\mathfrak{E}_\ell := \sum_{\substack{m_1 \cdots m_k n_1 \cdots n_{k'} \in I \\ n_1, \dots, n_{k'} \leq x^{1/4}}} u_{\ell_1}(m_1)\sigma_1(m_1) \cdots u_{\ell_k}(m_k)\sigma_k(m_k)\varrho_1(n_1) \cdots \varrho_{k'}(n_{k'})\Delta(am_1 \cdots m_k n_1 \cdots n_{k'}; R),$$

so that the sum  $\mathfrak{E}$  can be split as

$$\mathfrak{E} = \sum_{\ell \in \mathbb{N}^k} \mathfrak{E}_\ell.$$

Note that this last sum is in fact finite, since  $\mathfrak{E}_\ell$  becomes empty if the coordinates of  $\ell$  are large enough, namely if  $\ell_1, \dots, \ell_k \gg \log x$ . We will now estimate the sums  $\mathfrak{E}_\ell$  in different ways, depending on the sizes of the supports of the variables  $m_i$ .

**5B. Case I.** First assume that  $\ell$  has at least one coordinate, say  $\ell_1$ , satisfying  $M_{\ell_1} \geq x^{1/3+\eta}$ . Let  $m_0 := m_2 \cdots m_k n_1 \cdots n_{k'}$ . Denoting  $\sigma_1 = \chi_j$  for some  $j$ , we can use [Lemma 4.4](#) with  $X = ax$ ,  $b = am_0$  and  $M = M_{\ell_1}$  to get

$$\sum_{m_1 : m_0 m_1 \in I} u_{\ell_1}(m_1)\sigma_1(m_1)\Delta_h(am_0 m_1; R) \ll_{\varepsilon, A} x^\varepsilon \left( Da^{1/3}x^{1/3} + (am_0, hD^\infty) \frac{M_{\ell_1}}{x^{1/2}} + D^{1/2}R^{3/2} \right).$$

This leads to

$$\mathfrak{E}_\ell \ll_\varepsilon x^\varepsilon (Da^{1/3}x^{1-\eta} + (a, hD^\infty)(\log x)^{\omega(D)}x^{1/2} + x^{2/3-\eta}D^{1/2}R^{3/2}), \tag{5-7}$$

where we have made use of the fact that

$$\sum_{m_0 \leq x/M_{\ell_1}} (m_0, hD^\infty) \leq \sum_{\substack{D^* | D^\infty \\ D^* \leq x}} D^* \sum_{m_0 \leq x/(D^* M_{\ell_1})} (m_0, h) \ll_\varepsilon h^\varepsilon \frac{x}{M_{\ell_1}} \sum_{\substack{D^* | D^\infty \\ D^* \leq x}} 1 \ll_\varepsilon \frac{(\log x)^{\omega(D)}x^{1+\varepsilon}}{M_{\ell_1}}.$$

**5C. Case II.** Next assume that  $\ell$  has at least two nonzero coordinates, say  $\ell_1 \geq \ell_2 \geq 1$ . We can also assume that  $x^{1/4+\eta} \ll M_{\ell_1}$ ,  $M_{\ell_2} \ll x^{1/3+\eta}$ , since the case of larger  $M_{\ell_1}$  and  $M_{\ell_2}$  is already treated above. Let  $m_0 := m_3 \cdots m_k n_1 \cdots n_{k'}$ . We use [Lemma 4.5](#) with  $X = ax$  and  $b = am_0$ , which gives

$$\begin{aligned} \sum_{\substack{m_1, m_2 : \\ m_0 m_1 m_2 \in I}} u_{\ell_1}(m_1)\sigma_1(m_1)u_{\ell_2}(m_2)\sigma_2(m_2)\Delta_h(am_0 m_1 m_2; R) \\ \ll_{\varepsilon, A} (am_0, D^\infty)x^\varepsilon \left( D^{5/2}(axM_{\ell_1}M_{\ell_2})^{1/3} + (h, am_0)R^{3/2} \frac{M_{\ell_1}M_{\ell_2}}{a^{1/2}x^{1/2}} \right), \end{aligned}$$

so that altogether we are led to

$$\mathfrak{E}_\ell \ll_{\varepsilon, A} (a, h)(a, D^\infty)(\log x)^{\omega(D)}x^\varepsilon (D^{5/2}a^{1/3}x^{1-4/3\eta} + (a, h)R^{3/2}x^{1/2}). \tag{5-8}$$

**5D. Case III.** Finally, we need to consider the case, where  $\ell$  has at most one nonzero coordinate, say  $\ell_1$ , for which we have  $M_{\ell_1} \ll x^{\frac{1}{3}+\eta}$ . We split the sum  $\Xi_\ell$  into two parts,

$$\Xi_j =: \Xi_\ell^{(1)} + \Xi_\ell^{(2)},$$

according to whether  $n_1 \cdots n_{k'} > x^\eta$  or  $n_1 \cdots n_{k'} \leq x^\eta$ .

We look first at  $\Xi_\ell^{(1)}$ . We split this sum according to the value of

$$\mu = \min\{1 \leq \mu' \leq k' : n_1 \cdots n_{\mu'} > x^\eta\},$$

and write accordingly

$$\Xi_\ell^{(1)} =: \sum_{\mu=1}^{k'} \Xi_\ell^{(1)}(\mu).$$

After defining

$$\beta_m := \sum_{\substack{m_1 \cdots m_k n_{\mu+1} \cdots n_{k'} = m \\ n_{\mu+1}, \dots, n_{k'} \leq x^{1/4}}} u_{\ell_1}(m_1) \sigma_1(m_1) \cdots u_0(m_k) \sigma_k(m_k) \varrho_{\mu+1}(n_{\mu+1}) \cdots \varrho_{k'}(n_{k'}),$$

and

$$\gamma_n := \sum_{\substack{n_1 \cdots n_\mu = n \\ n_1 \cdots n_{\mu-1} \leq x^\eta, n_1, \dots, n_\mu \leq x^{1/4}}} \varrho_1(n_1) \cdots \varrho_\mu(n_\mu),$$

and renaming  $n \leftarrow n_1 \cdots n_\mu$  and  $m \leftarrow m_1 \cdots m_k n_{\mu+1} \cdots n_{k'}$ , we can write  $\Xi_\ell^{(1)}(\mu)$  as

$$\Xi_\ell^{(1)}(\mu) = \sum_{\substack{m, n: mn \in I \\ x^\eta < n \leq x^{1/4+\eta}}} \beta_m \gamma_n \Delta_h(amn; R).$$

Note that  $\gamma_n = 0$  if  $n > x^{1/4+\eta}$ . Moreover, we can bound the quantities  $\beta_m$  and  $\gamma_n$  by

$$|\beta_m| \leq \tau_{\|r\|_1+8T}(m), \quad |\gamma_n| \leq \tau_{\|r\|_1+4T}(n).$$

Hence we can apply [Lemma 4.6](#) with  $A \leftarrow 2\|r\|_1 + 8T$ , and we see that

$$\Xi_\ell^{(1)}(\mu) \ll \tau((a, h)) R^{-1/2} x (\log x)^{B_1} \quad \text{for } 1 \leq a, |h|, R \leq x^{\delta_1},$$

where  $\delta_1, B_1 > 0$  are certain constants which depend solely on  $\eta$  and  $A$ . Summing over  $\mu$ , we deduce

$$\Xi_\ell^{(1)} \ll_A \tau((a, h)) R^{-1/2} x (\log x)^{B_1} \|v\|_1 \quad \text{for } 1 \leq a, |h|, R \leq x^{\delta_1}. \quad (5-9)$$

The other sum  $\Xi_\ell^{(2)}$  can be estimated similarly — the role of the variables  $n_1, \dots, n_{k'}$  is now played by the variables  $m_2, \dots, m_k$ . Eventually, we get

$$\Xi_\ell^{(2)} \ll_A \tau((a, h)) R^{-1/2} x (\log x)^{B_2} \quad \text{for } 1 \leq a, |h|, R \leq x^{\delta_2}, \quad (5-10)$$

where  $\delta_2, B_2 > 0$  are certain constants which again depend solely on  $\eta$  and  $A$ .

**5E. Conclusion.** Grouping the different bounds (5-7)–(5-10), setting  $B := \max(B_1, B_2)$  and choosing  $\delta > 0$  small enough, we get

$$\mathfrak{E} \ll \tau((a, h)) D^{5/2} R^{-1/2} x(\log x)^{B+\omega(D)} \|v\|_1 \quad \text{for } 1 \leq a, |h|, R \leq x^\delta,$$

with the implicit constant depending only on  $A$  and  $T$ . This finally proves Proposition 5.1.

### 6. Interpolation to complex parameters

Let  $r_1, \dots, r_{\varphi(D)}$  be the residues mod  $D$  which are relatively prime to  $D$ . Any  $f \in \mathcal{F}_D^\omega(A)$  is given by

$$\sum_{n=1}^\infty \frac{f(n)}{n^s} = \prod_{j=1}^{\varphi(D)} \prod_{p \equiv r_j \pmod{D}} \left(1 + \frac{z_j}{p^s - 1}\right), \tag{6-1}$$

for  $\mathbf{z} = (z_1, \dots, z_{\varphi(D)}) \in \mathbb{C}^{\varphi(D)}$ , with  $|z_j| \leq A$ . After setting

$$\omega_r(n) := \#\{p \text{ prime} : p \mid n, p \equiv r \pmod{D}\}, \tag{6-2}$$

we can also write

$$f(n) = \sum_{n_1 \cdots n_{\varphi(D)} = n} \prod_{j=1}^{\varphi(D)} z_j^{\omega_{r_j}(n_j)}.$$

Our aim here is to show that the bound (2-4) holds for  $\Sigma_f(I; a, h; R)$ , for all  $f \in \mathcal{F}_D^\omega(A)$ . By Lemma 2.5 this will imply Proposition 2.1.

Let  $\chi_1, \dots, \chi_{\varphi(D)}$  be the Dirichlet characters mod  $D$ , let  $Q$  be the unitary matrix

$$Q := \frac{1}{\sqrt{\varphi(D)}} \begin{pmatrix} \chi_1(r_1) & \chi_2(r_1) & \cdots & \chi_{\varphi(D)}(r_1) \\ \chi_1(r_2) & \chi_2(r_2) & \cdots & \chi_{\varphi(D)}(r_2) \\ \vdots & \vdots & \ddots & \vdots \\ \chi_1(r_{\varphi(D)}) & \chi_2(r_{\varphi(D)}) & \cdots & \chi_{\varphi(D)}(r_{\varphi(D)}) \end{pmatrix},$$

and let  $M_Q : \mathbb{C}^{\varphi(D)} \rightarrow \mathbb{C}^{\varphi(D)}$  be the bijective linear map associated to  $Q$ .

Let  $K \geq 1$ . We define  $\mathcal{F}_D^\omega(A, K)$  to be the set of functions  $f \in \mathcal{F}_D^\omega(A)$  of the same form as in (6-1), but with the additional property that the parameters  $\mathbf{z}$  are given by

$$\mathbf{z} = M_Q(\mathbf{b})$$

for a tuple of rational numbers  $\mathbf{b} = (b_1, \dots, b_{\varphi(D)}) \in \mathbb{Q}^{\varphi(D)}$  satisfying

$$|b_j| \leq A \quad \text{and} \quad b_j = \frac{u_j}{v_j} \quad \text{with } u_j, v_j \in \mathbb{Z} \text{ and } |v_j| \leq K,$$

for all  $j = 1, \dots, \varphi(D)$ . By Proposition 5.1, Lemma 2.2 and Lemma 2.3, we deduce that the bound (2-4) holds for all  $f \in \mathcal{F}_D^\omega(A, K)$  in the following form.

**Proposition 6.1.** *Let  $A, D \geq 1$  be fixed. For  $K \geq 1, x \geq 3, I \subset [x/2, x]$  and  $f \in \mathcal{F}_D^\omega(A, K)$ , we have*

$$|\Sigma_f(I; a, h; R)| \leq CK\tau((a, h)) \frac{x(\log x)^B}{R^{1/2}} \quad \text{for } 1 \leq a, |h|, R \leq x^\delta, \quad (6-3)$$

where  $\delta > 0$  is some absolute constant, and where  $B, C > 0$  are constants which depend only on  $A$  and  $D$ .

Our goal is to interpolate this result to all functions in  $\mathcal{F}_D^\omega(A)$ . Let  $f \in \mathcal{F}_D^\omega(A)$  be fixed, with  $\mathbf{z}$  as in (6-1). For  $L \in [1, \infty]$ , we define two polynomials in the variables  $\mathbf{Z} = (Z_1, \dots, Z_{\varphi(D)})$  as follows,

$$P_L(\mathbf{Z}) := \sum_{\substack{n \in I \\ \forall j, \omega_{r_j}(n) \leq L}} \sum_{n_1 \cdots n_{\varphi(D)} = n} \prod_{j=1}^{\varphi(D)} Z_j^{\omega_{r_j}(n_j)} \Delta_h(an; R), \quad \tilde{P}_L(\mathbf{Z}) := P_L(M_Q(\mathbf{Z})).$$

By definition, both these polynomials have degree at most  $L$  in each variable. Furthermore, let

$$\mathbf{b} := M_Q^{-1}(\mathbf{Z}),$$

and note that  $\|\mathbf{b}\|_\infty \leq D^{1/2}A$ . Using this notation, we can now write the sum  $\Sigma_f(I; a, h; R)$  simply as

$$\sum_{n \in I} f(n) \Delta_h(an; R) = \tilde{P}_\infty(\mathbf{b}).$$

In order to have better control over the degree of  $\tilde{P}_\infty(\mathbf{Z})$ , we cut off all the terms of degree larger than some fixed real number  $L \geq 1$ . For a tuple  $\zeta = (\zeta_1, \dots, \zeta_{\varphi(D)})$  satisfying  $|\zeta_j| \leq AD^{1/2}$  and any real number  $E \geq 1$ , this leads to an error term of the following form,

$$\begin{aligned} |\tilde{P}_\infty(\zeta) - \tilde{P}_L(\zeta)| &\leq \sum_{\substack{n \in I \\ \omega(n) > L}} \tau_D(n) (AD)^{\omega(n)} |\Delta_h(an; R)| \\ &\leq E^{-L} \sum_{n \leq x} \tau_D(n) (ADE)^{\omega(n)} |\Delta_h(an; R)| \\ &\leq E^{-L} \left( \sum_{n \leq x} \tau_{AD^2E}(n)^2 \right)^{1/2} \left( \sum_{n \leq x} |\Delta_h(an; R)|^2 \right)^{1/2}. \end{aligned}$$

The different factors can be estimated via [Tenenbaum 1995, Theorem II.6.1], and Lemma 4.1, and we get

$$\begin{aligned} |\tilde{P}_\infty(\zeta) - \tilde{P}_L(\zeta)| &\ll E^{-L} (x(\log x)^{(ADE)^4 - 1})^{1/2} (x(\log x)^4 \tau((a, h))^2)^{1/2} \\ &\ll E^{-L} \tau((a, h)) x(\log x)^{(ADE)^4/2 + 2}, \end{aligned} \quad (6-4)$$

where the implicit constants depend at most on  $A, E$  and  $D$ .

Next, we set

$$\beta_\ell := \frac{2(\ell + 1) \lfloor AD^{1/2} \rfloor}{L + 1} - \lfloor AD^{1/2} \rfloor \quad \text{for } \ell = 0, \dots, L.$$

Obviously, all these numbers are bounded by  $|\beta_\ell| \leq AD^{1/2}$ , and are rational numbers with denominators not larger than  $L + 1$ . Furthermore, we have the bound

$$|\beta_{\ell_1} - \beta_{\ell_2}| \geq \frac{AD^{1/2}}{2L} |\ell_1 - \ell_2| \quad \text{for } \ell_1 \neq \ell_2.$$

For any tuple  $\ell = (\ell_1, \dots, \ell_{\varphi(D)}) \in \{0, \dots, L\}^{\varphi(D)}$ , denote  $\beta_\ell = (\beta_{\ell_1}, \dots, \beta_{\ell_{\varphi(D)}})$ . The value  $\tilde{P}_\infty(\beta_\ell)$  can be interpreted as an instance of the sum  $\Sigma_{\tilde{f}}(I; a, h; R)$  for an appropriate function  $\tilde{f} \in \mathcal{F}_D^\omega(AD, L + 1)$ ,

$$\tilde{P}_\infty(\beta_\ell) = \Sigma_{\tilde{f}}(I; a, h; R).$$

Hence, by Proposition 6.1 and the estimate in (6-4) we can deduce

$$\tilde{P}_L(\beta_\ell) \ll_{A,D} \tau((a, h))x(\log x)^{(ADE)^4/2+B} \left( \frac{L}{R^{1/2}} + \frac{1}{E^L} \right), \tag{6-5}$$

uniformly for  $1 \leq a, |h|, R \leq x^\delta$ .

By Lagrange interpolation, we bring  $\tilde{P}_L(\mathbf{b})$  into the following shape,

$$\tilde{P}_L(\mathbf{b}) = \sum_{\ell \in \{0, \dots, L\}^{\varphi(D)}} \tilde{P}_L(\beta_\ell) \prod_{j=1}^{\varphi(D)} \prod_{\substack{0 \leq i \leq L \\ i \neq \ell_j}} \frac{b_j - \beta_i}{\beta_{\ell_j} - \beta_i},$$

which is allowed since the Vandermonde determinant associated to  $(\beta_\ell)$  does not vanish. We can now estimate  $\tilde{P}_L(\mathbf{b})$  via the already known bound (6-5) for the expressions  $\tilde{P}_L(\beta_\ell)$ . Namely, we have

$$\begin{aligned} |\tilde{P}_L(\mathbf{b})| &\leq \sum_{\ell \in \{0, \dots, L\}^{\varphi(D)}} |\tilde{P}_L(\beta_\ell)| \prod_{j=1}^{\varphi(D)} \prod_{\substack{0 \leq i \leq L \\ i \neq \ell_j}} \frac{|b_j - \beta_i|}{|\beta_{\ell_j} - \beta_i|} \\ &\ll \tau((a, h))x(\log x)^{(ADE)^4/2+B} \left( \frac{L}{R^{1/2}} + \frac{1}{E^L} \right) (4L)^{L\varphi(D)} \sum_{\ell \in \{0, \dots, L\}^{\varphi(D)}} \prod_{j=1}^{\varphi(D)} \prod_{\substack{0 \leq i \leq L \\ i \neq \ell_j}} \frac{1}{|\ell_j - i|} \\ &\ll \tau((a, h))x(\log x)^{(ADE)^4/2+B} \left( \frac{L}{R^{1/2}} + \frac{1}{E^L} \right) \frac{(8L)^{L\varphi(D)}}{(L!)^{\varphi(D)}}, \end{aligned}$$

which after using Stirling’s approximation for the gamma function simplifies to

$$|\tilde{P}_L(\mathbf{b})| \ll \tau((a, h))x(\log x)^{(ADE)^4/2+B} \left( \frac{1}{R^{1/2}} + \frac{1}{E^L} \right) (4e)^{2DL},$$

with the implicit constant depending at most on  $A, E$  and  $D$ .

After adding all the terms we had cut off earlier, we are finally led to

$$\Sigma_f(I; a, h; R) \ll_{A,D,E} \tau((a, h))x(\log x)^{(ADE)^4/2+B} \left( \frac{1}{R^{1/2}} + \frac{1}{E^L} \right) (4e)^{2DL}.$$

With the choices

$$L := \frac{\log R}{12D \log(4e)} \quad \text{and} \quad E := (4e)^{6D},$$

and after reinterpreting the constant  $B$ , we get

$$\Sigma_f(I; a, h; R) \ll_{A,D} \tau((a, h)) \frac{x(\log x)^B}{R^{1/3}} \quad \text{for } 1 \leq a, |h|, R \leq x^\delta,$$

which is exactly the statement we wanted to prove.

### 7. Proof of Theorem 1.2 using Linnik's identity

We now sketch how Theorem 1.2 can alternatively be proven using Theorem 3.3. The details of the computations being very similar, we will restrict to discussing the main differences in the arguments.

As mentioned above, it is enough to consider the case  $f \in \mathcal{F}_D^t(A)$ , or in other words we can assume that  $f = \tau_{b_1}^{\chi_1} * \dots * \tau_{b_T}^{\chi_T}$ , where  $\chi_1, \dots, \chi_T$  are distinct Dirichlet characters mod  $D$ , and where  $b_1, \dots, b_T$  are complex numbers whose moduli are bounded by  $A$ . The sum in consideration is then given by

$$\Sigma_f(I; a, h; R) = \sum_{m_1 \dots m_T \in I} \tau_{b_1}^{\chi_1}(m_1) \dots \tau_{b_T}^{\chi_T}(m_T) \Delta_h(am_1 \dots m_T; R).$$

Here we replace each  $\tau_{b_j}^{\chi_j}(m_j)$  by its decomposition as given in Theorem 3.3 with  $K = 4$ , and after expanding the resulting expression, we end up with a linear combination of sums of the form

$$\Xi := \sum_{\substack{m_1 \dots m_k n_1 \dots n_T \in I \\ P^+(n_1 \dots n_T) \leq x^{1/4}}} \sigma_1(m_1) \dots \sigma_k(m_k) \rho_1(n_1) \dots \rho_T(n_T) \Delta(am_1 \dots m_k n_1 \dots n_T; R),$$

where each function  $\sigma_j$  is some Dirichlet character mod  $D$ , where each function  $\rho_j$  is equal to  $\tau_{b_j-\ell}^{\chi_j}$  for some  $j$  and  $\ell \in [0, 3]$ , and where  $k \leq 3T$ . We consider each sum  $\Xi$  separately.

To each factor  $\rho_j$  in the sum  $\Xi$  we apply Lemma 3.4 with  $y = x^{1/4}$  and  $w = x^\eta$  for some arbitrary, but fixed  $\eta \in (0, \frac{1}{24})$ . By compactity, it follows that for each  $j = 1, \dots, T$  there exist arithmetic functions  $\alpha_j$  and  $\beta_j$ , such that the sum  $\Xi$  can be written as

$$\Xi = \sum_{j=1}^T \Xi_j^{(1)} + \sum_{j=1}^T \Xi_j^{(2)} + \Xi^{(3)},$$

with

$$\Xi_j^{(1)} := \sum_{\substack{mn_1 \dots n_T \in I \\ P^+(n_1 \dots n_T) \leq x^{1/4} \\ n_1, \dots, n_{j-1} \leq x^\eta, n_j > x^{1/\eta} \\ \exists p^k \parallel n_j, p^k > x^{1/4}}} (\sigma_1 * \dots * \sigma_k)(m) \rho_1(n_1) \dots \rho_T(n_T) \Delta(amn_1 \dots n_T; R),$$

$$\begin{aligned} \mathfrak{E}_j^{(2)} &:= \sum_{\substack{mn_1 \cdots n_{j_1} n'_j n''_j n_{j+1} \cdots n_T \in I \\ P^+(n_1 \cdots n_{j_1} n'_j n''_j n_{j+1} \cdots n_T) \leq x^{1/4} \\ n_1, \dots, n_{j-1} \leq x^\eta, x^\eta < n'_j \leq x^{1/4+\eta}}} (\sigma_1 * \cdots * \sigma_k)(m) \left( \prod_{\substack{1 \leq k \leq T \\ k \neq j}} \rho_k(n_k) \right) \alpha_j(n'_j) \beta_j(n''_j) \Delta(amn_1 \cdots n_T; R), \\ \mathfrak{E}_j^{(3)} &:= \sum_{\substack{mn_1 \cdots n_T \in I \\ n_1, \dots, n_T \leq x^\eta}} (\sigma_1 * \cdots * \sigma_k)(m) \rho_1(n_1) \cdots \rho_T(n_T) \Delta(amn_1 \cdots n_T; R). \end{aligned}$$

The sums  $\mathfrak{E}_j^{(1)}$  can be bound trivially. Indeed, we note that if a prime power  $p^k > y$  divides  $n$ , then since  $P^+(n_j) \leq y$  we must have  $k \geq 2$ . Hence

$$\begin{aligned} \mathfrak{E}_j^{(1)} &\leq \sum_{\substack{n \in I \\ \exists p^k \mid n: p^k > x^{1/4}, k \geq 2}} \tau_{(A+6)T}(n) |\Delta(an; R)| \\ &\ll_\varepsilon \left( \sum_{n \in I} |\Delta(an; R)|^2 \right)^{1/2} \left( \sum_{\substack{n \in I \\ \exists p^k \mid n: p^k > x^{1/4}, k \geq 2}} 1 \right)^{1/2} \\ &\ll_{A,T} x^{1-1/17} \tau((a, h)), \end{aligned}$$

which is an acceptable error term.

Concerning the sums  $\mathfrak{E}_j^{(2)}$ , we can bound them following the arguments of Case III, Section 5D, since we have a variable localized in  $[x^\eta, x^{1/4+\eta}]$ , and since  $\frac{1}{4} + \eta < \frac{1}{3}$ . The remaining sum  $\mathfrak{E}^{(3)}$ , which is analogous to (5-6), can be estimated for all sufficiently small  $\eta > 0$  by the arguments of Sections 5B, 5C and 5D, according to the size of the involved variables. As a result, we get for these sums the estimate

$$\mathfrak{E}_j^{(2)}, \mathfrak{E}^{(3)} \ll_{A,T} \tau((a, h)) \frac{x(\log x)^{\mathcal{O}(1)}}{R^{1/2}}.$$

Together with the bound for  $\mathfrak{E}_j^{(1)}$ , this eventually proves Theorem 1.2.

### 8. Proof of Theorems 1.2, 1.3, 1.4 and 1.6

In this section we want to deduce Theorem 1.2 from Proposition 2.1, and afterwards apply this result to the problems mentioned in the introduction. Before doing so, we first need to prove an auxiliary result, which is concerned with bounds on average for functions in  $\mathcal{F}_D(A)$  twisted by a Dirichlet character.

**Lemma 8.1.** *Let  $f \in \mathcal{F}_D(A)$  and let  $B \geq 1$ . Then there exists a constant  $c > 0$ , such that, for all Dirichlet characters  $\chi \bmod q$  satisfying  $\text{cond}(\chi) \nmid D$  and  $q \leq (\log x)^B$ , we have*

$$\sum_{n \leq x} \chi(n) f(n) \ll x e^{-c\sqrt{\log x}}. \tag{8-1}$$

Both the constant  $c$  and the implicit constant depend at most on  $A, B$  and  $D$ .

*Proof.* Let  $F_\chi(s)$  be the Dirichlet series associated to the function  $\chi(n)f(n)$ . By [Lemma 2.4](#) we know that  $F_\chi(s)$  can be written as

$$F_\chi(s) = H_\chi(s) \prod_{\psi \bmod D} L(s, \chi\psi)^{b_\psi} \quad \text{for } \Re(s) > 1,$$

where  $H_\chi(s)$  is a holomorphic function in  $\Re(s) \geq \frac{1}{2} + \varepsilon$ , bounded in terms of  $A, D$  only.

Due to the assumption  $\text{cond}(\chi) \nmid D$  we know that none of the characters  $\chi\psi$  is principal, which means that none of the  $L$ -functions  $L(s, \chi\psi)$  has a pole at  $s = 1$ . It follows from Siegel's theorem that for any  $\delta > 0$  there exists a constant  $c(\delta)$  such that all  $L(s, \chi\psi)$  are zero-free in the region defined by the condition  $\Re(s) > 1 - \gamma(\Im(s))$ , where

$$\gamma(t) := \min \left\{ \frac{c(\delta)}{\log(qD(|t| + 2))}, \frac{c(\delta)}{(qD)^\delta} \right\}. \quad (8-2)$$

Using this zero-free region, the bound (8-1) follows using a standard contour integration argument; see e.g., [\[Montgomery and Vaughan 2007, Section 11.3\]](#).  $\square$

We now proceed to prove [Theorem 1.2](#). We set  $R = (\log x)^L$  where  $L \geq 1$  is some constant which depends only on  $A, B$  and  $D$ , and which we will determine at the very end. Note that in any case we can assume  $x$  to be large enough so that  $D \leq R$  is satisfied.

We start by splitting the sum  $D_f(x; a, h)$  into two parts as follows,

$$D_f(x; a, h) = D_f(\sqrt{x}; a, h) + \sum_{\sqrt{x} < n \leq x} f(n)\tau(an - h).$$

While the first sum can be estimated by trivial means, we can use [Proposition 2.1](#) to evaluate the second (after first dividing the range of summation into dyadic intervals). This eventually shows that there exists an absolute constant  $\delta > 0$ , and a constant  $B$  depending only on  $A$  and  $D$ , such that, for all  $1 \leq a, |h| \leq x^\delta$ ,

$$D_f(x; a, h) = \tilde{M}_f(x; a, h) + \mathcal{O}\left(\tau((a, h)) \frac{x(\log x)^B}{R^{1/3}}\right),$$

with

$$\tilde{M}_f(x; a, h) := \sum_{|h|/a < n \leq x} f(n)\tilde{\tau}_h(an; R).$$

It remains to evaluate this last sum.

After expanding  $\tilde{\tau}_h(an; R)$ , it can be written as

$$\tilde{M}_f(x; a, h) = 2 \sum_{q \leq \sqrt{ax}} \frac{1}{\varphi\left(\frac{q}{(h, q)}\right)} \sum_{\substack{\chi \bmod q/(h, q) \\ \text{cond } \chi \leq R}} \bar{\chi}\left(\frac{h}{(h, q)}\right) \sum_{\substack{q^2/a \leq n \leq x \\ (an, q) = (h, q)}} f(n)\chi\left(\frac{an}{(h, q)}\right) + \mathcal{O}(x^{\delta+\varepsilon}).$$

We now split the remaining sum into two parts, denoted by  $\tilde{M}_f^{(1)}(x; a, h)$  and  $\tilde{M}_f^{(2)}(x; a, h)$ , depending on whether  $\text{cond}(\chi) \mid D$  or not. A simple reordering of the sums shows that the first part is equal to



$M_f(x; a, h)$  as given in [Theorem 1.2](#). The second part can be written as

$$M_f^{(2)}(x; a, h) = 2 \sum_{t|(a,h)} \sum_{\substack{u|h/t \\ (u,a/t)=1}} \sum_{q \leq \sqrt{ax}/(tu)} \frac{1}{\varphi(q)} \sum_{\substack{\chi \bmod q \\ \text{cond } \chi \leq R \\ \text{cond}(\chi) \nmid D}} \bar{\chi}\left(\frac{h}{tu}\right) \chi\left(\frac{a}{t}\right) (S_{f,\chi}(x, u) - S_{f,\chi}\left(\frac{tu^2q^2}{a}, u\right)),$$

with  $S_{f,\chi}(x, u)$  given by

$$S_{f,\chi}(x, u) := \sum_{n \leq x/u} f(un) \chi(n).$$

This last sum can be estimated via [Lemma 8.1](#), namely we have

$$\begin{aligned} S_{f,\chi}(x, u) &= \sum_{\substack{u^*|u^\infty \\ u^* \leq \sqrt{x}}} f(uu^*) \chi(u^*) \sum_{\substack{n \leq x/(uu^*) \\ (n,u)=1}} f(n) \chi(n) + \mathcal{O}(x^{1/2+\varepsilon}) \\ &\ll x e^{-c\sqrt{\log x}} \sum_{u^* \leq \sqrt{x}} \frac{\tau_A(uu^*)}{uu^*} + x^{1/2+\varepsilon} \\ &\ll \frac{\tau_A(u)}{u} x (\log x)^A e^{-c\sqrt{\log x}}, \end{aligned}$$

for some constant  $c > 0$  depending on  $A, D$  and  $L$ . Hence

$$M_f^{(2)}(x; a, h) \ll \tau((a, h)) R x (\log x)^{A+2} e^{-c\sqrt{\log x}} \ll \tau((a, h)) x e^{-(c/2)\sqrt{\log x}}.$$

Eventually, we get

$$D_f(x; a, h) = M_f(x; a, h) + \mathcal{O}\left(\tau((a, h)) x \left(\frac{(\log x)^B}{R^{1/3}} + e^{-(c/2)\sqrt{\log x}}\right)\right),$$

and [Theorem 1.2](#) follows with the choice  $L = 3N + 3B$ .

**8A. Proof of Theorems 1.3, 1.4 and 1.6.** The applications mentioned in the introduction are essentially all immediate corollaries of [Theorem 1.2](#), except for the fact that it remains to evaluate the main terms. This is a rather tedious task, but can be done using standard techniques from analytic number theory, in particular the Selberg–Delange method, which is for example described in detail in [[Tenenbaum 1995](#), Chapter II.5]. In order to not further lengthen this article, we only want to indicate very briefly the main steps of the procedure.

In the case of [Theorem 1.3](#), the main term takes the form

$$M_{\tau_z}(x; 1, h) = 2 \sum_{q \leq \sqrt{x}} \frac{1}{\varphi\left(\frac{q}{(h,q)}\right)} \sum_{\substack{q^2 \leq n \leq x \\ (n,q)=(h,q)}} \tau_z(n),$$

which after a few simple transformations can be written as

$$M_{\tau_z}(x; 1, h) = \sum_{\substack{u|h, v|u^\infty \\ v \leq \sqrt{x}}} \tau_z(uv) \sum_{\substack{q \leq \sqrt{x}/u \\ (q, vh/u)=1}} \frac{D(x; uq, uv) - D(u^2q^2; uq, uv)}{\varphi(q)} + \mathcal{O}(x^{1/2+\varepsilon}), \tag{8-3}$$

where

$$D(y; r, t) := \sum_{\substack{n \leq y/t \\ (n,r)=1}} \tau_z(n).$$

This sum has been studied in detail in [Tenenbaum 1995, Chapter II.5]. In particular, following the proof of [loc. cit., Theorem II.5.2], we see that there exist complex numbers  $\mu_\ell^z(r, t)$  such that

$$D(y; r, t) = \frac{1}{2\pi i} \sum_{\ell=0}^L \frac{\mu_\ell^z(r, t)}{\Gamma(z-\ell)} \frac{y(\log y)^z}{(\log y)^{\ell+1}} + \mathcal{O}\left(\frac{(\log t)^{L+1}}{t} \frac{y(\log y)^z}{(\log y)^{L+2-\varepsilon}}\right),$$

where

$$\mu_\ell^z(r, t) := \Delta_s^\ell \left( \frac{\psi_s^z(r)}{t^s} \frac{(s-1)^z \zeta(s)^z}{s} \right) \quad \text{with} \quad \psi_s^z(r) := \prod_{p|r} \left( 1 - \frac{1}{p^s} \right)^z,$$

and where the differential operator  $\Delta_s^\ell$  is defined as

$$\Delta_s^\ell := \frac{1}{\ell!} \frac{\partial^\ell}{\partial s^\ell} \Big|_{s=1}.$$

By Leibniz's rule and the Taylor expansion of  $(s-1)\zeta(s)$  at 1, it remains to evaluate the sums

$$\sum_{\substack{q \leq \sqrt{x}/u \\ (q, vh/u)=1}} \frac{\Delta_s^\ell \psi_s^z(uq)}{\varphi(q)} \quad \text{and} \quad \sum_{\substack{q \leq \sqrt{x}/u \\ (q, vh/u)=1}} \frac{\Delta_s^\ell \psi_s^z(uq)}{\varphi(q)} \frac{q^2 (2 \log(uq))^z}{(2 \log(uq))^{\ell-1}}.$$

For the first sum this is a standard exercise in using counter integration, the result being

$$\sum_{\substack{q \leq \sqrt{x}/u \\ (q, vh/u)=1}} \frac{\Delta_s^\ell \psi_s^z(uq)}{\varphi(q)} = \Delta_s^\ell \operatorname{Res}_{w=0} \left( C_{s,w}^z \frac{\psi_s^z(u) \rho_w(\frac{vh}{u})}{\gamma_{s,w}^z(h)} \frac{x^{w/2}}{u^w} \frac{\zeta(w+1)}{w} \right) + \mathcal{O}\left(\frac{1}{x^{2/3-\varepsilon}}\right),$$

with

$$C_{s,w}^z := \prod_p \left( 1 + \frac{1}{(p-1)p^{w+1}} + \frac{\psi_s^z(p) - 1}{(p-1)p^w} \right),$$

and

$$\gamma_{s,w}^z(n) := \prod_{p|n} \left( 1 + \frac{p(\psi_s^z(p) - 1)}{p^{w+2} - p^{w+1} + 1} \right) \quad \text{and} \quad \rho_w(n) := \prod_{p|n} \left( 1 - \frac{p}{p^{w+2} - p^{w+1} + 1} \right).$$

An asymptotic formula for the second sum now follows via partial summation. After putting the resulting formulae back in (8-3) and completing the sum over  $v$ , this eventually leads to the main term described in

**Theorem 1.3.** In particular, the first coefficient is given by

$$\lambda_{h,0}(z) = \frac{1}{\Gamma(z)} \prod_{(p,h)=1} \left( 1 + \frac{(1-1/p)^{z-1} - 1}{p} \right) \cdot \prod_{p^\ell \parallel h} \left( 1 - \frac{1}{p} + \left( 1 - \frac{1}{p} \right)^{z+1} \sum_{j=1}^{\ell-1} \frac{(\ell-j)\tau_z(p^j)}{p^j} + \left( 1 - \frac{1}{p} \right)^{z-1} \frac{\tau_z(p^\ell)}{p^{\ell+1}} \right). \tag{8-4}$$

For [Corollary 1.5](#), we have from [[Narkiewicz 2004](#), Proposition 8.4, Theorem 8.6] that the characteristic function  $n \mapsto b_K(n)$  of the set  $\mathcal{N}_K$  is multiplicative with  $b(p) = 1$  if and only if  $\sum_{\chi \in X(K)} \chi(p) > 0$ , where  $X(K)$  is a subgroup of the Dirichlet characters modulo the discriminant  $D = \text{Disc}(K)$  and  $p \nmid D$ . The subgroup of residue classes  $a \pmod D$  such that  $\sum_{\chi \in X(K)} \chi(a) > 0$ , corresponding to the subgroup  $H$  in [[Narkiewicz 2004](#), Theorem 8.2], has density  $1/[K : \mathbb{Q}]$  inside  $(\mathbb{Z}/D\mathbb{Z})^\times$ . Thus we have a factorization

$$\sum_{n \geq 1} \frac{b_K(n)}{n^s} = \zeta(s)^{1/[K:\mathbb{Q}]} H(s)$$

where  $H$  is holomorphic and bounded in the strip  $\Re(s) \geq \frac{2}{3}$ . The rest of the argument follows the path described above. We leave the details to the reader.

In the case  $K = \mathbb{Q}(i)$ , the first coefficient is given by  $\beta_{h,0} = B_0 B(h)$ , where

$$B_0 := \frac{1}{\sqrt{2}} \prod_{p \equiv 3 \pmod 4} \left( 1 - \frac{1}{p^2} \right)^{-1/2}, \tag{8-5}$$

and

$$B(h) := \left( 1 + \frac{\chi_4(h^*)}{4h^\circ} \right) \prod_{\substack{p^\ell \parallel h \\ p \equiv 3 \pmod 4}} \left( 1 - \frac{1}{p+1} + \frac{(-1)^\ell}{p^\ell(p+1)} \right) \prod_{p \equiv 3 \pmod 4} \left( 1 + \frac{1}{p^2} \right),$$

with  $h^\circ := (h, 2^\infty)$ ,  $h^* := \frac{h}{h^\circ}$  and  $\chi_4$  the nonprincipal character mod 4.

Finally, the proof of [Theorem 1.6](#) rests upon the fact that

$$\sum_{\substack{|h| < n \leq x \\ \omega(n)=k}} \tau(n-h) = \frac{1}{k!} \frac{\partial^k}{\partial z^k} \mathfrak{E}_{x,h}(0) \quad \text{with} \quad \mathfrak{E}_{x,h}(z) := \sum_{|h| < n \leq x} z^{\omega(n)} \tau(n-h).$$

Since the function  $n \mapsto z^{\omega(n)}$  is an element of  $\mathcal{F}_1(A)$  for  $|z| \leq A$ , [Theorem 1.2](#) can again be applied in this case. After evaluating the arising main term in the same manner as described above, we see that there exist functions  $\gamma_{h,\ell}(z)$ , which are holomorphic in a neighborhood of  $z$ , such that

$$\mathfrak{E}_{x,h}(z) = x(\log x)^z \sum_{\ell=0}^L \frac{\gamma_{h,\ell}(z)}{(\log x)^\ell} + \mathcal{O}\left( \frac{x(\log x)^{\Re(z)}}{(\log x)^{L+1-\varepsilon}} \right).$$

At this point [Theorem 1.6](#) essentially follows by taking derivatives with respect to  $z$  on both sides. The procedure is however not completely straightforward, since we also need to have control over the error

term on the right hand side. In our case we can simply cite [Tenenbaum 1995, Theorem II.6.3], where a result of this type is proven in very large generality.

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# ONE-LEVEL DENSITY ESTIMATES FOR DIRICHLET $L$ -FUNCTIONS WITH EXTENDED SUPPORT

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ABSTRACT. We estimate the 1-level density of low-lying zeros of  $L(s, \chi)$  with  $\chi$  ranging over primitive Dirichlet characters of conductor  $\in [Q/2, Q]$  and for test functions whose Fourier transform is supported in  $(-2 - \frac{50}{1093}, 2 + \frac{50}{1093})$ . Previously any extension of the support past the range  $(-2, 2)$  was only known conditionally on deep conjectures about the distribution of primes in arithmetic progressions, beyond the reach of the Generalized Riemann Hypothesis (e.g Montgomery’s conjecture). Our work provides the first example of a family of  $L$ -functions in which the support is unconditionally extended past the “diagonal range” that follows from a straightforward application of the underlying trace formula (in this case orthogonality of characters). We also highlight consequences for non-vanishing of  $L(s, \chi)$ .

## 1. INTRODUCTION

Motivated by the problem of establishing the non-existence of Siegel zeros (see [CI02] for details), Montgomery [Mon73] investigated in 1972 the *vertical* distribution of the zeros of the Riemann zeta-function. He showed that under the assumption of the Riemann Hypothesis, for any smooth function  $f$  with  $\text{supp } \hat{f} \subset (-1, 1)$ ,

$$(1) \quad \lim_{T \rightarrow \infty} \frac{1}{N(T)} \sum_{T \leq \gamma, \gamma' \leq 2T} f\left(\frac{\log T}{2\pi} \cdot (\gamma - \gamma')\right) = \int_{\mathbb{R}} f(u) \cdot \left(\delta(u) + 1 - \left(\frac{\sin 2\pi u}{2\pi u}\right)^2\right) du$$

where  $N(T)$  denotes the number of zeros of the Riemann zeta-function up to height  $T$  and  $\gamma, \gamma'$  are ordinates of the zeros of the Riemann zeta-function, and  $\delta(u)$  is a Dirac mass at 0. Dyson famously observed that the right-hand side coincides with the pair correlation function of eigenvalues of a random Hermitian matrix.

Dyson’s observation leads one to conjecture that the spacings between the zeros of the Riemann zeta-function are distributed in the same way as spacings between eigenvalues of a large random Hermitian matrix. Subsequent work of Rudnick-Sarnak [RS94] provided strong evidence towards this conjecture by computing (under increasingly restrictive conditions) the  $n$ -correlations of the zeros of any given automorphic  $L$ -function. Importantly the work of Rudnick-Sarnak suggested that the distribution of the zeros of an automorphic  $L$ -function is universal and independent of the distribution of its coefficients [RS96].

For number theoretic applications, the distribution of the so-called “low-lying zeros”, that is zeros close to the central point is particularly interesting (see e.g [HB04, You06] for various applications; see also [GS18] and [Wat21], for instance, for results in a different direction). Following the work of Katz-Sarnak [KS99] and Iwaniec-Luo-Sarnak [ILS00], we believe that the distribution of these low-lying zeros is also universal and predicted by only a few random matrix ensembles (which are either symplectic, orthogonal or unitary).

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Specifically the work of Katz-Sarnak suggests that for any smooth function  $\phi$  and any natural “family” of automorphic objects  $\mathcal{F}$ ,

$$(2) \quad \frac{1}{\#\mathcal{F}} \sum_{\pi \in \mathcal{F}} \sum_{\gamma_\pi} \phi\left(\frac{\log \mathfrak{c}_\pi}{2\pi} \cdot \gamma_\pi\right) \xrightarrow{\#\mathcal{F} \rightarrow \infty} \int_{\mathbb{R}} \phi(x) K_{\mathcal{F}}(x) dx,$$

where  $\gamma_\pi$  are ordinates of the zeros of the L-function attached to  $\pi$ ,  $\mathfrak{c}_\pi$  is the analytic conductor of  $\pi$  and  $K_{\mathcal{F}}(x)$  is a function depending only on the “symmetry type” of  $\mathcal{F}$ . One may wish to consult [ILS00] and [SST16] for a more detailed discussion.

There is a vast literature providing evidence for (2) (see [MMR<sup>+</sup>16]). Similarly to Montgomery’s result (1) all of the results in the literature place a restriction on the support of the Fourier transform of  $\phi$ . This restriction arises from the limitations of the relevant trace formula (in some families it is not always readily apparent what this relevant trace formula is). In practice an application of the trace formula gives rise to so-called “diagonal” and “off-diagonal” terms. Trivially bounding the off-diagonal terms corresponds to what we call a “straightforward” application of the trace formula.

A central yet extremely difficult problem is to extend the support of  $\widehat{\phi}$  beyond what a “straightforward” application of the trace formula gives. In fact most works in which the support of  $\widehat{\phi}$  has been extended further rely on the assumption of various deep hypotheses about primes that sometimes lie beyond the reach of the Generalized Riemann Hypothesis (GRH).

For example, Iwaniec-Luo-Sarnak show that in the case of holomorphic forms of even weight  $\leq K$  one obtains unconditionally a result for  $\widehat{\phi}$  supported in  $(-1, 1)$  and that under the assumption of the Generalized Riemann Hypothesis this can be enlarged to  $(-2, 2)$  (it is observed in [DFS] that assuming GRH only for Dirichlet  $L$ -functions is sufficient). Iwaniec-Luo-Sarnak also show that this range can be pushed further to  $\text{supp } \widehat{\phi} \subset (-22/9, 22/9)$  under the additional assumption that, for any  $c \geq 1$ ,  $(a, c) = 1$  and  $\varepsilon > 0$ ,

$$\sum_{\substack{p \leq x \\ p \equiv a \pmod{c}}} e(2\sqrt{p}/c) \ll_{\varepsilon} x^{1/2+\varepsilon}.$$

A similar behaviour is observed on low-lying zeros of dihedral  $L$ -functions associated to an imaginary quadratic field [FI03], where an extension of the support is shown to be equivalent to an asymptotic formula on primes with a certain splitting behaviour.

Assuming GRH, Brumer [Bru92] studied the one-level density of the family of elliptic curves and proved a result for test functions supported in  $(-5/9, 5/9)$ ; this corresponds to the “diagonal” range for this family. Heath-Brown [HB04] improved this range to  $(-2/3, 2/3)$ , and Young [You06] pushed the support to  $(-7/9, 7/9)$ . One-level density estimates for this family have deep implications for average ranks of elliptic curves. In particular, the work of Young was the first to show that, under some reasonable conjectures, a positive proportion of elliptic curves have rank 0 or 1 and thus satisfy the rank part of the Birch and Swinnerton-Dyer conjecture<sup>1</sup>.

As another example, it follows for instance from minor modifications of [HR03, CLLR14] that in the family of primitive Dirichlet characters of modulus  $\leq Q$  one can estimate 1-level

<sup>1</sup>A stronger conclusion was later reached unconditionally by Bhargava and Shankar [BS15] through other methods.



densities unconditionally for  $\phi$  with  $\widehat{\phi}$  supported in  $(-2, 2)$ .<sup>2</sup> As a by-product of work of Fiorilli-Miller [FM15, Theorem 2.8], it follows that for any  $\delta \in (0, 2)$ , this support can be enlarged to  $(-2 - \delta, 2 + \delta)$  under the following “de-averaging hypothesis”

$$(3) \quad \sum_{Q/2 \leq q \leq Q} \left| \sum_{\substack{p \leq x \\ p \equiv 1 \pmod{q}}} \log p - \frac{x}{\varphi(q)} \right|^2 \ll Q^{-\delta/2} \sum_{Q/2 \leq q \leq Q} \sum_{\substack{(a, q) = 1 \\ p \equiv a \pmod{q}}} \left| \sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} \log p - \frac{x}{\varphi(q)} \right|^2.$$

In this paper we give a first example of a family of  $L$ -functions in which we can *unconditionally* enlarge the support past the “diagonal” range that follows from a straightforward application of the trace formula (in this case orthogonality of characters).

**Theorem 1.** *Let  $\Phi$  be a smooth function compactly supported in  $[1/2, 3]$ , and  $\phi$  be a smooth function such that  $\text{supp } \widehat{\phi} \subset (-2 - \frac{50}{1093}, 2 + \frac{50}{1093})$ . Then, as  $Q \rightarrow \infty$ ,*

$$(4) \quad \sum_q \Phi\left(\frac{q}{Q}\right) \sum_{\substack{\chi \pmod{q} \\ \text{primitive}}} \sum_{\gamma_\chi} \phi\left(\frac{\log Q}{2\pi} \gamma_\chi\right) = \widehat{\phi}(0) \sum_q \Phi\left(\frac{q}{Q}\right) \sum_{\substack{\chi \pmod{q} \\ \text{primitive}}} 1 + o(Q^2).$$

Here  $\frac{1}{2} + i\gamma_\chi$  correspond to non-trivial zeros of  $L(s, \chi)$  and since we do not assume the Generalized Riemann Hypothesis we allow the  $\gamma_\chi$  to be complex.

**Remark.** In stating the theorem we have, for technical simplicity, made a suitable approximation to the conductor  $\mathfrak{c}_\pi$  appearing in (2).

Note that  $\phi$ , initially defined on  $\mathbb{R}$ , is analytically continued to  $\mathbb{C}$  by compactness of  $\text{supp } \widehat{\phi}$ . Our arguments can be adapted to show that if  $\text{supp } \widehat{\phi} \subset (-2 - \frac{50}{1093} + \varepsilon, 2 + \frac{50}{1093} - \varepsilon)$  for some  $\varepsilon > 0$ , then the error term in (4) is  $O(Q^{2-\delta})$  with  $\delta = \delta(\varepsilon)$ , up to altering slightly the main terms: after applying the explicit formula as in section 2.2, include the terms of order  $\asymp Q^2 / \log Q$  into the main term instead of treating them as error terms.

We remark that we make no progress on the “de-averaging hypothesis” (3) of Fiorilli-Miller, which remains a difficult open problem. We estimate the original sum over primes in arithmetic progressions, on average over moduli, by a variant of an argument of Fouvry [Fou85] and Bombieri-Friedlander-Iwaniec [BF186] which is based on Linnik’s dispersion method. The GRH will be dispensed with by working throughout, as in [Dra15], with characters of large conductors.

The asymptotic formula (4) is expected to hold true without the extra averaging over  $q$ . This extra averaging over  $q$ , and the cancellation of arguments which comes along, play an important role in our arguments.

If the GRH is true for Dirichlet  $L$ -functions, then let any  $0 < \kappa < \frac{50}{1093}$  be fixed, and let  $\lambda > 1$  be small enough that  $\kappa' := 2(\lambda - 1) + \lambda\kappa \in (0, \frac{50}{1093})$  as well. Defining

$$\tilde{\phi}(x) = \lambda \left( \frac{\sin \pi(2 + \kappa)x}{\pi(2 + \kappa)x} \right)^2, \quad \phi = \tilde{\phi} * u$$

where  $u$  is a smooth, positive approximation of unity such that  $\phi(0) \geq \lambda^{-1} \tilde{\phi}(0) = 1$ , and using the inequality

$$1 - \sum_{\gamma_\chi} \phi\left(\frac{\log Q}{2\pi} \gamma_\chi\right) \leq \mathbf{1}\left(L\left(\frac{1}{2}, \chi\right) \neq 0\right),$$

<sup>2</sup>This is in fact the  $GL(1)$  analogue of the result of Iwaniec-Luo-Sarnak for holomorphic forms.

we deduce from Theorem 1 that the proportion of non-vanishing  $L(\frac{1}{2}, \chi)$  with  $\chi$  ranging over primitive characters of conductor in  $[Q/2, Q]$  is at least  $1 - \lambda(2 + \kappa')^{-1} = 1 - (2 + \kappa)^{-1}$  for any  $\kappa < \frac{50}{1093}$ . We record this consequence in the Corollary below.

**Corollary 2.** *Let  $\varepsilon \in (0, 10^{-7})$ . Assume the Generalized Riemann Hypothesis for Dirichlet  $L$ -functions. Then for all  $Q$  large enough, the proportion of primitive characters  $\chi$  with modulus  $\in [Q/2, Q]$  for which*

$$L(\frac{1}{2}, \chi) \neq 0$$

is at least

$$\frac{1}{2} + \frac{25}{2236} - \varepsilon > 0.51118.$$

Corollary 2 is related to a recent result of Pratt [Pra19] who showed unconditionally that the proportion of non-vanishing in this family is at least 0.50073. We note that both the arguments of [Pra19] and those presented here eventually rely on bounds of Deshouillers-Iwaniec [DI82] on cancellation in sums of Kloosterman sums.

**Notations.** We call a map  $f : \mathbb{R}_+ \rightarrow \mathbb{C}$  a *test function* if  $f$  is smooth and supported inside  $[\frac{1}{2}, 3]$ .

For  $w \in \mathbb{N}$ ,  $n \in \mathbb{Z}$  and  $R \geq 1$ , we let

$$\mathbf{u}_R(n, w) := \mathbf{1}_{n \equiv 1 \pmod{w}} - \frac{1}{\varphi(w)} \sum_{\substack{\chi \pmod{w} \\ \text{cond}(\chi) \leq R}} \chi(n).$$

Note the trivial bound

$$(5) \quad |\mathbf{u}_R(n, w)| \ll \mathbf{1}_{n \equiv 1 \pmod{w}} + \frac{R\tau(w)}{\varphi(w)}.$$

The symbol  $n \sim N$  in a summation means  $n \in [N, 2N] \cap \mathbb{Z}$ . We say that a sequence  $(\alpha_n)_n$  is *supported at scale  $N$*  if  $\alpha_n = 0$  unless  $n \sim N$ .

The letter  $\varepsilon$  will denote an arbitrarily small number, whose value may differ at each occurrence. The implied constants will be allowed to depend on  $\varepsilon$ .

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## 2. PROOF OF THEOREM 1

**2.1. Lemmas on primes in arithmetic progressions.** We will require two results about primes in arithmetic progressions. The first is a standard estimate, obtained from an application of the large sieve.

**Lemma 3.** *Let  $A > 0$ ,  $X, Q, R \geq 2$  satisfy  $1 \leq R \leq Q$  and  $X \geq Q^2/(\log Q)^A$ , and  $f$  be a test function with  $\|f^{(j)}\|_\infty \ll_j 1$ . Then*

$$(6) \quad \sum_{q \leq Q} \left| \sum_{n \in \mathbb{N}} f\left(\frac{n}{X}\right) \Lambda(n) \mathbf{u}_R(n, q) \right| \ll Q(\log Q)^{O(1)} \sqrt{X} \left(1 + \frac{\sqrt{X}}{RQ} + \frac{X^{3/8}}{Q}\right).$$

The implied constant depends at most on  $A$  and the implied constants in the hypothesis.

*Proof.* By Heath-Brown's combinatorial formula for primes [IK04, Proposition 13.3] (with  $K = 2$ ), we restrict to proving the bound with  $\Lambda(n)$  replaced by convolutions of type I and II, of the shape

$$\begin{aligned} \sum_{\substack{n=m\ell \\ m \sim M}} \alpha_m & \quad (M \ll X^{1/4}), \\ \sum_{\substack{n=m\ell \\ m \sim M}} \alpha_m \beta_\ell & \quad (X^{1/4} \ll M \ll X^{3/4}), \end{aligned}$$

where  $|\alpha_m| \ll (\log X)\tau_4(m)$  and the analogous bound holds for  $\beta_\ell$ ; here we noted that if  $m_1 \leq m_2 \leq \sqrt{X}$  and  $m_1 m_2 > X^{1/4}$ , then either  $X^{1/4} < m_1 m_2 \leq X^{3/4}$  or  $X^{1/4} \leq m_1 \ll X^{1/2}$ . We treat the type I case by the Polyá-Vinogradov inequality [IK04, Theorem 12.5], getting a bound  $O(MR^{3/2}(\log Q)^{O(1)})$ . We treat the type II case by the large sieve [IK04, Theorem 17.4], getting a contribution  $O(\sqrt{X}(\log Q)^{O(1)}(Q + \sqrt{M} + \sqrt{X/M} + \sqrt{X}R^{-1}))$ .  $\square$

The second estimate is substantially deeper and we defer its proof to Section 4.

**Proposition 4.** *Let  $\kappa \in (0, \frac{50}{1093})$  and  $\varepsilon > 0$ . Let  $\Psi$  and  $f$  be test functions,  $A > 0$ ,  $X, Q, W, R \geq 1$ , and  $b \in \mathbb{N}$ . Assume that*

$$\begin{aligned} \frac{Q^2}{(\log Q)^A} & \ll X \ll Q^{2+\kappa}, & X^{11/20}Q^{-1} & \leq R \leq Q^{2/3}X^{-2/9}, \\ b & \leq Q^\varepsilon & Q^{1-\varepsilon} & \ll W \ll Q, \end{aligned}$$

and that  $\|f^{(j)}\|_\infty, \|\Psi^{(j)}\|_\infty \ll_j 1$ . Then, if  $\varepsilon > 0$  is small enough in terms of  $\kappa$ , we have

$$\sum_{w \in \mathbb{N}} \Psi\left(\frac{w}{W}\right) \sum_{n \in \mathbb{N}} \Lambda(n) f\left(\frac{n}{X}\right) \mathbf{u}_R(n, bw) \ll Q^{1-\varepsilon} \sqrt{X}.$$

The implied constant depends at most on  $\kappa$ ,  $A$ , and the implied constants in the hypotheses.

*Proof.* See Section 4.  $\square$

**2.2. Explicit formula.** We let  $\kappa \in (0, \frac{50}{1093})$  be such that  $\text{supp } \widehat{\phi} \subset (-2 - \kappa, 2 + \kappa)$ .

We rewrite the left-hand side of (4) by applying the explicit formula, *e.g.* [Sic98, Theorem 2.2], where the quantity  $\Phi(\rho)$  there (not to be confused with our test function) is replaced by  $\phi(\frac{\rho-1/2}{2\pi i} \log Q)$ , so that  $F(x) = \frac{1}{\log Q} \widehat{\phi}(\frac{x}{\log Q})$ . For  $q > 1$  and  $\chi \pmod{q}$  primitive, we obtain

$$\begin{aligned} (7) \quad & \sum_{\substack{\rho \in \mathbb{C} \\ \text{Re}(\rho) \in (0,1) \\ L(\rho, \chi) = 0}} \phi\left(\frac{(\rho - \frac{1}{2}) \log Q}{2\pi i}\right) \\ & = O\left(\frac{1}{\log Q}\right) + \widehat{\phi}(0) \frac{\log q}{\log Q} - \frac{1}{\log Q} \sum_{n \geq 1} (\chi(n) + \bar{\chi}(n)) \frac{\Lambda(n)}{\sqrt{n}} \widehat{\phi}\left(\frac{\log n}{\log Q}\right), \end{aligned}$$

since the terms  $I, J$  appearing in [Sic98, Theorem 2.2] satisfy  $|I(\frac{1}{2}, b)| + |J(\frac{1}{2}, b)| \ll (\log Q)^{-1}$  for  $b \in \{0, \frac{1}{2}\}$  by reasoning similarly as in [Sic98, Lemma 3.1]. Let  $\Psi(x) = \Phi(x)x$ . Summing

(7) over  $\chi$  and  $q$  we see that to conclude it remains to show that

$$(8) \quad S_\phi(Q) := \sum_{q \in \mathbb{N}} \frac{1}{q} \Psi\left(\frac{q}{Q}\right) \sum_{\substack{\chi(q) \\ \text{primitive}}} \frac{1}{\log Q} \sum_{n \geq 1} (\chi(n) + \bar{\chi}(n)) \frac{\Lambda(n)}{\sqrt{n}} \widehat{\phi}\left(\frac{\log n}{\log Q}\right) = o(Q).$$

We will in fact obtain the following slightly stronger result.

**Proposition 5.** *Let  $\kappa \in (0, \frac{50}{1093})$ . For all  $Q$  large enough and  $\varepsilon > 0$  small enough in terms of  $\kappa$ , we have*

$$S_\phi(Q) = O\left(\frac{Q}{\log Q}\right).$$

The implied constant depends on  $\phi$  and  $\varepsilon$  at most.

We break down the proof of Proposition 5 into the following three sections.

**2.3. Orthogonality and partition of unity.** Applying character orthogonality for primitive characters (see the third display in the proof of Lemma 4.1 of [BM11]), we get

$$(9) \quad S_\phi(Q) = \frac{2}{\log Q} \sum_{v,w} \Psi\left(\frac{vw}{Q}\right) \frac{\mu(v)}{v} \frac{\varphi(w)}{w} \sum_{n \equiv 1 \pmod{w}} \frac{\Lambda(n)}{\sqrt{n}} \widehat{\phi}\left(\frac{\log n}{\log Q}\right).$$

Let  $V$  be any test function generating the partition of unity

$$\sum_{j \in \mathbb{Z}} V\left(\frac{x}{2^j}\right) = 1$$

for all  $x > 0$ . Inserting this in (9), we obtain

$$S_\phi(Q) = \frac{2}{\log Q} \sum_{\substack{j \in \mathbb{Z} \\ 1/2 \leq X := 2^j \leq 2Q^{2+\kappa}}} \sum_{v,w} \Psi\left(\frac{vw}{Q}\right) \frac{\mu(v)}{v} \frac{\varphi(w)}{w} \sum_{n \equiv 1 \pmod{w}} \frac{\Lambda(n)}{\sqrt{n}} V\left(\frac{n}{X}\right) \widehat{\phi}\left(\frac{\log n}{\log Q}\right).$$

Set  $f_j(x) = x^{-1/2} V(x) \widehat{\phi}\left(\frac{\log(2^j x)}{\log Q}\right)$  for  $\frac{1}{2} \leq 2^j \leq 2Q^{2+\kappa}$ . Differentiating the product, we have that for all  $k \geq 0$ , there exists  $C_{\phi,k} \geq 0$  such that  $\|f_j^{(k)}\|_\infty \leq C_{\phi,k}$  for all  $j$ . We deduce

$$S_\phi(Q) \ll \sup_{1 \ll X \ll Q^{2+\kappa}} X^{-1/2} \sup_f |T(Q, X)|,$$

where  $f$  varies among test functions subject to  $\|f^{(k)}\|_\infty \leq C_{\phi,k}$ , and

$$T(Q, X) := \sum_{v,w} \Psi\left(\frac{vw}{Q}\right) \frac{\mu(v)}{v} \frac{\varphi(w)}{w} \sum_{n \equiv 1 \pmod{w}} \Lambda(n) f\left(\frac{n}{X}\right).$$

We handle the very small values of  $X$  by the trivial bound

$$\sum_{n \equiv 1 \pmod{w}} \Lambda(n) f\left(\frac{n}{X}\right) \ll \log Q \sum_{\substack{X/2 < n < 3X \\ n \neq 1, n \equiv 1 \pmod{w}}} 1 \ll \frac{X \log Q}{w},$$

which implies

$$T(Q, X) \ll \frac{X \log Q}{Q} \sum_{vw > Q} 1 \ll X(\log Q)^2.$$

It will therefore suffice to show that for

$$Q^2/(\log Q)^6 \ll X \ll Q^{2+\kappa},$$

we have

$$T(Q, X) \ll \frac{\sqrt{X}Q}{\log Q}.$$

**2.4. Subtracting the main term.** We insert the coprimality condition  $(n, v) = 1$ . Since

$$\begin{aligned} & \sum_{v,w} \sum \Psi\left(\frac{vw}{Q}\right) \frac{\mu(v)}{v} \frac{\varphi(w)}{w} \sum_{\substack{n \equiv 1 \pmod{w} \\ (n,v) > 1}} \Lambda(n) f\left(\frac{n}{X}\right) \\ & \ll \sum_{v \ll Q} v^{-1} \sum_{\substack{p|v \\ 1 \leq k \ll \log X}} (\log p) \sum_{w|p^k-1} 1 \\ & \ll Q^{1+\varepsilon}, \end{aligned}$$

we obtain

$$T(Q, X) = \sum_{v,w} \sum \Psi\left(\frac{vw}{Q}\right) \frac{\mu(v)}{v} \frac{\varphi(w)}{w} \sum_{\substack{n \equiv 1 \pmod{w} \\ (n,v) = 1}} \Lambda(n) f\left(\frac{n}{X}\right) + O(Q^{1+\varepsilon}).$$

Let  $1 \leq R < Q/2$  so that  $R < vw$  for any  $v, w$  appearing in the sum. We replace the condition  $n \equiv 1 \pmod{w}$  by  $\mathbf{u}_R(n, w)$ . The difference is

$$\sum_q \frac{1}{q} \Psi\left(\frac{q}{Q}\right) \sum_{\substack{\chi \pmod{q} \\ r = \text{cond}(\chi) \leq R \\ r|q}} \sum_{(n,q)=1} \Lambda(n) f\left(\frac{n}{X}\right) \chi(n) \sum_{v|q/r} \mu(v) = 0$$

since  $r < q$  by our choice of  $R$ , so that

$$T(Q, X) = \sum_{v,w} \sum \Psi\left(\frac{vw}{Q}\right) \frac{\mu(v)}{v} \frac{\varphi(w)}{w} \sum_{(n,v)=1} \Lambda(n) f\left(\frac{n}{X}\right) \mathbf{u}_R(n, w) + O(Q^{1+\varepsilon}).$$

We next remove the coprimality condition on  $n$ , using the trivial bound (5). For the first term  $\mathbf{1}_{n \equiv 1 \pmod{w}}$  in  $\mathbf{u}_R(n, w)$ , this was already justified above. For the second term, we get

$$\ll RQ^{-1+\varepsilon} \sum_{\substack{v,w \\ vw \asymp Q}} \sum_{p|v} \log p \ll RQ^\varepsilon.$$

Since  $R \ll Q$ , both error terms are acceptable. We get

$$T(Q, X) = T(Q, X, R) + O(Q^{1+\varepsilon}),$$

where

$$T(Q, X, R) := \sum_{v,w} \sum \Psi\left(\frac{vw}{Q}\right) \frac{\mu(v)}{v} \frac{\varphi(w)}{w} \Delta(w),$$

$$(10) \quad \Delta(w) := \sum_n \Lambda(n) f\left(\frac{n}{X}\right) \mathbf{u}_R(n, w).$$

We are required to show that

$$(11) \quad T(Q, X, R) \ll \frac{\sqrt{X}Q}{\log Q}.$$

**2.5. Reduction to the critical range.** We now impose the additional conditions

$$(12) \quad Q^{\kappa/2+\varepsilon} \leq R \leq Q^{1/2}, \quad \kappa < 2/3.$$

Observe that this  $\kappa$  is the same as that appearing in the statement of Proposition 4. The condition  $\kappa < \frac{2}{3}$  is convenient for applying (6) below, but is rather loose since  $\kappa$  is ultimately required to be much smaller than  $\frac{2}{3}$ .

Let  $B \in [1, Q^{1/2}]$  be a parameter. In  $T(Q, X, R)$ , we write  $\frac{\varphi(w)}{w} = \sum_{b|w} \frac{\mu(b)}{b}$  and exchange summation, so that

$$\begin{aligned} T(Q, X, R) &\leq \sum_{b,v} \frac{1}{bv} \left| \sum_w \Psi\left(\frac{bvw}{Q}\right) \Delta(bw) \right| \\ &\ll (\log B)^2 \sup_{b,v \leq B} \left| \sum_w \Psi\left(\frac{bvw}{Q}\right) \Delta(bw) \right| + E_1 + E_2, \end{aligned}$$

where  $E_1$  (resp.  $E_2$ ) corresponds to the sum over  $b, v$  restricted to  $b > B$  (resp.  $v > B$ ). We recall that  $\text{supp } \Psi \subset [\frac{1}{2}, 3]$  by hypothesis. On the one hand, we have

$$\begin{aligned} E_1 &\ll \sum_{\substack{b,w \\ bw \leq 3Q \\ b > B}} \frac{1}{b} |\Delta(bw)| \\ &\ll Q^{\varepsilon/2} B^{-1} \sum_{q \leq 3Q} |\Delta(q)| \\ &\ll Q^{1+\varepsilon/2} \sqrt{X} B^{-1}, \end{aligned}$$

using (6) along with our hypotheses (12). On the other hand, we have

$$\begin{aligned} E_2 &\ll \sum_{\substack{b,w \\ bw \leq 3Q/B}} \frac{1}{b} |\Delta(bw)| \\ &\ll Q^{\varepsilon/2} \sum_{q \leq 3Q/B} |\Delta(q)| \\ &\ll Q \sqrt{X} (Q^{\varepsilon/2} B^{-1} + Q^{-\varepsilon}) \end{aligned}$$

again by (12) and (6); we have used the bounds  $Q^{-1+\varepsilon} \sqrt{X} R^{-1} \ll Q^{-\varepsilon}$  and  $Q^{-1+\varepsilon} X^{3/8} \ll Q^{-\varepsilon}$ , which follow from  $Q^{\kappa/2+\varepsilon} \leq R$  and  $\kappa < 2/3$  respectively upon reinterpreting  $\varepsilon$ .

Grouping the above, it will suffice to show that

$$\sum_w \Psi\left(\frac{bvw}{Q}\right) \Delta(bw) \ll Q^{1-\varepsilon} \sqrt{X}$$

uniformly for  $b, v \leq Q^\varepsilon$  and test functions  $\Psi$  and  $f$ . Assume now  $\kappa \in (0, \frac{50}{1093})$ . Then the conditions on  $R$  in (12) and in Proposition 4 overlap, so that we may apply Proposition 4

with  $W = \frac{Q}{bw}$ . This gives the above bound, and completes the proof of (11), hence of Proposition 5.

### 3. EXPONENTIAL SUMS ESTIMATES

In this section, we work out the modifications to be made to the arguments underlying [DI82] in order to exploit current knowledge on the spectral gap of the Laplacian on congruence surfaces [KS03]. We will follow the setting in Theorem 2.1 of [Dra17], since we will need to keep track of the uniformity in  $q_0$ . We also take the opportunity to implement the correction recently described in [BFI].

Let  $\theta \geq 0$  be a bound towards the Petersson-Ramanujan conjecture, in the sense of [Dra17, eq. (4.6)]. Selberg's 3/16 theorem corresponds to  $\theta \leq 1/4$ , and the Kim-Sarnak bound [KS03] asserts that  $\theta \leq 7/64$ .

**Proposition 6.** *Let the notations and hypotheses be as in [Dra17, Theorem 2.1]. Then*

$$\sum_{\substack{c \\ c \equiv c_0}} \sum_{\substack{d \\ d \equiv d_0 \pmod{q}}} \sum_{\substack{n \\ (qrd, sc)=1}} \sum_r \sum_s b_{n,r,s} g(c, d, n, r, s) e\left(n \frac{\overline{rd}}{sc}\right) \\ \ll_{\varepsilon, \varepsilon_0} (qCDNRS)^{\varepsilon + O(\varepsilon_0)} q^{3/2} K(C, D, N, R, S) \|b_{N,R,S}\|_2,$$

where  $\|b_{N,R,S}\|_2^2 = \sum_{n,r,s} |b_{n,r,s}|^2$ , and here

$$(13) \quad K(C, D, N, R, S)^2 = qCS(RS + N)(C + RD) + C^{1+4\theta} DS((RS + N)R)^{1-2\theta} + D^2NR.$$

*Remark 7.* The bound of Proposition 6 is monotonically stronger as  $\theta$  decreases, since the first term is larger than  $CDS(RS + N)R$ . Under the Petersson-Ramanujan conjecture for Maass forms, which predicts that  $\theta = 0$  is admissible, the second term in (13) is smaller than the first.

*Proof.* The proof of the proposition, as with all results of this type, relies on the Kuznetsov formula and large sieve inequalities for coefficients of automorphic forms. The application of the Kuznetsov formula requires one to understand the contribution of holomorphic forms, Eisenstein series, and Maass forms (whether the holomorphic forms appear depends on the sign of the variables inside the Kloosterman sum). We divide these forms into the *exceptional spectrum* and the *regular spectrum*. The exceptional spectrum consists of those (conjecturally non-existent) Maass forms whose eigenvalues  $t_f = \frac{1}{2} + it_f$  have  $t_f \in i\mathbb{R}$ . By the definition of  $\theta$  above we have that  $|t_f| \leq \theta$  for all  $f$  in the exceptional spectrum. The regular spectrum consists of everything that is not exceptional. The contribution of the regular spectrum is handled as in [Dra17], and does not require any modification here. We improve upon the analysis of [Dra17] in handling the exceptional spectrum by keeping track of the dependence on  $\theta$  (see the remark made in [Dra17, p. 703]). The statements of [Dra17] which are affected are Lemma 4.10, Proposition 4.12, Proposition 4.13 and the proof of Theorem 2.1. The treatment of the exceptional spectrum rests upon a weighted large sieve inequality. These weighted large sieve inequalities are proved, following [DI82], by an iterative procedure.

With the notations of [Dra17], the changes to be made are as follows :

— Lemma 4.10 bounds sums of the form

$$\sum_{\substack{q \leq Q \\ q_0 | q}} \sum_{\substack{f \in \mathcal{B}(q, \chi) \\ t_f \in i\mathbb{R}}} Y^{2|t_f|} \left| \sum_{N < n \leq 2N} n^{1/2} \rho_{f\infty}(n) \right|^2,$$

and serves to control the first step of the recursion. The bound

$$\sum_{\substack{q \leq Q \\ q_0 | q}} \sum_{\substack{f \in \mathcal{B}(q, \chi) \\ t_f \in i\mathbb{R}}} Y^{2|t_f|} \left| \sum_{N < n \leq 2N} n^{1/2} \rho_{f\infty}(n) \right|^2 \ll (QN)^\varepsilon (Qq_0^{-1} + N + (NY)^{1/2})N$$

may be replaced by the bound

$$\ll (QN)^\varepsilon (Qq_0^{-1} + N + (NY)^{2\theta} Q^{1-4\theta})N.$$

This does not require any change in the recursion argument, but merely to use the bound  $|t_f| \leq \theta$  in the very last step, page 278 of [DI82], whereby  $\sqrt{Y/Y_1}$  is replaced by  $(Y/Y_1)^{2\theta}$ .

— In Proposition 4.12 one bounds sums of the form

$$\sum_{\substack{m, n, r, s \\ (s, rq)=1}} a_m b_{n, r, s} \sum_{c \in \mathcal{C}(\infty, 1/s)} \frac{1}{c} \phi\left(\frac{4\pi\sqrt{mn}}{c}\right) S_{\infty, 1/s}(m, \pm n; c)$$

in terms of quantities  $L_{\text{reg}}$  and  $L_{\text{exc}}$ . In place of

$$L_{\text{exc}} = \left(1 + \sqrt{\frac{N}{RS}}\right) \sqrt{\frac{1+X^{-1}}{RS}} \left(\frac{MN}{RS+N}\right)^{1/4} \frac{\sqrt{RS}}{1+X} \sqrt{M} \|b_{N, R, S}\|_2$$

we claim the improved

$$L_{\text{exc}} = q_0^{\frac{1}{2}-2\theta} \left(1 + \sqrt{\frac{N}{RS}}\right) \left(\frac{1+X^{-1}}{RS}\right)^{2\theta} \left(\frac{MN}{RS+N}\right)^\theta \frac{\sqrt{RS}}{1+X} \sqrt{M} \|b_{N, R, S}\|_2.$$

To obtain this bound one uses the new bound for Lemma 4.10 and follows the arguments of [DI82, section 9.1].

— In Proposition 4.13, one bounds

$$\sum_{\substack{c, m, n, r, s \\ (sc, rq)=1}} b_{n, r, s} \bar{\chi}(c) g(c, m, n, r, s) e(mt) S(n\bar{r}, \pm m\bar{q}; sc)$$

in terms of quantities  $K_{\text{reg}}$  and  $K_{\text{exc}}$ . The term

$$K_{\text{exc}}^2 = C^3 S^2 \sqrt{R(N+RS)}$$

can be replaced by

$$K_{\text{exc}}^2 = C^{2+4\theta} S^2 (R(N+RS))^{1-2\theta}.$$

This is seen by using the new definition on  $L_{\text{exc}}$  in Proposition 4.12, and by keeping track of a factor  $q^{-1+2\theta}$  coming from the term  $(1+X^{-1})^{2\theta}/(1+X)$ .



— Finally, we modify the proof of Theorem 2.1 at two places. First, the bound for  $\mathcal{A}_0$  on page 706, as explained in [BFI], is wrong unless further hypotheses on  $(b_{n,r,s})$  are imposed. The correct bound in general is

$$\mathcal{A}_0 \ll q^{-2}(\log S)^2 D(NR)^{1/2} \|b_{N,R,S}\|_2,$$

and this yields the term  $D^2NR$  instead of  $D^2NRS^{-1}$ . Secondly, our new bound for  $K_{\text{exc}}$  in Proposition 4.13 gives a contribution  $C^{2+4\theta}S^2(R(RS+N))^{1-2\theta}$  instead of  $C^3S^2\sqrt{R(RS+N)}$  in the definition of  $L_{\text{exc}}^2$  and  $L^*(M_1)^2$  on p.707 of [Dra17]. This yields a term  $C^{1+4\theta}DS((N+RS)R)^{1-2\theta}$  instead of  $C^2DS\sqrt{(N+RS)R}$  in eq. (4.39) of [Dra17], and by following the rest of the arguments we deduce our claimed bound.  $\square$

#### 4. PRIMES IN ARITHMETIC PROGRESSIONS: PROOF OF PROPOSITION 4

The proof of Theorem 1 relies on Proposition 4 which for the convenience of the reader we recall below.

**Proposition.** *Let  $\kappa \in (0, \frac{50}{1093})$  and  $\varepsilon > 0$ . Let  $\Psi, f$  be test functions,  $A > 0, X, Q, W, R \geq 1$ , and  $b \in \mathbb{N}$ . Assume that*

$$\begin{aligned} \frac{Q^2}{(\log Q)^A} &\ll X \ll Q^{2+\kappa}, & X^{11/20}Q^{-1} &\leq R \leq Q^{2/3}X^{-2/9}, \\ b &\leq Q^\varepsilon & Q^{1-\varepsilon} &\ll W \ll Q, \end{aligned}$$

and that  $\|f^{(j)}\|_\infty, \|\Psi^{(j)}\|_\infty \ll_j 1$ . Then, if  $\varepsilon > 0$  is small enough in terms of  $\kappa$ , we have

$$\sum_{w \in \mathbb{N}} \Psi\left(\frac{w}{W}\right) \sum_{n \in \mathbb{N}} \Lambda(n) f\left(\frac{n}{X}\right) \mathbf{u}_R(n, bw) \ll Q^{1-\varepsilon} \sqrt{X}.$$

The implied constant depends at most on  $\kappa, A$ , and the implied constants in the hypotheses.

*Remark 8.* What is crucial in our statement is the size of the upper bound, which should be negligible with respect to  $Q\sqrt{X}$ . On the other hand, we are only interested in values of  $X$  larger than  $Q^2$ . This is in contrast with most works on primes in arithmetic progressions [FI83, BFI86, Zha14], where the main challenge is to work with values of  $X$  much smaller than  $Q^2$ , while only aiming at an error term which is negligible with respect to  $X$ . The main point is that in both cases, the large sieve yields an error term which is always too large (see [IK04, Theorem 17.4]), an obstacle which the dispersion method is designed to handle.

In what follows, we will systematically write

$$X = Q^{2+\varpi},$$

so that  $-o(1) \leq \varpi \leq \kappa + o(1)$  as  $Q \rightarrow \infty$ .

**4.1. Combinatorial identity.** We perform a combinatorial decomposition of the von Mangoldt function into sums of different shapes: Type  $d_1$  sums have a long smooth variable, Type  $d_2$  sums have two long smooth variables, and Type II sums have two rough variables that are neither too small nor too large. We accomplish this decomposition with the Heath-Brown identity and the following combinatorial lemma.

**Lemma 9.** Let  $\{t_j\}_{1 \leq j \leq J} \in \mathbb{R}$  be non-negative real numbers such that  $\sum_j t_j = 1$ . Let  $\lambda, \sigma, \delta \geq 0$  be real numbers such that

- $\delta < \frac{1}{12}$ ,
- $\sigma \leq \frac{1}{6} - \frac{\delta}{2}$ ,
- $2\lambda + \sigma < \frac{1}{3}$ .

Then at least one of the following must occur:

- (Type  $d_1$ ) There exists  $t_j$  with  $t_j \geq \frac{1}{3} + \lambda$ .
- (Type  $d_2$ ) There exist  $i, j, k$  such that  $\frac{1}{3} - \delta < t_i, t_j, t_k < \frac{1}{3} + \lambda$ , and

$$\sum_{t_j^* \notin \{t_i, t_j, t_k\}} t_j^* < \sigma.$$

- (Type II) There exists  $S \subset \{1, \dots, J\}$  such that

$$\sigma \leq \sum_{j \in S} t_j \leq \frac{1}{3} - \delta.$$

*Proof.* Assume that the Type  $d_1$  case and the Type II case both fail. Then for every  $j$  we have  $t_j < \frac{1}{3} + \lambda$ , and for every subset  $S$  of  $\{1, \dots, J\}$  we either have

$$\sum_{j \in S} t_j < \sigma$$

or

$$\sum_{j \in S} t_j > \frac{1}{3} - \delta.$$

Let  $s_1, \dots, s_K$  denote those  $t_j$  with  $\frac{1}{3} - \delta < t_j < \frac{1}{3} + \lambda$ . We will show that  $K = 3$ . Let  $t_j^*$  be any other  $t_j$ , so that  $t_j^* \leq \frac{1}{3} - \delta$ , and therefore  $t_j^* < \sigma$ . We claim that

$$\sum_j t_j^* < \sigma.$$

If not, then  $\sum_j t_j^* > \frac{1}{3} - \delta$ . By a greedy algorithm we can find some subcollection  $S^*$  of the  $t_j^*$  such that

$$\sigma < \sum_{j \in S^*} t_j^* \leq 2\sigma.$$

Since  $2\sigma \leq \frac{1}{3} - \delta$  this subcollection satisfies the Type II condition, in contradiction to our assumption.

Now we show that  $K = 3$ . Observe that  $K \geq 3$ , since if  $K \leq 2$  we have

$$1 = \sum_j t_j = \sum_{i=1}^K s_i + \sum_j t_j^* < 2\left(\frac{1}{3} + \lambda\right) + \sigma < 1.$$

Furthermore, we must have  $K \leq 3$ , since if  $K \geq 4$  we have

$$1 = \sum_j t_j \geq \sum_{i=1}^K s_i > 4\left(\frac{1}{3} - \delta\right) > 1.$$

This completes the proof. □

Using *e.g.* Heath-Brown's combinatorial identity [HB82], we deduce the following.

**Corollary 10.** *Let  $f$  be a test function,  $u : \mathbb{N} \rightarrow \mathbb{C}$  be any map, and  $X \geq 1$ . Then there exists a sequence  $(C_j)_{j \geq 0}$  of positive numbers, depending only on  $f$ , such that we have*

$$(14) \quad \left| \sum_{n \in \mathbb{N}} \Lambda(n) f\left(\frac{n}{X}\right) u(n) \right| \ll (\log X)^8 (T_1 + T_2 + T_{II}),$$

where

$$(15) \quad T_1 = \sup_{\substack{N \gg X^{1/3+\lambda} \\ MN \asymp X}} \sup_{\substack{g \in \mathcal{G} \\ \beta \in \mathcal{S}}} \left| \sum_{\substack{n \in \mathbb{N} \\ m \sim M}} g\left(\frac{n}{N}\right) \beta_m u(mn) \right|,$$

$$(16) \quad T_2 = \sup_{\substack{X^{1/3-\delta} \ll N_2 \leq N_1 \ll X^{1/3+\lambda} \\ MN_1 N_2 \asymp X}} \sup_{\substack{g_1, g_2 \in \mathcal{G} \\ \beta \in \mathcal{S}}} \left| \sum_{\substack{n_1, n_2 \in \mathbb{N} \\ m \sim M}} g_1\left(\frac{n_1}{N_1}\right) g_2\left(\frac{n_2}{N_2}\right) \beta_m u(mn_1 n_2) \right|,$$

$$(17) \quad T_{II} = \sup_{\substack{X^\sigma \ll N \ll X^{1/3-\delta} \\ MN \asymp X}} \sup_{\alpha, \beta \in \mathcal{S}} \left| \sum_{\substack{n \sim N \\ m \sim M}} \alpha_m \beta_n u(mn) \right|,$$

where the implied constants are absolute,  $\mathcal{G}$  is the set of test functions  $g$  satisfying  $\|g^{(j)}\|_\infty \leq C_j$  and  $\mathcal{S}$  is the set of sequences  $(\beta_n)$  satisfying  $|\beta_n| \leq d(n)^8$ .

*Proof.* By the Heath-Brown identity [IK04, Proposition 13.3], there exists bounded coefficients  $(c_J)_{1 \leq J \leq 4}$  such that

$$\Lambda(n) = \sum_{J=1}^4 c_J \sum_{\substack{m_1, \dots, m_J \\ n_1, \dots, n_J \\ n = m_1 \dots m_J n_1 \dots n_J \\ m_j \leq (3X)^{1/4}}} \log(n_1) \prod_j \mu(m_j)$$

for any  $n$  involved in the left-hand side of (14). Let  $\psi$  be a test function inducing a partition of unity in the sense that  $\sum_{j \in \mathbb{Z}} \psi\left(\frac{x}{2^j}\right) = 1$  for all  $x > 0$ . Then we have

$$\sum_{n \in \mathbb{N}} \Lambda(n) f\left(\frac{n}{X}\right) u(n) = \sum_{J=1}^4 c_J \sum_{(M_1, \dots, M_J, N_1, \dots, N_J) \in U_J} S(M_1, \dots, M_J, N_1, \dots, N_J),$$

$$S(M_1, \dots, N_J) = \sum_{m_1, \dots, n_J \in \mathbb{N}} \log(n_1) \left( \prod_j \psi\left(\frac{n_j}{N_j}\right) \right) \left( \prod_j \mu^*(m_j) \psi\left(\frac{m_j}{M_j}\right) \right) f\left(\frac{m_1 \dots n_J}{X}\right) u(m_1 \dots n_J),$$

where  $U_J$  is the set of  $2J$ -tuples of powers of 2 such that  $X/6 \leq M_1 \dots M_J N_1 \dots N_J \leq 6X$ , and  $\mu_*(m) = \mu(m)$  if  $m \leq (3X)^{1/4}$  and 0 otherwise. We abbreviated  $m_1 \dots n_J = m_1 \dots m_J n_1 \dots n_J$ . The set  $U_J$  has at most  $O((\log X)^{2J-1})$  elements. By Lemma 9, for each choice of  $J$  and  $(M_1, \dots, N_J) \in U_J$  we have either  $N \geq \frac{1}{6}X^{1/3+\lambda}$  for some  $N \in \{N_j\}$ , or  $\frac{1}{6}X^{1/3-\delta} \leq N', N'' \leq 6X^{1/3+\lambda}$  for some  $N', N'' \in \{N_j\}$ , or  $\frac{1}{6}X^\sigma \leq N \leq 6X^{1/3-\delta}$  for some subproduct  $N$  of  $N_j$  and  $M_j$  (here we used that for  $X$  large enough, we have  $(3X)^{1/4} <$

$\frac{1}{6}X^{1/3-\delta}$ ). Sorting the sum over  $J$  and  $(M_1, \dots, N_J)$  according to this trichotomy, and writing  $\log(n_1) = \log N_1 + \log(n_1/N_1)$ , the above is bounded in absolute values by

$$\ll (\log X)^8 (T_1^* + T_2^* + T_{\text{II}}^*),$$

$$T_1^* = \sup_{\substack{X/6 \leq MN \leq 6X \\ \frac{1}{6}X^{1/3+\lambda} \leq N \\ |r| \leq 8}} \sup_{\substack{g \in \{\psi, \psi \log\} \\ \beta \in \mathcal{S}}} \left| \sum_{\substack{n \in \mathbb{N} \\ m \sim M}} g\left(\frac{n}{N}\right) \beta_m f\left(\frac{mn}{2^r MN}\right) u(mn) \right|,$$

$$T_2^* = \sup_{\substack{X/6 \leq N_1 N_2 M \leq 6X \\ \frac{1}{6}X^{1/3-\delta} \leq N_1, N_2 \leq 6X^{1/3+\lambda} \\ |r| \leq 8}} \sup_{\substack{g_1, g_2 \in \{\psi, \psi \log\} \\ \beta \in \mathcal{S}}} \left| \sum_{\substack{n_1, n_2 \in \mathbb{N} \\ m \sim M}} g_1\left(\frac{n_1}{N_1}\right) g_2\left(\frac{n_2}{N_2}\right) \beta_m f\left(\frac{n_1 n_2 m}{2^r N_1 N_2 M}\right) u(n_1 n_2 m) \right|,$$

$$T_{\text{II}}^* = \sup_{\substack{X/6 \leq NM \leq 6X \\ \frac{1}{6}X^\sigma \leq N \leq 6X^{1/3-\delta} \\ |r| \leq 8}} \sup_{\alpha, \beta \in \mathcal{S}} \left| \sum_{\substack{m \sim M \\ n \sim N}} \alpha_m \beta_n f\left(\frac{mn}{2^r MN}\right) u(mn) \right|.$$

Here the conditions  $m \sim M$  and  $n \sim N$  in the sums were added by an additional bounded dichotomy (which is the reason for the presence of the sup over  $r$ ). Finally, letting  $\check{f}$  be the Mellin transform of  $f$ , we have by Mellin inversion  $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \check{f}(it) x^{-it} dt$ , and the map  $t \mapsto \check{f}(it)$  is of Schwartz class on  $\mathbb{R}$ . In particular, for  $M, N, r, g, \beta$  as in  $T_1^*$  we have

$$\left| \sum_{n \in \mathbb{N}, m \sim M} g\left(\frac{n}{N}\right) \beta_m f\left(\frac{mn}{2^r MN}\right) u(mn) \right| \ll \sup_{t \in \mathbb{R}} \left| \sum_{n \in \mathbb{N}, m \sim M} g_t\left(\frac{n}{N}\right) \beta_{m,t} u(mn) \right|$$

where  $g_t(x) = (1+t^2)\check{f}(it)x^{-it}g(x)$  (the factor  $1+t^2$  being included so that we could write a supremum) and  $\beta_{m,t} = m^{-it}\beta_m \in \mathcal{S}$ . We note that  $g_t$  is a test function satisfying  $\|g_t^{(j)}\|_\infty \ll C_j$  where  $C_j := \sup_{0 \leq k, \ell, m \leq j+2} \|t^k \check{f}(it)\|_\infty \|x^{-\ell} g^{(m)}(x)\|_\infty$  can be bounded in terms of  $f$  only. This yields the contribution of  $T_1$  in our claim. The contributions of  $T_2$  and  $T_{\text{II}}$  are obtained in the same way.  $\square$

In what follows, we successively consider  $T_1, T_2$  and  $T_{\text{II}}$ , which we specialize at

$$u(n) := \sum_{w \in \mathbb{N}} \Psi\left(\frac{w}{W}\right) \mathbf{u}_R(n, bw),$$

and we will denote

$$R = X^\rho.$$

**4.2. Type  $d_1$  sums.** We suppose  $M$  and  $N$  are given as in (15). The quantity we wish to bound is

$$T_1(M, N) = \sum_w \Psi\left(\frac{w}{W}\right) \sum_{\substack{m \sim M \\ (m, bw)=1}} \beta_m \left( \sum_{\substack{n \in \mathbb{N} \\ mn \equiv 1 \pmod{bw}}} g\left(\frac{n}{N}\right) - \frac{1}{\varphi(bw)} \sum_{\substack{\chi \pmod{bw} \\ \text{cond}(\chi) \leq R}} \chi(m) \sum_{(n, bw)=1} \chi(n) g\left(\frac{n}{N}\right) \right).$$

By Poisson summation and the classical bound on Gauss sums [IK04, Lemma 3.2], we have

$$\begin{aligned} \sum_{n \equiv \bar{m} \pmod{bw}} g\left(\frac{n}{N}\right) &= \frac{N}{bw} \widehat{g}(0) + \frac{N}{bw} \sum_{0 < |h| \leq W^{1+\varepsilon}/N} \widehat{g}\left(\frac{Nh}{bw}\right) e\left(\frac{\bar{m}h}{bw}\right) + O(Q^{-A}), \\ \frac{1}{\varphi(bw)} \sum_{(c,bw)=1} \chi(c) g\left(\frac{c}{N}\right) &= \frac{N}{bw} \widehat{g}(0) \mathbf{1}(\chi = \chi_0) + O\left(\frac{Q^\varepsilon R^{1/2}}{W}\right). \end{aligned}$$

Therefore,

$$T_1(M, N) = \frac{N}{b} \sum_w \frac{1}{w} \Psi\left(\frac{w}{W}\right) \sum_{\substack{(m,bw)=1 \\ m \sim M}} \beta_m \sum_{0 < |h| \leq W^{1+\varepsilon}/N} \widehat{g}\left(\frac{Nh}{bw}\right) e\left(\frac{\bar{m}h}{bw}\right) + O(MR^{3/2}Q^\varepsilon).$$

Our goal is to get cancellation in the exponential phases by summing over the smooth variable  $w$ . We apply the reciprocity formula

$$\frac{\bar{m}h}{bw} \equiv -\frac{\bar{b}wh}{m} + \frac{h}{mbw} \pmod{1},$$

which implies

$$\begin{aligned} T_1(M, N) &= \frac{N}{b} \sum_w \frac{1}{w} \Psi\left(\frac{w}{W}\right) \sum_{\substack{(m,bw)=1 \\ m \sim M}} \beta_m \sum_{0 < |h| \leq W^{1+\varepsilon}/N} \widehat{g}\left(\frac{Nh}{bw}\right) e\left(\frac{\bar{b}wh}{m}\right) \\ &\quad + O(MR^{3/2}Q^\varepsilon + Q^{1+\varepsilon}N^{-1}). \end{aligned}$$

We rearrange the sum as

$$\frac{N}{bW} \sum_{\substack{(m,b)=1 \\ m \sim M}} \beta_m \sum_{0 < |h| \leq W^{1+\varepsilon}/N} \sum_{(w,m)=1} \widehat{g}\left(\frac{Nh}{bw}\right) \frac{W}{w} \Psi\left(\frac{w}{W}\right) e\left(\frac{\bar{b}wh}{m}\right).$$

By partial summation and a variant of the Weil bound [Dra15, eq. (2.4)], the sum on  $w$  is

$$\ll ((h, m)WM^{-1} + \sqrt{(h, m)}\sqrt{M})Q^\varepsilon.$$

Summing over  $h$  and  $m$ , we obtain a bound

$$T_1(M, N) \ll Q^{1+\varepsilon} + M^{3/2}Q^\varepsilon + MR^{3/2}Q^\varepsilon.$$

This bound is acceptably small provided

$$\begin{aligned} N &\gg \left(\frac{X}{Q}\right)^{2/3+\varepsilon} = X^{\frac{1}{3} + \frac{\varpi}{3(2+\varpi)} + \frac{1+\varpi}{2+\varpi}\varepsilon}, \\ N &\gg \frac{X^{1/2}R^{3/2}}{Q^{1-2\varepsilon}} = X^{\frac{\varpi}{2(2+\varpi)} + \frac{3}{2}\rho + \frac{2\varepsilon}{2+\varpi}}. \end{aligned}$$

These inequalities are satisfied, for all sufficiently small  $\varepsilon > 0$ , under the assumptions

$$(18) \quad \lambda > \frac{\varpi}{3(2+\varpi)}, \quad \rho < \frac{4+\varpi}{9(2+\varpi)}.$$

We have proved the following.

**Lemma 11.** *Under the notations and hypotheses of Corollary 10, and assuming (18), we have*

$$T_1 \ll Q^{1-\varepsilon} \sqrt{X}.$$

The implied constant depends on  $\lambda, \rho$  and  $\varpi$ .

**4.3. Type  $d_2$  sums.** The treatment of the type  $d_2$  sums (16) is nearly identical to [BFI86, Section 14]. For convenience, we rename  $(N_1, N_2, M)$  into  $(M, N, L)$  so that we have  $MNL \asymp X$ . We wish to bound

$$\begin{aligned} T_2(M, N, L) &= \sum_{\ell \sim L} \beta_\ell \sum_{(w, \ell)=1} \Psi\left(\frac{w}{W}\right) \left( \sum_{\substack{m, n \\ \ell mn \equiv 1 \pmod{bw}}} g_1\left(\frac{m}{M}\right) g_2\left(\frac{n}{N}\right) \right. \\ &\quad \left. - \frac{1}{\varphi(bw)} \sum_{\substack{\chi \pmod{bw} \\ \text{cond}(\chi) \leq R}} \chi(\ell) \sum_{(mn, bw)=1} g_1\left(\frac{m}{M}\right) g_2\left(\frac{n}{N}\right) \chi(mn) \right). \end{aligned}$$

We perform Poisson summation on the  $m$ -sums to get

$$\begin{aligned} \sum_{m \equiv \bar{\ell} n \pmod{bw}} g_1\left(\frac{m}{M}\right) &= \frac{M}{bw} \sum_{|h| \leq H} \widehat{g}_1\left(\frac{Mh}{bw}\right) e\left(\frac{\bar{\ell} nh}{bw}\right) + O(Q^{-A}), \\ \sum_{(m, bw)=1} \chi(m) g_1\left(\frac{m}{M}\right) &= \frac{\varphi(bw)}{bw} M \widehat{g}_1(0) \mathbf{1}(\chi = \chi_0) + O(Q^\varepsilon R^{1/2}), \end{aligned}$$

where  $H = W^{1+\varepsilon} M^{-1}$ . The contribution of the error terms is

$$\ll LNR^{3/2} Q^\varepsilon.$$

The zero frequency of Poisson summation cancels out. For the non-zero frequencies we employ reciprocity in the form

$$e\left(\frac{\bar{\ell} nh}{bw}\right) = e\left(-\frac{\bar{b}wh}{\ell n}\right) + O\left(\frac{H}{LNW}\right),$$

and the error term contributes a quantity of size  $O(Q^{1+\varepsilon})$ . We therefore have (19)

$$\begin{aligned} T_2(M, N, L) &= \frac{M}{b} \sum_{\substack{\ell \sim L \\ (\ell, b)=1}} \beta_\ell \sum_{(w, \ell)=1} \frac{1}{w} \Psi\left(\frac{w}{W}\right) \sum_{(n, bw)=1} g_2\left(\frac{n}{N}\right) \sum_{0 < |h| \leq H} \widehat{g}_1\left(\frac{Mh}{bw}\right) e\left(-\frac{\bar{b}wh}{\ell n}\right) \\ &\quad + O(Q^{1+\varepsilon} + LNR^{3/2} Q^\varepsilon). \end{aligned}$$

We next separate the variables  $h$  and  $w$ . We change variables to write

$$\widehat{g}_1\left(\frac{Mh}{bw}\right) = \frac{w}{M} \int_{\mathbb{R}} g_1\left(\frac{wy}{M}\right) e\left(-\frac{hy}{b}\right) dy.$$

Since  $g_1$  and  $\Psi$  are test functions, the integral is restricted to  $y \asymp M/W$ . We move the integral to the outside to write the first term of the right-hand side of (19) as

$$(20) \ll \frac{M}{bW} \sup_{y \asymp M/W} \left| \sum_{\ell} \beta_\ell \sum_{0 < |h| \leq H} e\left(-\frac{hy}{b}\right) \sum_w \sum_n \Psi\left(\frac{w}{W}\right) g_1\left(\frac{wy}{M}\right) g_2\left(\frac{n}{N}\right) e\left(-\frac{\bar{b}wh}{\ell n}\right) \right|.$$

We then use [DI82, Theorem 12], amended as described in [BFI], more specifically, with the dictionary (the bold symbols denote the variables names from [DI82])

$$\begin{aligned} \mathbf{c}, \mathbf{C} &\leftrightarrow n, N, & \mathbf{d}, \mathbf{D} &\leftrightarrow w, W, \\ \mathbf{n}, \mathbf{N} &\leftrightarrow h, H, & \mathbf{r}, \mathbf{R} &\leftrightarrow b', b, \\ \mathbf{s}, \mathbf{S} &\leftrightarrow \ell, L, & \mathbf{b}_{\mathbf{n}, \mathbf{r}, \mathbf{s}} &\leftrightarrow \mathbf{1}_{b'=b} e(-hy/b) \beta_\ell. \end{aligned}$$

Since  $\lambda < 1/6$ , we have  $H \ll L$  if  $\varepsilon$  is sufficiently small. Therefore, with the same notations, we find the bounds

$$\begin{aligned} \mathbf{K}(\mathbf{C}, \mathbf{D}, \mathbf{N}, \mathbf{R}, \mathbf{S}) &\ll b(NL^2(N+W) + N^2WL^{3/2} + W^2H)^{1/2}, \\ \|\mathbf{b}_{\mathbf{N}, \mathbf{R}, \mathbf{S}}\|_2 &\ll L^\varepsilon (HL)^{1/2}. \end{aligned}$$

It will also be easier to sum up the bounds if we assume

$$(21) \quad N \ll W^{1+\varepsilon}.$$

We find

$$\begin{aligned} T_2(M, N, L) &\ll LNR^{3/2}Q^\varepsilon + Q^\varepsilon \left( \sqrt{X}L + \sqrt{M}NL^{5/4} + L^{1/2}W \right) \\ &\ll LNR^{3/2}Q^\varepsilon + Q^\varepsilon \left( \sqrt{X}L + \sqrt{M}NL^{5/4} \right), \end{aligned}$$

the second inequality following since  $L^{1/2}W \ll X^{1/2}L$ . This contribution is acceptable provided

$$(22) \quad M \gg X^{\frac{\varpi}{2(2+\varpi)} + \frac{3}{2}\rho + \varepsilon}, \quad MN \gg X^{\frac{1}{2} + \frac{\varpi}{2(2+\varpi)} + \varepsilon}$$

and

$$(23) \quad M^{3/2}N^{1/2} \gg X^{\frac{1}{2} + \frac{\varpi}{2+\varpi} + 2\varepsilon}.$$

The bounds (21)–(23) are satisfied if

$$(24) \quad \delta < \frac{1}{12} - \frac{\varpi}{2(2+\varpi)}, \quad \lambda < \frac{1}{6} - \frac{\varpi}{2(2+\varpi)}, \quad \rho < \frac{1}{6}.$$

We therefore conclude the following.

**Lemma 12.** *Under the notations and hypotheses of Corollary 10, and assuming (24), we have*

$$T_2 \ll Q^{1-\varepsilon} \sqrt{X}.$$

The implied constant depends on  $\lambda, \delta, \rho$  and  $\varpi$ .

**4.4. Type II sums.** In the type II case (17), we wish to prove the bound

$$T_{\text{II}}(M, N) := \sum_w \Psi\left(\frac{w}{W}\right) \sum_{m,n} \alpha_m \beta_n \mathbf{u}_R(mn, bw) \ll \sqrt{X} Q^{1-\varepsilon},$$

where  $\alpha$  is supported at scale  $M$ ,  $\beta$  is supported at scale  $N$ ,  $MN \asymp X$ , and  $X^\sigma \ll N \ll X^{1/3-\delta}$ . We have  $|\alpha_m| \leq \tau(m)^{O(1)}$ , and similarly for  $\beta$ . We use Linnik's dispersion method [Lin63], following closely [Fou85]; see also [BFI86, Section 10].

We interchange the order of summation and apply the triangle inequality, writing our sum as

$$|T_{\text{II}}(M, N)| \leq \sum_m \left| \sum_w \sum_n \right|.$$

Applying the Cauchy-Schwarz inequality, we arrive at

$$(25) \quad T_{\text{II}}(M, N)^2 \ll M(\log M)^{O(1)} \mathcal{D},$$

where

$$\mathcal{D} = \sum_m f\left(\frac{m}{M}\right) \left| \sum_{\substack{n, w \\ mn \equiv 1 \pmod{bw}}} \Psi\left(\frac{w}{W}\right) \beta_n - \frac{1}{\varphi(bw)} \sum_{\substack{\chi \pmod{bw} \\ \text{cond}(\chi) \leq R}} \sum_{\substack{n, w \\ mn, bw = 1}} \Psi\left(\frac{w}{W}\right) \beta_n \chi(mn) \right|^2.$$

Here  $f$  is some fixed, non-negative test function majorizing  $\mathbf{1}_{[1,2]}$ . It suffices to show that

$$\mathcal{D} \ll NQ^{2-\varepsilon}.$$

We open the square and arrive at

$$(26) \quad \mathcal{D} = \mathcal{D}_1 - 2 \operatorname{Re} \mathcal{D}_2 + \mathcal{D}_3,$$

say. We treat each sum  $\mathcal{D}_i$  in turn.

4.4.1. *Evaluation of  $\mathcal{D}_3$ .* By definition we have

$$\mathcal{D}_3 := \sum_m f\left(\frac{m}{M}\right) \sum_{\substack{w_1, w_2, n_1, n_2 \\ (mn_1, bw_1) = 1 \\ (mn_2, bw_2) = 1}} \sum_{\substack{\chi_1, \chi_2 \\ \chi_j \pmod{bw_j} \\ \text{cond}(\chi_j) \leq R}} \Psi\left(\frac{w_1}{W}\right) \Psi\left(\frac{w_2}{W}\right) \beta_{n_1} \overline{\beta_{n_2}} \frac{\chi_1(mn_1) \overline{\chi_2(mn_2)}}{\varphi(bw_1) \varphi(bw_2)}.$$

The computations in [Dra17, p. 711–712] can be directly quoted, putting formally

$$(27) \quad \gamma(q) = \mathbf{1}(b \mid q) \Psi(q/(bW)),$$

with the modification that  $\text{cond}(\chi_1 \overline{\chi_2}) \leq R^2$  (instead of  $R$ , as stated incorrectly in [Dra17]).

Writing  $H = Q^\varepsilon b[w_1, w_2] M^{-1}$ , we get

$$\begin{aligned} \mathcal{D}_3 &= \mathcal{M}_3 + O\left(Q^\varepsilon \sum_{\substack{w_1, w_2 \asymp W \\ n_1, n_2 \asymp N}} \frac{1}{\varphi(bw_1) \varphi(bw_2)} \sum_{\substack{\chi_1, \chi_2 \\ \text{cond}(\chi_j) \leq R}} \frac{M}{b[w_1, w_2]} \sum_{0 < |h| \leq H} R \sum_{d \mid (h, b[w_1, w_2])} d\right) \\ &= \mathcal{M}_3 + O(Q^\varepsilon N^2 R^5), \end{aligned}$$

where the main term is computed as in [Dra17, p. 712] to be

$$\mathcal{M}_3 := M \widehat{f}(0) \sum_{\substack{w_1, w_2, n_1, n_2 \\ (n_j, bw_j) = 1}} \sum_{\substack{\chi \text{ primitive} \\ \text{cond}(\chi) \leq R \\ \text{cond}(\chi) \mid b(w_1, w_2)}} \Psi\left(\frac{w_1}{W}\right) \Psi\left(\frac{w_2}{W}\right) \beta_{n_1} \overline{\beta_{n_2}} \chi(n_1 \overline{n_2}) \frac{\varphi(bw_1 w_2)}{bw_1 w_2 \varphi(bw_1) \varphi(bw_2)}.$$

The error term is acceptable provided

$$NR^5 \ll Q^{2-\varepsilon}.$$



Since  $N \ll X^{1/3}$  this is acceptable provided

$$(28) \quad \rho < \frac{4 - \varpi}{15(2 + \varpi)}.$$

4.4.2. *Evaluation of  $\mathcal{D}_2$ .* We have

$$\mathcal{D}_2 := \sum_{\substack{w_1, w_2, n_1, n_2 \\ (n_j, bw_j)=1}} \sum_{\substack{\chi \pmod{bw_2} \\ \text{cond}(\chi) \leq R}} \Psi\left(\frac{w_1}{W}\right) \Psi\left(\frac{w_2}{W}\right) \overline{\beta_{n_1}} \beta_{n_2} \frac{\chi(n_2)}{\varphi(bw_2)} \sum_{\substack{mn_1 \equiv 1 \pmod{bw_1} \\ (m, w_2)=1}} \chi(m) f\left(\frac{m}{M}\right).$$

The computations in [Dra17, p. 712–713] can be also quoted directly with the identification (27). We obtain

$$\mathcal{D}_2 = \mathcal{M}_3 + O(R^{3/2} N^2 Q^{1+\varepsilon}).$$

This is acceptable if

$$(29) \quad \rho < \frac{2}{3}\lambda + \frac{2(1 - \varpi)}{9(2 + \varpi)}.$$

4.4.3. *Evaluation of  $\mathcal{D}_1$ .* We have

$$\mathcal{D}_1 := \sum_{\substack{w_1, w_2, n_1, n_2 \\ (n_j, bw_j)=1 \\ n_1 \equiv n_2 \pmod{b}}} \Psi\left(\frac{w_1}{W}\right) \Psi\left(\frac{w_2}{W}\right) \beta_{n_1} \overline{\beta_{n_2}} \sum_{mn_j \equiv 1 \pmod{bw_j}} f\left(\frac{m}{M}\right).$$

We need to separate the variables  $w_1, w_2, n_1, n_2$  from each other, and this requires a subdivision of the variables. We decompose these variables uniquely, following [FR18], as follows:

$$\begin{cases} d = (n_1, n_2), \\ n_1 = dd_1\nu_1, \text{ with } d_1 \mid d^\infty \text{ and } (d, \nu_1) = 1, \\ n_2 = d\nu_2, \\ q_0 = (w_1, w_2), \\ w_i = q_0q_i, i \in \{1, 2\}. \end{cases}$$

The summation conditions imply

$$(dd_1\nu_1, q_0q_1) = (d\nu_2, q_0q_2) = 1.$$

We therefore have

$$\begin{aligned} \mathcal{D}_1 &= \sum_{(d,b)=1} \sum_{d_1 \mid d^\infty} \sum_{(q_0,d)=1} \mathcal{D}_1(d, d_1, q_0), \\ \mathcal{D}_1(\dots) &= \sum_{\substack{q_1, q_2, \nu_1, \nu_2 \\ (d\nu_1, \nu_2) = (q_1, q_2) = 1 \\ (q_1q_2, d) = (\nu_1, d) = 1 \\ (\nu_1, q_1) = (\nu_2, q_2) = (\nu_1\nu_2, bq_0) = 1 \\ d_1\nu_1 \equiv \nu_2 \pmod{bq_0}}} \Psi\left(\frac{q_0q_1}{W}\right) \Psi\left(\frac{q_0q_2}{W}\right) \beta_{dd_1\nu_1} \overline{\beta_{d\nu_2}} \sum_{\substack{mdd_1\nu_1 \equiv 1 \pmod{bq_0q_1} \\ md\nu_2 \equiv 1 \pmod{bq_0q_2}}} f\left(\frac{m}{M}\right). \end{aligned}$$

Using smooth partitions of unity we break the variables into dyadic ranges:  $d \asymp D, d_1 \asymp D_1, q_0 \asymp Q_0$ . The contribution from  $d \asymp D$  and  $d_1 \asymp D_1$  is

$$\begin{aligned} &\ll Q^\varepsilon M \sum_{d \asymp D} \sum_{\substack{d_1 | d^\infty \\ d_1 \asymp D_1}} \sum_{\substack{\nu_1 \asymp N/dd_1 \\ \nu_2 \asymp N/d}} \sum_{\nu_2 \asymp N/d} |\beta_{dd_1\nu_1}| |\beta_{d\nu_2}| \ll Q^\varepsilon MN^2 \sum_{d \asymp D} \frac{1}{d^2} \sum_{d_1 | d^\infty} \frac{\tau(d_1)^{O(1)}}{d_1} \left(\frac{d_1}{D_1}\right)^{1-\varepsilon^2} \\ &\ll Q^\varepsilon MN^2 D_1^{-1+\varepsilon^2} D^{-1}, \end{aligned}$$

where the sum over  $q_0, q_1$  was bounded by  $O(\tau_3(|mdd_1\nu_1 - 1|)) = O(Q^\varepsilon)$ , likewise for the sum over  $q_2$  (note that  $md\nu_2 \neq 1$  and  $mdd_1\nu_1 \neq 1$ ). This bound is acceptable provided

$$(30) \quad DD_1 \gg \frac{X}{Q^{2-\varepsilon}},$$

so we may henceforth assume  $DD_1 \ll XQ^{-2+\varepsilon}$ .

The contribution from  $q_0 \asymp Q_0$  is

$$\begin{aligned} &\ll Q^\varepsilon \sum_{q_0 \asymp Q_0} \sum_{q_1 \asymp Q/q_0} \sum_{\substack{n_1 \equiv n_2 \pmod{q_0} \\ n_j \asymp N}} \sum_{\substack{m \asymp M \\ m \equiv \bar{n}_1 \pmod{q_0 q_1}}} 1 \\ &\ll Q^\varepsilon M \sum_{q_0 \asymp Q_0} \sum_{q_1 \asymp Q/q_0} \frac{1}{q_0 q_1} \sum_{\substack{n_1 \equiv n_2 \pmod{q_0} \\ n_j \asymp N}} 1 \\ &\ll Q^\varepsilon (MN^2 Q_0^{-1} + MN), \end{aligned}$$

where in the first line the sum over  $q_2$  was again bounded by  $\tau(|md\nu_2 - 1|)$ . This is acceptable provided

$$(31) \quad N \gg \frac{X}{Q^{2-\varepsilon}}, \quad \text{and} \quad Q_0 \gg \frac{X}{Q^{2-\varepsilon}},$$

so we may henceforth assume  $Q_0 \ll XQ^{-2+\varepsilon}$ .

We use Poisson summation, following [Dra17, pp. 714–716]. Let

$$\tilde{q} = bq_0 q_1 q_2, \quad \mu \equiv \begin{cases} \overline{dd_1\nu_1} \pmod{bq_0 q_1}, \\ \overline{d\nu_2} \pmod{bq_0 q_2}. \end{cases}$$

Note that  $\tilde{q} \geq \frac{1}{2}W \gg Q^{1-\varepsilon}$ . With  $H = \tilde{q}^{1+\varepsilon} M^{-1} \ll Q^{2+\varepsilon}/(q_0 M)$ , we get for any fixed  $A > 0$

$$(32) \quad \sum_{m \equiv \mu \pmod{\tilde{q}}} f\left(\frac{m}{M}\right) = \frac{M}{\tilde{q}} \sum_{|h| \leq H} \hat{f}\left(\frac{hM}{\tilde{q}}\right) e\left(\frac{\mu h}{\tilde{q}}\right) + O(Q^{-A}).$$

The zero frequency in (32) contributes the main term, which, after summing over  $d, d_1, q_0$  (and reintegrating the values  $DD_1, Q_0$  larger than  $XQ^{-2+\varepsilon}$  which were discarded earlier), is given by

$$\mathcal{M}_1 := \frac{M}{b} \hat{f}(0) \sum_{\substack{w_1, w_2, n_1, n_2 \\ (n_j, bw_j)=1 \\ n_1 \equiv n_2 \pmod{b(w_1, w_2)}}} \Psi\left(\frac{w_1}{W}\right) \Psi\left(\frac{w_2}{W}\right) \beta_{n_1} \overline{\beta_{n_2}} \frac{1}{[w_1, w_2]}.$$

The error term in (32) induces in  $\mathcal{D}_1(d, d_1, q_0)$  a contribution

$$\ll Q^{-10} N^2,$$

and therefore in  $\mathcal{D}_1$  a contribution  $O(1)$ , which is acceptable.

We solve the congruence conditions on  $\mu$  by writing

$$d_1\nu_1 - \nu_2 = bq_0t, \quad \mu dd_1\nu_1 = 1 + bq_0q_1\ell, \quad \mu d\nu_2 = 1 + bq_0q_2m,$$

with  $t, \ell, m \in \mathbb{Z}$ . We deduce

$$\mu dt = q_1\ell - q_2m, \quad t = q_1\nu_2\ell - q_2d_1\nu_1m.$$

Then we have the equalities, modulo  $\mathbb{Z}$ ,

$$\begin{aligned} \frac{\mu}{\tilde{q}} &= \frac{\mu}{bq_0q_1q_2} = \frac{1}{dd_1\nu_1bq_0q_1q_2} + \frac{\ell}{dd_1\nu_1q_2} \\ &\equiv \frac{1}{dd_1\nu_1bq_0q_1q_2} + \frac{\ell dd_1}{\nu_1q_2} + \frac{\ell \overline{\nu_1q_2}}{dd_1} \\ &\equiv \frac{1}{dd_1\nu_1bq_0q_1q_2} + \frac{tq_1\nu_2\overline{dd_1}}{\nu_1q_2} - \frac{\overline{bq_0q_1\nu_1q_2}}{dd_1} \\ &\equiv \frac{1}{dd_1\nu_1bq_0q_1q_2} + \frac{d_1\nu_1 - \nu_2}{bq_0} \frac{\overline{q_1\nu_2\overline{dd_1}}}{\nu_1q_2} - \frac{\overline{bq_0q_1\nu_1q_2}}{dd_1}. \end{aligned}$$

By estimating trivially the first term, we have

$$(33) \quad e\left(\frac{h\mu}{\tilde{q}}\right) = e\left(h \frac{d_1\nu_1 - \nu_2}{bq_0} \frac{\overline{q_1\nu_2\overline{dd_1}}}{\nu_1q_2} - \frac{\overline{hbq_0q_1\nu_1q_2}}{dd_1}\right) + O\left(\frac{Hq_0}{NW^2}\right).$$

The error term here is  $\ll Q^\varepsilon X^{-1}$ , which contributes to  $\mathcal{D}_1(d, d_1, q_0)$  a quantity

$$\frac{Q^{2+\varepsilon}N}{Xq_0^2dd_1} \left(1 + \frac{N}{d}\right),$$

and upon summing over  $(d, d_1, q_0)$ , this contributes to  $\mathcal{D}_1$  a quantity  $O(Q^{2+\varepsilon}N^2X^{-1})$ . This error is acceptable if

$$(34) \quad N \ll Q^{2-\varepsilon}.$$

Then we insert the first term of (33) in (32), and insert the Fourier integral. The non-zero frequencies contribute a term

$$\begin{aligned} \mathcal{R}_1(d, d_1, q_0) &:= \frac{Mq_0}{bW^2} \int \sum_{\substack{q_1, q_2, \nu_1, \nu_2 \\ (d\nu_1, \nu_2) = (q_1, q_2) = 1 \\ (q_1q_2, d) = (\nu_1, d) = 1 \\ (\nu_1, q_1) = (\nu_2, q_2) = (\nu_1\nu_2, bq_0) = 1 \\ d_1\nu_1 \equiv \nu_2 \pmod{bq_0}}} \sum_{0 < |h| \leq H} \Psi\left(\frac{q_0q_1}{W}\right) \Psi\left(\frac{q_0q_2}{W}\right) \beta_{dd_1\nu_1} \overline{\beta_{d\nu_2}} \times \\ &\quad \times f\left(t \frac{q_0^2q_1q_2}{W^2}\right) e\left(h \frac{d_1\nu_1 - \nu_2}{bq_0} \frac{\overline{q_1\nu_2\overline{dd_1}}}{\nu_1q_2} - \frac{\overline{hbq_0q_1\nu_1q_2}}{dd_1}\right) e\left(\frac{-htMq_0}{bW^2}\right) dt. \end{aligned}$$

So far, we have obtained under the conditions (31) and (34) the bound

$$\mathcal{D}_1 = \mathcal{M}_1 + \mathcal{R}_1 + O(NQ^{2-\varepsilon}),$$

$$\mathcal{R}_1 := \sum_{\substack{Q_0, DD_1 \ll QX^{-2+\varepsilon} \\ Q, D, D_1 \text{ dyadic}}} \sum_{\substack{d \asymp D \\ d_1 \asymp D_1 \\ q_0 \asymp Q_0}} \mathcal{R}(d, d_1, q_0).$$

We now restrict the summation over  $q_1, q_2$  in residue classes both modulo  $dd_1$ , to account for the oscillatory factors. Let  $\lambda_1, \lambda_2 \in (\mathbb{Z}/dd_1\mathbb{Z})^\times$ , and

$$b_{\mathbf{n}, \mathbf{r}, \mathbf{s}} = \sum_{\substack{\nu_1 \\ \nu_2 \\ \nu_1 = \mathbf{s} \\ \nu_2 dd_1 = \mathbf{r} \\ (d\nu_1, \nu_2) = (\nu_1 \nu_2, bq_0) = 1 \\ (\nu_1, d) = 1 \\ d_1 \nu_1 \equiv \nu_2 \pmod{bq_0}}} \sum_{\substack{0 < |h| \leq H \\ h(d_1 \nu_1 - \nu_2) = bq_0 \mathbf{n}}} \beta_{dd_1 \nu_1} \overline{\beta_{d\nu_2}} e\left(-\frac{hbq_0 \lambda_1 \nu_1 \lambda_2}{dd_1} - \frac{htMq_0}{bW^2}\right),$$

$$g(\mathbf{c}, \mathbf{d}, \mathbf{n}, \mathbf{r}, \mathbf{s}) = \Psi\left(\frac{q_0 \mathbf{c}}{W}\right) \Psi\left(\frac{q_0 \mathbf{d}}{W}\right) f\left(\frac{tq_0^2 \mathbf{c} \mathbf{d}}{W^2}\right).$$

Then

$$\mathcal{R}_1(d, d_1, q_0) = \frac{Mq_0}{bW^2} \int_{t \asymp f^1} \sum_{\lambda_1, \lambda_2 \pmod{dd_1}^*} \tilde{\mathcal{R}}_1(t, (\lambda_j)) dt,$$

$$\tilde{\mathcal{R}}_1(t, (\lambda_j)) = \sum_{\substack{\mathbf{n}, \mathbf{r}, \mathbf{s}, \mathbf{c}, \mathbf{d} \\ \mathbf{c} \equiv \lambda_1, \mathbf{d} \equiv \lambda_2 \pmod{dd_1} \\ (\mathbf{s} \mathbf{c}, \mathbf{r} \mathbf{d} b d d_1) = 1}} b_{\mathbf{n}, \mathbf{r}, \mathbf{s}} g(\mathbf{c}, \mathbf{d}, \mathbf{n}, \mathbf{r}, \mathbf{s}) e\left(\frac{\overline{\mathbf{n} \mathbf{r} \mathbf{d}}}{\mathbf{s} \mathbf{c}}\right).$$

We apply Proposition 6, with sizes given by

$$\mathbf{C} = \mathbf{D} = \frac{W}{q_0}, \quad \mathbf{S} = \frac{N}{dd_1}, \quad \mathbf{R} = Nd_1, \quad \mathbf{N} = \frac{HN}{dbq_0}.$$

Let  $X = Q^2 Y$ , so  $Y = Q^\varpi$ . Note that

$$\mathbf{R} \mathbf{S} \asymp N^2 D^{-1}, \quad \mathbf{N} \ll Q^\varepsilon N^2 Y^{-1} D^{-1} Q_0^{-2} \ll Q^\varepsilon \mathbf{R} \mathbf{S}, \quad \mathbf{C} \ll Q^\varepsilon \mathbf{R} \mathbf{D}.$$

We get

$$\tilde{\mathcal{R}}_1(t, \lambda_j) \ll Q^\varepsilon (DD_1)^{3/2} K \|b_{\mathbf{N}, \mathbf{R}, \mathbf{S}}\|_2,$$

where

$$Q^{-\varepsilon} K^2 \ll Q^2 N^4 D^{-1} D_1 Q_0^{-2} + Q^{2+4\theta} N^{4-6\theta} D^{-2+2\theta} D_1^{-2\theta} Q_0^{-2-4\theta} + Q^2 N^3 Y^{-1} D^{-1} D_1 Q_0^{-4}.$$

To bound the term  $\|b_{\mathbf{N}, \mathbf{R}, \mathbf{S}}\|_2$ , we assume

$$(35) \quad XQ^{-2+\varepsilon} = o(N),$$

so that  $D = o(N)$  by virtue of the line below (30), and the case  $d_1 \nu_1 = \nu_2$  never occurs in  $b_{\mathbf{n}, \mathbf{r}, \mathbf{s}}$ . Then

$$\begin{aligned} \|b_{\mathbf{N}, \mathbf{R}, \mathbf{S}}\|_2^2 &\leq \sum_{\substack{\nu_1, \nu_2, h \\ d_1 \nu_1 \equiv \nu_2 \pmod{q_0} \\ 0 < |h| < H}} |\beta_{dd_1 \nu_1} \beta_{d\nu_2}|^2 \ll \frac{Q^{2+\varepsilon}}{Q_0 M} \frac{N}{DD_1} \left(\frac{N}{DQ_0} + 1\right) \\ &\ll Q^\varepsilon (N^3 Y^{-1} D^{-2} D_1^{-1} Q_0^{-2} + N^2 Y^{-1} D^{-1} D_1^{-1} Q_0^{-1}). \end{aligned}$$

We deduce

$$\tilde{\mathcal{R}}_1(t, (\lambda_j)) \ll Q^\varepsilon \sum_{k=1}^6 Q^{\eta_{k,1}} N^{\eta_{k,2}} Y^{\eta_{k,3}} D^{\eta_{k,4}} D_1^{\eta_{k,5}} Q_0^{\eta_{k,6}},$$

where for each  $k$ ,  $\eta_k = (\eta_{k,\ell})_{1 \leq \ell \leq 6}$  is given by

$$\{\eta_k\} = \left\{ \begin{pmatrix} 1 \\ 3 \\ -1/2 \\ 1/2 \\ 3/2 \\ -3/2 \end{pmatrix}, \begin{pmatrix} 1 \\ 7/2 \\ -1/2 \\ 0 \\ 3/2 \\ -2 \end{pmatrix}, \begin{pmatrix} 2\theta + 1 \\ 3 - 3\theta \\ -1/2 \\ \theta \\ 1 - \theta \\ -2\theta - 3/2 \end{pmatrix}, \begin{pmatrix} 2\theta + 1 \\ 7/2 - 3\theta \\ -1/2 \\ \theta - 1/2 \\ 1 - \theta \\ -2\theta - 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 5/2 \\ -1 \\ 1/2 \\ 3/2 \\ -5/2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -1 \\ 0 \\ 3/2 \\ -3 \end{pmatrix} \right\}.$$

Summing over  $\lambda_j$ , integrating over  $t$ , and multiplying by  $\frac{Mq_0}{bW^2} \ll Q^\varepsilon N^{-1} Y Q_0$ , we get

$$\mathcal{R}_1(d, d_1, q_0) \ll Q^\varepsilon \sum_{k=1}^6 Q^{\eta_{k,1}} N^{\eta_{k,2}-1} Y^{\eta_{k,3}+1} D^{\eta_{k,4}+2} D_1^{\eta_{k,5}+2} Q_0^{\eta_{k,6}+1}.$$

We sum over  $d, d_1$  and  $q_0$  in dyadic intervals of lengths  $D, D_1$  and  $Q_0$ , obtaining

$$\sum_{\substack{d \asymp D \\ d_1 \asymp D_1, d_1 | d^\infty \\ q_0 \asymp Q_0 \\ (d,b)=(q_0,d)=1}} \mathcal{R}_1(d, d_1, q_0) \ll Q^\varepsilon \sum_{k=1}^6 Q^{\eta_{k,1}} N^{\eta_{k,2}-1} Y^{\eta_{k,3}+1} D^{\eta_{k,4}+3} D_1^{\eta_{k,5}+2} Q_0^{\eta_{k,6}+2}.$$

Finally we sum this dyadically over  $Q_0, D, D_1$  subject to  $Q_0 + DD_1 \ll YQ^\varepsilon$ . We get

$$\mathcal{R}_1 \ll Q^\varepsilon \sum_{k=1}^6 Q^{\eta_{k,1}} N^{\eta_{k,2}-1} Y^{\eta_{k,3}+1+\max(0,\eta_{k,6}+2)+\max(0,\eta_{k,4}+3,\eta_{k,5}+2)}.$$

Here, the terms for  $k = 5, 6$  are majorized by the term  $k = 1$ , therefore,

$$\mathcal{R}_1 \ll Q^\varepsilon \sum_{k=1}^4 Q^{\theta_{k,1}} N^{\theta_{k,2}} Y^{\theta_{k,3}},$$

where

$$\{\theta_k\} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 9/2 \end{pmatrix}, \begin{pmatrix} 1 \\ 5/2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1+2\theta \\ 2-3\theta \\ 4-\theta \end{pmatrix}, \begin{pmatrix} 1+2\theta \\ 5/2-3\theta \\ 7/2-\theta \end{pmatrix} \right\}.$$

We conclude that

$$\mathcal{D}_1 = \mathcal{M}_1 + O(Q^{2-\varepsilon} N)$$

on the condition  $N \ll Q^{-\varepsilon} \min(QY^{-9/2}, Q^{2/3}Y^{-8/3}, Q^{\frac{1-2\theta}{1-3\theta}}Y^{-\frac{4-\theta}{1-3\theta}}, Q^{2/3}Y^{-\frac{7-2\theta}{3(1-2\theta)}})$ . Upon using  $\theta \leq 7/64$ , these conditions are implied by

$$(36) \quad N \ll X^{-\varepsilon} \min(X^{\frac{2-9\varpi}{2(2+\varpi)}}, X^{\frac{50-249\varpi}{43(2+\varpi)}}, X^{\frac{50-217\varpi}{75(2+\varpi)}}),$$

and the above hypotheses (31), (34), (35).

4.4.4. *Main terms.* The main terms  $\mathcal{M}_1$  and  $\mathcal{M}_3$ , which are real numbers by the symmetry  $n_1 \leftrightarrow n_2$ , combine to form

$$\begin{aligned} \mathcal{M}_1 - \mathcal{M}_3 &= M\widehat{f}(0) \sum_{w_1, w_2} \sum \Psi\left(\frac{w_1}{W}\right) \Psi\left(\frac{w_2}{W}\right) \frac{1}{b[w_1, w_2] \varphi(b(w_1, w_2))} \\ &\quad \times \sum_{\substack{\chi \text{ prim} \\ \text{cond}(\chi) > R \\ \text{cond}(\chi) | b(w_1, w_2)}} \sum_{\substack{n_1, n_2 \\ (n_j, bw_j) = 1}} \beta_{n_1} \overline{\beta_{n_2}} \chi(n_1) \overline{\chi(n_2)}. \end{aligned}$$

We may quote the computations in [Dra17, p. 717], again with the identification (27), to obtain

$$|\mathcal{M}_3 - \mathcal{M}_1| \ll Q^\varepsilon M(N + N^2 R^{-2}) \ll Q^\varepsilon (X + NXR^{-2}).$$

This is acceptable provided

$$(37) \quad N \gg Q^{\varpi + \varepsilon}, \quad R \gg Q^{\frac{\varpi}{2} + \varepsilon}.$$

4.4.5. *Conclusion.* The hypotheses (28), (29), (31), (34), (35), (36) and (37) are all satisfied if

$$(38) \quad \varpi < 1/8, \quad \varpi < \sigma < \frac{1}{3} - \delta < \frac{1}{3} - \frac{242\varpi}{75(2 + \varpi)}, \quad \frac{\varpi}{2(2 + \varpi)} < \rho < \frac{1}{9} - \frac{\varpi}{3(2 + \varpi)}.$$

We therefore conclude the following.

**Lemma 13.** *Under the notations and hypotheses of Corollary 10, assuming (38), we have*

$$T_{II} \ll \sqrt{X} Q^{1 - \varepsilon}.$$

4.5. **Proof of Proposition 4.** We combine Lemmas 11, 12, 13 and 9. Setting  $\sigma = \varpi + \varepsilon$  and recalling that  $\varpi < 1/8$ , we obtain the conditions

$$\begin{aligned} \frac{\varpi}{3(2 + \varpi)} < \lambda < \frac{1}{6} - \frac{\varpi}{2}, \\ \frac{242\varpi}{75(2 + \varpi)} < \delta < \frac{1}{12} - \frac{\varpi}{2(2 + \varpi)}, \\ \frac{\varpi}{2(2 + \varpi)} < \rho < \frac{1}{9} - \frac{\varpi}{3(2 + \varpi)}. \end{aligned}$$

The third is implied by our hypothesis on  $R$ . The first two can be satisfied whenever  $-o(1) \leq \varpi < \frac{50}{1093} - o(1)$ . This proves Proposition 4.

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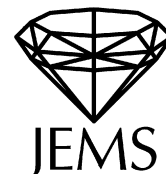
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# Niveau de répartition des polynômes quadratiques et crible majorant pour les entiers friables

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**Abstract.** We obtain new estimates on the level of distribution of the set  $\{Q(n)\}$  where  $Q \in \mathbb{Z}[X]$  is irreducible quadratic, for well-factorable moduli, improving a result due to Iwaniec. As a by-product of our arguments, we study the Chebyshev problem of estimating  $\max\{P^+(n^2 - D), n \leq x\}$  and make explicit, in Deshouillers–Iwaniec’s state-of-the-art result, the dependence on the Selberg eigenvalue conjecture.

Combined with the construction of an upper-bound sieve for numbers free of large factors, we obtain new upper bounds for the quantity  $\Psi_Q(x, y) = |\{n \leq x : p \mid Q(n) \Rightarrow p \leq y\}|$  for  $Q \in \mathbb{Z}[X]$  linear or quadratic.

**Keywords.** Polynomial values, quadratic congruences, arithmetic progressions, friable numbers

## 1. Introduction

### 1.1. Niveau de répartition de suites polynomiales et problème de Tchebychev

Soit  $Q$  un polynôme à coefficients entiers. La question d’estimer les “sommes de congruence”  $A_d(x) := |\{n \in \mathbb{Z} \cap [1, x] : Q(n) \equiv 0 \pmod{d}\}|$ , lorsque  $x \rightarrow +\infty$ , le plus uniformément possible par rapport à l’entier  $d \geq 1$ , est au cœur de la théorie multiplicative des nombres. Toute réponse partielle à cette question permet, en conjonction avec les méthodes de cribles [HR74, FI10], d’approcher la fréquence avec laquelle  $Q$  prend des valeurs sous une contrainte multiplicative : par exemples des valeurs premières [HR74, Theorem 2.6], ou ayant un grand facteur premier [Ten90b], ou encore n’ayant que des petits facteurs premiers, cas sur lequel nous nous concentrerons ci-dessous.

Dans le présent travail, nous nous intéressons à ce problème lorsque  $Q$  est quadratique, dans le cas particulier important des modules  $d$  pondérés par un poids “bien factorisable” au sens de la théorie du crible linéaire, qui a notamment été étudié dans [Iwa78].

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**Théorème 1.1.** Soient  $\eta > 0$ ,  $x \geq 1$ ,  $Q \leq x^{1+25/178-\eta}$ , et  $(\lambda(q))$  une fonction arithmétique bornée et “bien factorisable” au sens de [FI83, p. 199]. Soit  $D \in \mathbb{Z}$  qui n’est pas un carré d’entier, et  $V : \mathbb{R} \rightarrow \mathbb{C}$  une fonction lisse à support compact inclus dans  $\mathbb{R}_+^*$ . Alors

$$\sum_{q \leq Q} \lambda(q) \left( \sum_{\substack{n \in \mathbb{N} \\ q|n^2-D}} V\left(\frac{n}{x}\right) - x \widehat{V}(0) \frac{\rho(q)}{q} \right) \ll_{\eta, V, D} x^{1-\eta/3} \tag{1.1}$$

avec  $\widehat{V}(0) = \int_{\mathbb{R}} V(t) dt$  et  $\rho(q) := |\{v \pmod q : v^2 \equiv D \pmod q\}|$ .

Le gain  $25/178 = 1/7,12$  améliore le gain correspondant  $1/15$  de [Iwa78] (voir aussi [LO12]), ainsi que le gain  $1/9$  qui découlerait d’une utilisation de la conjecture  $R^*$  de Hooley sur les sommes d’exponentielles incomplètes (cf. [Iwa78, p. 185]). Pour obtenir cette amélioration, nous faisons appel à des majorations de type “grand crible” sur les coefficients de Fourier des formes cuspidales de  $GL_2$ , adaptant un argument de Tóth [Tót00], en utilisant une récente extension [Dra17] par le second auteur des majorations de sommes d’exponentielles de Deshouillers–Iwaniec [DI83].

Les méthodes qui sous-tendent le Théorème 1.1 s’appliquent naturellement au problème de Tchebychev de minorer la fonction  $P_D(x) = P^+(\prod_{x < n \leq 2x} (n^2 - D))$ , pour l’historique duquel nous référons à [Hoo67] (voir aussi [Dar15]). Le dernier résultat en date sur cette question précise, dû à Deshouillers et Iwaniec [DI82], implique en particulier que

$$P_D(x) \geq x^{1,2024}$$

pour tout  $x$  suffisamment grand. D’un autre côté, la conjecture de Selberg sur les valeurs propres du Laplacien hyperbolique (cf. [DI83, section 1.3]) permettrait d’obtenir, pour tout  $\varepsilon > 0$  suffisamment petit, la valeur  $\sqrt{3/2} - \varepsilon \geq 1, 2247$  en exposant. Nos résultats permettent de rendre explicite, dans les arguments de Deshouillers–Iwaniec [DI82], la dépendance vis-à-vis de la conjecture de Selberg.

**Théorème 1.2.** Pour  $\theta \in [0, 1/4]$ , définissons  $\kappa(\theta) \in [1, 2]$  comme l’unique réel satisfaisant

$$\int_1^{\kappa(\theta)} \frac{t dt}{1 - 2\theta t} = \frac{1}{4(1 - 2\theta)}.$$

Pour tout  $\varepsilon > 0$  et  $D \in \mathbb{Z}$  qui n’est pas un carré d’entier, nous avons

$$P_D(x) \gg_{\varepsilon, D} x^{\kappa(\theta)-\varepsilon}$$

pour tout  $\theta \geq 0$  qui est admissible pour la conjecture de Ramanujan–Selberg. En particulier,  $\theta = 7/64$  convient [Kim03]; nous avons donc pour  $x$  suffisamment grand

$$P_D(x) \geq x^{1,2182}. \tag{1.2}$$

### 1.2. Crible majorant pour les entiers friables

Notre application principale qui motive le Théorème 1.1 concerne la majoration de la fréquence avec laquelle  $Q$  prend des valeurs sans grand facteur premier. Nous ne supposons plus nécessairement que  $Q$  est quadratique. Dans le présent travail, nous améliorons les résultats connus dans le cas où  $Q$  est linéaire ou produit de deux facteurs linéaires, et nous obtenons les premiers résultats non triviaux sur cette question lorsque  $Q$  est quadratique irréductible.

On dit qu'un entier  $n$  est  $y$ -friable si, et seulement si, son plus grand facteur premier noté  $P^+(n)$  est  $\leq y$ . Nous adopterons la convention  $P^+(1) = 1$ . Les travaux d'Hildebrand et Tenenbaum [HT86] complétés par ceux de Saias [Sai89] permettent d'évaluer asymptotiquement  $\Psi(x, y) := \text{card } S(x, y)$  où  $S(x, y) := \{n \leq x : P^+(n) \leq y\}$  dans un large domaine en  $y$ .

Nous souhaitons estimer la densité des entiers  $y$ -friables dans la suite  $\{Q(n)\}$ . Malheureusement seul le cas des polynômes de degré 1 a été pour l'instant complètement résolu. Nous pourrions nous reporter à [FT91, Bre98] pour l'uniformité en fonction des coefficients. Dans le cas du degré  $\geq 2$ , même une majoration du bon ordre de grandeur semble intéressant et permettrait d'importantes applications.

Lorsque  $Q$  est un polynôme de  $\mathbb{Z}[X]$ , posons

$$\Psi_Q(x, y) := \text{card } \{n \leq x : P^+(Q(n)) \leq y\}.$$

Puisque l'on peut trivialement se ramener au cas des polynômes sans facteur carré, nous factorisons

$$Q(X) := \prod_{j=1}^r Q_j(X) \tag{1.3}$$

où  $Q_j$  sont des polynômes irréductibles non proportionnels de  $\mathbb{Z}[X]$  de degré  $d_j$ . Martin [Mar02] a conjecturé l'équivalent asymptotique

$$\Psi_Q(x, y) \sim x \prod_{j=1}^r \frac{\Psi(x^{d_j}, y)}{x^{d_j}}$$

dans un large domaine en  $y$ . En désignant par  $\varrho$  la fonction de Dickmann et en utilisant [Hil86], lorsque  $y \geq \exp\{(\log \log x)^{3/5+\varepsilon}\}$ , nous espérons donc

$$\Psi_Q(x, y) \sim x \prod_{j=1}^r \varrho(d_j u) \quad \text{où } u := (\log x)/\log y.$$

Lorsque  $(d_1, r) \neq (1, 1)$ , cette relation n'a été établie que dans certains cas, et lorsque  $u$  est restreint à de petits intervalles bornés. Nous renvoyons à la section 4.3 du survol [Gra08].

C'est donc à l'aune de cette prévision que l'on pourra mesurer la qualité des majorations que nous établirons. Ainsi, nous conjecturons

$$\Psi_Q(x, y) = x \rho(u)^{\sum_{j=1}^r d_j + o(1)} \tag{1.4}$$

lorsque  $u$  tend vers l'infini dans un large domaine en  $y$ .

Lorsque  $Q$  est un polynôme en plusieurs variables, il est possible d'obtenir certaines estimations asymptotiques et nous renvoyons le lecteur intéressé aux articles [BBDT12, Lac15, Lac18]. Dans le cas d'un polynôme univarié, qui nous intéresse ici, des minoration du bon ordre de grandeur  $\Psi_Q(x, y) \gg x$  sont établies par Dartyge, Martin et Tenenbaum [DMT01] pour des petites valeurs de  $u$ . Dans le cas particulier d'un produit de deux formes linéaires, Hildebrand [Hil85b] obtient une minoration du bon ordre de grandeur ; nous renvoyons à [BET90] pour une minoration effective. Enfin, lorsque  $Q$  est un produit quelconque de formes linéaires, une minoration qualitative mais dépendant explicitement de  $u$  est obtenue dans [BW98].

L'objectif du présent travail est de montrer des majorations de  $\Psi_Q(x, y)$  qui approchent aussi près que possible la taille conjecturée (1.4). Une conséquence de nos résultats simple à énoncer est la suivante.

**Théorème 1.3.** *Soit  $Q$  un polynôme de  $\mathbb{Z}[X]$  fixé avec la factorisation (1.3) et  $d_1 \leq \dots \leq d_r$ . Soit  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  telle que  $\lim_{x \rightarrow +\infty} \psi(x) = +\infty$ . Nous avons*

$$\Psi_Q(x, y) \ll x^{\varrho(u)^{c_Q+o(1)}}$$

lorsque  $x$  tend vers l'infini,  $(\log x)^{\psi(x)} \leq y \leq x$  et

$$\begin{aligned} c_Q &= \frac{8}{5} & (d_1 = d_2 = 1, r \geq 2), \\ c_Q &= \frac{3}{2} & (d_1 = 1, d_2 \geq 2), \\ c_Q &= 1 + \frac{25}{178} & (d_1 = 2), \\ c_Q &= 1 & (d_1 \geq 3, [\text{Kh}64]). \end{aligned}$$

Nous notons que  $\frac{25}{178} = \frac{1}{7,12} \approx 0,1404$ . Le dernier cas permet de retrouver un résultat de Khmyrova [Kh64], précisé ultérieurement par Timofeev [Tim77]. L'estimation du Théorème 1.3 dans le cas  $d_1 = 2$  est, à notre connaissance, la première faisant intervenir une valeur  $c_Q > 1$ .

Chacune de ces majorations repose sur une inégalité essentiellement du type

$$1_{P^+(n) \leq y}(n) \leq \sum_{\substack{d|n \\ P^+(d) \leq y \\ R/y < d \leq R}} 1 \quad (1.5)$$

pour tout  $R \geq n$ . Le choix du paramètre  $R$  doit être optimisé en fonction des résultats d'équirépartition à notre disposition. Dans le premier cas, il s'agira d'une rapide adaptation des récents travaux du second auteur, dans le deuxième, du théorème de Harper [Har12] de type Bombieri–Vinogradov pour les entiers friables. Dans le cas quadratique, nous utiliserons une variante du Théorème 1.1. Dans le dernier cas, nous choisirons  $R = x$  sans pouvoir utiliser de résultats spécifiques d'équirépartition.

Notons aussi que d'après [GoY06] (voir également [Sch68]), un polynôme  $Q$  avec la factorisation (1.3) et de degré au moins 2, ne peut pas prendre des valeurs trop friables : en effet, il découle du corollaire 4 de [GoY06] que  $P^+(Q(n)) \gg (\log_2 n)(\log_3 n)/\log_4 n$

lorsque  $n$  tend vers l'infini.<sup>1</sup> Ainsi, par exemple, la  $y$ -friabilité de  $n$  et celle de  $n + 1$  ne sont pas indépendantes lorsque  $y$  prend des très petites valeurs par rapport à la taille de  $n$ .

À titre d'illustration de l'efficacité de notre méthode dans le cas de deux facteurs linéaires, nous appliquons ces estimations à l'étude de l'ensemble  $\mathcal{C}$  des entiers divisibles par le carré de leur plus grand facteur premier,

$$\mathcal{C} := \{n \geq 1 : P^+(n)^2 \mid n\}.$$

Il est aisé d'établir que lorsque  $x \rightarrow +\infty$ ,  $|\mathcal{C} \cap [1, x]| = xe^{-(1+o(1))\sqrt{2(\log x) \log \log x}}$ . L'ensemble des entiers  $n$  tels que  $(n, n + 1) \in \mathcal{C}^2$  est en revanche beaucoup plus délicat à étudier. Nous établissons la majoration suivante, qui améliore [DKDL13].

**Corollaire 1.4.** *Lorsque  $x$  tend vers l'infini, l'estimation*

$$|\{n \leq x : (n, n + 1) \in \mathcal{C}^2\}| \leq xe^{-(c+o(1))\sqrt{2(\log x) \log \log x}} \quad (1.6)$$

a lieu avec  $c = 4/\sqrt{5} \approx 1,789$ .

**Remarque.** Dans [DKDL13], les auteurs obtiennent  $c = 25/24 \approx 1,042$ . Conjecturellement, la valeur optimale attendue est  $c = 2$ . Cela correspond à l'heuristique que les événements  $n \in \mathcal{C}$  et  $n + 1 \in \mathcal{C}$  surviennent de façon statistiquement indépendante.

Une autre application concerne la minoration du nombre d'entiers friables dans les petits intervalles établie au théorème 5 de [Gou17]. Lorsque  $\varepsilon \in ]0, 1/6[$ , il existe  $u_0 = u_0(\varepsilon)$  tel que lorsque  $x \geq 2$ ,  $u_0 \leq u \leq (\log x)^{1/6-\varepsilon}$  et  $\varrho(u)^{-(3+\varepsilon)} \leq h \leq \sqrt{x}$ , l'on ait

$$\Psi(x + h\sqrt{x}, x^{1/u}) - \Psi(x, x^{1/u}) \geq \varrho(u)^2 \frac{h\sqrt{x}}{(\log x)^3}.$$

La méthode repose de manière cruciale sur des majorations de cardinal d'ensembles d'entiers  $n$  tels que les valeurs de deux formes affines en  $n$  soient simultanément friables. Les majorations obtenues au Théorème 1.1 avec un exposant  $c_Q = 8/5$  dans le cas  $r = 2$ ,  $d_1 = d_2 = 1$  permettent d'obtenir des résultats intéressants à ce sujet et de remplacer l'exposant  $3 + \varepsilon$  par  $3/2 + \varepsilon$  dans la minoration de  $h$  ci-dessus. La valeur conjecturelle  $c_Q = 2$  fournirait un exposant  $1/2 + \varepsilon$ .

### Notations et plan

Lorsque  $(a, q) = 1$ , nous notons  $\bar{a} \pmod{q}$  la classe inverse de  $a$  modulo  $q$ . Le conducteur d'un caractère de Dirichlet  $\chi$  est noté  $\text{cond } \chi$ . Étant donnés  $m, n \in \mathbb{N}_{>0}$ , nous notons  $m \mid n^\infty$  lorsque tous les facteurs premiers de  $m$  divisent  $n$ . Ainsi, nous notons  $(m, n^\infty)$  le plus grand diviseur de  $m$  dont tous les facteurs premiers divisent  $n$ . La lettre  $\varrho$  désigne la fonction de Dickman, tandis que la notation  $\rho_Q(q)$  pour un polynôme  $Q$  et un entier  $q$  désigne le nombre de racines de  $Q$  modulo  $q$ . Nous noterons  $\rho_Q(q) = \rho(q)$  lorsque le contexte sera clair. Dans les sections 8.1.9, 8.1.10 et 8.1.11 uniquement, la

<sup>1</sup> Ici  $\log_k$  désigne la  $k$ -ième itérée du logarithme.

notation  $(\rho_{f,a}(n))$  désigne la suite des coefficients de Fourier d'une forme de Maass  $f$ . Pour  $k \geq 1$ , nous définissons la fonction diviseur généralisée  $\tau_k$  par  $\sum_n \tau_k(n)n^{-s} = \zeta(s)^k$  pour tout  $s > 1$ . Nous utiliserons l'abréviation  $e(x) = e^{2\pi i x}$  ( $x \in \mathbb{R}$ ).

Pour tout  $u \geq 1$ , nous notons

$$H(u) := \exp\left\{\frac{u}{(\log 2u)^2}\right\}.$$

Pour un ensemble  $\mathcal{I} \subset \mathbb{R}_+$ , nous notons

$$\Psi(\mathcal{I}, y; q, a) := |\{n \in \mathcal{I} : P^+(n) \leq y, n \equiv a \pmod{q}\}|,$$

$$\Psi_q(\mathcal{I}, y) := |\{n \in \mathcal{I} : P^+(n) \leq y, (n, q) = 1\}|,$$

$$\Psi(\mathcal{I}, y; \chi) := \sum_{n \in \mathcal{I}, P^+(n) \leq y} \chi(n).$$

Nous utiliserons à plusieurs reprises l'estimation

$$\Psi(x, y) = x \varrho(u) e^{O_\varepsilon(u)} \quad ((\log x)^{1+\varepsilon} \leq y \leq x), \tag{1.7}$$

qui découle directement des formules (2.6) et (2.7) de [HT86].

Le plan de ce travail est le suivant. Dans la section 2, nous établissons des résultats d'équirépartition des entiers friables en progressions arithmétiques de module friable. Dans la section 3, nous définissons une certaine fonction  $\vartheta(q)$ , qui jouera dans notre contexte le rôle d'un crible majorant pour les entiers friables. Dans la section 4, nous considérons le cas  $d_1 = 1$ , où nous mettons en jeu les résultats de la section 2. Dans la section 5, nous considérons le cas quadratique  $d_1 = 2$ , en supposant acquis le Théorème 1.1. Dans la section 6, nous étudions le cas  $d_1 \geq 3$ . Dans la section 7, nous déduisons le Corollaire 1.4. Enfin, dans la section 8 nous prouvons le Théorème 1.1.

## 2. Équirépartition en progressions arithmétiques

Les travaux de Harper et du second auteur permettent de démontrer une approximation de  $\Psi(\mathcal{I}, y_1; q, a)$  par  $\Psi_q(\mathcal{I}, y_1)/\varphi(q)$  en moyenne sur les modules  $q$ . En vue de nos applications, nous montrons dans cette section une version de ces résultats qui est restreinte aux modules  $q$  qui sont  $y_2$ -friables.

**Théorème 2.1.** *Soient  $\varepsilon, A > 0$  et  $k \geq 1$ . Il existe  $C, \delta > 0$  tels que pour  $x^\varepsilon \leq Q \leq x^{3/5-\varepsilon}$  et  $(x, \eta, y_1, y_2) \in \mathbb{R}^4$  avec  $(\log x)^C \leq y_1, y_2 \leq x, y_1 \leq x^{1/C}$  et  $\eta \in [e^{-\delta\sqrt{\log y}}, 1]$ , et pour tout entier non nul  $q_0 \leq x^\delta$  avec  $(q_0, a_1 a_2) = 1$  et  $P^+(q_0) \leq y_2$ , l'on ait*

$$\sum_{\substack{Q/2 < q \leq Q \\ P^+(q) \leq y_2 \\ (q, a_1 a_2) = 1}} \tau_k(q) \left| \Psi([x, (1 + \eta)x], y_1; q_0 q, a_1 \overline{a_2}) - \frac{\Psi_{q_0 q}([x, (1 + \eta)x], y_1)}{\varphi(q_0 q)} \right| \ll_{\varepsilon, A, k} \eta \Psi(x, y_1) \frac{\Psi(Q, y_2)}{\varphi(q_0) Q} \frac{e^{O_k(u_2)}}{(\log x)^A} \tag{2.1}$$

uniformément pour  $0 < |a_1|, |a_2| \leq x^\delta$ , où l'on a posé  $u_2 := (\log x)/\log y_2$ .

De plus, lorsque  $Q \leq x^{1/2-\varepsilon}$ , l'on a

$$\sum_{\substack{Q/2 < q \leq Q \\ P^+(q) \leq y_2}} \tau_k(q) \max_{(a, q_0q)=1} \left| \Psi([x, (1+\eta)x], y_1; q_0q, a) - \frac{\Psi_{q_0q}([x, (1+\eta)x], y_1)}{\varphi(q_0q)} \right| \ll_{\varepsilon, A, k} \eta \Psi(x, y_1) \frac{\Psi(Q, y_2)}{\varphi(q_0)Q} \frac{e^{O_k(u_2)}}{(\log x)^A}. \quad (2.2)$$

*Démonstration.* Les arguments de [Har12], notamment l'inégalité (3.1) et les calculs qui la suivent, montrent que pour prouver la borne annoncée, il suffit de montrer que

$$\sum_{\substack{Q/2 < q \leq Q \\ P^+(q) \leq y_2}} \frac{\tau_k(q)}{\varphi(q_0q)} \sum_{\substack{\chi \pmod{q_0q} \\ 1 < \text{cond } \chi \leq x^\delta}} |\Psi([x, (1+\eta)x], y_1; \chi)| \ll_{\varepsilon, A, k} \frac{\eta \Psi(x, y_1) \Psi(Q, y_2) e^{O_k(u_2)}}{\varphi(q_0)Q (\log x)^A} \quad (2.3)$$

où  $\chi$  désigne un caractère de Dirichlet, et  $\text{cond } \chi$  est le conducteur de  $\chi$ . La section 3.3 de l'article de Harper [Har12] traite du cas  $y_2 = Q$  et  $\eta = 1$ . Les changements nécessaires pour tenir compte des conditions  $P^+(q) \leq y_2$  sont mentionnés dans [LV18, Dra14]. Nous nous contentons donc d'indiquer que le point essentiel est que pour tout  $r \leq x^\delta$ ,

$$\sum_{\substack{Q/2 < q \leq Q \\ P^+(q) \leq y_2 \\ r|q}} \frac{\tau_k(q)}{\varphi(q)} \ll (e^{u_2} \log x)^{O_k(1)} \frac{\tau_k(r) r^{1-\alpha_2}}{\varphi(r)} \frac{\Psi(Q, y_2)}{Q}. \quad (2.4)$$

Ici,  $\alpha_2 = \alpha(x, y_2)$  et  $u_2 = (\log x)/\log y_2$ . Lorsque  $C$  est choisi suffisamment grand, par l'inégalité triangulaire et les calculs de la section 3.3 de [Har12], nous obtenons une borne

$$\ll_{\varepsilon} \frac{\Psi(x, y_1) \Psi(Q, y_2)}{\varphi(q_0)Q} (e^{u_2} \log x)^{O_k(1)} \{y_1^{-2\delta} + e^{-2\delta\sqrt{\log x}}\}$$

pour la contribution à la somme (2.3) provenant des caractères dont la série de Dirichlet n'a pas de zéro "de Siegel", c'est-à-dire trop proche de 1 (voir [Har12, Section 3.3, définition de  $\mathcal{G}_1$ ] pour la définition exacte). Étant donnée l'hypothèse  $\eta \geq e^{-\delta\sqrt{\log y}}$ , nous concluons que ces caractères fournissent une contribution acceptable.

Il reste alors à traiter la contribution  $\mathcal{S}_1$  au membre de gauche de (2.3) des caractères éventuels de conducteurs au plus  $\min\{y^\eta, e^{\eta\sqrt{\log x}}\}$  dont la série de Dirichlet possède un zéro réel supérieur à  $1 - M/\min\{\log y, \sqrt{\log x}\}$  pour des constantes absolues adéquates  $\eta$  et  $M$ . Si de tels caractères existent, ils sont tous induits par un unique caractère primitif  $\chi_1$ , de conducteur  $r_1 \leq \min\{y_1^b, e^{b\sqrt{\log x}}\}$  (notés respectivement  $\chi_{\text{bad}}^*$  et  $r_{\text{bad}}$  dans [Har12]), pour un certain réel  $b > 0$  :

$$\mathcal{S}_1 := \sum_{\substack{Q/2 < q \leq Q \\ P^+(q) \leq y_2, r_1|q}} \frac{\tau_k(q)}{\varphi(q_0q)} \sum_{\substack{\chi \pmod{q_0q} \\ \chi_1 \text{ induit } \chi}} |\Psi([x, (1+\eta)x], y_1; \chi)|. \quad (2.5)$$



Tout d’abord, nous observons que l’inégalité triangulaire permet de nous ramener trivialement au contexte de la section 3.3 de [Har12], ce qui fournit la borne

$$|\Psi([x, (1 + \eta)x], y_1; \chi)| \ll_\varepsilon \Psi(x, y_1)(\log x)^2 \tag{2.6}$$

pour  $C$  suffisamment grand en fonction de  $\varepsilon$ . Cette borne découle directement de la première équation, page 16 de [Har12] dans le cas  $x^{1/(\log \log x)^2} \leq y$ . Dans le cas complémentaire, nous utilisons l’inégalité de la dernière équation, page 17 de [Har12] ainsi que

$$\sum_{p|q_0} p^{-\alpha} \leq \sum_{p \leq p_*} p^{-\alpha} \ll \log \log p_* + p_*^{1-\alpha} \quad (\pi(p_*) = \omega(q_0)), \tag{2.7}$$

et la majoration élémentaire  $p_* \ll \log q_0 \ll \log x$ , qui implique  $p_*^{1-\alpha} \ll y_1^{(1-\alpha)/3} \ll u_1^{1/2}$ . Les majorations (2.4), (2.6) et la minoration  $r_1 \gg_A (\log x)^{10A}$  provenant du théorème de Siegel [Ten15, théorème II.8.33] fournissent

$$\mathcal{S}_1 \ll_{\varepsilon, A, k} \Psi(x, y_1) \frac{\Psi(Q, y_2)}{\varphi(q_0)Q} \frac{e^{O_k(u_2)}}{r_1^{1/2} H(u_1)^\delta (\log x)^A}. \tag{2.8}$$

Cette borne est acceptable si  $r_1 > e^{\delta\sqrt{\log y_1}}$ , ou bien si  $H(u_1) > e^{\sqrt{\log y_1}}$ .

Nous pouvons donc supposer dans ce qui suit que  $r_1 \leq e^{\delta\sqrt{\log y_1}}$  et  $H(u_1) \leq e^{\sqrt{\log y_1}}$ . En particulier, nous avons  $y_1 \gg_\varepsilon e^{(\log x)^{2/3-\varepsilon}}$ .

Nous reprenons les arguments de la section 6 de [FT91], et en particulier les équations (6.29) et (6.36). Celles-là fournissent

$$\Psi([x, (1 + \eta)x], y_1; \chi_1) \ll |F(\log((1 + \eta)x)) - F(\log x)| + \Psi(x, y_1)e^{-2\delta\sqrt{\log y_1}}$$

avec

$$F(t) := e^{\beta t} \int_0^\infty y_1^{-\beta v} \left\{ \omega\left(\frac{t}{\log y_1} - v\right) - e^{-v} \right\} K(y_1^v) dv, \quad K(t) := \sum_{n \leq t} \chi_1(n),$$

où  $\omega$  désigne la fonction de Buchstab [FT91, équation (6.5)] et  $\beta$  désigne le zéro exceptionnel, proche de 1, de  $L(s, \chi_1)$ . Pour  $e^t \notin y_1\mathbb{N}$ , nous avons

$$F'(t) = \beta F(t) + e^{\beta t} \int_0^\infty y_1^{-\beta v} \omega'\left(\frac{t}{\log y_1} - v\right) K(y_1^v) dv - y_1^\beta K(e^t/y_1)$$

où  $\omega'$  est prolongée par continuité à droite aux points où  $\omega$  est non dérivable. En utilisant le lemme 6.2 de [FT91], l’inégalité de Pólya–Vinogradov [Ten15, théorème II.8.10] et le fait que  $\eta \leq 1$ , il vient

$$\begin{aligned} &|F(\log((1 + \eta)x)) - F(\log x)| \\ &\ll \eta \left( x^\beta \int_0^\infty y_1^{-\beta v} \varrho(u_1 - v) H(u_1)^{-\delta} e^{2\delta v} |K(y_1^v)| dv + y_1^\beta \sqrt{r_1} \log r_1 \right) \ll \eta x^\beta \varrho(u_1) \end{aligned}$$



où nous avons utilisé l'hypothèse  $y_1 \leq x^{1/C}$  pour majorer le terme  $y_1^\beta \sqrt{r_1} \log r_1$ . Nous déduisons

$$\Psi([x, (1 + \eta)x], y_1; \chi_1) \ll \Psi(x, y_1) \{ \eta x^{\beta-1} \log x + e^{-\delta \sqrt{\log y_1}} \}.$$

Nous insérons cela dans (2.5), en utilisant encore (2.4) et (2.7). Nous obtenons

$$\mathcal{S}_1 \ll_{\varepsilon, A, k} \Psi(x, y_1) \frac{\Psi(Q, y_2)}{\varphi(q_0) Q} e^{O_k(u_2)} \{ \eta (\log x)^{-A} + e^{-\delta \sqrt{\log y_1}} \}. \quad (2.9)$$

En réinterprétant le paramètre  $\delta$ , la conjonction des deux bornes (2.8) et (2.9) implique donc la borne annoncée

$$\mathcal{S}_1 \ll_{\varepsilon, A, k} \eta \Psi(x, y_1) \frac{\Psi(Q, y_2)}{\varphi(q_0) Q} \frac{e^{O_k(u_2)}}{(\log x)^A}. \quad \square$$

### 3. Crible majorant pour les entiers friables

La fonction de  $n$  au membre de droite de l'inégalité (1.5) fonctionne de façon analogue à un crible, en ce qu'elle majore la fonction indicatrice d'un ensemble d'entiers au moyen d'informations sur les diviseurs de petite taille des éléments de cet ensemble. Dans cette section nous établissons la version précise de l'inégalité (1.5) que nous utiliserons.

**Proposition 3.1.** *Pour tous  $\varepsilon, \kappa > 0$ , il existe  $u_0 > 0$  tel que lorsque  $(R, y) \in \mathbb{R}^2$ ,  $(\log R)^{2+\varepsilon} \leq y \leq R^{1/u_0}$  et  $D \in \mathbb{N}$ , il existe une fonction  $\vartheta = \vartheta_{R, y, D, \kappa} : \mathbb{N} \rightarrow \mathbb{R}$  ayant les propriétés suivantes.*

- (i)  $|\vartheta(q)| \leq \tau_3(q)$  pour tout  $q \in \mathbb{N}$ .
- (ii)  $\vartheta(q) = 0$  si  $q \notin [R, y^4 R]$ ,  $(q, D) > 1$  ou  $P^+(q) > y$ .
- (iii) Pour tout  $n \in \mathbb{N}$ ,

$$\mathbf{1}_{P^+(n) \leq y} \leq \sum_{q|n} \vartheta(q)$$

si  $n/(n, D^\infty) \geq R$ , et  $\sum_{q|n} \vartheta(q) \geq 0$  dans tous les cas.

- (iv) Pour  $m \in \mathbb{N}$ , et  $f$  une fonction multiplicative avec  $0 \leq f(p^\nu) \leq \kappa$ ,

$$\left| \sum_{(q, m)=1} \frac{\vartheta(q) f(q)}{q} \right| \leq e^{O_{\varepsilon, \kappa}((\log R)/\log y)} \varrho \left( \frac{\log R}{\log y} \right).$$

La constante implicite dépend au plus de  $\varepsilon$  et  $\kappa$ .

*Démonstration.* Soit  $m := n/(n, D^\infty)$ , supposons  $m \geq R$  et écrivons  $m = p_1 \cdots p_J$  où les nombres  $p_j$  sont premiers et  $p_j \leq p_{j+1}$  pour tout  $j$ . En raisonnant comme dans [Khm64, Vau89], considérons la suite  $(c_j)_0^J$  de diviseurs de  $m$  définie par  $c_0 = 1$  et, pour  $j \geq 1$ ,  $c_j = p_1 \cdots p_j$ . Puisque par hypothèse  $c_J = m \geq R > 1$ , nous avons nécessairement  $c_{j-1} < R \leq c_j$  pour un certain  $j \geq 1$ . Notons  $d_0$  l'entier  $c_j$  ainsi formé. Il

s'agit d'un diviseur de  $n$ , premier avec  $D$ , satisfaisant les conditions  $R \leq d_0 < RP^+(d_0)$  et  $P^-(m/d_0) \geq P^+(d_0)$ . Nous avons donc

$$\mathbf{1}_{P^+(n) \leq y} \leq \sum_{\substack{R \leq d_0 < RP^+(d_0) \\ P^+(d_0) \leq y \\ d_0 | n, (d_0, D) = 1}} \mathbf{1}_{P^-(m/d_0) \geq P^+(d_0)}. \tag{3.1}$$

Cette majoration implique bien (1.5). Il est cependant important, pour les grandes valeurs de  $y$ , de ne pas oublier la condition  $P^-(m/d_0) \geq P^+(d_0)$ , au risque d'engendrer un facteur supplémentaire de l'ordre de  $\log y$  dans nos applications. Pour contourner cela, pour chaque entier  $d_0$  apparaissant dans (3.1), nous introduisons des coefficients de crible majorant  $(\xi_{d_1})$  de niveau  $y^2$  et dimension  $\kappa$ , pour les nombres premiers  $p < P^+(d_0)$ , en suivant la construction donnée par exemple dans [FI78, lemme 5] où  $s$  et  $z$  sont remplacés respectivement par 2 et  $P^+(d_0)$ . En particulier, on a  $\xi_1 = 1$ ,  $|\xi_{d_1}| \leq 1$ , et

$$\xi_{d_1} \neq 0 \implies \begin{cases} d_1 \leq y^2, \\ P^+(d_1) < P^+(d_0), \\ \mu(d_1)^2 = 1. \end{cases}$$

Enfin, ces coefficients vérifient la majoration uniforme [FI78, lemme 5]

$$\sum_{(d_1, m) = 1} \frac{\xi_{d_1} f(d_1)}{d_1} \ll_{\kappa} \prod_{\substack{p < P^+(d_0) \\ p \nmid m}} \left(1 + \frac{f(p)}{p}\right)^{-1} \quad (m \in \mathbb{N}),$$

lorsque la fonction  $f$  est multiplicative avec  $0 \leq f(p^v) \leq \kappa$ . Notons que ces propriétés impliquent bien

$$\mathbf{1}_{P^-(m/d_0) \geq P^+(d_0)} \leq \sum_{\substack{d_1 | n/d_0 \\ (d_1, d_0 D) = 1}} \xi_{d_1}.$$

Nous insérons cette majoration dans (3.1). En posant, pour tout  $q \in \mathbb{N}$ ,

$$\vartheta(q) := \sum_{\substack{q = d_0 d_1 \\ R \leq d_0 < RP^+(d_0) \\ P^+(d_0) \leq y \\ (d_0 d_1, D) = (d_0, d_1) = 1}} \xi_{d_1}, \tag{3.2}$$

il résulte de ce qui précède que

$$\mathbf{1}_{P^+(n) \leq y} \leq \sum_{\substack{R \leq d_0 < RP^+(d_0) \\ P^+(d_0) \leq y \\ d_0 | n, (d_0, D) = 1}} \sum_{\substack{d_1 | n/d_0 \\ (d_1, d_0 D) = 1}} \xi_{d_1} = \sum_{q | n} \vartheta(q).$$

Il reste à montrer la propriété (iv) qui porte sur une majoration en moyenne. Nous avons

$$S := \sum_{(q,m)=1} \frac{\vartheta(q)f(q)}{q} = \sum_{\substack{R \leq d_0 < RP^+(d_0) \\ (d_0,m)=1}} \frac{f(d_0)}{d_0} \sum_{(d_1,md_0D)=1} \frac{\xi_{d_1} f(d_1)}{d_1} \\ \ll_{\kappa} \sum_{\substack{R \leq d_0 < RP^+(d_0) \\ (d_0,m)=1}} \frac{f(d_0)}{d_0} \prod_{\substack{p < P^+(d_0) \\ p \nmid md_0D}} \left(1 + \frac{f(p)}{p}\right)^{-1}.$$

Nous séparons la somme sur  $d_0$  suivant  $\ell = P^+(d_0)$ ,  $\ell^v \parallel d_0$ . Nous obtenons

$$S \ll_{\kappa} \sum_{\substack{\ell \text{ premier} \\ \ell \leq y \\ \ell^{v-1} \leq R}} \frac{f(\ell^v)}{\ell^v} \prod_{\substack{p < \ell \\ p \nmid mD}} \left(1 + \frac{f(p)}{p}\right)^{-1} \sum_{\substack{R/\ell^v \leq d < R/\ell^{v-1} \\ P^+(d) < \ell \\ (d,mD)=1}} \frac{f(d)}{d} \prod_{p|d} \left(1 + \frac{f(p)}{p}\right),$$

Notons  $\beta_{\ell} = \alpha(R/\ell, \ell)$  si  $\ell \geq (\log R)^{2+\varepsilon}$ , et  $\beta_{\ell} = \frac{1+\varepsilon}{2+\varepsilon} > \frac{1}{2}$  sinon. La somme sur  $d$  ci-dessus est

$$\leq \left(\frac{R}{\ell^v}\right)^{-1+\beta_{\ell}} \sum_{\substack{P^+(d) < \ell \\ (d,mD)=1}} \frac{f(d)}{d^{\beta_{\ell}}} \prod_{p|d} \left(1 + \frac{f(p)}{p}\right) \ll_{\varepsilon, \kappa} \left(\frac{R}{\ell^v}\right)^{-1+\beta_{\ell}} \prod_{\substack{p < \ell \\ p \nmid mD}} \left(1 + \frac{f(p)}{p^{\beta_{\ell}}}\right)$$

où nous avons utilisé le fait que  $\beta_{\ell} > \min\left\{\frac{3}{4}, \frac{1}{2} + \frac{\varepsilon}{10}\right\}$  lorsque  $u_0$  (donc  $R$ ) est suffisamment grand. Nous avons donc

$$S \ll \sum_{\substack{\ell \text{ premier} \\ \ell \leq y \\ \ell^{v-1} \leq R}} \left(\frac{R}{\ell^v}\right)^{-1+\beta_{\ell}} \frac{1}{\ell^v} \prod_{\substack{p < \ell \\ p \nmid mD}} \left(1 + O_{\kappa}\left(\frac{p^{1-\beta_{\ell}} - 1}{p}\right)\right) \\ \ll \sum_{\substack{\ell \text{ premier} \\ \ell \leq y}} \left(\frac{R}{\ell}\right)^{-1+\beta_{\ell}} \frac{1}{\ell} \prod_{\substack{p < \ell \\ p \nmid mD}} \left(1 + O_{\kappa}\left(\frac{p^{1-\beta_{\ell}} - 1}{p}\right)\right). \quad (3.3)$$

Le produit eulérien est borné par

$$\exp\left\{O_{\kappa}\left(\sum_{p < \ell} \frac{p^{1-\beta_{\ell}} - 1}{p}\right)\right\} = \exp\left\{O_{\kappa}\left(\min\left\{\frac{\log R}{\log \ell}, \frac{\ell^{1/(2+\varepsilon)}}{\log \ell}\right\}\right)\right\}. \quad (3.4)$$

en utilisant [BT05, lemme 3.6]. Nous insérons cette majoration dans le membre de droite de (3.3), et notons  $S_1 + S_2$  la somme résultante, avec  $S_1$  la contribution de  $\ell \leq (\log R)^{2+\varepsilon}$  et  $S_2$  la contribution complémentaire. Ainsi, nous avons

$$S_1 \ll R^{-1/(2+\varepsilon)} \sum_{\substack{\ell \text{ premier} \\ \ell \leq (\log R)^{2+\varepsilon}}} \frac{e^{O_{\kappa}(\ell^{1/(2+\varepsilon)}/\log \ell)}}{\ell^{(1+\varepsilon)/(2+\varepsilon)}} \ll R^{-1/(2+\varepsilon) + O_{\kappa}(1/\log \log R)}.$$

Considérons ensuite  $S_2$ , en notant  $v_\ell := (\log R)/\log \ell$  et  $u := (\log R)/\log y$ . Nous avons ainsi

$$S_2 \ll \sum_{\substack{\ell \text{ premier} \\ (\log R)^{2+\varepsilon} < \ell \leq y}} \frac{e^{O_\kappa(v_\ell)}}{\ell(R/\ell)^{1-\beta_\ell}}.$$

Nous utilisons ensuite l'estimation, déduite de la formule (7.8) de [HT86],

$$(R/\ell)^{\beta_\ell-1} = e^{O_\kappa(v_\ell)} \frac{\Psi(R/\ell, \ell)}{R/\ell}.$$

En découpant la somme en intervalles  $y^{1/(j+1)} < \ell \leq y^{1/j}$ , nous obtenons

$$\begin{aligned} \sum_{\substack{\ell \text{ premier} \\ (\log R)^{2+\varepsilon} < \ell \leq y}} \frac{e^{O_\kappa(v_\ell)}}{\ell(R/\ell)^{1-\beta_\ell}} &\leq \frac{1}{R} \sum_{\substack{j \geq 1 \\ (\log R)^{2j} \leq y}} e^{O_\kappa(ju)} \sum_{\substack{\ell \text{ premier} \\ \ell \leq y^{1/j}}} \Psi(R/\ell, \ell) \\ &= \frac{1}{R} \sum_{\substack{j \geq 1 \\ (\log R)^{2j} \leq y}} e^{O_\kappa(ju)} \Psi(R, y^{1/j}) \end{aligned}$$

par l'identité de Buchstab [Buc37]. Nous utilisons ensuite l'estimation (1.7), qui fournit

$$\Psi(R, y^{1/j}) \ll e^{O_\kappa(u)} \left( \frac{e^{O_\kappa(1)}}{u \log u} \right)^{(j-1)u} \Psi(R, y),$$

pour conclure que  $S_2 \ll e^{O_\kappa(u)} \Psi(R, y)/R$  lorsque  $u_0$  est suffisamment grand. Lorsque  $y \geq (\log R)^{2+2\varepsilon}$ , nous avons  $R^{\frac{1+\varepsilon}{2+\varepsilon} + O(1/\log \log R)} \ll_\varepsilon \Psi(R, y)$ , et ainsi

$$S_1 + S_2 \ll_\varepsilon e^{O_\kappa(u)} \frac{\Psi(R, y)}{R} \ll e^{O_\kappa(u)} \varrho(u)$$

comme annoncé, en utilisant encore l'estimation (1.7). □

### 4. Cas d'un facteur linéaire

Nous étudions dans cette section le cas  $d_1 = 1$  et  $r \geq 2$ .

#### 4.1. Cas $d_2 = 1$

Le cas de deux facteurs linéaires présente une structure particulière, puisque dans ce cas nous pouvons utiliser des résultats d'équirépartition spécifiques.

**Théorème 4.1.** *Soit  $\varepsilon > 0$  fixé. Il existe  $C, \delta > 0$  tels que pour des entiers  $a, b, c, d$  de valeurs absolues au plus  $x^\delta$ , avec  $a, c > 0, \Delta := ad - bc \neq 0$  et  $(a, c) = 1$ , l'on ait*

$$\begin{aligned} \text{card} \{n \in [x, (1 + \eta)x] : P^+(an + b) \leq y_1, P^+(cn + d) \leq y_2\} \\ \ll_\varepsilon \eta \Psi(x, y_1) \varrho(u_2)^{3/5-\varepsilon} \end{aligned} \quad (4.1)$$

pour  $(\log x)^C \leq y_1 \leq y_2 \leq x, e^{-\delta \sqrt{\log y_1}} \leq \eta \leq 1$  et  $y_2 \leq y_1^C$ . Ici,  $u_j := (\log x)/\log y_j$ .

De cette estimation, combinée avec le théorème 2(ii) et le corollaire 2 de Hildebrand et Tenenbaum [HT86], nous déduisons le corollaire suivant.

**Corollaire 4.2.** *Pour tout  $\varepsilon > 0$ , il existe  $C, \delta > 0$  pouvant dépendre de  $\varepsilon$  tels que lorsque  $(\log x)^C \leq y \leq x$ ,  $e^{-\delta\sqrt{\log y}} \leq \eta \leq 1$  et  $1 \leq a, |b| \leq x^\delta$ , l'on ait*

$$|\{n \in [x, (1 + \eta)x] : P^+(n(an + b)) \leq y\}| \ll_\varepsilon \eta x \varrho(u)^{8/5-\varepsilon}.$$

*Démonstration du Théorème 4.1.* Nous pouvons supposer que  $(a, b) = (c, d) = 1$  sans perte de généralité. Supposons de plus  $y_2 \leq x^{1/C}$  puisque dans le cas contraire la borne énoncée est triviale. Notons  $S$  le membre de gauche de (4.1). Nous écrivons

$$S = S_1 + S_2,$$

où  $S_1$  est la contribution des entiers  $n$  satisfaisant  $\prod_{p^v \parallel cn+d, p \nmid ac\Delta} p^v \geq x^{3/5-\varepsilon}$ , et  $S_2$  la contribution complémentaire.

Nous majorons d'abord  $S_1$ . La Proposition 3.1 avec  $D = ac|\Delta|$ ,  $R = x^{3/5-\varepsilon}$  et  $\kappa = 2$  fournit l'existence d'une fonction  $\vartheta$  satisfaisant les conditions (i)–(iv) de la Proposition 3.1, en particulier

$$S_1 \leq \sum_q \vartheta(q) \sum_{\substack{x \leq n \leq (1+\eta)x \\ P^+(an+b) \leq y_1 \\ q \mid cn+d}} 1.$$

La relation de congruence s'écrit  $c(an + b) \equiv -\Delta \pmod{aq}$ . Par construction de  $\vartheta$ , nous avons  $(q, ac\Delta) = 1$ . Notons  $X_1 := ax + b$ ,  $X_2 := (1 + \eta)ax + b$  et  $\mathcal{I} = [X_1, X_2]$ ; on remarque que  $(X_2 - X_1)/X_1 \asymp \eta$ . Alors

$$S_1 \leq \sum_{(q, ac\Delta)=1} \vartheta(q) \Psi(\mathcal{I}, y_1; aq, -\bar{c}\Delta). \quad (4.2)$$

Nous appliquons l'estimation (2.1) du Théorème 2.1. Considérons d'abord le terme principal  $T_1$  qui en résulte. On obtient

$$T_1 = \sum_{(q, ac\Delta)=1} \vartheta(q) \frac{\Psi_{aq}(\mathcal{I}, y_1)}{\varphi(aq)} = \sum_{\substack{m \in \mathcal{I} \\ P^+(m) \leq y_1 \\ (m, a)=1}} \sum_{(q, mac\Delta)=1} \frac{\vartheta(q)}{\varphi(aq)}.$$

L'estimation (iv) de la Proposition 3.1 permet de majorer la somme sur  $q$  par

$$\frac{e^{O(u_2)} \Psi(R, y_2)}{\varphi(a) R}.$$

Nous avons donc

$$T_1 \ll \frac{e^{O(u_2)} \Psi(R, y_2)}{\varphi(a) R} \Psi_a(2ax + b, y_1).$$

Le théorème 2.4 de [BT05] et le théorème 4 de [Hil85a] permettent d'écrire

$$\begin{aligned} \Psi_a(2ax + b, y_1) &\ll \prod_{p|a, p \leq y_1} (1 - p^{-\alpha(x, y_1)}) \Psi(2ax + b, y_1) \\ &\ll a \prod_{p|a, p \leq y_1} (1 - p^{-1}) \Psi(x, y_1). \end{aligned}$$

Le théorème de Mertens et les inégalités  $y_1 \geq \log x$  et  $\omega(a) \ll \log x$  fournissent aisément  $\prod_{p|a, p \leq y_1} (1 - p^{-1}) \ll \varphi(a)/a$ . Nous avons donc obtenu

$$T_1 \ll e^{O(u_2)} \frac{\Psi(R, y_2)}{R} \Psi(x, y_1).$$

Puisque  $e^{O(u_2)} \Psi(R, y_2)/R \ll_\varepsilon \varrho(u_2)^{3/5-2\varepsilon}$ , nous avons bien la majoration requise, en utilisant (1.7).

Nous écrivons ensuite

$$U_1 := S_1 - T_1 \leq \sum_{(q, ac\Delta)=1} |\vartheta(q)| \left| \Psi(\mathcal{I}, y_1; aq, -\bar{c}\Delta) - \frac{\Psi_{aq}(\mathcal{I}, y_1)}{\varphi(aq)} \right|.$$

La Proposition 3.1(i) fournit

$$U_1 \leq \sum_{\substack{R \leq q \leq y^4 R \\ P^+(q) \leq y_2 \\ (q, ac\Delta)=1}} \tau_3(q) \left| \Psi(\mathcal{I}, y_1; aq, -\bar{c}\Delta) - \frac{\Psi_{aq}(\mathcal{I}, y_1)}{\varphi(aq)} \right|.$$

L'estimation (2.1) du Théorème 2.1 montre que la somme sur  $q$  est

$$\ll_{\varepsilon, A} \eta \frac{\Psi(ax + b, y_1)}{a} \frac{\Psi(R, y_2)}{R} \frac{e^{O(u_2)}}{(\log x)^A}$$

pour tout  $A > 0$ . On en déduit, de même que précédemment et lorsque  $C$  est choisi suffisamment grand,

$$U \ll_\varepsilon \eta \Psi(x, y_1) \varrho(u_2)^{3/5-\varepsilon}. \tag{4.3}$$

Ce majorant est de nouveau de l'ordre de grandeur de la borne annoncée.

Enfin, concernant la contribution  $S_2$ , nous avons trivialement

$$S_2 \leq \sum_{\substack{n \leq 2x \\ P^+(cn+d) \leq y \\ (cn+d, (ac|\Delta|)^\infty) \geq x^{1/5}}} 1 \leq \sum_{\substack{n \leq 2x^{1+\delta} \\ (n, (ac|\Delta|)^\infty) \geq x^{1/5}}} 1 \ll x^{1-\delta} \tag{4.4}$$

si  $\delta$  est suffisamment petit. Cela est bien de l'ordre de grandeur annoncé en vertu de nos hypothèses sur  $(\eta, y_1, y_2)$ . □

4.2. Cas  $d_2 > 1$ 

**Théorème 4.3.** Soient  $\varepsilon > 0$ ,  $(a, b) \in \mathbb{N}^2$  et  $Q \in \mathbb{Z}[X]$  fixés. Il existe  $C, \delta > 0$  tels que lorsque  $a \neq 0$  et  $Q$  est de degré  $g \geq 1$ , l'on ait

$$|\{n \leq x : P^+((an + b)Q(n)) \leq y\}| \ll_{\varepsilon, Q, a, b} x^{\varrho(u)^{3/2-\varepsilon}} \quad (4.5)$$

pour  $(\log x)^C \leq y \leq x$ .

*Démonstration.* Nous supposons sans perte de généralité que  $Q$  irréductible, non proportionnel à  $aX + b$ , que  $(a, b) = 1$  et que  $y \leq x^{1/10}$ . Soit  $D^*$  le discriminant de  $Q$ , ainsi que

$$D := a^{g+1}Q(-b/a)D^*. \quad (4.6)$$

De même que dans la preuve du Théorème 4.1, nous écrivons le membre de gauche de (4.5) sous la forme  $S_1 + S_2$ , avec  $S_1$  la contribution des  $n$  tels que  $Q(n)/(Q(n), D^\infty) \geq x^{1/2-\varepsilon}$ , et  $S_2$  la contribution complémentaire.

Nous étudions d'abord  $S_1$ . Nous utilisons la Proposition 3.1 avec  $R = x^{1/2-\varepsilon}$ , l'entier  $D$  défini en (4.6) et  $\kappa = g$ . Nous obtenons alors

$$\begin{aligned} S_1 &\leq \sum_q \vartheta(q) \sum_{\substack{n \leq x \\ P^+(an+b) \leq y \\ Q(n) \equiv 0 \pmod{q}}} 1 \\ &= O_\varepsilon(x^\varepsilon |b|) + \sum_q \vartheta(q) \sum_{\substack{\lambda \pmod{q} \\ Q(\lambda) \equiv 0 \pmod{q}}} \Psi(ax + b, y; aq, a\lambda + b). \end{aligned}$$

Notons que nécessairement  $(a, a\lambda + b) = 1$ . De plus, on a  $(q, a\lambda + b) \mid a^g Q(-b/a)$ . Par définition de  $D$  et la Proposition 3.1(ii), cela fournit  $(q, a\lambda + b) = 1$ ; nous sommes donc en mesure d'appliquer l'estimation (2.2). Notons

$$\rho_Q(q) := |\{\lambda \pmod{q} : Q(\lambda) \equiv 0 \pmod{q}\}|.$$

Nous écrivons

$$S_1 \ll_\varepsilon x^\varepsilon |b| + T_1 + U_1,$$

où

$$\begin{aligned} T_1 &:= \sum_q \vartheta(q) \rho_Q(q) \frac{\Psi_{aq}(ax + b, y)}{\varphi(aq)}, \\ U_1 &:= \sum_{\substack{R \leq q \leq y^4 R \\ P^+(q) \leq y \\ (q, D) = 1}} \tau_3(q) \rho_Q(q) \max_{(c, q) = 1} \left| \Psi(ax + b, y; aq, c) - \frac{\Psi_q(ax + b, y)}{\varphi(aq)} \right|. \end{aligned}$$

Notons que  $\rho_Q(q) \leq g^{\omega(q)}$  lorsque  $(q, D^*) = 1$  (voir par exemple la formule (2.5) de [Hen12]). Nous avons d’une part, grâce à la Proposition 3.1(iv),

$$T_1 = \frac{1}{\varphi(a)} \sum_{\substack{m \in S(ax+b,y) \\ (m,a)=1}} \sum_{(q,m)=1} \frac{\vartheta(q)\rho_Q(q)}{\varphi(q)} \ll_{Q,a} e^{O(u)} \frac{\Psi(ax+b,y)\Psi(R,y)}{R} \\ \ll_{\varepsilon,Q,a,b} x_Q(u)^{3/2-2\varepsilon},$$

qui est de l’ordre de grandeur annoncé, quitte à diminuer  $\varepsilon > 0$ . D’autre part, l’estimation (2.2) fournit, pour tout  $A > 0$  fixé,

$$U_1 \ll_{\varepsilon,a} \frac{e^{O(u)}\Psi(ax+b,y)\Psi(R,y)}{(\log x)^A R},$$

ce qui est à nouveau l’ordre de grandeur annoncé.

En ce qui concerne  $S_2$ , qui est pour rappel la contribution au membre de gauche de (4.5) des entiers  $n$  avec  $Q(n)/(Q(n), D^\infty) \leq R$ , nous notons tout d’abord que pour  $x$  suffisamment grand, nous avons  $Q(n) \geq x^{g-1/2}$  pour  $n \geq x^{1-1/(3g)}$ . De plus, on a pour une certaine constante  $C = C_Q$  la majoration  $\rho_Q(m) \leq C^{\omega(m)}$  (voir par exemple la formule (2.14) de [Ten90a]). Ainsi, de façon similaire à (4.4), nous avons

$$S_2 \leq \sum_{\substack{m|D^\infty \\ m \geq x^{g-1}}} \sum_{\substack{n \leq x \\ m|Q(n)}} 1 \ll_Q \sum_{\substack{m|D^\infty \\ x \leq m \ll x^g}} C^{\omega(m)} \left(\frac{x}{m} + 1\right) \ll_Q x^{1/2},$$

par la majoration de Rankin et l’inégalité  $\sum_{m|D^\infty} C^{\omega(m)} m^{-1/(3g)} \ll_Q 1$ . □

### 5. Cas d’un facteur quadratique, $d_1 = 2$

Dans le cas quadratique, nous montrons la majoration suivante, dont l’exposant dépasse le seuil  $c_Q = 1$ .

**Théorème 5.1.** *Soit  $\delta$  réel donné,  $0 < \delta < 25/178$ , et  $Q \in \mathbb{Z}[X]$  quadratique irréductible. Lorsque  $C > 0$  est suffisamment grande en fonction de  $\delta$  et  $Q$ , nous avons*

$$|\{n \leq x : P^+(Q(n)) \leq y\}| \ll_{\delta,Q} x_Q(u)^{1+\delta}. \tag{5.1}$$

pour  $(\log x)^C \leq y \leq x$ .

Notre démonstration de ce résultat utilise l’équirépartition des racines des congruences quadratiques, due notamment à Hooley [Hoo63]. Notons

$$\theta \in [0, 7/64] \tag{5.2}$$

une borne en direction de la conjecture de Selberg–Ramanujan [IK04, chapitre 15.5]. La majoration  $\theta \leq 7/64$  est due à Kim et Sarnak [Kim03]. Nous montrons le résultat suivant.



**Théorème 5.2.** Soient  $\varepsilon > 0$ ,  $x, M, N \geq 1$ ,  $MN \leq x^2$ , et  $(a_m), (b_n)$  deux suites complexes bornées en module par 1. Soient  $D \in \mathbb{Z}$  qui n'est pas un carré d'entier, et  $V : \mathbb{R} \rightarrow \mathbb{C}$  une fonction lisse à support compact inclus dans  $\mathbb{R}_+^*$ . Alors

$$\sum_{M \leq m \leq 2M} \sum_{\substack{N \leq n \leq 2N \\ (n,m)=1}} a_m b_n \left( \sum_{\substack{k \in \mathbb{N} \\ mn|k^2 - D}} V\left(\frac{k}{x}\right) - x \widehat{V}(0) \frac{\rho(mn)}{mn} \right) \ll_{\varepsilon, D, V} x^{1/2+\varepsilon} M^{1/2} + x^{1+\varepsilon} N^{3/2-\theta} M^{-1/4+\theta/2}, \quad (5.3)$$

où nous avons noté  $\widehat{V}(\xi) = \int_{\mathbb{R}} V(t) e(-t\xi) dt$  pour  $\xi \in \mathbb{R}$ .

Notons que nous avons en toute circonstance la majoration triviale  $O(x^\varepsilon MN)$ . En utilisant  $\theta \leq 7/64$ , et avec le choix  $M \leq x^{1-\eta/2}$ ,  $N \leq x^{25/178-\eta/2}$ , nous obtenons le Théorème 1.1 cité dans l'introduction. Le premier résultat de ce type est dû à Iwaniec, qui considère dans [Iwa78] le cas  $D = -4$  et montre un exposant de répartition  $1 + 1/15$ . Une extension à un polynôme quadratique quelconque est montrée par Lemke Oliver dans [LO12]. Iwaniec mentionne que la conjecture  $(R^*)$  de Hooley, qui porte sur des majorations de sommes courtes d'exponentielles, permettrait d'obtenir l'exposant  $1/9$ ; notons que  $25/178 > 1/9$ . Notre amélioration est basée sur l'utilisation dans ce contexte de méthodes issues de la théorie spectrale des formes automorphes [DFI95, Tót00]. La conjecture de Selberg  $\theta = 0$  entraînerait la validité de la majoration (5.1) pour tout  $\delta < 1/6$ .

Le Théorème 5.2 peut être généralisé au cas d'un polynôme quadratique irréductible quelconque, cependant dans le présent travail nous tirons parti de la possibilité de nous réduire *a priori* au cas de  $X^2 - D$ , ce qui clarifie la présentation.

La preuve du Théorème 5.2 étant indépendante du reste du présent travail, nous reportons sa démonstration à la section 8.

*Démonstration du Théorème 5.1 à partir du Théorème 5.2.* Soient  $\eta, \delta > 0$  des paramètres avec  $\delta \leq 1/6$ . Nous posons  $Q(X) = aX^2 + bX + c$ , et notons  $S$  la somme du membre de gauche de (5.1). Nous supposons sans perte de généralité que  $4a \leq y \leq x^\eta$ , la borne annoncée étant triviale dans le cas contraire. De plus, par un découpage dyadique, nous pouvons supposer que les entiers  $n$  comptés dans  $S$  satisfont  $x/2 < n \leq x$ . Enfin, la relation  $4a(aX^2 + bX + c) = (2aX + b)^2 + 4ac - b^2$  nous permet de supposer sans perte de généralité que  $Q(X) = X^2 - D$ , où  $D = b^2 - 4ac$  n'est pas un carré d'entier. Nous posons  $D^* = 2D$ .

De même que dans la preuve du Théorème 4.3, nous écrivons le membre de gauche de (5.1) comme  $S_1 + S_2$ , où  $S_1$  est la contribution des entiers  $n$  tels que

$$\prod_{\substack{p^v \parallel n^2 - D \\ p \nmid D^*, v \leq 2}} p^v \geq x^{6/5}, \quad (5.4)$$

et  $S_2$  est la contribution complémentaire. Lorsque l'entier  $n$  est compté dans  $S_2$ , l'une des deux inégalités suivantes est satisfaite :

$$\prod_{\substack{p^v \parallel n^2 - D \\ p \mid D^*}} p^v \gg_D x^{1/5} \quad \text{ou} \quad \prod_{\substack{p^v \parallel n^2 - D \\ v \geq 3}} p^v \geq x^{1/5}.$$

Le nombre des entiers satisfaisant la première inégalité est majoré par  $O_D(x^{9/10})$  de façon similaire à la preuve du Théorème 4.3. Le nombre des entiers satisfaisant la seconde inégalité est majoré, en suivant le raisonnement de [Ten90a, lemme 3.7], par

$$\sum_{\substack{x^{1/5} \leq m \ll x^2 \\ p^v \parallel m \Rightarrow v \geq 3}} \sum_{\substack{n \leq x \\ m \mid n^2 - D}} 1 \ll_D \sum_m C^{\omega(m)} \left( \frac{x}{m} + 1 \right) \ll_{\varepsilon, D} x^{13/15 + \varepsilon} + x^{2/3 + \varepsilon}. \quad (5.5)$$

Cela fournit un terme d'erreur acceptable.

Nous considérons maintenant  $S_1$ . Nous fixons une fonction  $V$  de classe  $C^\infty$  sur  $\mathbb{R}_+$  satisfaisant

$$\mathbf{1}_{[1/2, 1]} \leq V \leq \mathbf{1}_{[1/3, 2]}.$$

Nous avons ainsi

$$S_1 \leq \sum_{\substack{P^+(n^2 - D) \leq y \\ n \text{ vérifie (5.4)}}} V\left(\frac{n}{x}\right).$$

Nous appliquons la Proposition 3.1 avec les paramètres  $R = x^{1+\delta}$  (nous rappelons que  $\delta \in [0, 1/6]$ ),  $\kappa = 2$  pour la fonction  $f(q) = \rho_Q(q)$ , et avec  $n$  remplacé par  $\prod_{p^v \parallel n^2 - D, p \nmid D, v \leq 2} p^v$ . Cette dernière quantité est bien supérieure à  $R$  grâce à l'hypothèse (5.4). Nous obtenons donc l'existence d'une fonction  $\vartheta$  satisfaisant les propriétés (i)–(iv) de la Proposition 3.1. Puisque  $\mathbf{1}_{P^+(n) \leq y} \leq \mathbf{1}_{P^+(\prod_{p^v \parallel n^2 - D, p \nmid D, v \leq 2} p^v) \leq y}$ , nous en déduisons

$$S_1 \leq \sum_{\substack{n \leq x \\ n \text{ vérifie (5.4)}}} V\left(\frac{n}{x}\right) \sum_{q \mid n^2 - D}^b \vartheta(q) \leq O_D(x^{14/15}) + \sum_q^b \vartheta(q) \sum_{\substack{n \in \mathbb{N} \\ q \mid n^2 - D}} V\left(\frac{n}{x}\right), \quad (5.6)$$

où le symbole  $\sum^b$  indique que la somme est restreinte aux entiers  $q$  tels que  $p^v \parallel q \Rightarrow v \leq 2$ . Dans la seconde inégalité, nous avons réintégré les entiers  $n$  qui ne satisfont pas (5.4), à l'aide de la borne (5.5). Soit  $S'_1$  la somme du membre de droite de (5.6). Le terme principal attendu est

$$T'_1 := x \widehat{V}(0) \sum_q^b \frac{\vartheta(q) \rho(q)}{q} \ll e^{O(u)} x \frac{\Psi(x^{1+\delta}, y)}{x^{1+\delta}}.$$

Nous étudions ensuite la différence

$$S'_1 - T_1 = \sum_q^b \vartheta(q) \left( \sum_{\substack{n \in \mathbb{N} \\ q \mid n^2 - D}} V\left(\frac{n}{x}\right) - x \widehat{V}(0) \frac{\rho(q)}{q} \right). \quad (5.7)$$

Nous rappelons que  $\vartheta$  est définie en (3.2), et nous considérons la variable de sommation  $d_0$  qui y intervient. Soient  $Q_1, Q_2 \geq 1$  deux réels avec  $Q_1 Q_2 = R$ . Puisque  $d_0 \geq R$ , nous pouvons factoriser de façon unique  $d_0 = m\ell$ , avec

$$\begin{cases} Q_2 \leq \ell < Q_2 P^-(\ell)^\nu & (P^-(\ell)^\nu \parallel \ell), \\ P^+(m) < P^-(\ell). \end{cases}$$

Notons que par notre restriction sur la somme  $\sum_q^b$ , nous avons  $P^-(\ell)^\nu \leq y^2$ . Nous avons donc

$$\vartheta(q) = \sum_{\substack{q=m\ell d_1 \\ R \leq m\ell < R P^+(\ell) \\ Q_2 \leq \ell < Q_2 P^+(\ell)^\nu \\ P^+(m) < P^-(\ell), P^+(\ell) \leq y \\ (m\ell d_1, D^*) = (m\ell, d_1) = 1}} \xi_{d_1}$$

où l'entier  $\nu$  dans la somme est déterminé par  $P^+(\ell)^\nu \parallel \ell$ . Nous remarquons que dans cette somme, le poids de crible  $\xi_{d_1}$  dépend implicitement de  $P^+(m\ell) = P^+(\ell)$ ; ainsi, il ne dépend pas de  $m$ . Nous séparons les variables  $m$  et  $\ell$  au moyen, par exemple, du lemme 13.11 de [IK04] (avec  $z = x$ ), puis nous renommons  $q_1 \leftarrow m$  et  $q_2 \leftarrow \ell d_1$ . Nous obtenons donc l'existence d'une fonction  $W : \mathbb{R}^3 \rightarrow \mathbb{C}$  telle que pour toute fonction  $F : \mathbb{R} \rightarrow \mathbb{C}$ ,

$$\int_{\mathbb{R}^3} |W(\mathbf{t})| d\mathbf{t} \ll (\log x)^3, \quad \sum_q^b \vartheta(q) F(q) = \int_{\mathbb{R}^3} W(\mathbf{t}) T_F(\mathbf{t}) d\mathbf{t},$$

où nous avons posé  $\mathbf{t} = (t_1, t_2, t_3)$  et

$$\begin{aligned} T_F(\mathbf{t}) &:= \sum_{\substack{y^{-2} Q_1 \leq q_1 \leq y Q_1 \\ Q_2 \leq q_2 \leq y^2 Q_2 \\ (q_1, q_2) = 1}} \alpha_{q_1, \mathbf{t}} \beta_{q_2, \mathbf{t}} F(q_1 q_2), \\ \alpha_{q_1, \mathbf{t}} &:= \mathbf{1}_{\substack{(q_1, D^*) = 1 \\ p^\nu \parallel q_1 \Rightarrow \nu \leq 2}} q_1^{i(t_1 - t_2)} P^+(q_1)^{-it_3}, \\ \beta_{q_2, \mathbf{t}} &:= \mathbf{1}_{\substack{(q_2, D^*) = 1 \\ p^\nu \parallel q_2 \Rightarrow \nu \leq 2}} \sum_{\substack{q_2 = \ell d_1 \\ (\ell, d_1) = 1 \\ Q_1 < \ell \leq Q_1 P^+(\ell)^\nu \\ P^+(\ell) \leq y}} \xi_{d_1} \ell^{i(t_1 - t_2)} P^+(\ell)^{it_2} P^-(\ell)^{it_3}. \end{aligned}$$

Nous appliquons cela avec  $F(q)$  remplacé par la parenthèse intérieure du membre de droite de (5.7). Uniformément pour  $\mathbf{t} \in \mathbb{R}^3$ , et  $(Q'_1, Q'_2) \in \mathbb{R}^2$  satisfaisant  $y^{-2} Q_1/2 \leq Q'_1 \leq y Q_1$  et  $Q_2/2 \leq Q'_2 \leq y^2 Q_2$ , le Théorème 5.2 ainsi que la borne  $\tau(q_2) \ll Q_2^\varepsilon$  fournissent

$$\begin{aligned} \sum_{\substack{Q'_1 < q_1 \leq 2Q'_1 \\ Q'_2 < q_2 \leq 2Q'_2 \\ (q_1, q_2) = 1}} \alpha_{q_1, \mathbf{t}} \beta_{q_2, \mathbf{t}} &\left( \sum_{\substack{n \in \mathbb{N} \\ q_1 q_2 | n^2 - D}} V\left(\frac{n}{x}\right) - x \widehat{V}(0) \frac{\rho(q_1 q_2)}{q_1 q_2} \right) \\ &\ll_{\varepsilon, D} y^{O(1)} x^{1+\varepsilon} (x^{-1/2} Q_1^{1/2} + Q_1^{-1/4+\theta/2} Q_2^{3/2-\theta}). \end{aligned}$$

Nous optimisons par le choix  $Q_1 = x^{1-K\eta}$  et  $Q_2 = x^{\kappa-K\eta}$  avec  $\kappa = \frac{1-2\theta}{6-4\theta}$  et  $K > 0$  une constante absolue suffisamment grande. Cela fixe donc

$$\delta = 1 + \frac{1 - 2\theta}{6 - 4\theta} - 2K\eta. \tag{5.8}$$

Notre hypothèse  $y \leq x^\eta$ , une somme dyadique sur  $Q'_1$  et  $Q'_2$ , et une intégration par rapport à  $(t_1, t_2, t_3)$  fournissent donc

$$S'_1 - T'_1 \ll_D x^{1+O(\eta)-\frac{1}{2}K\eta} \ll_D x^{1-\eta}$$

si  $K$  est suffisamment grande. Ce terme d'erreur est acceptable pour  $y \geq (\log x)^C$ , avec  $C = C(\eta)$ . Nous concluons en utilisant la borne de Kim–Sarnak  $\theta \leq 7/64$  [Kim03] dans (5.8), et en choisissant  $\eta$  suffisamment petit.  $\square$

### 6. Le cas $d_1 \geq 3$

Dans cette section, nous utilisons ce qui précède pour retrouver des résultats de Khmyrova [Khm64] et Timofeev [Tim77], à l'uniformité en  $Q$  près.

**Théorème 6.1.** *Soit  $Q \in \mathbb{Z}[X]$  irréductible de degré  $g \geq 3$ . Il existe alors  $C > 0$  tel que*

$$|\{n \leq x : P^+(Q(n)) \leq y\}| \leq e^{O_Q(u)} x_Q(u) \tag{6.1}$$

pour  $(\log x)^C \leq y \leq x$ . La constante implicite et  $C$  dépendent au plus de  $Q$ .

*Démonstration.* Soit  $D$  le discriminant de  $Q$ , et  $S$  le membre de gauche de (6.1). De façon similaire à précédemment, nous nous restreignons à compter la contribution  $S_1$  à  $S$  constituée des entiers  $n$  tels que  $Q(n)/(Q(n), D^\infty) \geq x$ . Nous appliquons alors la Proposition 3.1 avec  $\kappa = \deg(Q)$  et  $R = x/y^5$ . Nous avons alors

$$\begin{aligned} S_1 &\leq \sum_q \vartheta(q) \sum_{\substack{\lambda \pmod q \\ Q(\lambda) \equiv 0 \pmod q}} \sum_{\substack{n \leq x \\ n \equiv \lambda \pmod q}} 1 \\ &\ll x \left| \sum_q \frac{\vartheta(q) \rho_Q(q)}{q} \right| + \sum_{\substack{x/y^5 \leq q \leq x/y \\ P^+(q) \leq y}} \tau_3(q) \rho_Q(q) \leq e^{O_Q(u)} x_Q(u) \end{aligned}$$

comme annoncé.  $\square$

### 7. Application aux entiers $n$ tels que $P^+(n)^2 \mid n$

Dans cette section, nous prouvons le Corollaire 1.4. Notons

$$\mathcal{L} := e^{\sqrt{(\log x) \log \log x}}.$$

**Lemme 7.1.** Soient  $\varepsilon > 0$ ,  $\varepsilon \leq a < b \leq \varepsilon^{-1}$  et  $\alpha, \beta > 0$  donnés. Lorsque  $x$  tend vers l'infini,

$$\sum_{\mathcal{L}^a \leq p \leq \mathcal{L}^b} \frac{\varrho\left(\alpha \frac{\log x}{\log p}\right)}{p^{1+\beta}} \ll_{\varepsilon, \alpha, \beta} \mathcal{L}^{-\gamma+o(1)} \quad \text{avec} \quad \gamma := \inf_{t \in [a, b]} \left( \frac{\alpha}{2t} + \beta t \right).$$

*Démonstration.* En notant  $u_p = (\log x)/\log p$ , nous avons

$$\sum_{\mathcal{L}^a \leq p \leq \mathcal{L}^b} \frac{\varrho\left(\alpha \frac{\log x}{\log p}\right)}{p^{1+\beta}} \ll \mathcal{L}^{o(1)} \sum_{\mathcal{L}^a \leq p \leq \mathcal{L}^b} \frac{p^{-\beta} u_p^{-\alpha u_p}}{p} \ll \mathcal{L}^{-\gamma+o(1)} \sum_{\mathcal{L}^a \leq p \leq \mathcal{L}^b} \frac{1}{p} \ll \mathcal{L}^{-\gamma+o(1)}. \quad \square$$

*Démonstration du Corollaire 1.4.* Nous remarquons tout d'abord qu'en procédant à un découpage dyadique, il suffit de montrer la majoration (1.6) avec la contrainte supplémentaire

$$x/2 < n \leq x. \quad (7.1)$$

Notons  $E(x) = |\{x/2 < n \leq x : P^+(n)^2 \mid n, P^+(n+1)^2 \mid n+1\}|$ . Pour toute paire  $(p, q)$  de nombres premiers distincts, nous notons  $E_{p,q}(x)$  la contribution à  $E(x)$  des entiers  $n$  tels que  $p = P^+(n)$  et  $q = P^+(n+1)$ , de sorte que

$$E(x) = 2 \sum_{\substack{p, q \text{ premiers} \\ p < q}} E_{p,q}(x). \quad (7.2)$$

Nous avons la borne triviale

$$E_{p,q}(x) \leq \min\{\Psi(x/p^2, p), \Psi(x/q^2, q)\}.$$

La majoration

$$\sum_{p \leq \mathcal{L}^{1/6} \text{ ou } p \geq \mathcal{L}^3} \Psi(x/p^2, p) \ll x \mathcal{L}^{-3} + \Psi(x, \mathcal{L}^{1/6}) \ll x \mathcal{L}^{-2\sqrt{2}}$$

permet donc de restreindre la somme (7.2) à  $p, q \in [\mathcal{L}^{1/6}, \mathcal{L}^3]$ . Nous fixons dans ce qui suit une telle paire  $(p, q)$ .

Le théorème de Bézout nous assure l'existence et l'unicité d'une paire  $(r, s)$  d'entiers tels que  $q^2 r - p^2 s = 1$ , avec  $1 \leq r \leq p^2$  et  $1 \leq s \leq q^2$ . Les conditions  $p^2 \mid n$  et  $q^2 \mid n+1$  nous permettent de paramétrer  $n$  sous la forme

$$n = p^2(s + q^2 m), \quad n + 1 = q^2(r + p^2 m).$$

Au vu de la condition (7.1), nous avons  $x/(3(pq)^2) \leq m \leq x/(pq)^2$ . Ainsi,

$$E_{p,q}(x) \leq \left| \left\{ m \in \left[ \frac{x}{3(pq)^2}, \frac{x}{(pq)^2} \right] : P^+(q^2 m + s) \leq p, P^+(p^2 m + r) \leq q \right\} \right|.$$

Le Théorème 4.1 (avec  $\Delta = 1$ ) fournit donc

$$E_{p,q}(x) \ll_{\varepsilon} \mathcal{L}^{o(1)} \frac{x}{(pq)^2} \frac{\Psi(x, p)}{x} \frac{\Psi(x^{3/5-\varepsilon}, q)}{x^{3/5-\varepsilon}}.$$

Nous sommions cette borne sur  $(p, q)$  avec  $p < q$ , et notons  $\kappa = 3/5 - \varepsilon$ . Lorsque  $x$  tend vers l'infini, le Lemme 7.1 fournit d'une part

$$\sum_{p < q \leq \mathcal{L}^3} \frac{\Psi(x^\kappa, q)}{q^2 x^\kappa} \ll \sum_{p < q \leq \mathcal{L}^3} \frac{\varrho\left(\kappa \frac{\log x}{\log q}\right)}{q^2} \ll \begin{cases} \mathcal{L}^{-\sqrt{2\kappa}+o(1)}, & p \leq \mathcal{L}^{\sqrt{\kappa/2}}, \\ \mathcal{L}^{o(1)} \frac{\varrho\left(\kappa \frac{\log x}{\log p}\right)}{p}, & p \geq \mathcal{L}^{\sqrt{\kappa/2}}, \end{cases}$$

et d'autre part

$$\begin{aligned} \sum_{\mathcal{L}^{1/6} \leq p \leq \mathcal{L}^{\sqrt{\kappa/2}}} \frac{\varrho\left(\frac{\log x}{\log p}\right)}{p^2} &\ll \mathcal{L}^{-\sqrt{1/(2\kappa)}-\sqrt{\kappa/2}+o(1)}, \\ \sum_{\mathcal{L}^{\sqrt{\kappa/2}} < p \leq \mathcal{L}^3} \frac{\varrho\left(\frac{\log x}{\log p}\right)\varrho\left(\kappa \frac{\log x}{\log p}\right)}{p^3} &\ll \mathcal{L}^{-2\sqrt{1+\kappa}+o(1)}. \end{aligned}$$

Puisque  $2\sqrt{1+\kappa} \leq (\sqrt{2} + 1/\sqrt{2})\sqrt{\kappa} + 1/\sqrt{2\kappa}$ , nous obtenons

$$E(x) \ll x \mathcal{L}^{-2\sqrt{2}} + \sum_{\mathcal{L}^{1/6} \leq p < q \leq \mathcal{L}^3} E_{p,q}(x) \ll_\varepsilon \mathcal{L}^{-2\sqrt{1+\kappa}+o(1)}.$$

Le résultat annoncé suit en faisant  $\varepsilon$  tendre vers 0. □

### 8. Niveau de répartition de $\{n^2 - D\}$

Dans cette section, nous montrons le Théorème 5.2. Celui-ci découle immédiatement de la proposition suivante, grâce à l'inégalité de Cauchy–Schwarz.

**Proposition 8.1.** *Soient  $\varepsilon > 0$ ,  $x, M, N \geq 1$ ,  $MN \leq x^2$ ,  $(b_n) \in \mathbb{C}^{\mathbb{N}}$  avec  $\|b\|_\infty \leq 1$ ,  $D \in \mathbb{Z}$  qui n'est pas un carré d'entier, et  $V : \mathbb{R} \rightarrow \mathbb{C}$  une fonction lisse à support compact inclus dans  $\mathbb{R}_+^*$ . Alors*

$$\sum_{M < m \leq 2M} \left| \sum_{\substack{N < n \leq 2N \\ (n,m)=1}} b_n \left( \sum_{\substack{k \in \mathbb{N} \\ mn|k^2-D}} V\left(\frac{k}{x}\right) - x \widehat{V}(0) \frac{\rho(mn)}{mn} \right) \right|^2 \ll_{\varepsilon, V, D} \left( 1 + x \left( \frac{M}{N^2} \right)^{-3/2+\theta} \right) x^{1+\varepsilon}. \quad (8.1)$$

**Remarque.** Le membre de gauche de (8.1) est majoré trivialement par  $M^{-1}x^{2+\varepsilon}$ , ce qui permet en particulier de supposer d'emblée que  $M \geq N^2$ , et justifie l'intérêt d'avoir une valeur de  $\theta$  aussi petite que possible.

La proposition précédente sera déduite du lemme suivant, qui concerne l'équirépartition des racines de congruences quadratiques.

**Lemme 8.2.** Soient  $(q, r, d) \in \mathbb{N}^3$  avec  $(q, 2Dr) = 1$  et  $d \mid q$ ,  $\lambda \pmod{d}$  une classe inversible, et  $\omega \pmod{d}$  une classe de résidus telle que  $\omega^2 \equiv D \pmod{d}$ . Soient  $M \gg qd$ ,  $f$  une fonction lisse à support compact inclus dans  $\mathbb{R}_+^*$ , satisfaisant

$$\|f^{(j)}\|_\infty \ll_j 1, \quad (8.2)$$

soient  $0 \leq \alpha < \beta < 1$  et

$$P_f(M; q, r, d, \lambda, \omega, \alpha, \beta) := \sum_{\substack{(m, \Omega) \in \mathcal{D} \\ \alpha \leq \frac{\Omega}{mq} < \beta}} f\left(\frac{m}{M}\right), \quad (8.3)$$

où  $\mathcal{D}$  est l'ensemble des paires  $(m, \Omega)$  telles que

$$(m, qr) = 1, \quad m \equiv \lambda \pmod{d}, \quad \Omega^2 \equiv D \pmod{mq}, \quad \Omega \equiv \omega \pmod{d}.$$

Alors pour tout  $\varepsilon > 0$ , l'on a

$$\begin{aligned} P_f(M; q, r, d, \lambda, \omega, \alpha, \beta) \\ = A_f(M; q, r, d, \alpha, \beta) + O_{\varepsilon, D, f}((qrM)^\varepsilon d^{3/4} (qd)^{1/2-\theta} M^{1/2+\theta}). \end{aligned} \quad (8.4)$$

Ici, le terme principal  $A_f$  est défini par

$$A_f(M; q, r, d, \alpha, \beta) := (\beta - \alpha) M \widehat{f}(0) C_D \frac{A(qr) \rho(q/(q, d^\infty))}{\varphi(d)},$$

où  $A(qr) := \prod_{p|qr} (1 + 1/p)^{-1}$  et  $C_D > 0$  est une constante qui ne dépend que de  $D$ . La constante implicite ne dépend que de  $\varepsilon$ ,  $D$ , et des constantes implicites dans (8.2).

Ce résultat découle des majorations de sommes d'exponentielles suivantes.

**Lemme 8.3.** Sous les hypothèses et notations du Lemme 8.2, les majorations suivantes ont lieu pour tout  $\varepsilon > 0$ .

(1) Pour  $1 \leq |h| \leq qd^{1/2}$ ,

$$\begin{aligned} \sum_{(m, \Omega) \in \mathcal{D}} f\left(\frac{m}{M}\right) e\left(\frac{h\Omega}{mq}\right) \\ \ll_{\varepsilon, D, f} |h| (qr)^\varepsilon + (rM)^\varepsilon d^{3/4} (qd, h)^\theta (qd)^{1/2-\theta} M^{1/2+\theta}. \end{aligned} \quad (8.5)$$

(2) Pour  $1/2 \leq H \ll qM$ ,

$$\begin{aligned} \frac{1}{H} \sum_{H < |h| \leq 2H} \left| \sum_{(m, \Omega) \in \mathcal{D}} f\left(\frac{m}{M}\right) e\left(\frac{h\Omega}{mq}\right) \right| \\ \ll_{\varepsilon, D, f} H (qr)^\varepsilon + (rM)^\varepsilon d^{3/4} (qd)^{1/2-\theta} M^{1/2+\theta}. \end{aligned} \quad (8.6)$$

(3) Supposons  $d = 1$ ,  $1/2 \leq Q \ll M$ ,  $1/2 \leq H \ll QM$ ,  $t \in [0, 1]$ , et soit  $\mathcal{I} \subset [H, 2H]$  un intervalle, et  $(f_q)_{Q < q \leq 2Q}$  une suite de fonctions satisfaisant (8.2) et  $f_q(v) \neq 0 \Rightarrow v \asymp 1$  uniformément en  $q$ . Alors

$$\frac{1}{Q} \sum_{\substack{Q < q \leq 2Q \\ (q, 2Dr)=1}} \left| \frac{1}{H} \sum_{h \in \mathcal{I}} e(th) \sum_{(m, \Omega) \in \mathcal{D}} f_q\left(\frac{m}{M}\right) e\left(\frac{h\Omega}{mq}\right) \right| \ll_{\varepsilon, D, f} H(Qr)^\varepsilon + (rM)^\varepsilon \{M^{1/2} + H^{-1/2} Q^{1/2-\theta} M^{1/2+\theta}\}. \tag{8.7}$$

Les constantes implicites ne dépendent que de  $\varepsilon$ ,  $D$ , et des constantes implicites dans (8.2).

**Remarques.** Dans le cas  $d = 1$ , en utilisant la borne de Selberg  $\theta \leq 1/4$  (cf. [DI83, théorème 4]), nous retrouvons, à l’uniformité en  $h$  près, un résultat de Duke, Friedlander et Iwaniec [DFI95, formule (25)] et Tóth [Tót00, formule (15)].

Les majorations (8.5) et (8.6) sont également valables pour  $M \ll qd$ , mais elles sont alors moins précises que la majoration triviale  $O_{\varepsilon, D, f}(q^\varepsilon M^{1+\varepsilon})$ .

### 8.1. Démonstration du Lemme 8.3

Nous nous concentrons dans un premier temps sur la majoration (8.5). Nous supposons sans perte de généralité que  $h > 0$ , quitte à considérer le nombre complexe conjugué.

8.1.1. *Coprimauté.* Soit  $S'(M, q, \lambda)$  le membre de gauche de (8.5). Une inversion de Möbius fournit

$$S'(M, q, \lambda) = \sum_{\substack{\ell | qr \\ (\ell, d)=1}} \mu(\ell) S(M/\ell, q\ell, \lambda\bar{\ell}) \tag{8.8}$$

où  $\mu$  désigne la fonction de Möbius et

$$S(M, q, \lambda) := \sum_{\substack{m \in \mathbb{N} \\ m \equiv \lambda \pmod{d}}} \sum_{\substack{\Omega \in \mathbb{N} \\ \alpha m q \leq \Omega < \beta m q \\ \Omega^2 \equiv D \pmod{mq} \\ \Omega \equiv \omega \pmod{d}}} f\left(\frac{m}{M}\right) e\left(\frac{h\Omega}{mq}\right).$$

Il nous suffira donc de montrer que  $S(M, q, \lambda)$  est majorée par le membre de droite de (8.5).

8.1.2. *Correspondance de Gauss.* Soit

$$\mathcal{Q}_D := \{Q(X, Y) = AX^2 + 2BXY + CY^2 : (A, B, C) \in \mathbb{Z}^3, B^2 - AC = D\}.$$

Pour  $Q \in \mathcal{Q}_D$ , nous notons  $(A(Q), B(Q), C(Q))$  les coefficients dans l’écriture ci-dessus. Le groupe  $\Gamma = \text{PSL}_2(\mathbb{Z})$  agit sur  $\mathcal{Q}_D$  par

$$\sigma Q(x, y) = Q((x, y)\sigma) \quad (\sigma \in \Gamma)$$



où le produit dans le membre de droite est le produit matriciel. En particulier,

$$\begin{aligned} B(\sigma Q) &= \alpha\gamma A + (\alpha\delta + \beta\gamma)B + \beta\delta C, \\ C(\sigma Q) &= \gamma^2 A + 2\gamma\delta B + \delta^2 C = Q(\gamma, \delta) \end{aligned} \quad (8.9)$$

si  $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ . En raisonnant de façon identique à [DFI95, p. 427] (voir aussi [Kow04, section 6.1]), nous obtenons

$$S(M, q, \lambda) = \sum_{Q \in \Gamma \backslash \mathcal{Q}_D} \sum_{\substack{\sigma \in \Gamma_\infty \backslash \Gamma / \Gamma_Q \\ \mathcal{P}(\sigma)}} f\left(\frac{C(\sigma Q)}{qM}\right) e\left(\frac{hB(\sigma Q)}{C(\sigma Q)}\right), \quad (8.10)$$

où  $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$  et  $\mathcal{P}(\sigma)$  désigne la propriété

$$\mathcal{P}(\sigma) \iff \begin{cases} C(\sigma Q) \equiv \lambda q \pmod{qd}, \\ B(\sigma Q) \equiv \omega \pmod{d}, \end{cases}$$

et  $\Gamma_Q \subset \Gamma$  est le stabilisateur de  $Q$ .

*8.1.3. Localisation des variables.* Soit  $\sigma = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix}$  un élément générique en indice de la somme du membre de droite de (8.10). Nous introduisons, suivant Tóth [Tót00, lemme 4.2], une fonction  $\Psi : \Gamma \rightarrow \mathbb{R}$  qui permet d'encoder le quotient par  $\Gamma_Q$ . Cette fonction satisfait  $\sum_{\tau \in \Gamma_Q} \Psi(\sigma\tau) = 1$  pour tout  $\sigma \in \Gamma$ . Dans le cas  $D < 0$ , la fonction  $\Psi$  est constante, et dans le cas contraire  $\Psi(\sigma)$  est une fonction  $C^\infty$  du rapport  $\delta/\gamma$ . Nous nous donnons aussi une fonction lisse  $w : \mathbb{R} \rightarrow \mathbb{R}_+$  satisfaisant

$$\mathbf{1}_{|t| \leq 1/2} \leq w(t) \leq \mathbf{1}_{|t| \leq 2}, \quad w(t) + w(1/t) = 1$$

pour  $t \neq 0$ . Nous insérons le poids  $w(\gamma/\delta) + w(\delta/\gamma)$  dans le membre de droite de (8.10). Dans la contribution du terme  $w(\gamma/\delta)$ , nous appliquons aux sommes sur  $\sigma$  et  $Q$  les involutions changeant  $Q(X, Y)$  en  $\tilde{Q} = Q(Y, X)$ , et  $\sigma$  en  $\tilde{\sigma} = \begin{pmatrix} -\beta & -\alpha \\ \delta & \gamma \end{pmatrix}$ . Nous avons alors

$$B(\tilde{\sigma}\tilde{Q}) = -B(\sigma Q), \quad C(\tilde{\sigma}\tilde{Q}) = C(\sigma Q).$$

Nous obtenons

$$S(M, q, \lambda) = S(h, \Psi_1) + S(-h, \Psi_2) \quad (8.11)$$

avec

$$(\Psi_1(\sigma), \Psi_2(\sigma)) := (w(t)\Psi(t), w(t)\Psi(1/t)) \quad (\sigma = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix} \in \Gamma_\infty \backslash \Gamma, t = \gamma/\delta), \quad (8.12)$$

$$F_{\Psi, Q}(\sigma) := \Psi(\sigma) f\left(\frac{C(\sigma Q)}{qM}\right), \quad (8.13)$$

$$S(h, \Psi) := \sum_{Q \in \Gamma \backslash \mathcal{Q}_D} \sum_{\substack{\sigma \in \Gamma_\infty \backslash \Gamma \\ \mathcal{P}(\sigma)}} F_{\Psi, Q}(\sigma) e\left(\frac{hB(\sigma Q)}{C(\sigma Q)}\right). \quad (8.14)$$

Nous fixons dorénavant  $\Psi \in \{\Psi_1, \Psi_2\}$ , en notant que cette fonction (donc la fonction  $F_{\Psi, Q}$  associée) est nulle dès que  $|t| \geq 2$  (avec la notation (8.12)).

8.1.4. *Simplification de la phase.* Nous écrivons la définition (8.14) comme  $S(h, \Psi) = \sum_{Q \in \Gamma \backslash \mathcal{Q}_D} S_Q(h, \Psi)$ . La somme sur  $Q$  est finie et son nombre de termes dépend au plus de  $D$ . Il nous suffira donc de borner  $S_Q(h, \Psi)$  séparément pour chaque  $Q$ . Pour  $\sigma \in \Gamma$ , nous définissons

$$\phi_\sigma = \frac{\alpha}{\gamma} \in \mathbb{R}/\mathbb{Z} \quad (\sigma = \begin{pmatrix} \alpha & * \\ \gamma & * \end{pmatrix}, \gamma \neq 0).$$

L'identité, due à Hooley [Hoo63, formule (27)],

$$e\left(\frac{hB(\sigma Q)}{C(\sigma Q)}\right) = e(h\phi_\sigma) + O(h(qM)^{-1}) \quad (8.15)$$

est alors établie de façon similaire au lemme 4.3 de Toth [Tó00]. En remplaçant dans  $S_Q(h, \Psi)$ , nous obtenons

$$S_Q(h, \Psi) = T_Q(h, F_\Psi, Q) + O(h) \quad (8.16)$$

avec

$$T_Q(h, F) := \sum_{\substack{\sigma \in \Gamma_\infty \backslash \Gamma \\ \mathcal{P}(\sigma)}} F(\sigma) e(h\phi_\sigma).$$

8.1.5. *Conditions de congruence.* Nous décomposons par le sous-groupe de congruence de Hecke  $\Gamma_0(qd)$  pour obtenir

$$T_Q(h, \Psi) = \sum_{\substack{\tau \in \Gamma_\infty \backslash \Gamma_0(qd) \\ \sigma \in \Gamma_0(qd) \backslash \Gamma \\ \mathcal{P}(\tau\sigma)}} F(\tau\sigma) e(h\phi_{\tau\sigma}).$$

Si  $\tau = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(qd)$ , alors les relations (8.9) ainsi que  $qd \mid \gamma$  montrent que

$$\mathcal{P}(\tau\sigma) \iff \begin{cases} q \mid C(\sigma Q), \\ \delta^2 q^{-1} C(\sigma Q) \equiv \lambda \pmod{d}, \\ B(\sigma Q) \equiv \omega \pmod{d}. \end{cases}$$

Puisque  $(\lambda, d) = 1$ , la seconde condition est détectée par des caractères de Dirichlet, ce qui fournit

$$T_Q(h, \Psi) = \frac{1}{\varphi(d)} \sum_{\chi \pmod{d}} \overline{\chi(\lambda)} \sum_{\substack{\sigma \in \Gamma_0(qd) \backslash \Gamma \\ \mathcal{P}^*(\sigma)}} \chi(q^{-1} C(\sigma Q)) U_{Q, \sigma}(h, \Psi) \quad (8.17)$$

avec

$$U(h, \Psi) = U_{Q, \sigma}(h, \Psi) = \sum_{\tau \in \Gamma_\infty \backslash \Gamma_0(qd)} \overline{\vartheta(\tau)} F(\tau\sigma) e(h\phi_{\tau\sigma}),$$

où  $\mathcal{P}^*(\sigma)$  désigne maintenant les conditions

$$\mathcal{P}^*(\sigma) \iff \begin{cases} q \mid C(\sigma Q), \\ B(\sigma Q) \equiv \omega \pmod{d}, \end{cases} \quad (8.18)$$

et  $\vartheta$  dénote le caractère central défini par

$$\vartheta(\tau) = \overline{\chi}^2(\delta) \quad (\tau = \begin{pmatrix} * & * \\ * & \delta \end{pmatrix} \in \Gamma_0(qd)).$$

8.1.6. *Rappels concernant les sommes de Kloosterman généralisées.* Dans cette section, nous rappelons quelques faits sur les sommes de Kloosterman. Nous référons aux chapitres 2 et 4 de [Iwa97] pour les définitions. Soient deux pointes  $\mathfrak{a}, \mathfrak{b} \in \mathbb{P}^1(\mathbb{R})$  pour l'action de  $\Gamma_0(qd)$ , de stabilisateurs  $\Gamma_{\mathfrak{a}}$  et  $\Gamma_{\mathfrak{b}}$  et matrices d'échelle  $\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{b}} \in \mathrm{PSL}_2(\mathbb{R})$ , c'est-à-dire telles que

$$\Gamma_{\mathfrak{a}} = \sigma_{\mathfrak{a}} \Gamma_{\infty} \sigma_{\mathfrak{a}}^{-1}, \quad \Gamma_{\mathfrak{b}} = \sigma_{\mathfrak{b}} \Gamma_{\infty} \sigma_{\mathfrak{b}}^{-1}.$$

Une pointe est équivalente sous l'action de  $\Gamma_0(qd)$  à une unique pointe  $\mathfrak{a}' = u/v$  avec

$$v \geq 1, \quad v \mid qd, \quad (u, v) = 1, \quad 1 \leq u \leq (v, qd/v).$$

Nous pouvons donc définir la *largeur* de la pointe  $\mathfrak{a}$  comme le nombre

$$w_{\mathfrak{a}} = q/(q, v^2). \quad (8.19)$$

Nous associons à  $(\mathfrak{a}, \mathfrak{b})$  l'ensemble de modules

$$\mathcal{C}(\mathfrak{a}, \mathfrak{b}) := \left\{ c \in \mathbb{R}_+^* : \exists \alpha, \beta, \delta \in \mathbb{R}, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}} \right\}.$$

Pour tout  $c \in \mathcal{C}(\mathfrak{a}, \mathfrak{b})$  et  $(m, n) \in \mathbb{Z}^2$ , nous définissons la somme de Kloosterman

$$S_{\mathfrak{a}\mathfrak{b}}(m, n; \gamma) := \sum_{\substack{\delta \in [0, \gamma[ \\ \begin{pmatrix} \alpha & * \\ \gamma & \delta \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1} \Gamma_0(qd) \sigma_{\mathfrak{b}}}} \overline{\vartheta} \left( \sigma_{\mathfrak{a}} \begin{pmatrix} \alpha & * \\ \gamma & \delta \end{pmatrix} \sigma_{\mathfrak{b}}^{-1} \right) e \left( \frac{\alpha m + \delta n}{\gamma} \right). \quad (8.20)$$

Nous renvoyons à la section 4.1.1 de [Dra17] pour plus de détails, notamment sur la dépendance de  $S_{\mathfrak{a}\mathfrak{b}}(m, n; c)$  vis-à-vis des matrices d'échelle  $(\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{b}})$ . Nous utiliserons dans ce travail les faits suivants.

**Lemme 8.4.** *Soit  $\sigma \in \Gamma_0(qd) \setminus \Gamma$  satisfaisant les conditions  $\mathcal{P}^*(\sigma)$  définies en (8.18).*

- (1) *Le nombre de telles classes  $\sigma$  est majoré par  $O(d\tau(q))$ .*
- (2) *Supposons que la pointe  $\mathfrak{a} = \sigma \infty$  soit équivalente à  $u/v$  avec  $v \mid q$ ,  $1 \leq u < v$  et  $(u, v) = 1$ . Alors  $v \mid Q(0, 1)^2$ , en particulier  $v = O_Q(1)$ , et*

$$w_{\mathfrak{a}} = qd/(qd, v^2) \asymp_Q qd. \quad (8.21)$$

- (3) *L'ensemble de modules  $\mathcal{C}(\infty, \mathfrak{a})$  s'écrit*

$$\mathcal{C}(\infty, \mathfrak{a}) = \{w_{\mathfrak{a}}^{1/2} vm : m \in \mathbb{Z}, (m, qd/v) = 1\}.$$

(4) Lorsque  $\gamma = w_\alpha^{1/2}vm \in \mathcal{C}(\infty, \mathfrak{a})$ , la somme de Kloosterman  $S_{\infty\alpha}(h, n; \gamma)$  admet l'expression

$$S_{\infty\alpha}(h, n; \gamma) = \sum_{\substack{\alpha \pmod{vm} \\ \delta \pmod{u[v, v']m} \\ \delta \equiv m \pmod{uv'} \\ (\delta - m, um) = u \\ \alpha\delta \equiv u \pmod{vm}}} \bar{\vartheta} \begin{pmatrix} * & * \\ * & \delta \end{pmatrix} e\left(\frac{h\alpha}{vm} + \frac{n\delta}{u[v, v']m}\right),$$

où l'on a noté  $v' := qd/v$ . Ici, les matrices d'échelles choisies sont

$$\sigma_\infty = \text{Id}, \quad \sigma_\alpha = \begin{pmatrix} u\sqrt{w_\alpha} & 0 \\ v\sqrt{w} & (u\sqrt{w})^{-1} \end{pmatrix}.$$

(5) Nous avons la borne triviale

$$|S_{\infty\alpha}(h, n; \gamma)| \leq \frac{v}{(v, v')} (m, u)m \ll_Q m. \tag{8.22}$$

(6) Lorsque  $n = 0$ , nous avons

$$|S_{\infty\alpha}(h, 0; \gamma)| \leq \tau(2m)^{O_{\mathfrak{a}, \varrho}(1)}(dh, m). \tag{8.23}$$

La démonstration de ce lemme, qui est indépendante du reste de la démonstration du Lemme 8.3, est reportée à la section 8.5.

8.1.7. *Complétion de sommes.* Dans la somme  $U(h, \Psi)$ , nous changeons  $\tau$  en  $\tau\sigma^{-1}$ , de sorte que

$$U(h, \Psi) = \sum_{\tau \in \Gamma_\infty \backslash \Gamma_0(qd)\sigma} \vartheta(\tau\sigma^{-1})F(\tau)e(h\phi_\tau).$$

La pointe  $\mathfrak{a} = \sigma\infty$  est équivalente à  $u/v$  pour un certain  $v \mid qd$  et  $(u, v) = 1$ , et cette écriture est unique si l'on impose  $1 \leq u \leq (v, qd/v)$ . Nous posons temporairement

$$\tau_\alpha := \begin{pmatrix} w_\alpha^{1/2} & 0 \\ 0 & w_\alpha^{-1/2} \end{pmatrix}, \quad \sigma_\alpha := \sigma\tau_\alpha,$$

de sorte que le stabilisateur  $\Gamma_\alpha \subset \Gamma_0(qd)$  de  $\mathfrak{a}$  vérifie  $\Gamma_\alpha = \sigma_\alpha\Gamma_\infty\sigma_\alpha^{-1}$ . Dans la somme du membre de droite de (8.24), nous remplaçons encore  $\tau$  par  $\tau\tau_\alpha^{-1}$  en remarquant que cela laisse la quantité  $\phi_\tau$  inchangée. Nous obtenons

$$U(h, \Psi) = \sum_{\tau \in \Gamma_\infty \backslash \Gamma_0(qd)\sigma_\alpha} \vartheta(\tau\sigma_\alpha^{-1})F(\tau\tau_\alpha^{-1})e(h\phi_\tau). \tag{8.24}$$

À ce stade, nous remarquons que  $(v, qd/v) \mid q$ . En particulier, la pointe  $\mathfrak{a}$  est singulière pour  $\vartheta$ , ce qui signifie

$$\vartheta(\tau) = 1 \quad (\tau \in \Gamma_\alpha).$$

Nous séparons la somme sur  $\tau$  dans le membre de droite de (8.24) suivant les classes à droite modulo  $\Gamma_\infty$ . Notons que pour tout  $\omega \in \Gamma_\infty$ , nous avons  $\phi_{\tau\omega} \equiv \phi_\tau \pmod{1}$ , ainsi que

$$\vartheta(\tau\omega\sigma_\alpha^{-1}) = \vartheta(\tau\sigma_\alpha^{-1})\vartheta(\sigma_\alpha\omega\sigma_\alpha^{-1}) = \vartheta(\tau\sigma_\alpha^{-1}).$$

Nous obtenons

$$U(h, \Psi) = \sum_{\tau \in \Gamma_\infty \backslash \Gamma_0(qd)\sigma_\alpha / \Gamma_\infty} \vartheta(\tau\sigma_\alpha^{-1}) e(h\phi_\tau) \sum_{k \in \mathbb{Z}} F\left(\tau \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \tau_\alpha^{-1}\right).$$

Étant données les relations (8.12) et (8.13), la fonction

$$t \mapsto F\left(\tau \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \tau_\alpha^{-1}\right)$$

est lisse, à support compact, et ne dépend que de la ligne inférieure de  $\tau$ . Si  $\tau = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix}$ , alors

$$F\left(\tau \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \tau_\alpha^{-1}\right) = F\left(\begin{pmatrix} * & * \\ \gamma w_\alpha^{-1/2} & \gamma(t + \delta/\gamma)w_\alpha^{1/2} \end{pmatrix}\right).$$

La formule de Poisson fournit

$$\sum_{k \in \mathbb{Z}} F\left(\tau \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \tau_\alpha^{-1}\right) = e\left(\frac{n\delta}{\gamma}\right) \sum_{n \in \mathbb{N}} G(\gamma, n),$$

où

$$G(\gamma, n) := \int_{\mathbb{R}} F\left(\begin{pmatrix} * & * \\ \gamma w_\alpha^{-1/2} & \gamma t w_\alpha^{1/2} \end{pmatrix}\right) e(-nt) dt.$$

En utilisant la définition (8.20), nous obtenons finalement

$$U(h, \Psi) = \sum_{n \in \mathbb{Z}} \sum_{\gamma \in \mathcal{C}(\infty, \alpha)} S_{\infty\alpha}(h, n; \gamma) G(\gamma, n). \quad (8.25)$$

*8.1.8. Localisation et préparation des variables.* Nous rappelons que  $w_\alpha \asymp_Q qd$ . Par définition de  $F = F_{\Psi, Q}$ , nous avons

$$G(\gamma, n) = \int_{\mathbb{R}} \Psi(tw_\alpha) f\left(\frac{Q(\gamma, \gamma tw_\alpha)}{qw_\alpha M}\right) e(-nt) dt.$$

Lorsque l'intégrand est non nul, nous avons  $|t| \leq 2w_\alpha^{-1}$  et  $\gamma \asymp_{Q, f} q(dM)^{1/2}$ . En intégrant par parties (cf. le lemme 5.1 de [Tó00]), il vient

$$G(\gamma, n) \ll_j (qd)^{j-1} n^{-j} \quad (j \in \mathbb{N}). \quad (8.26)$$

Cette majoration dépend aussi des constantes implicites dans (8.2); cette dépendance ne sera pas explicitée afin d'alléger la notation.

Posons  $N_1 := qd(Mq)^\eta$ . Dans le membre de droite de (8.25), nous isolons la contribution  $U_0$  (resp.  $U_1$ ) provenant de  $n = 0$  (resp.  $|n| > N_1$ ). Les bornes (8.22), (8.23) et (8.26) permettent d'écrire

$$|U_0(h, \Psi)| \ll_{\varepsilon, D} (qd)^{-1} \sum_{\substack{\gamma \in \mathcal{C}(\infty, \mathfrak{a}) \\ \gamma \asymp q(dM)^{1/2}}} |S_{\infty \mathfrak{a}}(h, 0; \gamma)| \ll (Mqh)^\varepsilon d^{-1} q^{-1/2} M^{1/2}, \tag{8.27}$$

$$|U_1(h, \Psi)| \ll_{j, D} (qM)^{3/2} \left(\frac{qd}{N}\right)^{j-1} \ll_{\eta, D} (qM)^{-10}$$

en choisissant  $j = j(\eta)$  suffisamment grand. Ces deux termes d'erreur sont bien de l'ordre du membre de droite de (8.5).

D'autre part, la formule de Faà di Bruno montre que la fonction  $G$  vérifie

$$\frac{\partial^{k+\ell}}{\partial x^{\ell_1} \partial y^{\ell_2}} G(x, y) \Big|_{\substack{x=\gamma \\ y=n}} \ll_{\ell_1, \ell_2} \gamma^{-\ell_1} (qd)^{-\ell_2-1}.$$

De même que (8.26), cette majoration dépend également des constantes implicites dans (8.2).

Nous introduisons une partition de l'unité pour la variable  $n$ ,

$$G(\gamma, n) = \sum_{0 \leq k \leq K} G_{2^k}(\gamma, n),$$

où  $K \leq 2 + (\log N_1)/\log 2$ , et pour  $1 \leq N \leq N_1$ , la fonction  $G_N(\gamma, n)$  est lisse par rapport aux deux variables, nulle en dehors de  $n \in [N/2, 2N]$  et satisfait

$$\frac{\partial^{k+\ell}}{\partial x^k \partial y^\ell} G_N(x, y) \Big|_{\substack{x=\gamma \\ y=n}} \ll (qd)^{-1} \gamma^{-k} (\min\{qd, N\})^{-\ell} \ll (qM)^{O(\eta\ell)} (qd)^{-1} \gamma^{-k} N^{-\ell}. \tag{8.28}$$

En accord avec cette décomposition, nous écrivons

$$\sum_{0 < |n| \leq N_1} \sum_{\gamma \in \mathcal{C}(\infty, \mathfrak{a})} S_{\infty \mathfrak{a}}(h, n; \gamma) G(\gamma, n) = \sum_{0 \leq k \leq K} V_{2^k}, \tag{8.29}$$

$$V_N := \sum_{N/2 \leq |n| \leq 2N} \sum_{\gamma \in \mathcal{C}(\infty, \mathfrak{a})} S_{\infty \mathfrak{a}}(h, n; \gamma) G_N(\gamma, n). \tag{8.30}$$

Nous posons ensuite

$$F(x, \xi) := \int_{\mathbb{R}} G_N\left(\frac{4\pi\sqrt{h|y|}}{x}, y\right) e(y\xi) dy,$$

$$G_N(\gamma, n) := \int_{\mathbb{R}} F\left(\frac{4\pi\sqrt{h|n|}}{\gamma}, \xi\right) e(-n\xi) d\xi.$$

L'intégrale définissant  $F$  est à support sur  $y \asymp N$ , et lorsque  $F(x, \xi) \neq 0$ , nous avons nécessairement  $x \asymp X := (hN/q^2dM)^{1/2}$ . La formule de Faà di Bruno implique encore

$$\partial_{k0} F(x, \xi) \ll_k (qM)^{O(\eta)} X^{-k} \frac{(qd)^{-1}N}{1 + (N\xi)^2}.$$

Ici la constante implicite dans  $O(\eta)$  est indépendante de  $k$ . Nous posons finalement

$$\phi_\xi(x) := X(1 + (N\xi)^2)qd(xN)^{-1}(qM)^{-\varpi} F(x, \xi)$$

pour un certain réel positif  $\varpi = O(\eta)$ , de sorte que la fonction  $x \mapsto \phi_\xi(x)$  soit lisse, à support sur  $x \asymp X$  et satisfasse

$$\sup_{\xi \in \mathbb{R}} \|\phi_\xi^{(j)}\| \ll_j X^{-j}. \quad (8.31)$$

Ici encore la borne dépend des constantes implicites dans (8.2). Cela ne sera plus rappelé dans la suite. Nous avons alors

$$V_N = 4\pi(qM)^\varpi d^{-1/2} M^{1/2} \int_{\mathbb{R}} \frac{N}{1 + (N\xi)^2} W_N(\xi) d\xi, \quad (8.32)$$

où nous avons posé

$$W_N(\xi) := \sum_{N/2 \leq |n| \leq 2N} a_n \sum_{\gamma \in \mathcal{C}(\infty, \alpha)} \frac{S_{\infty\alpha}^{(\xi)}(h, n; \gamma)}{\gamma} \phi_\xi\left(\frac{4\pi\sqrt{h|n|}}{\gamma}\right), \quad (8.33)$$

ainsi que  $a_n = \sqrt{|n|/N}$ , et où nous avons intégré le facteur  $e(-n\xi)$  dans la matrice d'échelle de  $\infty$  (ce qui est indiqué par la notation  $S_{\infty\alpha}^{(\xi)}$ ).

*8.1.9. Utilisation de la formule de Kuznetsov.* Nous majorons  $W_N$  pour chaque  $\xi$ . Nous omettrons la quantité  $\xi$  de la notation, et n'utiliserons de  $(a_n)$  et  $\phi$  que la borne  $|a_n| \leq 1$ , les majorations (8.31) et le fait que  $\phi(x) \neq 0$  entraîne  $x \asymp X$ , où nous rappelons que  $X \asymp (hN/q^2dM)^{1/2}$ .

Pour chaque  $n \in [N/2, 2N]$ , nous appliquons la formule de Kuznetsov (lemme 4.5 de [Dra17], avec  $\kappa = 0$ ). Nous obtenons

$$\sum_{\gamma \in \mathcal{C}(\infty, \alpha)} \frac{S_{\infty\alpha}(h, n; \gamma)}{\gamma} \phi\left(\frac{4\pi\sqrt{hn}}{\gamma}\right) = \mathcal{H}_{h,n}^+ + \mathcal{E}_{h,n}^+ + \mathcal{M}_{h,n}^+ \quad (n > 0),$$

$$\sum_{\gamma \in \mathcal{C}(\infty, \alpha)} \frac{S_{\infty\alpha}(h, n; \gamma)}{\gamma} \phi\left(\frac{4\pi\sqrt{h|n|}}{\gamma}\right) = \mathcal{E}_{h,n}^- + \mathcal{M}_{h,n}^- \quad (n < 0),$$

où

$$\mathcal{M}_{h,n}^+ := \sum_{f \in \mathcal{B}(q, \chi)} \frac{\tilde{\phi}(t_f)}{\cosh(\pi t_f)} (hn)^{1/2} \overline{\rho_{f\infty}(h)} \rho_{f\alpha}(n),$$

et  $\mathcal{M}_{h,n}^-$ ,  $\mathcal{E}_{h,n}^\pm$ ,  $\mathcal{H}_{h,n}^+$  sont données par des expressions similaires. Ici, l'ensemble  $\mathcal{B}(q, \chi)$  désigne une base orthonormée de formes de Maass paraboliques  $f$ , chacune étant fonction propre du Laplacien hyperbolique, de valeur propre associée  $\lambda_f = 1/4 + t_f^2$  et de coefficients de Fourier  $\rho_{fa}(n)$ . Nous référons à la section 4.1.2 de [Dra17] pour les définitions précises et la normalisation. Nous avons  $t_f \in \mathbb{R} \cup [-i\theta, i\theta]$ , où nous rappelons que  $\theta \leq 7/64$  grâce à Kim et Sarnak [Kim03], et que la conjecture de Selberg–Ramanujan prédit que  $\theta = 0$ . La transformée  $\tilde{\phi}$  dans l'expression ci-dessus est donnée par

$$\tilde{\phi}(t) = \frac{2\pi i}{\sinh(\pi t)} \int_0^\infty (J_{2it}(x) - J_{-2it}(x))\phi(x) \frac{dx}{x}$$

où  $J_\nu(x)$  désigne la fonction de Bessel. La transformée  $\tilde{\phi}$  satisfait les bornes énoncées au lemme 4.4 de [Dra17] (voir le lemme 2.4 de [Top17] pour des bornes plus précises). Dans le cas présent, nous avons  $X \ll (qM)^{\eta/2}$ , donc

$$|\tilde{\phi}(t)| \ll \begin{cases} (qM)^{2\eta}(1 + |t|)^{-3}, & t \in \mathbb{R}, \\ (Mq)^{\eta/2}(q^2dM/hN)^{|t|}, & t \in [-i/4, i/4]. \end{cases}$$

La quantité  $\mathcal{E}_{h,n}^\pm$  (resp.  $\mathcal{H}_{h,n}^+$ ) correspond à la contribution des séries d'Eisenstein non holomorphes (resp. à la contribution des formes holomorphes de poids  $\geq 2$ ). Nous étudions en détail le cas  $\mathcal{M}^+$ , les autres termes étant traités de façon similaire.

Notre traitement diffère selon que nous prenons la moyenne sur  $h$  ou pas.

8.1.10. *Le cas  $(h, q)$  fixé.* Nous séparons dans  $\mathcal{M}_{h,n}^+$  la contribution des fonctions  $f \in \mathcal{B}(q, \chi)$  avec  $t_f \in \mathbb{R}$  de celles avec  $t_f \in i\mathbb{R}$ . En accord avec cette décomposition, nous écrivons

$$W_N = \sum_{N/2 \leq n \leq 2N} a_n \mathcal{M}_{h,n}^+ = \mathcal{M}_{h,N}^{\text{rég}} + \mathcal{M}_{h,N}^{\text{exc}}, \tag{8.34}$$

où la notation correspond à ‘régulier’ et ‘exceptionnel’. L'inégalité de Cauchy–Schwarz fournit

$$|\mathcal{M}_{h,N}^{\text{rég}}| \leq (\mathcal{M}_h^{\text{rég}} \mathcal{M}_N^{\text{rég}})^{1/2} \tag{8.35}$$

avec

$$\begin{aligned} \mathcal{M}_h^{\text{rég}} &:= \sum_{\substack{f \in \mathcal{B}(q, \chi) \\ t_f \in \mathbb{R}}} \frac{|\tilde{\phi}(t_f)|}{\cosh(\pi t_f)} h |\rho_{f\infty}(h)|^2, \\ \mathcal{M}_N^{\text{rég}} &:= \sum_{\substack{f \in \mathcal{B}(q, \chi) \\ t_f \in \mathbb{R}}} \frac{|\tilde{\phi}(t_f)|}{\cosh(\pi t_f)} \left| \sum_{N/2 \leq n \leq 2N} a_n n^{1/2} \rho_{fa}(n) \right|^2. \end{aligned}$$

Pour majorer  $\mathcal{M}_h^{\text{rég}}$ , nous faisons appel au lemme 2.7 de [Top17], soit

$$\mathcal{M}_h^{\text{rég}} \ll_\varepsilon (qhM)^\varepsilon \left\{ 1 + (qd, h)^{1/2} \frac{h^{1/2}}{qd^{1/2}} \right\}.$$



Pour majorer  $\mathcal{M}_N^{\text{rég}}$ , nous utilisons l'inégalité de grand crible (proposition 4.7 de [Dra17])

$$\mathcal{M}_N^{\text{rég}} \ll_{\varepsilon} (qM)^{\varepsilon} N \left\{ 1 + \frac{N}{qd^{1/2}} \right\}. \quad (8.36)$$

Notre hypothèse  $h \ll q$  et  $N \leq N_1$  implique donc

$$\mathcal{M}_{h,N}^{\text{rég}} \ll_{\eta} (qM)^{O(\eta)} q^{1/2} d^{3/4}. \quad (8.37)$$

Pour chaque  $h$ , nous avons par l'inégalité de Cauchy–Schwarz

$$|\mathcal{M}_{h,N}^{\text{exc}}| \ll (\mathcal{M}_h^{\text{exc}} \mathcal{M}_N^{\text{exc}})^{1/2}, \quad (8.38)$$

$$\mathcal{M}_h^{\text{exc}} := \sum_{\substack{f \in \mathcal{B}(q, \chi) \\ t_f \in i\mathbb{R}}} |\tilde{\phi}(t_f)|^2 h |\rho_{f\infty}(h)|^2, \quad \mathcal{M}_N^{\text{exc}} := \sum_{\substack{f \in \mathcal{B}(q, \chi) \\ t_f \in i\mathbb{R}}} \left| \sum_{N/2 \leq n \leq 2N} a_n n^{1/2} \rho_{f\mathfrak{a}}(n) \right|^2. \quad (8.39)$$

L'inégalité de grand crible fournit encore

$$\mathcal{M}_N^{\text{exc}} \ll_{\varepsilon} (qM)^{\varepsilon} N \left\{ 1 + \frac{N}{qd^{1/2}} \right\}. \quad (8.40)$$

Pour  $\mathcal{M}_h^{\text{exc}}$ , nous utilisons le lemme 2.9 de [Top17],

$$\mathcal{M}_h^{\text{exc}} \ll_{\eta} (qhM)^{O(\eta)} \{(qd, h)MN^{-1}\}^{2\theta}.$$

Notre hypothèse  $N \leq N_1$  implique donc

$$\mathcal{M}_{h,N}^{\text{exc}} \ll_{\eta} (qhM)^{O(\eta)} d^{1/4} (qd, h)^{\theta} M^{\theta} (qd)^{1/2-\theta}.$$

Le membre de droite de cette inégalité est supérieur à celui obtenu en (8.37). Nous obtenons donc au final

$$\sum_{N/2 \leq n \leq 2N} a_n \mathcal{M}_{h,n}^+ \ll_{\eta} (qhM)^{O(\eta)} d^{1/4} (qd, h)^{\theta} M^{\theta} (qd)^{1/2-\theta}.$$

La même majoration est valable dans le cas de  $\mathcal{M}_{h,n}^-$ , tandis que les termes  $\mathcal{E}^{\pm}$  et  $\mathcal{H}^+$ , sont majorés par une quantité de l'ordre du membre de droite de (8.37). Nous avons ainsi

$$W_N \ll_{\eta} (qM)^{O(\eta)} d^{1/4} (qd, h)^{\theta} M^{\theta} (qd)^{1/2-\theta}.$$

Nous insérons cela dans (8.32) puis (8.29), ce qui fournit, avec les majorations (8.27) et en choisissant  $\eta > 0$  arbitrairement petit,

$$U(h, \Psi) \ll_{\varepsilon} (qM)^{\varepsilon} d^{-1/4} (qd, h)^{\theta} M^{\theta} (qd)^{1/2-\theta}.$$

Nous reportons cela dans (8.17), en utilisant le point (i) du Lemme 8.4 pour majorer le cardinal de la somme sur  $\sigma$ . Nous obtenons

$$T_Q(h, \Psi) \ll_{\varepsilon} (qM)^{\varepsilon} d^{3/4} (qd, h)^{\theta} M^{\theta} (qd)^{1/2-\theta},$$

ce qui fournit la majoration (8.5) grâce à (8.16), (8.11) et (8.8) successivement.

8.1.11. *Majoration en moyenne sur  $h$ .* Nous justifions dans cette section la majoration (8.6). Lorsque  $H \leq qd^{1/2}$ , nous nous contentons de prendre la moyenne sur  $h$  de l'estimation (8.5) établie dans les sections précédentes. Nous supposons donc dorénavant que  $H > qd^{1/2}$ .

Soit  $(c_h) \in \mathbb{C}^{\mathbb{N}}$  une suite telle que  $|c_h| \leq 1$  et

$$\left| \sum_{(m,\Omega) \in \mathcal{D}} f\left(\frac{m}{M}\right) e\left(\frac{h\Omega}{mq}\right) \right| = c_h \sum_{(m,\Omega) \in \mathcal{D}} f\left(\frac{m}{M}\right) e\left(\frac{h\Omega}{mq}\right).$$

Les coefficients  $(c_h)$  dépendent au plus de  $(h, q, d, \lambda, \omega, M, f)$ .

Rappelant la définition (8.34), il nous suffira d'établir les majorations

$$\mathcal{M}_{H,N}^{\text{rég}} := \frac{1}{H} \sum_{H < h \leq 2H} c_h \mathcal{M}_{h,N}^{\text{rég}} \ll_{\eta} (qHM)^{O(\eta)} d^{3/4} q^{1/2}, \tag{8.41}$$

$$\mathcal{M}_{H,N}^{\text{exc}} := \frac{1}{H} \sum_{H < h \leq 2H} c_h \mathcal{M}_{h,N}^{\text{exc}} \ll_{\eta} (qHM)^{O(\eta)} d^{1/4} M^{\theta} (qd)^{1/2-\theta}. \tag{8.42}$$

Les bornes (8.36) et (8.40) sont toujours valables en moyenne sur  $h$ , puisqu'elles ne dépendent pas de  $h$ .

Dans le cas de (8.41), un raisonnement similaire à (8.35) nous ramène à étudier

$$\mathcal{M}_H^{\text{rég}} := H^{-2} \sum_{\substack{f \in \mathcal{B}(q,\chi) \\ t_f \in \mathbb{R}}} \frac{|\tilde{\phi}(t_f)|}{\cosh(\pi t_f)} \left| \sum_{H < h \leq 2H} c_h \sqrt{h} \rho_{f\infty}(h) \right|^2.$$

L'inégalité de grand crible fournit

$$\mathcal{M}_H^{\text{rég}} \ll_{\varepsilon} (qHM)^{\varepsilon} H^{-1} \left\{ 1 + \frac{H}{qd^{1/2}} \right\} \ll_{\varepsilon} (qHM)^{\varepsilon},$$

d'où l'on déduit la borne (8.41).

Concernant (8.42), en raisonnant de façon similaire à (8.38), nous nous ramenons à étudier

$$\mathcal{M}_H^{\text{exc}} := H^{-2} \sum_{\substack{f \in \mathcal{B}(q,\chi) \\ t_f \in i\mathbb{R}}} |\tilde{\phi}(t_f)|^2 \left| \sum_{H < h \leq 2H} c_h \sqrt{h} \rho_{f\infty}(h) \right|^2.$$

Nous appliquons l'inégalité de grand crible pour le spectre exceptionnel, lemme 4.8 de [Dra17], en tirant parti de la borne de Kim–Sarnak (voir la remarque qui précède la section 4.3 de [Dra17]). Nous obtenons

$$\begin{aligned} \mathcal{M}_H^{\text{exc}} &\ll_{\eta} H^{-1} (qM)^{O(\eta)} \left( 1 + \left(\frac{qM}{N}\right)^{2\theta} \right) \left( 1 + d^{1/2} \left(\frac{H}{qd}\right)^{1-2\theta} \right) \\ &\ll_{\eta} (qM)^{O(\eta)} (q^2 d)^{2\theta-1/2} \left(\frac{M}{N}\right)^{2\theta}. \end{aligned}$$

Cela suffit pleinement à démontrer l'estimation (8.42). La formule (8.6) en est déduite de façon identique au cas de  $h$  fixé.

8.1.12. *Majoration en moyenne sur  $h$  et sur  $q$ .* Supposons maintenant que  $d = 1$ , et que  $c_h = e(th)\mathbf{1}_{h \in \mathcal{I}}$  pour un certain intervalle  $\mathcal{I} \subset [H, 2H]$ . Nous suivons les arguments des sections précédentes, en encodant le facteur  $e(th)$  dans la matrice d'échelle de  $\infty$ , ce qui nous ramène à l'estimation de

$$\mathcal{M}_H^{\text{exc}}(q) = H^{-2} \sum_{\substack{f \in \mathcal{B}(q, \mathbf{1}) \\ t_f \in i\mathbb{R}}} |\tilde{\phi}(t_f)|^2 \left| \sum_{h \in \mathcal{I}} \sqrt{h} \rho_{f\infty}(h) \right|^2.$$

Nous sommes cela sur  $q \in [Q, 2Q]$ , et utilisons l'inégalité de grand crible pondérée de Deshouillers–Iwaniec, [DI83, théorème 7]. Nous obtenons

$$\frac{1}{Q} \sum_{Q < q \leq 2Q} \mathcal{M}_H^{\text{exc}}(q) \ll_{\varepsilon, \eta} M^{O(\eta)} H^{-1+\varepsilon} \left\{ 1 + \frac{H}{Q} + \left( \frac{M}{N} \right)^{2\theta} \right\}.$$

Avec  $2\theta$  remplacé par  $\frac{1}{2}$ , cela découle directement du théorème 7 de [DI83]. L'équation qui précède est aisément justifiée en notant qu'à la conclusion de la preuve du théorème 7, page 278 de [DI83], la quantité  $\sqrt{Y/Y_1}$  peut être remplacée par  $(Y/Y_1)^{2\theta}$ . La conclusion de la preuve suit de façon identique au cas de  $h$  fixé.

**Remarques.** Lorsque  $H$  est grand, le terme d'erreur que nous obtenons est légèrement meilleur que celui annoncé en (8.6). Cela n'a pas d'influence pour l'application que nous considérons ici.

Les facteurs  $h$  et  $H$  apparaissant aux premiers termes des membres de droite de (8.5)–(8.7) peuvent être améliorés en opérant une intégration par parties au lieu de l'approximation triviale (8.15).

## 8.2. Démonstration du Lemme 8.2

Du Lemme 8.3, nous déduisons par une technique standard d'analyse de Fourier l'estimation

$$\begin{aligned} & P_f(M; q, r, d, \lambda, \omega, \alpha, \beta) \\ &= (\beta - \alpha) P_f(M; q, r, d, \lambda, \omega, 0, 1) + O_{\varepsilon, D, f}((qM)^\varepsilon d^{3/4} (qd)^{1/2-\theta} M^{1/2+\theta}). \end{aligned} \quad (8.43)$$

Nous omettons les détails, qui sont similaires aux pages 179 et 180 de [Iwa78]. La seule différence avec notre traitement tient aux termes supplémentaires  $h$  et  $H$  dans les membres de droite de (8.5) et (8.6), qui imposent le choix  $\Delta = (q + M^{1/2})^{-1}$  dans l'argument d'Iwaniec. Cela induit un terme d'erreur supplémentaire de l'ordre de

$$\Delta^{-1} + M\Delta \ll q + M^{1/2} \ll q^{1/2-\theta} M^{1/2+\theta},$$

qui est acceptable.

Nous nous concentrons donc sur le traitement du terme principal. Nous aurons besoin du lemme suivant.

**Lemme 8.5.** Soient  $x \in \mathbb{R}$  avec  $x \geq 1$ ,  $D \in \mathbb{Z}$  qui n'est pas un carré d'entier,  $(q, d) \in \mathbb{N}^2$  avec  $q \geq 1$ ,  $(q, 2D) = 1$ ,  $d \mid q$  et  $\lambda \pmod{d}$  avec  $(\lambda, d) = 1$ . Notons  $\chi_D = \left(\frac{D}{\cdot}\right)$  le symbole de Kronecker, et  $\varkappa_D(n) := (1 * \chi_D)(n)$ . Alors

$$\sum_{\substack{n \leq x \\ (n,q)=1 \\ n \equiv \lambda \pmod{d}}} \varkappa_D(n) = \frac{x}{\varphi(d)} \frac{\varphi(q)}{q} \prod_{p \mid q} \left(1 - \frac{\chi_D(p)}{p}\right) L(1, \chi_D) + O_{\varepsilon, D}(x^{1/2} q^\varepsilon).$$

*Démonstration.* Cela découle facilement du principe de l'hyperbole de Dirichlet. □

Rappelons que pour  $p \nmid 2D$ , nous avons  $\rho(p) = 1 + \left(\frac{D}{p}\right) = \varkappa_D(p)$ . Nous écrivons  $\rho = \varkappa_D * h_D$ , de sorte que la fonction  $h_D$  vérifie  $\sum_{\ell} |h_D(\ell)| \ell^{-1/2-\varepsilon} \ll_{\varepsilon, D} 1$ . Lorsque  $M \geq 1$  et  $(\lambda, d) = 1$ , nous en déduisons, à l'aide du Lemme 8.5 et d'une intégration par parties,

$$\begin{aligned} \sum_{\substack{(m,q)=1 \\ m \equiv \lambda \pmod{d}}} f\left(\frac{m}{M}\right) \rho(m) &= \sum_{\substack{(\ell,q)=1 \\ \ell \ll M}} h_D(\ell) \sum_{\substack{(n,q)=1 \\ n \equiv \lambda \bar{\ell} \pmod{d}}} f\left(\frac{n\ell}{M}\right) \varkappa_D(n) \\ &= \frac{1}{\varphi(d)} \frac{\varphi(q)}{q} L(1, \chi_D) M \widehat{f}(0) \sum_{(\ell,q)=1} \frac{h_D(\ell)}{\ell} + O_{\varepsilon, D, f}(q^\varepsilon M^{1/2+\varepsilon}). \end{aligned} \tag{8.44}$$

Notons

$$C_D := L(1, \chi_D) \sum_{\ell \geq 1} \frac{h_D(\ell)}{\ell} = \sum_{\ell \geq 1} \frac{(\rho * \mu)(\ell)}{\ell}.$$

Nous obtenons

$$L(1, \chi_D) \frac{\varphi(q)}{q} \sum_{(\ell,q)=1} \frac{h_D(\ell)}{\ell} = C_D \prod_{p \mid q} \left(1 + \frac{1}{p}\right)^{-1}. \tag{8.45}$$

Nous revenons maintenant à l'estimation du terme principal du membre de droite de (8.43). Le théorème des restes chinois et les relations (8.44) et (8.45) avec  $q$  remplacé par  $qr$  fournissent

$$\begin{aligned} P_f(M; q, r, d, \lambda, \omega, 0, 1) &= \sum_{\substack{(m,qr)=1 \\ m \equiv \lambda \pmod{d}}} f\left(\frac{m}{M}\right) \sum_{\substack{\Omega \pmod{qm} \\ \Omega^2 \equiv D \pmod{qm} \\ \Omega \equiv \omega \pmod{d}}} 1 \\ &= \rho_{\omega, d}(q) \sum_{\substack{(m,qr)=1 \\ m \equiv \lambda \pmod{d}}} f\left(\frac{m}{M}\right) \rho(m) \\ &= C_D \prod_{p \mid qr} \left(1 + \frac{1}{p}\right)^{-1} \frac{\rho_{\omega, d}(q)}{\varphi(d)} M \widehat{f}(0) + O_{\varepsilon, D, f}(M^{1/2+\varepsilon} q^\varepsilon), \end{aligned}$$

où nous avons noté, pour tout  $\omega \pmod{d}$  avec  $\omega^2 \equiv D \pmod{d}$ ,

$$\rho_{\omega,d}(q) := \sum_{\substack{\Omega \pmod{q} \\ \Omega^2 \equiv D \pmod{q} \\ \Omega \equiv \omega \pmod{d}}} 1.$$

Il est aisé de voir que  $\rho_{\omega,d}(q) = \rho(q)$  si  $d = 1$ , et pour tout  $p \nmid 2D$ ,  $1 \leq \delta \leq v$ ,  $\rho_{\omega,p^\delta}(p^\nu) = 1$  par le lemme de Hensel. Nous en déduisons que  $\rho_{\omega,d}(q) = \rho(q/(q, d^\infty))$  indépendamment de  $\omega$ . Cela conclut la démonstration du Lemme 8.2.

### 8.3. Démonstration de la Proposition 8.1

**8.3.1. Première réduction.** Nous remarquons tout d'abord que la borne triviale  $x^{2+\varepsilon}/M$  pour membre de gauche de (8.1) nous permet de supposer sans perte de généralité que  $x \geq M$ .

Dans le but de simplifier la preuve de la Proposition 8.1, nous nous ramenons à supposer que la suite  $(b_n)$  est à support sur les entiers impairs premiers avec  $D$ . Supposons donc dans un premier temps que l'estimation (8.1) est valable pour de telles suites. Notant

$$r_D(x; q) := \sum_{\substack{k \in \mathbb{N} \\ q \mid k^2 - D}} V\left(\frac{k}{x}\right) - x \widehat{V}(0) \frac{\rho(q)}{q}, \quad (8.46)$$

nous avons, grâce à l'inégalité de Cauchy–Schwarz et la majoration  $\sum_{\substack{1 \leq v \leq 2N \\ v \mid (2D)^\infty}} 1 \ll x^\varepsilon$ ,

$$\begin{aligned} \sum_{M < m \leq 2M} \left| \sum_{\substack{N < n \leq 2N \\ (n,m)=1}} b_n r_D(x; mn) \right|^2 &= \sum_{M < m \leq 2M} \left| \sum_{\substack{1 \leq v \leq 2N \\ v \mid (2D)^\infty}} \sum_{\substack{N/v < n \leq 2N/v \\ (n, 2Dm)=1}} b_{vn} r_D(x; vmn) \right|^2 \\ &\ll_\varepsilon x^\varepsilon \sum_{M < m \leq 2M} \sum_{\substack{1 \leq v \leq 2N \\ v \mid (2D)^\infty}} \left| \sum_{\substack{N/v < n \leq 2N/v \\ (n, 2Dm)=1}} b_{vn} r_D(x; vmn) \right|^2 \\ &\ll_\varepsilon x^\varepsilon \sum_{\substack{v \leq 2N \\ v \mid (2D)^\infty}} \sum_{vM < m \leq 2vM} \left| \sum_{\substack{N/v < n \leq 2N/v \\ (n, 2Dm)=1}} b_{vn} r_D(x; mn) \right|^2. \end{aligned}$$

La majoration (8.1) appliquée pour chaque  $v$  au membre de droite fournit l'estimation voulue.

Nous supposons donc dans ce qui suit que la suite  $(b_n)$  est à support sur les entiers  $n$  tels que  $(n, 2D) = 1$ .

**8.3.2. Interprétation d'une congruence.** Nous suivons les arguments des pages 180–183 de [Iwa78]. Pour cela nous devons modifier la construction de la classe  $c \pmod{[n_1, n_2]}$ , page 183 de [Iwa78], en raison du fait que dans notre situation,  $(b_n)$  n'est pas supposée à support sur les entiers sans facteur carré.

**Lemme 8.6.** Soient  $m, n_1, n_2, \ell_1, \ell_2 \geq 1$  donnés avec  $(2mD, n_1n_2) = 1$ . Définissons

$$d := (n_1, n_2)/(n_1, n_2, \ell_1 - \ell_2),$$

et supposons que

$$(m(\ell_1 - \ell_2))^2 \equiv 4D \pmod{d}. \tag{8.47}$$

Alors il existe  $c \in \mathbb{Z}$  avec  $0 \leq c < [n_1, n_2]$  tel que les ensembles

$$\mathcal{D}_1 := \left\{ v \in \mathbb{Z} \cap [0, m[ : \begin{array}{l} v^2 \equiv D \pmod{m} \\ (m\ell_j + v)^2 \equiv D \pmod{n_j} \quad (j \in \{1, 2\}) \end{array} \right\}$$

et

$$\mathcal{D}_2 := \left\{ \Omega \in \mathbb{Z} \cap [cm, (c + 1)m[ : \begin{array}{l} \Omega^2 \equiv D \pmod{m[n_1, n_2]} \\ \Omega \equiv m\left(c - \frac{1}{2}(\ell_1 + \ell_2)\right) \pmod{d} \end{array} \right\}$$

soient en bijection.

**Remarque.** Les ensembles  $\mathcal{D}_1$  et  $\mathcal{D}_2$  sont vides si la condition (8.47) n'est pas satisfaite.

*Démonstration du Lemme 8.6.* Notons

$$n_j := \prod_p p^{v_j(p)} \quad (j \in \{1, 2\}),$$

Nous définissons  $c \in \mathbb{Z}$ ,  $0 \leq c < [n_1, n_2]$ , comme l'unique entier satisfaisant, pour tout  $p$ ,

$$c \equiv \begin{cases} \ell_1 \pmod{p^{v_1(p)}} & \text{si } v_1(p) \geq v_2(p), \\ \ell_2 \pmod{p^{v_2(p)}} & \text{sinon.} \end{cases}$$

À tout  $v \in \mathbb{Z} \cap [0, m[$ , nous associons  $\Omega(v) = cm + v \in [cm, (m + 1)c[$ . Cette application est bijective, et il nous suffit de vérifier que  $\Omega(\mathcal{D}_1) = \mathcal{D}_2$ . Supposons  $v \in \mathcal{D}_1$ , et soit  $\Omega = \Omega(v)$ . Puisque  $(m, [n_1, n_2]) = 1$ , il suffit de vérifier la congruence  $\Omega^2 \equiv D$  modulo  $m$  et  $[n_1, n_2]$ , séparément. Nous avons  $\Omega \equiv v \pmod{m}$ , ce qui fournit bien  $\Omega^2 \equiv D \pmod{m}$ . Pour tout  $p$ , nous avons

$$\Omega \equiv \ell_j m + v \pmod{p^{v_j(p)}}$$

avec  $j = 1$  si  $v_1(p) \geq v_2(p)$ , et  $j = 2$  sinon. Nous obtenons dans les deux cas  $\Omega^2 \equiv D \pmod{p^{v_j(p)}}$ , donc  $\Omega^2 \equiv D \pmod{[n_1, n_2]}$ . La condition  $\Omega \equiv m\left(c - \frac{1}{2}(\ell_1 + \ell_2)\right) \pmod{d}$  découle facilement du fait que

$$(m\ell_1 + v)^2 \equiv (m\ell_2 + v)^2 \pmod{(n_1, n_2)}.$$

Supposons ensuite  $\Omega \in \mathcal{D}_2$  donné, et posons  $v = \Omega - mc$ . La relation  $v^2 \equiv D \pmod{m}$  est alors immédiate. Soit ensuite  $p$  fixé, notons  $v_j = v_j(p)$  et supposons  $v_1 \geq v_2$  (le cas complémentaire étant traité de façon identique). Nous avons alors

$$c \equiv \ell_1 \pmod{p^{v_1}}, \quad \Omega^2 \equiv D \pmod{p^{v_1}},$$

ce qui fournit directement la relation  $(m\ell_1 + v)^2 \equiv D \pmod{p^{v_1}}$ . Nous avons par ailleurs

$$(m\ell_2 + v)^2 \equiv \Omega^2 - 2m(\ell_1 - \ell_2)\Omega + (m(\ell_1 - \ell_2))^2 \pmod{p^{v_2}}.$$

Par hypothèse,  $\Omega^2 \equiv D \pmod{p^{v_2}}$ . Ensuite,

$$\Omega \equiv m\left(c - \frac{1}{2}(\ell_1 + \ell_2)\right) \equiv \frac{1}{2}m(\ell_1 - \ell_2) \pmod{\frac{p^{v_2}}{(p^{v_2}, \ell_1 - \ell_2)}},$$

ce qui fournit

$$2m(\ell_1 - \ell_2)\Omega \equiv (m(\ell_1 - \ell_2))^2 \pmod{p^{v_2}}.$$

Nous en déduisons  $(m\ell_2 + v)^2 \equiv D \pmod{p^{v_2}}$ . Nous avons donc  $v \in \mathcal{D}_1$ .  $\square$

**8.3.3. Méthode de dispersion.** Nous développons le carré dans le membre de gauche de (8.1). En accord avec [Iwa78], nous posons

$$Y(m) := \sum_{\substack{N < n \leq 2N \\ (n, m) = 1}} b_n \frac{\rho(n)}{n}.$$

Fixons aussi une fonction lisse  $f : \mathbb{R} \rightarrow \mathbb{R}$  telle que  $\mathbf{1}_{1 \leq t \leq 2} \leq f(t) \leq \mathbf{1}_{1/2 \leq t \leq 3}$ . Enfin, nous rappelons la notation (8.46). Le membre de gauche de (8.1) est majoré par

$$\begin{aligned} & \sum_m f\left(\frac{m}{M}\right) \left| \sum_{\substack{N < n \leq 2N \\ (n, m) = 1}} b_n r_D(x; mn) \right|^2 \\ &= \sum_m f\left(\frac{m}{M}\right) \left| \sum_{\substack{N < n \leq 2N \\ (n, m) = 1}} b_n \sum_{\substack{k \in \mathbb{N} \\ q|k^2 - D}} V\left(\frac{k}{x}\right) - x \widehat{V}(0) \frac{\rho(q)}{q} r_D(x; mn) \right|^2 \\ &= \sum_m f\left(\frac{m}{M}\right) \left| \sum_{\substack{0 \leq v < m \\ v^2 \equiv D \pmod{m}}} \left( \sum_{\substack{N < n \leq 2N \\ (n, m) = 1}} b_n \sum_{\substack{k \in \mathbb{N} \\ k \equiv v \pmod{m} \\ k^2 \equiv D \pmod{n}}} V\left(\frac{k}{x}\right) - x \widehat{V}(0) \frac{\rho(n)}{mn} \right) \right|^2 \\ &\leq \sum_m f\left(\frac{m}{M}\right) \rho(m) \sum_{\substack{0 \leq v < m \\ v^2 \equiv D \pmod{m}}} \left| \sum_{\substack{N < n \leq 2N \\ (n, m) = 1}} b_n \sum_{\substack{k \in \mathbb{N} \\ k \equiv v \pmod{m} \\ k^2 \equiv D \pmod{n}}} V\left(\frac{k}{x}\right) - x \widehat{V}(0) \frac{\rho(n)}{mn} \right|^2 \\ &\ll_{\varepsilon} x^{\varepsilon} (S_1 - 2x \overline{\widehat{V}(0)} \operatorname{Re} S_2 + |x \widehat{V}(0)|^2 S_3) \end{aligned} \tag{8.48}$$

avec

$$S_j := \sum_m f\left(\frac{m}{M}\right) \sum_{\substack{0 \leq v < m \\ v^2 \equiv D \pmod{m}}} T_j(m),$$

$$T_1(m) := \sum_{\substack{N < n_1, n_2 \leq 2N \\ (n_1 n_2, m) = 1}} b_{n_1} \overline{b_{n_2}} \sum_{\substack{k_1, k_2 \in \mathbb{N} \\ k_j \equiv v \pmod{m} \\ k_j^2 \equiv D \pmod{n_j}}} V\left(\frac{k_1}{x}\right) \overline{V\left(\frac{k_2}{x}\right)},$$

$$T_2(m) := \frac{\overline{Y(m)}}{m} \sum_{\substack{N < n \leq 2N \\ (n, m) = 1}} b_n \sum_{\substack{k \in \mathbb{N} \\ k \equiv v \pmod{m} \\ k^2 \equiv D \pmod{n}}} V\left(\frac{k}{x}\right), \quad T_3(m) := \left(\frac{Y(m)}{m}\right)^2.$$

8.3.4. *Estimation de  $S_3$ .* Nous avons

$$S_3 = \frac{1}{M^2} \sum_{N < n_1, n_2 \leq 2N} b_{n_1} \overline{b_{n_2}} \frac{\rho(n_1)\rho(n_2)}{n_1 n_2} \sum_{(m, n_1 n_2) = 1} \frac{M^2}{m^2} f\left(\frac{m}{M}\right) \rho(m).$$

Notant  $g_1(t) := t^{-2} f(t)$ , la somme en  $m$  du membre de droite vaut

$$P_{g_1}(M; 1, n_1 n_2, 1, 1, 1, 0, 1).$$

Nous obtenons donc

$$S_3 = P_3 + O_{\varepsilon, D}(x^\varepsilon M^{-3/2+\theta}), \tag{8.49}$$

avec

$$P_3 := C_D M^{-1} \left( \int_{\mathbb{R}} t^{-2} f(t) dt \right) \sum_{N < n_1, n_2 \leq 2N} b_{n_1} \overline{b_{n_2}} A(n_1 n_2) \frac{\rho(n_1)\rho(n_2)}{n_1 n_2}.$$

8.3.5. *Estimation de  $S_2$ .* Nous avons

$$S_2 = \sum_{N < n_1, n_2 \leq 2N} b_{n_1} \overline{b_{n_2}} \frac{\rho(n_2)}{n_2} \sum_{(m, n_1 n_2) = 1} \frac{1}{m} f\left(\frac{m}{M}\right) \sum_{\substack{0 \leq v < m \\ v^2 \equiv D \pmod{m}}} \sum_{\substack{k \in \mathbb{N} \\ k \equiv v \pmod{m} \\ k^2 \equiv D \pmod{n_1}}} V\left(\frac{k}{x}\right).$$

Nous écrivons  $k = m\ell + v$  avec  $\ell \geq 0$  et  $\ell \ll x/m$ . Nous avons alors

$$V\left(\frac{m\ell + v}{x}\right) = V\left(\frac{m\ell}{x}\right) + O\left(\frac{m}{x}\right),$$

ce qui fournit, de même que dans [Iwa78, formule (11)], l'approximation  $S_2 = S'_2 + O(x^\varepsilon)$  avec

$$S'_2 := \sum_{N < n_1, n_2 \leq 2N} b_{n_1} \overline{b_{n_2}} \frac{\rho(n_2)}{n_2} \sum_{(m, n_1 n_2) = 1} \frac{1}{m} f\left(\frac{m}{M}\right) \sum_{\substack{\ell \geq 0 \\ 0 \leq v < m \\ v^2 \equiv D \pmod{m} \\ (m\ell + v)^2 \equiv D \pmod{n_1}}} V\left(\frac{m\ell}{x}\right).$$

Les supports de  $f$  et  $V$  impliquent que les entiers  $\ell$  fournissant une contribution non nulle à  $S'_2$  proviennent d'un intervalle d'entiers  $I$  tels que  $\ell \asymp x/M$  pour tout  $\ell \in I$ . Pour



tout  $n_2$  avec  $\rho(n_2) \neq 0$ , nous posons  $n'_2 = n_2/(n_2, n_1^\infty)$ . Soit  $c \in \mathbb{N} \cap [0, n_1[$  l'unique entier satisfaisant  $c \equiv \ell \pmod{n_1}$ . Nous avons une bijection

$$\left\{ v \in \mathbb{N} \cap [0, m[ : \begin{array}{l} v^2 \equiv D \pmod{m} \\ (m\ell + v)^2 \equiv D \pmod{n_1} \end{array} \right\} \\ \rightarrow \left\{ \Omega \in \mathbb{N} \cap [0, mn_1[ : \begin{array}{l} \Omega^2 \equiv D \pmod{mn_1} \\ cm \leq \Omega < (c+1)m \end{array} \right\}$$

donnée par  $v \mapsto mc + v$ . Ainsi,

$$S'_2 = \frac{1}{M} \sum_{N < n_1, n_2 \leq 2N} b_{n_1} \overline{b_{n_2}} \frac{\rho(n_2)}{n_2 \rho(n_2/(n_2, n_1^\infty))} \sum_{\ell \in I} \sum_{(m, n_1 n_2)=1} g_{2, \ell} \left( \frac{m}{M} \right) \sum_{\substack{\Omega \in \mathbb{N} \\ \Omega^2 \equiv D \pmod{mn_1} \\ cm \leq \Omega < (c+1)m}} 1$$

avec  $g_{2, \ell}(t) := t^{-1} f(t) V(t\ell M/x)$ , qui satisfait l'hypothèse (8.2). La somme sur  $(m, \Omega)$  est exactement  $P_{g_{2, \ell}}(M; n_1, n'_2, 1, 1, 1, \frac{c}{n_1}, \frac{c+1}{n_1})$ , le Lemme 8.2 fournit donc

$$S'_2 = P_2 + O_{\varepsilon, D}(x^\varepsilon N^{-3/2-\theta} M^{-1/2+\theta})$$

avec

$$P_2 := C_D \sum_{N < n_1, n_2 \leq 2N} b_{n_1} \overline{b_{n_2}} \frac{\rho(n_1)\rho(n_2)}{n_1 n_2} A(n_1 n_2) \int t^{-1} f(t) \sum_{\ell \in \mathbb{Z}} V\left(\frac{\ell t M}{x}\right) dt.$$

Uniformément pour  $t \in \text{supp } f$ , nous utilisons

$$\sum_{\ell \in \mathbb{Z}} V\left(\frac{\ell t M}{x}\right) = \frac{x}{Mt} \widehat{V}(0) + O(1),$$

ce qui fournit  $P_2 = x \widehat{V}(0) P_3 + O_{\varepsilon, D}(x^\varepsilon)$ , et finalement

$$S_2 = x \widehat{V}(0) P_3 + O_{\varepsilon, D}(x^\varepsilon \{1 + N^{-3/2-\theta} M^{-1/2+\theta}\}). \quad (8.50)$$

**8.3.6. Estimation de  $S_1$  et conclusion.** Dans la somme  $S_1$ , nous posons  $k_j = m\ell_j + v$  avec  $\ell_j \geq 0$ , ainsi

$$S_1 = \sum_{N < n_1, n_2 \leq 2N} b_{n_1} \overline{b_{n_2}} \sum_{\ell_1, \ell_2 \geq 0} \sum_{(m, n_1 n_2)=1} f\left(\frac{m}{M}\right) \\ \times \sum_{\substack{0 \leq v < m \\ v^2 \equiv D \pmod{m} \\ (m\ell_j + v)^2 \equiv D \pmod{n_j}}} V\left(\frac{m\ell_1 + v}{x}\right) \overline{V\left(\frac{m\ell_2 + v}{x}\right)}.$$

Nous remplaçons le produit  $V(\dots)\overline{V(\dots)}$  par  $V(m\ell_1/x)\overline{V(m\ell_2/x)}$ . L'erreur induite dans  $S_1$  est  $O_{\varepsilon,D}(x^{1+\varepsilon})$ , de sorte que  $S_1 = S'_1 + O_{\varepsilon,D}(x^{1+\varepsilon})$  avec

$$S'_1 := \sum_{N < n_1, n_2 \leq 2N} b_{n_1} \overline{b_{n_2}} \sum_{\ell_1, \ell_2 \geq 0} \sum_{(m, n_1 n_2) = 1} f\left(\frac{m}{M}\right) \times V\left(\frac{m\ell_1}{x}\right) \overline{V\left(\frac{m\ell_2}{x}\right)} \sum_{\substack{0 \leq v < m \\ v^2 \equiv D \pmod{m} \\ (m\ell_j + v)^2 \equiv D \pmod{n_j}}} 1. \tag{8.51}$$

Pour chaque  $(n_1, n_2, \ell_1, \ell_2)$ , la somme sur  $v$  est exprimée au moyen du Lemme 8.6. Nous posons  $q := [n_1, n_2]$ ,  $d := (n_1, n_2)/(n_1, n_2, \ell_1 - \ell_2)$ , et

$$\mathcal{L} := \{\lambda \pmod{d} : (\lambda(\ell_1 - \ell_2))^2 \equiv 4D \pmod{d}\}.$$

Puisque  $(d, 2D) = 1$ , nous avons  $\mathcal{L} = \emptyset$  si  $(\ell_1 - \ell_2, d) > 1$ , et  $|\mathcal{L}| = \rho(d)$  sinon. Nous supposons donc que la somme sur  $(\ell_1, \ell_2)$  est restreinte à  $(\ell_1 - \ell_2, d) = 1$ . La somme sur  $(m, v)$  dans le membre de droite de (8.51) vaut

$$\sum_{\lambda \in \mathcal{L}} P_{g_3}\left(M; q, d, \lambda, \omega_\lambda, \frac{c}{q}, \frac{c+1}{q}\right) \quad (\omega_\lambda := \lambda(c - \frac{1}{2}(\ell_1 + \ell_2)))$$

avec  $g_3(t) := f(t)V(t\ell_1 M/x)\overline{V(t\ell_2 M/x)}$ . Puisque  $|\mathcal{L}| = \rho(d)$  et  $\rho(d)\rho(q/(q, d^\infty)) = \rho(q)$ , le Lemme 8.2 fournit

$$S'_1 = P_1 + O_{\varepsilon,D}\left(x^{1+\varepsilon} + x^{2+\varepsilon}\left(\frac{N^2}{M}\right)^{3/2-\theta}\right)$$

avec

$$P_1 = C_D M \sum_{N < n_1, n_2 \leq 2N} b_{n_1} \overline{b_{n_2}} \sum_{\substack{\ell_1, \ell_2 \geq 0 \\ (\ell_1 - \ell_2, d) = 1}} \frac{\rho(q)}{q} \frac{A(q)}{\varphi(d)} \int_{\mathbb{R}} f(t) V\left(\frac{t\ell_1 M}{x}\right) \overline{V\left(\frac{t\ell_2 M}{x}\right)} dt. \tag{8.52}$$

Notons temporairement  $n_0 = (n_1, n_2)$ . Rappelons que  $d = n_0/(n_0, \ell_1 - \ell_2)$ . Pour  $X \gg 1$ , nous avons

$$\begin{aligned} \sum_{\substack{\ell_1, \ell_2 \in \mathbb{N} \\ (\ell_1 - \ell_2, n_0) = n_0/d \\ (\ell_1 - \ell_2, d) = 1}} V\left(\frac{\ell_1}{X}\right) \overline{V\left(\frac{\ell_2}{X}\right)} &= \mathbf{1}_{(d, n_0/d) = 1} \sum_{\ell \in \mathbb{N}} V\left(\frac{\ell}{X}\right) \sum_{\substack{k \in \mathbb{Z} \\ (k, d) = 1}} \overline{V\left(\frac{\ell + kn_0/d}{X}\right)} \\ &= \mathbf{1}_{(d, n_0/d) = 1} \left\{ \frac{\varphi(d)}{n_0} |X \widehat{V}(0)|^2 + O_{\varepsilon,D}(d^\varepsilon X) \right\}. \end{aligned} \tag{8.53}$$

Notons que la propriété  $\rho(p^\nu) = \rho(p) \in \{0, 2\}$  (pour  $p \nmid 2D$ ,  $\nu \geq 1$ ) implique

$$\rho([n_1, n_2]) \sum_{\substack{d|(n_1, n_2) \\ (d, (n_1, n_2)/d)=1}} 1 = \rho([n_1, n_2]) 2^{\omega((n_1, n_2))} = \rho(n_1)\rho(n_2). \quad (8.54)$$

Nous insérons l'estimation (8.53) avec  $X = x/(Mt)$  dans le membre de droite de (8.52) (nous rappelons que l'hypothèse supplémentaire  $M \leq x$  a été justifiée à la section 8.3.1). Les facteurs  $\varphi(d)$  se compensent, et la relation (8.54) nous permet de déduire  $P_1 = P'_1 + O(x^{1+\varepsilon})$  avec

$$P'_1 = \frac{|x\widehat{V}(0)|^2}{M} C_D \sum_{N < n_1, n_2 \leq 2N} b_{n_1} \overline{b_{n_2}} \frac{\rho(n_1)\rho(n_2)}{n_1 n_2} A(n_1 n_2) \int_{\mathbb{R}} t^{-2} f(t) dt.$$

Nous avons donc  $P'_1 = |x\widehat{V}(0)|^2 P_3$ , et finalement

$$S_1 = |x\widehat{V}(0)|^2 P_3 + O_{\varepsilon, D} \left( x^{1+\varepsilon} + x^{2+\varepsilon} \left( \frac{N^2}{M} \right)^{3/2-\theta} \right). \quad (8.55)$$

En insérant les estimations (8.49), (8.50), et (8.55) dans (8.48), nous obtenons l'estimation annoncée (8.1). Cela conclut la preuve de la Proposition 8.1.

#### 8.4. Démonstration du Théorème 1.2

Nous déduisons dans cette section le Théorème 1.2, à partir de la majoration (8.7). Nous suivons les arguments et les notations des sections 4 et 5 de [DI82]. Nous considérons

$$R_H(x, P, D) = \sum_{D < d \leq 2D} \lambda_d \sum_{0 < |h| \leq H} \sum_{m \equiv 0 \pmod{d}} \frac{C(m) \log m}{m} \sum_{v^2 \equiv D \pmod{m}} \widehat{b} \left( \frac{h}{m} \right) e \left( -\frac{hv}{m} \right),$$

où  $D \leq x^{1/2}$ ,  $P \in [x, x^2]$ ,  $\eta > 0$  est arbitraire,  $H = Px^{-1+\eta}$ ,  $b$  est une fonction lisse à support compact inclus dans  $[x, 2x]$ , telle que  $\|b^{(j)}\|_{\infty} \ll_j x^{-j}$ ,  $C$  est une fonction lisse à support compact inclus dans  $[P, 4P]$ , telle que  $\|C^{(j)}\|_{\infty} \ll P^{-j}$ , et  $(\lambda_d)$  une suite de coefficients avec  $|\lambda_d| \leq 1$ . Nous insérons la définition

$$\frac{1}{m} \widehat{b} \left( \frac{h}{m} \right) = \int_{\mathbb{R}} e(-ht) b(mt) dt.$$

Posant  $M := P/D$  et  $f_{d,t}(v) := C(Mvd) \log(Mvd) b(Mvdt)$ , nous obtenons

$$|R_H(x, P, D)| \ll x P^{-1} \sup_{|t| \in [x/(4P), 2x/P]} \sum_{D < d \leq 2D} \left| \sum_{0 < |h| \leq H} e(th) \sum_m f_{d,t} \left( \frac{m}{M} \right) \sum_{v^2 \equiv D \pmod{m}} e \left( -\frac{hv}{md} \right) \right|.$$

Nous avons bien  $\|f_{d,t}^{(j)}\|_\infty \ll_j 1$ ,  $D \ll M$  et  $H \ll MD$ . Nous sommes donc en mesure d'appliquer la majoration (8.7) à chaque sous-somme dyadique  $H_1 < h \leq 2H_1$ , pour  $1/2 \leq H_1 \leq H$ . Nous obtenons

$$R_H(x, P, D) \ll x^{1+\varepsilon+O(\eta)} P^{-1} D \sup_{1/2 \leq H_1 \leq H} H_1 \{H_1 + M^{1/2} + H_1^{-1/2} D^{1/2-\theta} M^{1/2+\theta}\} \\ \ll x^{\varepsilon+O(\eta)} \{x^{-1} DP + (DP)^{1/2} + x^{1/2} P^\theta D^{1-2\theta}\}.$$

Cela est  $O(x^{1-\eta})$  si  $D \leq x^{-K\eta} \min\{x^2 P^{-1}, x^{1/(2-4\theta)} P^{-\theta/(1-2\theta)}\}$  et  $K$  est une constante suffisamment grande. Cette majoration de  $D$ , en conjonction avec les arguments de la section 8 de [DI82], fournit le résultat annoncé.

### 8.5. Démonstration du Lemme 8.4

Écrivons  $\sigma \equiv \begin{pmatrix} u & * \\ v & r \end{pmatrix}$  avec  $r \in \mathbb{Z}$ . Les classes  $\Gamma_0(qd) \backslash \Gamma$  sont en bijection avec  $\mathbb{P}^1(\mathbb{Z}/qd\mathbb{Z})$ , la correspondance étant donnée par  $\sigma \mapsto [v : r]$ . La condition  $q \mid C(\sigma Q)$  correspond alors à  $q \mid Q(v, r)$ .

La relation  $q \mid C(\sigma Q) = Q(v, r)$  implique  $v \mid Q(v, r)^2$ . Cependant, on a la relation  $Q(v, r) \equiv Q(0, 1)r^2 \pmod{v}$  et  $(r, v) = 1$ , de sorte que finalement  $v \mid Q(0, 1)^2$ .

Le calcul explicite de  $\mathcal{C}(\infty, \mathfrak{a})$  et de  $S_{\infty\mathfrak{a}}(h, n; \gamma)$  est un calcul élémentaire similaire à la section 2.2 de Deshouillers–Iwaniec [DI83]. Nous omettons les détails. La majoration (8.22) en découle par l'inégalité triangulaire, et en notant que la condition  $\alpha\delta \equiv u \pmod{vm}$  détermine  $\alpha \pmod{vm/(u, m)}$ .

Pour la preuve de (8.23), nous utilisons le théorème des restes chinois. Soit  $p$  un nombre premier, et notons

$$p^\mu \parallel m, \quad p^\lambda \parallel u, \quad p^v \parallel v, \quad p^{v'} \parallel v', \quad p^\Delta \parallel d.$$

Nos hypothèses  $(v, u) = (v', m) = 1$  impliquent donc

$$\mu > 0 \implies v' = 0, \quad \lambda > 0 \implies v = 0, \quad \Delta \leq \max\{v, v'\}.$$

Le théorème des restes chinois montre que l'on peut se ramener à établir la majoration

$$S_p(h) := \sum_{\substack{\alpha \pmod{p^{v+\mu}} \\ \delta \pmod{p^{\lambda+\mu+\max\{v, v'\}}} \\ \delta \equiv m \pmod{p^{\lambda+v'}} \\ (\delta - m, p^{\lambda+\mu}) = p^\lambda \\ \alpha\delta \equiv u \pmod{p^{v+\mu}}} \chi_p(\delta) e\left(\frac{h\alpha}{p^{v+\mu}}\right) \ll_Q (p^\Delta h, p^\mu), \tag{8.56}$$

où  $\chi_p$  est un caractère modulo  $p^\Delta$ . Le changement de variables  $\delta \leftarrow m + \delta p^{\lambda+v'}$  transforme le membre de gauche en

$$S_p(h) = \sum_{\substack{\alpha \pmod{p^{v+\mu}} \\ \delta \pmod{p^{\mu+\max\{v-v', 0\}}} \\ (\delta, p^\mu) = 1 \\ \alpha(m + \delta p^{\lambda+v'}) \equiv u \pmod{p^{v+\mu}}} \chi_p(m + \delta p^{\lambda+v'}) e\left(\frac{h\alpha}{p^{v+\mu}}\right).$$

Nous écartons d'abord le cas  $\mu \leq \max\{\lambda, \nu\}$ , en nous contentant la borne triviale

$$S_p(h) \ll_Q 1,$$

qui découle du fait que  $u, v \ll_Q 1$ .

Supposons ensuite  $\mu > \max\{\lambda, \nu\} \geq 0$ , en particulier,  $\nu' = 0$ . Considérons d'abord le cas  $\nu = 0$ . Dans ce cas  $\Delta = 0$ , donc le caractère est trivial et notre somme se simplifie en

$$S_p(h) = \sum_{\substack{\alpha \pmod{p^\mu} \\ (\alpha, p)=1}} e\left(\frac{h\alpha}{p^\mu}\right) \sum_{\substack{\delta \pmod{p^\mu} \\ \alpha\delta \equiv u/p^\lambda \pmod{p^{\mu-\lambda}}}} 1 = p^\lambda c_{p^\mu}(h),$$

où  $c_r(h) = \sum_{b \pmod{r}, (b,r)=1} e(hb/r)$  est la somme de Ramanujan. Nous obtenons donc

$$|S_p(h)| \leq p^\lambda(h, p^\mu).$$

Considérons ensuite le cas  $\nu > 0$ . Cela implique  $\lambda = 0$  et  $\Delta \leq \nu$ , et donc

$$S_p(h) = \sum_{\substack{\alpha \pmod{p^{\nu+\mu}} \\ \delta \pmod{p^{\nu+\mu}} \\ (\delta, p)=1 \\ \alpha(m+\delta) \equiv u \pmod{p^{\nu+\mu}}}} \chi(m+\delta) e\left(\frac{h\alpha}{p^{\nu+\mu}}\right) = \chi(u) \sum_{\substack{\alpha \pmod{p^{\nu+\mu}} \\ (\alpha, p)=1}} \chi(\bar{\alpha}) e\left(\frac{h\alpha}{p^{\nu+\mu}}\right),$$

qui est une somme de Gauss (cf. [IK04, lemme 3.2]). Nous avons ainsi

$$|S_p(h)| \leq 2(p^\Delta h, p^{\nu+\mu}).$$

Nous obtenons dans tous les cas de figure la borne (8.56), ce qui conclut la preuve.  $\square$

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# Limit laws for rational continued fractions and value distribution of quantum modular forms

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## Abstract

We study the limiting distributions of Birkhoff sums of a large class of cost functions (observables) evaluated along orbits, under the Gauss map, of rational numbers in  $(0, 1]$  ordered by denominators. We show convergence to a stable law in a general setting, by proving an estimate with power-saving error term for the associated characteristic function. This extends results of Baladi and Vallée on Gaussian behaviour for costs of moderate growth. We apply our result to obtain the limiting distribution of values of several key examples of quantum modular forms. We obtain the Gaussian behaviour of central values of the Esterman function  $\sum_{n \geq 1} \tau(n) e^{2\pi i n x} / \sqrt{n}$  ( $x \in \mathbb{Q}$ ), a problem for which known approaches based on Eisenstein series have been so far ineffective. We give a new proof, based on dynamical systems, that central modular symbols associated with a holomorphic cusp form for  $SL(2, \mathbb{Z})$  have a Gaussian distribution, and give the first proof of an estimate for their probabilities of large deviations. We also recover a result of Vardi on the convergence of Dedekind sums to a Cauchy law, using dynamical methods.

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# 1 | VALUE DISTRIBUTION OF QUANTUM MODULAR FORMS

Let  $\Gamma \subset SL(2, \mathbb{Z})$  be a cofinite Fuchsian subgroup, which acts of functions on  $\mathbb{P}^1(\mathbb{Q})$  by the weight- $k$  ‘slash operator’

$$f|_k\gamma(x) := (cx + d)^{-k} f(\gamma x) \quad \text{if } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

where  $\gamma x = \frac{ax+b}{cx+d}$  is the Möbius transformation.

In their simplest guise, quantum modular forms, introduced by Zagier [121] (see [119] for early examples), denote the set of functions

$$f : \mathbb{P}^1(\mathbb{Q}) \setminus S \rightarrow \mathbb{C},$$

for some finite set  $S$ , satisfying a form of modularity, in the purposely vague sense that for all  $\gamma \in \Gamma$ , the function of  $x \in \mathbb{P}^1(\mathbb{Q}) \setminus (S \cup \gamma^{-1}S)$  defined by

$$h_\gamma(x) := f(x) - f|_k\gamma(x), \tag{1.1}$$

has some regularity property. Part of the research effort has focused on constructing examples in interrelated ways:

- Generating series associated with combinatorial sequences: Fishburn matrices [21], unimodal sequences [19, 70], partition theory [91] (we note that in the latter, quantum modularity is actually a crucial tool for the asymptotic estimation of partition-related sequences),
- Radial limits of modular objects (mock theta function, quasi-modular forms) defined on the hyperbolic disk [22, 30, 45, 46, 120],
- Eichler integrals, periods of modular functions [20, 23]; state integrals involving the quantum dilogarithm function [37, 49, 73],
- Describing the homology of spaces of cusp forms [26, 27, 29],
- Kashaev knot invariants and Nahm sums [13, 50, 121],
- Correlations of the fractional part functions appearing as covariances in the Nyman–Beurling reformulation of the Riemann hypothesis [6, 10, 11, 77],
- Diophantine approximation and multifractal analysis [66, 100].

In the present paper, we are concerned with the following problem: given a quantum modular form  $f$ , how do the multi-sets

$$\{f(x), x \in \mathbb{Q} \cap (0, 1], \text{denom}(x) \leq Q\}, \tag{1.2}$$

appropriately normalized, distribute as  $Q \rightarrow \infty$ ? This topic is tightly related to weak limits of partial sums of certain arithmetic functions, which goes back to Hardy–Littlewood [55], and has been since then periodically revisited: we mention in particular the works [67, 81, 115] on theta sums, and [7, 79] on cotangent sums. These works are all concerned with instances of QMF of *non-zero* weight.

Our interest in this question comes from the statistical study of additive twists central values of  $L$  functions, which as we will see are *weight-zero* QMF related to the third item described above

(periods of modular functions). We are aware of two occurrences of this setting in the literature. The first is a result of Vardi [114] on the existence of a limiting distribution for Dedekind sums. The second is a recent result of Petridis and Risager [96] on the distribution of modular symbols, which was motivated by conjectures of Mazur–Rubin [85] and Stein [108]. Both results exploit a close connection with Fourier analysis of the modular surface (the spectral analysis of the hyperbolic Laplacian). It is unclear how to extend these methods to more general QMF.

In the present paper, we present an approach, based on dynamical systems and the spectral properties of a family of transfer operators, which allows us to answer the question (1.2) for essentially all level 1 (that is,  $\Gamma = SL(2, \mathbb{Z})$ ) and weight 0 QMF.

We denote  $\sigma = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$  and  $\tau = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$  the two usual generators of  $SL(2, \mathbb{Z})$ , so that the associated period functions (1.1) with weight  $k = 0$  are given by

$$\begin{aligned} h_\tau(x) &= f(x) - f(x+1), \\ h_\sigma(x) &= f(x) - f(-1/x). \end{aligned} \tag{1.3}$$

Our result may be stated informally as follows.

**Theorem 1.1.** *Let  $f$  be a weight 0 QMF for  $SL(2, \mathbb{Z})$ , in the sense that  $h_\sigma$  extends to a Hölder-continuous function on  $(0, 1]$  with some regular growth behaviour at 0, and  $h_\tau = 0$ . Then, up to a suitable renormalization, the multi-sets (1.2) become equidistributed according to a stable law, which is characterized by the growth of  $h_\sigma$  at 0.*

The hypothesis are stated more precisely in Theorem 3.1. The restriction to  $x \in (0, 1]$  and the assumption  $h_\tau = 0$  are made to clarify the statement, but are inessential (cf. footnote 2, p. 8) and are natural in applications. However, the restriction to  $\Gamma = SL(2, \mathbb{Z})$  is important in our argument. What is required is that the action of  $\Gamma$  on  $\mathbb{P}^1(\mathbb{R})$  can be induced into an expanding Markov map (the Gauss map in the case  $\Gamma = SL(2, \mathbb{Z})$ ). The restriction to weight zero QMF is also natural, as the problem for non-zero weights is of different nature and typically simpler; see [14] for more details.

We will find in practice that any bound of the shape  $h_\sigma(x) = O(x^{-1/\alpha})$  as  $x \rightarrow 0$  ( $\alpha > 2$ ) ensures convergence to a Gaussian law. A bound of type  $h_\sigma(x) \asymp x^{-1/\alpha+o(1)}$  for some  $\alpha \in (0, 2)$ , will typically imply the convergence to a stable law of parameter  $\alpha$ .

Due to the relative weakness in its hypotheses, Theorem 1.1 applies to a wide class of QMF. In the next section, using Theorem 1.1, we answer the distribution question for several arithmetic invariants. We expect further applications to follow in the future.

## 2 | APPLICATIONS

For all  $Q \geq 1$ , we endow the set

$$\Omega_Q := \{x = a/q, 1 \leq a \leq q \leq Q, (a, q) = 1\} \subset \mathbb{Q} \cap (0, 1]$$

with the uniform probability measure  $\mathbb{P}_Q$ , and we denote  $\mathbb{E}_Q$  the associated expectation,

$$\mathbb{E}_Q(f(x)) = |\Omega_Q|^{-1} \sum_{x \in \Omega_Q} f(x).$$

## 2.1 | Central values of the Estermann function

Let

$$\tau(n) = \sum_{d|n} 1 \quad (n \geq 1)$$

denote the divisor function. The Estermann function, introduced in 1930 [41], is defined by

$$D(s, x) := \sum_{n \geq 1} \frac{\tau(n)e(nx)}{n^s}$$

for  $\operatorname{Re}(s) > 1$ , and extended by analytic continuation otherwise. It was initially introduced in relation with the shifted divisor problem  $\sum_n \tau(n)\tau(n+1)$ . Its functional equation still serves as a basic tool to derive Voronoï summation formulae, which are then used in conjunction with the circle method to study moments of  $L$ -function and their arithmetics applications: we mention the proportion of critical zeroes of  $\zeta$  [31], the binary divisor problems [88], and non-vanishing of central  $L$ -values [78].

We mention a further connection with moments of Dirichlet  $L$ -functions  $L(s, \chi) = \sum_{n \geq 1} \chi(n)n^{-s}$ . By [8, Theorem 5], the twisted second moment of Dirichlet  $L$ -functions satisfies

$$\begin{aligned} M(a, q) &:= \frac{1}{q^{1/2}} \sum_{\chi \pmod{q}} \left| L\left(\frac{1}{2}, \chi\right) \right|^2 \chi(a) \\ &= \operatorname{Re} D\left(\frac{1}{2}, \frac{a}{q}\right) + \operatorname{Im} D\left(\frac{1}{2}, \frac{a}{q}\right) + O(q^{-1/2}) \end{aligned} \quad (2.1)$$

for  $q$  prime and  $q \nmid a$ . From this expression, we see that the second moment  $\sum_{a \pmod{q}} |M(a, q)|^2$  is essentially the fourth moment of Dirichlet  $L$ -functions  $\sum_{\chi \pmod{q}} |L(\frac{1}{2}, \chi)|^4$ , whose full evaluation in [118] lies at the threshold of current techniques of analytic number theory (see [16, 17] for further work on this topic). This fits in the general problem of understanding the distribution of central values of  $L$ -functions and their twists, which is a fundamental topic in analytic number theory [32, 56, 99, 103, 104]. Up to now, essentially all known results have been obtained by the moments method.

Using Theorem 1.1, we obtain the following Central Limit Theorem.

**Theorem 2.1.** *For all  $\varepsilon > 0$ ,  $Q \geq 3$ , and all rectangle  $\mathcal{R} \subset \mathbb{C}$ , we have*

$$\mathbb{P}_Q \left( \frac{D\left(\frac{1}{2}, x\right)}{\sigma(\log Q)^{\frac{1}{2}} (\log \log Q)^{\frac{3}{2}}} \in \mathcal{R} \right) = \int_{v_1 + iv_2 \in \mathcal{R}} \frac{e^{-(v_1^2 + v_2^2)/2} dv_1 dv_2}{2\pi} + O_\varepsilon \left( \frac{1}{(\log \log Q)^{1-\varepsilon}} \right),$$

where  $\sigma = 1/\pi$ .

We will obtain this result as an application of Theorem 1.1, using the fact, proved in [8], that  $D(\frac{1}{2}, x)$  is a weight-0 QMF, the associated function  $h_\sigma$  (1.3) being a  $(\frac{1}{2} - \varepsilon)$ -Hölder continuous function on  $(0, 1]$ .

The simplicity with which we will deduce Theorem 2.1 from Theorem 1.1 contrasts with the fact that other methods appear to be completely ineffective with this problem. The moments method (and therefore also the approach of [92] described in the following section) cannot be applied, due to the presence of a negligible proportion of  $x \in \Omega_Q$  with abnormally large continued fraction coefficients, whose contribution dominates the integer moments of  $D(\frac{1}{2}, x)$ . In fact, all moments of  $D(\frac{1}{2}, x)$  and  $M(a, q)$  have recently been computed in [9]: starting already from the second they grow faster than what is suggested by Theorem 2.1.

Another tentative approach to Theorem 2.1 consists in obtaining a limiting distribution result for  $\sum_{n \geq 1} \tau(n)e(nz)$ , where  $z = x + iy; x \in [0, 1]$  is chosen at random and  $y \rightarrow 0^+$ , and transferring these properties to its discrete counterpart  $D(\frac{1}{2}, x) \approx \sum_{n \leq q^2} \tau(n)e(nx)n^{-1/2}$ , for  $x = a/q \in \mathbb{Q}$ , by Fourier expansion. This method is employed for the incomplete Gauss sum in [36], based on [81]. In our setting, however, connecting the continuous and the discrete averages raises several additional difficulties, among which the problem of dealing with a divergent second moment as well as those coming from considering a central value rather than an object corresponding to an  $L$ -value off the line. Also the incomplete Gauss sums are not Gaussian distributed, but rather are distributed as the push-forward measure by a theta series. This suggests a difference in nature between the two problems and, notwithstanding the technical difficulties, it suggests this method could not be adapted to our case.

It would be interesting to obtain a statement analogous to Theorem 2.1 for the values  $M(a, q)$ , however, the identity (2.1) as stated holds only for  $q$  prime; the corresponding identity for generic  $q$  involves a multiplicative convolution, which is not clearly accountable for using the present method. However, we still believe  $M(a, q)$  is distributed according to a normal law.

**Conjecture 2.2.** *As  $q \rightarrow \infty$  along primes, the multi-set*

$$\left\{ \frac{M(a, q)}{(\log Q)^{1/2}(\log \log Q)^{3/2}}, 1 \leq a < q \right\}$$

*become distributed according to a dilated centred Gaussian.*

## 2.2 | Modular symbols

Out of many possible ways, modular symbols [80] can be seen as elements of the space of linear forms on  $S_k(\Gamma_0(N))$ , the vector space of cusp forms of weight  $k \equiv 0 \pmod{2}$  and level  $N \geq 1$ , spanned by the Shimura integrals

$$f \mapsto \langle x \rangle_{f,m} := \frac{(2\pi i)^m}{(m-1)!} \int_x^{i\infty} f(z)(z-x)^{m-1} dz =: \langle x \rangle_{f,m}^+ + i \langle x \rangle_{f,m}^-$$

for any  $1 \leq m \leq k-1$ ,  $x \in \mathbb{Q}$ ,  $f \in S_k(\Gamma_0(N))$ , and where  $\langle x \rangle_{f,m}^\pm \in \mathbb{R}$ . (In this section and Section 9.1 only, the letter  $f$  will denote a holomorphic cusp form).

Up to an explicit factor, the value  $\langle x \rangle_{f,m}$  is also the special value  $L(f, x, m)$  of the analytic continuation of the  $L$ -function

$$L(f, x, s) := \sum_{n=1}^{\infty} a_n e(nx) n^{-s} \quad (\text{Re } s > k),$$

where we write  $f(z) = \sum_{n \geq 1} a_n e(nz)$  for  $\text{Im}(z) > 0$ . Being at the intersection of the geometric, modular and arithmetic aspects of  $\Gamma_0(N)$ , modular symbols received a considerable amount of interest, for example, for computing with modular forms [34, 107]. The central values of the Estermann function  $D(\frac{1}{2}, x)$ , which was the subject of the previous section, may be interpreted as a regularized modular symbol for the derivative  $\frac{\partial}{\partial s} E_2(z, s)|_{s=1/2}$  of the Eisenstein series [64, chapter 3.5].

The symbol associated with the central value  $m = k/2$  plays a particular rôle and is the focus of this section. Motivated by the *abc*-conjecture, the question of the size of modular symbols was initially asked by Goldfeld [51] and later studied by Petridis and Risager [94, 95]. Interest in this question was recently revived by questions of Mazur, Rubin and Stein [85, 108], motivated in part by the growth of ranks of elliptic curves. In particular, for  $f$  a fixed form of weight  $k = 2$  (and so  $m = 1$ ) and  $x \in (0, 1]$  varying along rationals of reduced denominator  $q$ , with  $q \rightarrow \infty$ , Mazur and Rubin predicted that

$$\left\{ \frac{\langle a/q \rangle_{f,1}^{\pm}}{\sqrt{\log q}}, 1 \leq a \leq q, (a, q) = 1 \right\}$$

becomes asymptotically distributed according to a suitably dilated normal law. To our knowledge, only the first and second moments have been computed [18, Chapter 9].

The situation changes with an additional average over  $q \leq Q$ . Then the Central Limit Theorem for weight  $k = 2$ , level  $N \geq 1$  forms has been proved by Petridis and Risager [96], by using the spectral analysis of twisted Eisenstein series and the location of eigenvalues of the hyperbolic Laplacian.

Using Theorem 1.1, we will prove the following Central Limit Theorem for modular symbols associated with forms of level  $N = 1$  and arbitrary weight  $k \geq 12$ .

**Theorem 2.3.** *Let  $k \geq 12$  be even, and  $f \in S_k(SL(2, \mathbb{Z})) \setminus \{0\}$  be fixed. Then for all  $Q \geq 2$  and rectangle  $\mathcal{R} \subset \mathbb{C}$ , we have*

$$\mathbb{P}_Q \left( \frac{\langle x \rangle_{f, \frac{k}{2}}}{\sigma_f \sqrt{\log Q}} \in \mathcal{R} \right) = \int_{v_1 + iv_2 \in \mathcal{R}} \frac{e^{-(v_1^2 + v_2^2)/2} dv_1 dv_2}{2\pi} + O\left(\frac{1}{\sqrt{\log Q}}\right), \quad (2.2)$$

with  $\sigma_f^2 = \frac{3(4\pi)^k}{\pi\Gamma(k)} \|f\|_k^2$ , where  $\|f\|_k$  is the weight- $k$  Petersson norm of  $f$ .

For example, this result applies for  $k = 12$  with  $f$  being the discriminant modular form  $\Delta(z)$ . The error term is optimal and uniform with respect to  $\mathcal{R}$ .

Using Theorem 1.1, we will also deduce the following statement, which can be interpreted as a Large Deviations Principle.

**Theorem 2.4.** *Let  $f$  be fixed as in Theorem 2.3. For any  $\varepsilon > 0$ , we have*

$$\limsup_{Q \rightarrow \infty} \mathbb{P}_Q \left( \left| \langle x \rangle_{f, \frac{k}{2}} \right| > \varepsilon \log Q \right)^{1/\log Q} < 1. \quad (2.3)$$



We will deduce Theorems 2.3 and 2.4 from Theorem 1.1 by showing that the map  $x \mapsto \langle x \rangle_{f, \frac{k}{2}}$  is a weight 0 QMF with the period function  $h_\sigma$  (1.3) being Hölder-continuous on  $[0,1]$ . We will estimate all small exponential moments, from which the Large Deviations Principle will follow.

Theorem 2.3 is related to three works, two of which have been carried out around the same time as the present.

- The work of Petridis and Risager [96] was concerned with the case of weight  $k = 2$  forms, which is relevant to the conjectures of Mazur-Rubin and Stein (cusp forms associated with elliptic curves). Their method is based on the analysis of twisted Eisenstein series. Although the authors do not seem to mention it, their method is capable of obtaining a Large Deviations Principle. The assumption  $k = 2$  is, however, crucial in their approach, since only in this case the analogues of the function  $h_\sigma$  in (1.3) are constant (this translates into the independence of the multiplier system defined in [96, p. 7] with respect to  $z_0$ ).
- Very recently, Nordentoft [92] has obtained the Central Limit Theorem for arbitrary weight  $k \geq 2$  and level  $N \geq 1$  modular forms. Specializing to  $N = 1$ , this gives in particular an independent proof of Theorem 2.3. Similarly as [96], the spectral analysis of twisted Eisenstein series play a central rôle, however, his argument is based on a completely different construction (due to the ‘independence with respect to  $z_0$ ’ obstruction mentioned above). As a result, it crucially relies on the consideration of moments, and falls short of establishing a Large Deviations Principle (Theorem 2.4).
- Also very recently, Lee and Sun [74] have independently obtained a proof of the main theorem of [96], for weight  $k = 2$ , by a method similar to the one we pursue here. This is achieved by considering a certain twisted version of the Gauss map, keeping track at each iteration of a coset in  $\Gamma_0(N) \backslash SL_2(\mathbb{Z})$ . On the other hand, for  $k = 2$  the period functions (the analogues of  $h_\sigma$  in (1.3)) are constant. The technical difficulties they are faced with are of very different nature from the ones we encounter here. By mixing the methods presented here with those of [74], it is plausible that the main results of [92], for arbitrary weight and levels, could be recovered by dynamical methods.

The difference in the magnitude of the variance between Theorems 2.1 and 2.4 is noteworthy, and is due to the presence, or not, of a pole the associated  $L$ -functions ( $\zeta^2(s)$  and  $L(f, s)$ , respectively) at  $s = 1$ .

We have considered only the modular symbols  $\langle x \rangle_{f, m}$  at the central value  $m = k/2$ . The behaviour for  $m \neq k/2$  is very different and simpler: for instance, when  $m > k/2$ , the series  $L(f, x, m)$  extends to a continuous function of  $x \in \mathbb{R}$ , and the values  $\{\langle x \rangle_{f, m}, x \in \Omega_Q\}$  become distributed according to the push-forward  $(L(f, \cdot, m))_*(d\nu)$  of the Lebesgue measure. We return to this question in more details in [14].

The original conjectures of Mazur–Rubin and Stein were concerned with the case of a single  $q$ . There does not seem to be an effective way, with our method or those of [74, 92, 96], to isolate rationals of fixed denominators, whence all results proven so far rely on the extra average over  $q$ .

### 2.3 | Kashaev invariants of the $4_1$ knot and sums of continued fraction coefficients

Our next application is motivated by a question in [121]. To a knot  $K$  and an integer  $n \geq 2$ , we associate the  $n$ -coloured Jones polynomial [48, section 1], which is a Laurent polynomial

$J_{K,n}(\mathbf{q}) \in \mathbb{Z}[[\mathbf{q}]]$  defined by a combinatorial construction through a diagram representation of  $K$ . For any root of unity  $\mathbf{q} \in \mathbb{C}$ , we define  $J_{K,0}(\mathbf{q}) := J_{K,n}(\mathbf{q})$  where  $n \geq 1$  is the order of  $\mathbf{q}$ . In [89], it was shown that the sequence  $(J_{K,0}(e^{2\pi i/n}))_{n \geq 1}$  is the Kashaev invariant of  $K$  [68] (the full function can be reconstructed by the action of the Galois group). In the case of the  $4_1$  knot (or ‘figure-eight’ knot), the simplest hyperbolic knot, we have explicitly [121]

$$\mathbb{J}(x) := J_{4_1,0}(e(x)) = \sum_{m=0}^{\infty} |1 - e^{2\pi i x}|^2 \dots |1 - e^{2\pi i m x}|^2, \quad (x \in \mathbb{Q}).$$

Note that for each given  $x \in \mathbb{Q}$ , the sum is finite. In this case, Zagier’s modularity conjecture, stated precisely in [121], predicts that  $x \mapsto \log \mathbb{J}(x)$  is a weight 0 QMF: the difference

$$h(x) := \log \mathbb{J}(-1/x) - \log \mathbb{J}(x),$$

which is depicted in [121, Figure 4] is expected to behave ‘nicely’ with respect to  $x$ , although not continuously. Conjecturally, we expect  $h(x) \sim C/x$  as  $x \rightarrow 0$ , where  $C = \text{Vol}(4_1)/2\pi$  and  $\text{Vol}(4_1) = 2.02 \dots$  is the hyperbolic volume of  $4_1$ ; Kashaev’s volume conjecture is the case  $x = 1/n, n \in \mathbb{N}$ , which is known in this case [2].

A proof of Zagier’s conjecture for the  $4_1$  knot has been announced by Garoufalidis and Zagier [50]. In [13], we obtained independently another proof, complemented by a reciprocity formula relative to a transformation of another kind, by which we deduced the following asymptotic estimate: for  $\lambda \geq 0$ , denote

$$\Sigma_\lambda(x) = \sum_{j=1}^{r(x)} a_j^\lambda, \quad x \in \mathbb{Q} \cap (0, 1], \quad x = [0; a_1, \dots, a_{r(x)}], \quad a_{r(x)} > 1, \quad (2.4)$$

the sum of continued fraction coefficients of  $x$ . Then by [13], we have

$$\log \mathbb{J}(x) \sim C \Sigma_1(x) \quad \text{as} \quad \frac{\Sigma_1(x)}{r(x)} \rightarrow \infty. \quad (2.5)$$

This is in accordance with the conjectured behaviour of  $h(x)$  as  $x \rightarrow 0$ .

The map  $x \mapsto \Sigma_\lambda(x)$ , suitably extended to  $\mathbb{Q}$ , is a weight 0 QMF with associated period function (1.3) satisfying  $h_\sigma(x) = \lfloor 1/x \rfloor^\lambda$  for  $x \in \mathbb{Q} \cap (0, 1]$ . Using Theorem 1.1, we compute in Theorem 9.4 below the distribution of  $\Sigma_\lambda$  for all  $\lambda \geq 0$ , extending work of [3, 4]. In particular, in the case  $\lambda = 1$ , we obtain the following result.

**Theorem 2.5.** *Let  $G_1$  denote the cumulative distribution function of the stable law  $S_1(\frac{6}{\pi}, 1, 0)$ , that is  $G_1(v) := \int_{-\infty}^v g_1(x) dx$  where*

$$g_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} e^{-\frac{6}{\pi}|t| - \frac{12}{\pi^2}it \log |t|} dt,$$

and let  $\gamma_0$  be the Euler–Mascheroni constant. Then, as  $Q \rightarrow \infty$ ,

$$\mathbb{P}_Q \left( \frac{\Sigma_1(x)}{\log Q} - \frac{\log \log Q - \gamma_0}{\pi^2/12} \leq v \right) = G_1(v) + O \left( \frac{1}{(\log Q)^{1-\varepsilon}} \right). \quad (2.6)$$



Theorem 2.5 answers a question in [44], and echoes a result of Heinrich [58] which obtained a similar convergence for Lebesgue-almost all  $x \in [0, 1]$ . In particular, (2.6) implies that the generic time complexity of the subtractive algorithm for the GCD is asymptotically  $(1 + o(1))\frac{12}{\pi^2}(\log Q) \log \log Q$  on pairs of coprime numbers at most  $Q$ , as  $Q \rightarrow \infty$ . This is in sharp contrast with the average time complexity, which is  $(1 + o(1))\frac{6}{\pi^2}(\log Q)^2$  (the latter is known even with a single average over numerators, see [117]). Note that, contrary to the case of the Estermann function (Theorem 2.1), this discrepancy between the typical size and the average size is consistent with fact that the stable law  $S_1(\frac{6}{\pi}, 1, 0)$  has a divergent first moment.

Combined with (2.5), Theorem 2.5 implies the following law of large numbers for the values of  $\log \mathbb{J}(x)$ .

**Corollary 2.6.** *As  $Q \rightarrow \infty$ , we have*

$$\log \mathbb{J}(x) \sim \frac{12}{\pi^2} C \log Q \log \log Q$$

for a proportion  $1 - o(1)$  of fraction  $x \in \mathbb{Q} \cap (0, 1]$  of denominators at most  $Q$ .

The typical size of  $\log \mathbb{J}(x)$  among fractions  $x$  with  $\text{denom}(x) \leq Q$  is therefore much less than that of  $\log \mathbb{J}(1/Q) \sim CQ$  (by (2.5)).

We expect a convergence in law analogous to (2.6) for the values  $\log \mathbb{J}(x)$ , however, the error term which we obtain in (2.5) is not precise enough to carry this out. This issue is discussed more precisely in [13], along with the case of other knots.

## 2.4 | Dedekind sums

For all coprime integers  $a$  and  $q$  with  $q \geq 1$ , the Dedekind sum

$$s\left(\frac{a}{q}\right) := \sum_{h=1}^{q-1} \left(\left(\frac{ha}{q}\right)\right) \left(\left(\frac{h}{q}\right)\right), \quad ((x)) := \begin{cases} \{x\} - 1/2 & (x \notin \mathbb{Z}), \\ 0 & (\text{otherwise}) \end{cases}$$

is a rational number of modulus at most  $q/12 + 1/2$ . They appear naturally in the multiplier system attached to the Dedekind  $\eta$  function; we refer to the monograph [98] for further properties and references. The value distribution of  $s(x)$  on average over rational  $x$  has been studied from several points of view [25, 60, 113, 114].

From Theorem 1.1, we will deduce a short proof of the following convergence to a Cauchy law, which is the main result of [114].

**Theorem 2.7** [114]. *Uniformly for  $v \in \mathbb{R}$  and  $Q \geq 2$ , we have*

$$\mathbb{P}_Q\left(\frac{s(x)}{\log Q} \leq \frac{v}{2\pi}\right) = \frac{1}{\pi} \int_{-\infty}^v \frac{dy}{1 + y^2} + O\left(\frac{1}{(\log Q)^{1-\varepsilon}}\right). \tag{2.7}$$

This statement will easily follow from Theorem 1.1 by noting that  $x \mapsto s(x)$  is a weight 0 QMF, with associated period function  $h_\sigma$  (1.3) being roughly  $[1/x]$ .<sup>†</sup> This last fact is a consequence of the reciprocity formula for  $s(x)$  [98, Chapter 2, Theorem 1].

<sup>†</sup> Note that we obtain a different distribution than in Theorem 2.5. This is due to the fact in that case the period relation is more precisely  $f(x) + f(-1/x) = [1/x]$  rather than  $f(x) - f(-1/x) = [1/x]$ .

The proof of [114] builds on the close connection between  $s(x)$  and the multiplier system associated to the  $\eta$  function, which brings the problem to an analysis of twisted Poincaré series on the modular surface, which are in turn studied by means of the Kuznetsov trace formula. By contrast, our arguments are dynamical in nature, and use little arithmetic information beyond the group structure of  $SL(2, \mathbb{Z})$ .

### 3 | OVERVIEW

#### 3.1 | Reduction to dynamical analysis

We now overview the arguments underlying Theorem 1.1. Suppose  $f : \mathbb{Q} \rightarrow \mathbb{C}$  is such that the functions defined through (1.1) satisfy  $h_\tau = 0$  and  $h_\sigma$  extends to a continuous function on  $(0, 1]$ . To simplify the presentation, in this section only, we assume that  $f$  is an even function.

The starting point is the remark that the action of  $\sigma$  and  $\tau$  on  $\mathbb{P}^1(\mathbb{Q})$  can be induced into an expanding Markov map on  $(0, 1]$ , namely the Gauss map  $T : x \mapsto \{1/x\}$ , where the braces denote the fractional part. More precisely, by periodicity and since  $f$  is even, we have  $f(x) = h_\sigma(x) + f(\{1/x\})$ , which after iteration (Euclid's algorithm) yields<sup>†</sup>

$$f(x) = \sum_{j=1}^{r(x)} h_\sigma(T^{j-1}(x)) + f(0) \quad (x \in (0, 1]), \quad (3.1)$$

where  $r(x) \geq 0$  is minimal subject to  $T^{r(x)}(x) = 0$ .

This expression relates the values of  $f$  to Birkhoff sums associated with the Gauss map  $T$ . This map is known to have good mixing properties [33, p. 174], by which we expect the sum (3.1) to behave like a sum of independent random variables. Precise limiting behaviour may be obtained through the study of spectral properties of transfer operators associated with  $T$ : this is an important theme in smooth dynamics and stationary Markov chains [1, 24, 39, 47, 62, 101, 116]. We refer in particular to [24] and to the introduction of [1] for an extensive historical account and references. Among maps of the interval, the Gauss map has been particularly studied because of its link with the analysis [82, 84, 86, 97] of geodesic flows on the surface  $SL(2, \mathbb{Z}) \backslash \mathfrak{h}$  (where  $\mathfrak{h}$  is the upper-half plane).

A prominent example is given by [24, Theorem 8.1]: suppose  $\phi : [0, 1] \rightarrow \mathbb{R}$  is of bounded variations and not of the shape  $c + k - k \circ T$ , for some function  $k$  of bounded variations and some constant  $c \in \mathbb{R}$ , and let

$$S_N(\phi, x) := \sum_{j=1}^N \phi(T^j(x)).$$

Then for some constants  $\mu_\phi \in \mathbb{R}$  and  $\sigma_\phi \in \mathbb{R}_+^*$ , uniformly for  $v \in \mathbb{R}$ ,

$$\mathbb{P} \left( \frac{S_N(\phi, x) - \mu_\phi N}{\sigma_\phi \sqrt{N}} \leq v \right) = \Phi(v) + O \left( \frac{1}{\sqrt{N}} \right), \quad \Phi(v) := \int_{-\infty}^v \frac{e^{-t^2}}{\sqrt{2\pi}} dt$$

<sup>†</sup> Without the assumption  $h_\tau = 0$ , a similar formula holds: we have in general  $f(x) = \sum_{j=1}^{r(x)} H(T^{j-1}(x)) + f(0)$ , where  $H(x) := h_\sigma(x) - \sum_{i=1}^{\lfloor 1/x \rfloor} h_\tau(1/x - i)$  for  $x \in (0, 1]$ , and the ensuing analysis still applies.

as  $N \rightarrow \infty$ , where  $x \in (0, 1)$  is chosen uniformly according to the Lebesgue measure. The implied constant may depend only on  $\phi$ . This relies on the spectral analysis of perturbations of the Gauss–Kuzmin–Wirsing transfer operator

$$\mathbb{H}[f](x) = \sum_{n \geq 1} \frac{1}{(n+x)^2} f\left(\frac{1}{n+x}\right).$$

We are interested in the case when  $x$  is a rational chosen at random in  $\Omega_Q$ , and the Birkhoff sum in (3.1) is over the full orbit, its length varying with  $x$ . Denote

$$S_\phi(x) := \sum_{j=1}^{r(x)} \phi(T^{j-1}(x)). \quad (3.2)$$

In [112], Vallée has shown that the expectations of  $S_\phi(x)$  satisfy

$$\mathbb{E}_Q(S_\phi(x)) = \mu_\phi \log Q + \nu_\phi + O(Q^{-\delta})$$

for functions  $\phi$  which are constant on each interval  $(\frac{1}{n+1}, \frac{1}{n})$  ( $n \geq 1$ ) and under some growth condition at 0. Here the number  $\delta > 0$  is absolute and the implied constant may depend on  $\phi$ . The numbers  $\mu_\phi, \nu_\phi$  depend only on  $\phi$ , and in fact

$$\mu_\phi = \frac{12 \log 2}{\pi^2} \int_0^1 \phi(x) \xi(x) dx, \quad \xi(x) := \frac{1}{(1+x) \log 2}.$$

The quantity  $\mu_\phi$ , and the measure  $\xi(x) dx$ , are essentially the projection of  $\phi$  on the eigenspace of  $\mathbb{H}$  associated to its dominant eigenvalue 1. An important point is that this question was studied within the framework of dynamical methods. The argument uses the construction of a suitable generating series involving the quasi-inverse  $(\text{Id} - \mathbb{H}_\tau)^{-1}$  of twisted transfer operators

$$\mathbb{H}_\tau[f] = \sum_{n \geq 1} \frac{1}{(n+x)^{2+i\tau}} f\left(\frac{1}{n+x}\right), \quad (3.3)$$

for arbitrary large  $\tau \in \mathbb{R}$  (by contrast with the continuous setting when  $x$  was chosen uniformly at random in  $[0, 1]$ , which involves perturbations of a single fixed operator). This construction crucially relies on the fact that the denominator  $q(x)$  of  $x$  can be detected by means of the Birkhoff sum  $\log q(x) = -\sum_{j=1}^{r(x)} \log(T^{j-1}(x))$ . Earlier approaches [38, 57], restricted to  $\phi = 1$ , involved number-theoretic methods based on bounds on algebraic exponential sums.

The approach of [112] was developed by Baladi and Vallée [4], who proved that for  $\phi$  constant on each interval of the shape  $(\frac{1}{n+1}, \frac{1}{n})$  and under a logarithmic growth condition  $\phi(x) = O(\log(2/x))$ , the Laplace transform satisfies the ‘quasi-powers expansion’

$$\mathbb{E}_Q(e^{wS_\phi(x)}) = \exp \{U(w) \log Q + V(w) + O(Q^{-\delta})\} \quad (3.4)$$

for  $w$  in some complex neighbourhood of 0; the holomorphic functions  $U, V$ , the number  $\delta > 0$  and the implied constant may depend on  $\phi$ . By Hwang’s theorem [61], this has a number of consequences in terms of the limiting distribution, and among them, an effective central limit

theorem: if  $\phi$  is real, non-identically zero and as above, then for some numbers  $\mu_\phi \in \mathbb{R}$ ,  $\sigma_\phi > 0$  and all  $v \in \mathbb{R}$ , we have

$$\mathbb{P}_Q \left( \frac{S_\phi(x) - \mu_\phi \log Q}{\sigma_\phi (\log Q)^{1/2}} \leq v \right) = \int_{-\infty}^v \frac{e^{-t^2/2} dt}{\sqrt{2\pi}} + O \left( \frac{1}{\sqrt{\log Q}} \right) \quad (t \in \mathbb{R}).$$

The power-saving error term in (3.4) depends on proving a pole-free strip for the quasi-inverse of the twists (3.3), which can be viewed as a ‘quasi-Riemann hypothesis’ for the generating function of interest<sup>†</sup>. The approach of [4] extends seminal work of Dolgopyat [40] to the case of an expanding interval map with an infinite partition (see also [5]).

Motivated by our application (3.1), we extend the methods of [4] in two directions. In the first direction, we consider cost functions  $\phi : (0, 1] \rightarrow \mathbb{R}$  which are not necessarily constant by parts. We will require that  $\phi$  can be extended to a Hölder continuous function, with some uniform exponent, on each interval  $[\frac{1}{n+1}, \frac{1}{n}]$ .

The second direction we wish to consider is cost functions  $\phi(x)$  having a possibly divergent first or second moment, say  $\phi(x) = x^{-1/2}$ , or  $\phi(x) = \lfloor 1/x \rfloor$ , with the consequence that the limit law will be stable, but not necessarily Gaussian anymore [4, p. 384]. This is a well-known theme in the theory of sums of independent random variables; see [75, Chapter VI], or [42, Chapters VI.1 and XVII.5-6]. The corresponding phenomenon for sums of continued fractions coefficients in the continuous setting (Lebesgue-almost all  $x \in (0, 1)$ ) has been investigated by elementary means by Lévy [76] (see also [58]), and later by transfer operator methods [1, 53, 54, 109]. In fact, this falls into the general ‘countable Markov–Gibbs’ framework of [1], where it is referred to as the ‘distributional limit’ problem. A large part of later work has focused on non-uniformly hyperbolic maps; we refer to the survey [52] and the references therein.

We investigate the corresponding question in the discrete setting ( $x$  at random in  $\Omega_Q$ ). We evaluate with effective, power-saving error terms the characteristic function

$$\mathbb{E}_Q(e^{itS_\phi(x)})$$

for  $t$  in a real neighbourhood of 0, under hypotheses which essentially reduce to the boundedness of some positive absolute moment,  $\int_0^1 |\phi(x)|^{\alpha_0} dx < \infty$  for some  $\alpha_0 > 0$ .

### 3.2 | Statement of the main distributional result

For technical reasons, we will work in a more general setting. For a parameter  $\kappa \in [0, 1]$ , a real interval  $\mathcal{I}$  and a normed space  $X$ , let  $H^\kappa(\mathcal{I}, X)$  denotes the set of functions  $\mathcal{I} \rightarrow X$  such that the Hölder semi-norm

$$\|f\|_{(\kappa)} := \sup_{x, y \in \mathcal{I}, x \neq y} \frac{\|f(x) - f(y)\|}{|x - y|^\kappa} \quad (3.5)$$

is finite.

We let  $\mathcal{H} := \{h : [0, 1] \rightarrow \mathbb{R} \mid \exists n \geq 1 \text{ s.t. } h(x) = \frac{1}{n+x}\}$  be the inverse branches of the Gauss map, and  $\mathcal{H}^\ell := \{h_1 \circ \dots \circ h_\ell \mid h_j \in \mathcal{H}\}$ .

<sup>†</sup> The same set of idea imply a zero-free strip for the Selberg zeta function of the full modular group [90].

We fix an integer  $m \geq 1$ , and  $m$  functions  $\phi_1, \dots, \phi_m : [0, 1] \rightarrow \mathbb{R}^d$ . We extend this definition by periodicity, letting  $\phi_j := \phi_{j \pmod m}$  for all  $j \geq 1$ , and we define a function on  $\mathbb{Q} \cap (0, 1]$  by letting  $S(1) := 0$  and

$$S(x) := \sum_{j=1}^{r(x)} \phi_j(T^{j-1}(x)), \quad (x \in \mathbb{Q} \cap (0, 1))$$

where we recall that

$$r(x) := \min\{j \geq 0, T^j(x) = 0\}.$$

For  $x \in (0, 1)$  with  $r(x) \geq m$ , we define

$$\phi(x) := \sum_{j=1}^m \phi_j(T^{j-1}(x)), \tag{3.6}$$

We make the following hypotheses.

- (1) ( $\kappa_0$ -Hölder continuity) For each  $n \geq 1$  and  $1 \leq j \leq m$ , the function  $\phi_j$  can be extended as a  $\kappa_0$ -Hölder continuous function on the interval  $[\frac{1}{n+1}, \frac{1}{n}]$ .
- (2) (Norm  $\alpha_0$ th moment) For each  $1 \leq j \leq m$ , we have

$$\sum_{h \in \mathcal{H}} |h'(0)| \|\phi_j|_{h(\mathcal{I})}\|_{\infty}^{\alpha_0} < \infty. \tag{3.7}$$

- (3) (Hölder  $\lambda_0$ th moment) For each  $1 \leq j \leq m$ , we have

$$\sum_{h \in \mathcal{H}} |h'(0)| \|\phi_j|_{h(\mathcal{I})}\|_{(\kappa_0)}^{\lambda_0} < \infty. \tag{3.8}$$

For all  $\mathbf{t} \in \mathbb{R}^d$ , we denote  $\|\mathbf{t}\| := \|\mathbf{t}\|_{\infty}$ . Finally, let

$$\mathfrak{F}_{\phi}(\mathbf{t}) := \int_0^1 (e^{i\langle \mathbf{t}, \phi(x) \rangle} - 1) \xi(x) dx.$$

**Theorem 3.1.** *Let  $\kappa_0, \alpha_0, \lambda_0 > 0$  be given with  $\kappa_0, \lambda_0 \leq 1$ , and  $\phi : (0, 1] \rightarrow \mathbb{R}^d$  satisfying the conditions (3.7)–(3.8). There exists  $t_0 > 0, \delta > 0$ , and functions  $U, V : \{\mathbf{t} \in \mathbb{R}^d, \|\mathbf{t}\| \leq t_0\} \rightarrow \mathbb{C}$  such that for all  $\mathbf{t} \in \mathbb{R}^d$  with  $\|\mathbf{t}\| \leq t_0$ , we have*

$$\mathbb{E}_{\mathbb{Q}} \left( e^{i\langle \mathbf{t}, S(x) \rangle} \right) = \exp \{ U(\mathbf{t}) \log Q + V(\mathbf{t}) + O(Q^{-\delta}) \}, \tag{3.9}$$

and

$$U(\mathbf{t}) = \frac{12 \log 2}{m\pi^2} \mathfrak{F}_{\phi}(\mathbf{t}) + O(\|\mathbf{t}\|^2 + \|\mathbf{t}\|^{2\alpha_0 - \epsilon}), \tag{3.10}$$

$$V(\mathbf{t}) = O(\|\mathbf{t}\| + \|\mathbf{t}\|^{\alpha_0 - \epsilon}).$$

If moreover  $\alpha_0 > 1$ , then there exists a real  $d \times d$  matrix  $C_\phi$  such that

$$U(\mathbf{t}) = \frac{12 \log 2}{m\pi^2} \mathfrak{F}_\phi(\mathbf{t}) + \mathbf{t}^T C_\phi \mathbf{t} + O(\mathbf{t}^3 + |\mathbf{t}|^{1+\alpha_0-\varepsilon}) \quad (3.11)$$

with  $\mathbf{t}$  interpreted as a column vector and  $\mathbf{t}^T$  its transpose. The numbers  $\delta$ ,  $t_0$  and the implied constant depend at most on  $\alpha_0$ ,  $\kappa_0$ ,  $\lambda_0$ ,  $\varepsilon$  and on an upper-bound for the left-hand sides of (3.7)–(3.8).

*Remark.*

- It is important to note that the actual values of  $\kappa_0$  and  $\lambda_0$  only affect the statement up to the value of  $t_0$ ,  $\delta$  and the implied constants. In particular, if  $\phi$  is  $C^1$  on each interval  $[\frac{1}{n+1}, \frac{1}{n}]$  and if there exists  $C \geq 1$  such that  $\|d_x \phi\| = O(x^{-C})$  for  $x \in (0, 1]$ , then (3.8) is satisfied with  $\kappa_0 = 1$  and any  $\lambda_0 < 1/C$ .
- In our applications, we will have  $\|\phi(x)\| \asymp \|\phi(y)\|$  for all  $x, y$  in  $h(I)$ , uniformly in  $h \in \mathcal{H}^m$ . In this situation, (3.7) is equivalent to  $\int_0^1 \|\phi(x)\|^{\alpha_0} dx < \infty$ .
- In typical applications, the quantity  $\mathfrak{F}_\phi(\mathbf{t})$  can be evaluated by standard methods, and the results relevant for the present paper are worked out in [12].
- The results of [4] are stated in a formalism which includes the Gauss map as a special case. The most crucial assumption, at least as far as one is interested in power-saving error terms, is the ‘uniform non-integrability’ assumption [4, p. 357] (see [87] for a qualitative result, not using the UNI condition). In the present work, we do not use any more specific properties of the Gauss map; with suitable modifications, the arguments presented here apply to the centred and odd Euclidean algorithms [4, Figure 1] as well.
- A generalization in a different direction of Baladi–Vallée’s results, for maps associated to a reduction algorithm in congruence subgroups, has very recently and independently been obtained by Lee and Sun [74]<sup>†</sup>.

The main feature of Theorem 3.1 we use is the relative weakness of the hypotheses on  $\phi$ . This is important in our applications, where often little is known on  $\phi$  besides the regularity properties, and rough bounds on the Hölder norms. To obtain this uniformity, we systematically use Hölder spaces, not only because of the regularity of  $\phi$ , but also in order to dampen the oscillations of  $e^{i\langle \mathbf{t}, \phi \rangle}$  (see (4.3)). This is the main reason why arbitrarily small values of  $\lambda_0$  are admissible for Theorem 3.1.

The shape of the asymptotic estimates (3.9)–(3.11) are characteristic of infinitely divisible distribution [42, Chapter XVII.2]. The informally stated Theorem 1.1 follows, in all the cases we consider, by an evaluation of  $\mathfrak{F}_\phi(\mathbf{t})$  and the Berry–Esseen inequality [110, Theorem II.7.16], [43, Theorem IX.5].

The variety of situations we consider prevents us from making a clear and concise set of hypotheses on  $h_\sigma$  which would make Theorem 1.1 rigorous. Besides working out in details the applications, we restrict to illustrating the case  $m = 1$  and  $\alpha_0 > 2$  by the following Central Limit Theorem for the Birkhoff sums  $S_\phi$  defined in (3.2), which recovers and extends in particular [4, Theorem 3.(a)] (see also [3, Remark 1.3]). We recall that  $\Phi$  is the cumulative distribution function of the standard normal law.

<sup>†</sup> Another way to study levels  $N > 1$  would be by building an expanding map out of Atkin–Lehner homographies, but our attempts to construct such a map satisfying the UNI condition were not successful.



**Corollary 3.2.** *Assume that the bound*

$$\sum_{n \geq 1} \frac{1}{n^2} \left( \sup_{x \in [\frac{1}{n+1}, \frac{1}{n}]} |\phi(x)|^{\alpha_0} + \sup_{x, y \in [\frac{1}{n+1}, \frac{1}{n}]} \frac{|\phi(x) - \phi(y)|^{\lambda_0}}{|x - y|^{\kappa_0 \lambda_0}} \right) < \infty.$$

holds for some  $\alpha_0 > 2$  and  $\lambda_0, \kappa_0 > 0$ . Suppose that  $\phi$  is not of the form  $c \log + f - f \circ T$  for a function  $f : [0, 1] \rightarrow \mathbb{R}$  and  $c \in \mathbb{R}$ . Then, for some  $\sigma > 0$  and

$$\mu = \frac{12}{\pi^2} \int_0^1 \frac{\phi(x) dx}{1 + x}, \tag{3.12}$$

we have

$$\mathbb{P}_Q \left( \frac{S_\phi(x) - \mu \log Q}{\sigma \sqrt{\log Q}} \leq v \right) = \Phi(v) + O \left( \frac{1}{\sqrt{\log Q}} + \frac{1}{(\log Q)^{\alpha_0/2 - 1 - \varepsilon}} \right) \tag{3.13}$$

uniformly in  $v \in \mathbb{R}$ .

A variation on the argument shows that the milder hypothesis  $\alpha_0 = 2$  implies the estimate (3.13) with a qualitative error term  $o(1)$  as  $Q \rightarrow \infty$ . For any  $k \in \mathbb{N}_{>0}$ , under the condition  $\alpha_0 > k$ , a variation on the arguments also leads to an estimate with power-saving error term for  $\mathbb{E}_Q(S_\phi(x)^k)$ .

## Notations

For any function  $f(s, t)$  of two real or complex variables, and all  $k, \ell \geq 0$ , we let  $\partial_{k, \ell} f := \frac{\partial^{k+\ell}}{\partial s^k \partial t^\ell} f$  whenever this function is defined.

We recall that the semi-norm  $\|f\|_{(\kappa)}$  is defined in (3.5). The Landau symbol  $f = O(g)$  means that there is a constant  $C \geq 0$  for which  $|f| \leq Cg$  whenever  $f$  and  $g$  are defined. The notation  $f \ll g$  means  $f = O(g)$ . If the constant depends on a parameter, say  $\varepsilon$ , this is indicated in subscript, for example,  $f = O_\varepsilon(g)$  or  $f \ll_\varepsilon g$ .

## 4 | LEMMAS

### 4.1 | Hölder constants

We compile here several facts we will use on the Hölder norms  $\|f\|_{(\kappa)}$  for  $f \in H^\kappa(I, \mathbb{C})$ .

(1) For  $f, g \in H^\kappa$ , we have

$$\|fg\|_{(\kappa)} \leq \|f\|_{(\kappa)} \|g\|_\infty + \|f\|_\infty \|g\|_{(\kappa)}. \tag{4.1}$$

This follows by writing  $f g(x) - f g(y) = (f(x) - f(y))g(x) + f(y)(g(x) - g(y))$ .

(2) For  $g \in H^1$  and  $f \in H^\kappa(g(I))$ , we have  $f \circ g \in H^\kappa$  and

$$\|f \circ g\|_{(\kappa)} \leq \|g\|_{(1)}^\kappa \|f\|_{g(I)}\|_{(\kappa)}. \quad (4.2)$$

This follows by writing  $\frac{|f(g(x))-f(g(y))|}{|x-y|^\kappa} = \left| \frac{g(x)-g(y)}{x-y} \right|^\kappa \frac{|f(g(x))-f(g(y))|}{|g(x)-g(y)|^\kappa}$ .

(3) For  $\lambda \in [\kappa, 1]$  and  $f \in H^{\kappa/\lambda}$  real, we have

$$\|e^{if}\|_{(\kappa)} \leq \|f\|_{(\kappa/\lambda)}^\lambda. \quad (4.3)$$

This follows by writing  $\frac{|e^{if(x)}-e^{if(y)}|}{|x-y|^\kappa} = \frac{|e^{i(f(x)-f(y))}-1|}{|x-y|^\kappa} \leq \frac{|f(x)-f(y)|^\lambda}{|x-y|^\kappa}$ .

(4) For  $0 < \kappa < \lambda$  and  $f \in H^\lambda$ , we have

$$\|f\|_{(\kappa)} \leq \|f\|_{(0)}^{1-\kappa/\lambda} \|f\|_{(\lambda)}^{\kappa/\lambda}. \quad (4.4)$$

This follows by writing  $\frac{|f(x)-f(y)|}{|x-y|^\kappa} = |f(x)-f(y)|^{1-\kappa/\lambda} \left( \frac{|f(x)-f(y)|}{|x-y|^\lambda} \right)^{\kappa/\lambda}$ .

## 4.2 | Oscillating integrals

We will require the following analogue of van der Corput's lemma. We let

$$I = [0, 1].$$

**Lemma 4.1.** *Let  $\Delta, \kappa > 0$ . Assume that  $\Psi : I \rightarrow \mathbb{R}$  is  $C^1$  with  $\Psi' \geq \Delta$  and that  $\Psi'$  is monotonous on  $I$ . Let  $g \in H^\kappa$ . Then*

$$\int_0^1 g(x)e^{i\Psi(x)} dx \ll \frac{\|g\|_\infty}{\Delta} + \frac{\|g\|_{(\kappa)}}{\Delta^\kappa}.$$

*Proof.* The lemma is obtained by combining the methods of the usual van der Corput Lemma ([105, Proposition VIII.2, p. 332]), and the bound on Fourier coefficients of a Hölder continuous function ([106, example 15, p. 92]); we restrict to mentioning the main steps. We change variables and let

$$h(x) = \frac{g \circ \Psi^{-1}(x)}{\Psi' \circ \Psi^{-1}(x)}.$$

Let  $R = (\Psi(I) \setminus (\Psi(I) - \pi)) \cup (\Psi(I) \setminus (\Psi(I) + \pi))$ . We have

$$\begin{aligned} \int_0^1 g(x)e^{i\Psi(x)} dx &= \int_{\Psi(I)} h(x)e^{ix} dx \\ &= O\left(\int_R \frac{\|g\|_\infty dx}{\Psi' \circ \Psi^{-1}(x)}\right) - \frac{1}{2} \int_{\Psi(I) \cap (\Psi(I) - \pi)} (h(x + \pi) - h(x))e^{ix} dx. \end{aligned}$$



Now, on the one hand,

$$\begin{aligned}
 h(x + \pi) - h(x) &= \frac{g \circ \Psi^{-1}(x + \pi) - g \circ \Psi^{-1}(x)}{\Psi' \circ \Psi^{-1}(x)} \\
 &\quad + (g \circ \Psi^{-1})(x + \pi) \left( \frac{1}{\Psi' \circ \Psi^{-1}(x + \pi)} - \frac{1}{\Psi' \circ \Psi^{-1}(x)} \right) \\
 &\ll \frac{\|g\|_{(\kappa)} \|\Psi^{-1}\|_{(1)}^\kappa}{\Psi' \circ \Psi^{-1}(x)} + \|g\|_\infty \left| \frac{1}{\Psi' \circ \Psi^{-1}(x + \pi)} - \frac{1}{\Psi' \circ \Psi^{-1}(x)} \right|
 \end{aligned}$$

by (4.2) on the first term, and on the other hand,

$$\int_{\Psi(I) \cap (\Psi(I) - \pi)} \left| \frac{1}{\Psi' \circ \Psi^{-1}(x + \pi)} - \frac{1}{\Psi' \circ \Psi^{-1}(x)} \right| dx \leq \int_R \frac{dx}{\Psi' \circ \Psi^{-1}(x)} = O(1/\Delta)$$

by monotonicity. We conclude using  $\|\Psi^{-1}\|_{(1)} \ll 1/\Delta$ . □

## 5 | PROPERTIES OF THE TRANSFER OPERATOR

In this section and the following ones, all implied constants in the notations  $O(\dots)$  and  $\ll$  may depend on  $\alpha_0, \kappa_0, \lambda_0, m$  and an upper-bound for the values of (3.7)–(3.8). Additional dependencies will be indicated in subscript.

**Definition.** Let

$$\kappa := \min \left( \frac{1}{3}, \frac{1}{2} \kappa_0 \lambda_0 \right), \tag{5.1}$$

where we recall that  $\kappa_0$  is the Hölder exponent of  $\phi$  on each interval  $(\frac{1}{n+1}, \frac{1}{n})$ . For all  $\mathbf{t} \in \mathbb{R}^d$  and  $s \in \mathbb{C}$  with  $\text{Re}(s) > 1$ , define an operator  $\mathbb{H}_{s,\mathbf{t}}^{(j)}$  acting on  $H^\kappa([0, 1], \mathbb{R})$  by

$$\begin{aligned}
 \mathbb{H}_{s,\mathbf{t}}^{(j)}[f](x) &= \sum_{n \geq 1} \frac{e^{i\langle \mathbf{t}, \phi_j(1/(n+x)) \rangle}}{(n+x)^s} f\left(\frac{1}{n+x}\right) \\
 &= \sum_{h \in \mathcal{H}} e^{i\langle \mathbf{t}, \phi_j \circ h(x) \rangle} |h'(x)|^{s/2} (f \circ h)(x).
 \end{aligned}$$

When  $\mathbf{t} = (0, \dots, 0)$ , this is independent of  $j$ , in which case we drop the notation  $(j)$ . We abbreviate further

$$\mathbb{H}_s := \mathbb{H}_{s,0}, \quad \mathbb{H} := \mathbb{H}_{2,0}.$$

Define the norm and the semi-norm

$$\|f\|_0 := \|f/\xi\|_\infty, \quad \|f\|_1 := \|f/\xi\|_{(\kappa)}.$$

Here we recall that  $\xi(x) = \frac{1}{\log 2} \frac{1}{1+x}$ . We equip  $H^\kappa([0, 1])$  with the norms

$$\|f\|_{1,\beta} := \|f\|_0 + \beta^{-\kappa} \|f\|_1, \quad \text{where } \beta > 0.$$

For  $\mathbf{t} \in \mathbb{R}^d$  and  $s \in \mathbb{C}$ ,  $\operatorname{Re}(s) > 1$ , and  $0 \leq j \leq m$ , let

$$\Pi_{s,\mathbf{t}}^{(j)} := \mathbb{H}_{s,\mathbf{t}}^{(j)} \cdots \mathbb{H}_{s,\mathbf{t}}^{(1)},$$

with  $\Pi_{s,\mathbf{t}}^{(0)} = \operatorname{Id}$ . In what follows, we will often abbreviate

$$\Pi_{s,\mathbf{t}} := \Pi_{s,\mathbf{t}}^{(m)},$$

and define

$$\mathfrak{Z}(x) := \prod_{0 \leq j \leq m-1} T^j(x), \quad g_{s,\mathbf{t}}(x) := e^{i\langle \mathbf{t}, \phi(x) \rangle} \mathfrak{Z}(x)^{s-2}. \quad (5.2)$$

Note that  $g_{s,\mathbf{t}} \circ h$  belongs to  $H^{\kappa_0}([0, 1])$  for all  $h \in \mathcal{H}^m$ . Moreover, since  $|T'(x)| = x^{-2}$  whenever  $T$  is differentiable at  $x$ , for all  $h \in \mathcal{H}^m$ , we have

$$|h'| = \prod_{j=0}^{m-1} |T' \circ T^{j-1}|^{-1} = \mathfrak{Z}(x)^2,$$

and therefore

$$\Pi_{s,\mathbf{t}} : f \mapsto \mathbb{H}^m[g_{s,\mathbf{t}}f]. \quad (5.3)$$

## 5.1 | Properties at the central point

By [24] (Section 2.2, Proposition 4.1 and Theorem 4.2), along with the Ruelle–Perron–Frobenius theorem (see [83]), we have that the operator  $\mathbb{H}_{2,0}$  acting on  $H^\kappa([0, 1])$  is quasi-compact. It has 1 as a simple eigenvalue, and no other eigenvalue of modulus  $\geq 1$ . The projection associated with the eigenvalue 1 is given by

$$\mathbb{P}_{2,0}[f](x) = \left( \int_{[0,1]} f \, d\nu \right) \xi(x),$$

where  $\nu$  is the Lebesgue measure. In particular, we have

$$\mathbb{H}_{2,0}[\xi] = \xi.$$

Since  $\Pi_{2,0}^{(m)} = \mathbb{H}_{2,0}^m$ , we obtain the existence of a linear operator  $\mathbb{N}_{2,0}$  acting on  $H^\kappa$ , such that

$$\Pi_{2,0}^{(m)} = \mathbb{P}_{2,0} + \mathbb{N}_{2,0}, \quad (5.4)$$

and additionally the spectral radius of  $\mathbb{N}_{2,0}$  satisfies  $\text{srd}(\mathbb{N}_{2,0}) < 1$  and  $\mathbb{P}_{2,0}\mathbb{N}_{2,0} = \mathbb{N}_{2,0}\mathbb{P}_{2,0} = 0$ . We will use on several occasions that for  $f$  continuous by parts,

$$\int_{[0,1]} \mathbb{H}_{2,0}[f] \, d\nu = \int_{[0,1]} f \, d\mu. \tag{5.5}$$

### 5.2 | Dominant spectral properties

In the sequel, we will repeatedly use the following facts. We let  $\mathcal{H}^* := \cup_{m \geq 0} \mathcal{H}^m$ .

- The bounded distortion property [4, equation (3.1)]: for all  $h \in \mathcal{H}^*$ , we have  $|h''| \ll |h'|$  with a uniform constant, in particular, independently of the depth of  $h$ . This implies that for all  $x \in [0, 1]$  and  $h \in \mathcal{H}^*$ ,

$$|h'(x)| \asymp |h'(0)|. \tag{5.6}$$

- For all  $q > \frac{1}{2}$ , we have

$$\sum_{h \in \mathcal{H}} \|h'\|_\infty^q < \infty.$$

- The ‘contracting ratio’ property of the inverse branches [4, bound 2.4, Figure 1]: for some  $\rho \in [0, 1)$ , we have the following uniform bound for all  $\ell \in \mathbb{N}$  and  $h \in \mathcal{H}^\ell$ :

$$\|h\|_{(1)} \ll \rho^\ell. \tag{5.7}$$

**Lemma 5.1.**

- For  $\text{Re}(s) = \sigma > 1$ , we have

$$\Pi_{s,t}[f] \leq \Pi_{\sigma,0}[|f|].$$

- For all  $\sigma > 1$ , there exists  $A_\sigma > 0$  such the map  $\sigma \mapsto A_\sigma$  is Lipschitz-continuous and decreasing,  $A_2 = 1$ , and we have the bounds on operator norms

$$\|\mathbb{H}_{\sigma,0}\|_0 \leq A_\sigma, \quad \|\Pi_{\sigma,0}\|_0 \leq A_\sigma^m. \tag{5.8}$$

In particular,  $A_\sigma \leq e^{O(2-\sigma)}$  for  $\sigma \in (1, 2]$ .

- For some  $\rho < 1$ , all  $k \in \mathbb{N}$  and all  $f \in H^k$  with  $f \geq 0$ , we have

$$\|\Pi_{2,0}^k[f]\|_0 \ll \int_{[0,1]} f \, d\nu + \rho^k \|f\|_0. \tag{5.9}$$

The implied constant is absolute.

*Proof.*

- The first statement is trivial by the triangle inequality.

– By a direct computation, we have

$$\|\mathbb{H}_\sigma\|_0 \leq A_\sigma := \sup_{x \in [0,1]} \frac{1}{\xi(x)} \sum_{n \geq 1} \frac{1}{(n+x)^\sigma} \xi\left(\frac{1}{n+x}\right).$$

The properties we require of  $A_\sigma$  are readily verified.

– The third item follows from (5.4) with any fixed  $\rho \in (\text{srd}(\mathbb{N}_{2,0}), 1)$ , by the definition of the spectral radius.  $\square$

### 5.3 | Spectral gap at $t = 0$ and $\tau \neq 0$

**Lemma 5.2.** *For  $\tau \neq 0$ , we have  $\|\mathbb{H}_{2+i\tau}\|_0 < 1$ , and so similarly for  $\Pi_{2+i\tau,0}$ .*

*Proof.* This is a well-studied phenomenon; see [93, Proposition 6.1]. Our exact statement for  $\mathbb{H}_{2+i\tau}$  acting on a space of holomorphic functions is proved in [111, p. 476] (see also [87, Proposition 3.2]), however, an inspection of the proof shows that it actually holds for  $\mathbb{H}_{2+i\tau}$  acting on  $\mathcal{C}([0, 1])$ , and therefore also on  $H^k$ .  $\square$

### 5.4 | Perturbation

**Lemma 5.3.** *For all  $\varepsilon > 0$ , there exists  $\delta, t_0 > 0$  such that for  $1 \leq j \leq m$ , the following holds.*

– For  $\sigma \geq 2 - \delta$  and  $\|\mathbf{t}\| \leq t_0$ , we have

$$\left\| \Pi_{s,\mathbf{t}}^{(j)} - \Pi_{s,0}^{(j)} \right\|_0 \ll_\varepsilon \|\mathbf{t}\| + \|\mathbf{t}\|^{\alpha_0 - \varepsilon}. \quad (5.10)$$

– For  $2 - \delta < \sigma \leq 3$  and all  $\tau \in \mathbb{R}$ , we have

$$\left\| \Pi_{\sigma+i\tau,0}^{(j)} - \Pi_{2+i\tau,0}^{(j)} \right\|_0 \ll |\sigma - 2|. \quad (5.11)$$

– For  $\tau_1, \tau_2 \in \mathbb{R}$ , we have

$$\left\| \Pi_{2+i\tau_1,0}^{(j)} - \Pi_{2+i\tau_2,0}^{(j)} \right\|_0 \ll |\tau_1 - \tau_2|. \quad (5.12)$$

*Proof.* We assume  $j = m$ , the general case being essentially identical.

– We have

$$\begin{aligned} \left\| \Pi_{s,\mathbf{t}} - \Pi_{s,0} \right\|_0 &= \sup_{\|f\|_0=1} \left\| \mathbb{H}^m[(g_{s,\mathbf{t}} - g_{s,0})f] \right\|_0 \\ &\ll \sum_{h \in H^m} |h'(0)| \left\| (g_{s,\mathbf{t}} - g_{s,0}) \circ h \right\|_\infty. \end{aligned}$$

However, for all  $x \in (0, 1]$ , by the bounded distortion property (5.6), we have

$$|g_{s,t}(h(x)) - g_{s,0}(h(x))| \ll |h'(0)|^{\sigma/2-1} |e^{i\langle t, \phi(x) \rangle} - 1| \ll |h'(0)|^{\sigma/2-1} (\|t\| \|\phi(x)\|)^\alpha$$

with  $\alpha = \min(1, \alpha_0 - \varepsilon)$ . By Hölder’s inequality, we deduce

$$\begin{aligned} \|\Pi_{s,t} - \Pi_{s,0}\|_0 &\ll \|t\|^\alpha \sum_{h \in \mathcal{H}^m} |h'(0)|^{\sigma/2} \|\phi \circ h\|_\infty^\alpha \\ &\ll \|t\|^\alpha \left( \sum_{h \in \mathcal{H}^m} |h'(0)| \|\phi|_{h(\mathcal{I})}\|_\infty^{\alpha_0} \right)^{\alpha/\alpha_0} \left( \sum_{h \in \mathcal{H}^m} |h'(0)|^q \right)^{1-\alpha/\alpha_0} \end{aligned} \tag{5.13}$$

with  $q = \frac{\sigma\alpha_0 - 2\alpha}{2(\alpha_0 - \alpha)}$ . Picking  $\delta = \frac{1}{2}(1 - \frac{\alpha}{\alpha_0}) = O(\varepsilon)$  ensures that  $q > \frac{1}{2}$ , and with our hypothesis (3.7) and the bounded distortion property (5.6), we obtain

$$\begin{aligned} \sum_{h \in \mathcal{H}^m} |h'(0)| \|\phi|_{h(\mathcal{I})}\|_\infty^{\alpha_0} &\ll \sum_{j=1}^m \sum_{h=h_1 \circ \dots \circ h_m \in \mathcal{H}^m} |h'_1(0) \cdots h'_m(0)| \|\phi_j \circ T^{j-1}|_{h(\mathcal{I})}\|_\infty^{\alpha_0} \\ &\ll \sum_{j=1}^m \sum_{h_j \in \mathcal{H}} |h'_j(0)| \|\phi_j|_{h_j(\mathcal{I})}\|_\infty^{\alpha_0} < \infty. \end{aligned} \tag{5.14}$$

Therefore, both sums in (5.13) are bounded in terms of  $\varepsilon$  and  $\phi$  only, and we deduce the claimed bound  $\|\Pi_{s,t} - \Pi_{s,0}\| \ll_\varepsilon \|t\|^\alpha$ .

– Proceeding as above, we find

$$\begin{aligned} \|\Pi_{\sigma+i\tau,0} - \Pi_{2+i\tau,0}\|_0 &\ll \sum_{h \in \mathcal{H}^m} |h'(0)| \|\mathfrak{Z}^{\sigma-2} - 1 \circ h\|_\infty \\ &\ll |\sigma - 2| \sum_{h \in \mathcal{H}^m} (1 + |\log |h'(0)||) |h'(0)|^{\max(\sigma/2, 1)}. \end{aligned}$$

For any  $\sigma > 1$ , the last sum is finite, so that our statement follows for any fixed  $\delta \in (0, 1)$ .

– Once again proceeding again as above, we have

$$\|\Pi_{2+i\tau_1,0} - \Pi_{2+i\tau_2,0}\|_0 \ll \sum_{h \in \mathcal{H}^m} |h'(0)| \|\mathfrak{Z}^{i(\tau_1 - \tau_2)} - 1 \circ h\|_\infty.$$

Letting  $\tau = \tau_1 - \tau_2$ , we insert the inequality  $\|\mathfrak{Z}^{i\tau} - 1 \circ h\|_\infty \ll |\tau|(1 + |\log |h'(0)||)$ . The resulting sum over  $h$  being absolutely bounded, we deduce  $\|\Pi_{2+i\tau_1,0} - \Pi_{2+i\tau_2,0}\|_0 \ll |\tau|$  as required. □

### 5.5 | First estimate on $\|\Pi_{s,t}\|_1$

The following is a weak form of [4, Lemma 2] (which is referred to, there, as a Lasota–Yorke type inequality). We recall that the Hölder exponent  $\kappa$  was defined in (5.1).

**Lemma 5.4.** For all  $\delta \in (0, \kappa)$ , there exists  $\rho \in [0, 1)$  such that for  $\sigma \geq 2 - \delta$ ,  $\mathbf{t} \in \mathbb{R}^d$  with  $\|\mathbf{t}\| \leq 1$ , and  $\tau \in \mathbb{R}$ , we have

$$\|\Pi_{s,\mathbf{t}}[f]\|_1 \leq O(1 + |s|^\kappa)\|f\|_0 + \rho\|f\|_1,$$

*Proof.* Let  $f \in H^\kappa$ . We write

$$\frac{1}{\xi}\Pi_{s,\mathbf{t}}[f] = \sum_{h \in \mathcal{H}^m} \frac{1}{\xi}(\xi \circ h)|h'|^{s/2} e^{i\langle \mathbf{t}, \phi \circ h \rangle} \left( \frac{f}{\xi} \circ h \right).$$

Splitting as a sum of differences, we obtain

$$\left\| \frac{1}{\xi}\Pi_{s,\mathbf{t}}[f] \right\|_{(\kappa)} \leq \|f\|_0 \sum_{h \in \mathcal{H}^m} \left\| \frac{1}{\xi}(\xi \circ h)|h'|^{s/2} e^{i\langle \mathbf{t}, \phi \circ h \rangle} \right\|_{(\kappa)} + \rho_\sigma \|f\|_1,$$

where

$$\rho_\sigma := \left\| \sum_{h \in \mathcal{H}^m} \frac{1}{\xi}(\xi \circ h)|h'|^{\sigma/2} w_h \right\|_\infty, \quad w_h(x) = \sup_{0 \leq y \leq 1} \left| \frac{h(x) - h(y)}{x - y} \right|^\kappa = |h'(x)h'(0)|^{\kappa/2}.$$

This last equality follows from the fact that each  $h \in \mathcal{H}^* = \cup_{m \geq 0} \mathcal{H}^m$  is a homography associated with an element of  $GL_2(\mathbb{Z})$  with non-negative entries,  $h(x) = \frac{ax+b}{cx+d}$ , and so the supremum above is  $\sup_y |(cx+d)(cy+d)|^{-1} = |h'(0)h'(x)|^{-1/2}$ . Since  $|h'(0)| \leq 1$  by the chain rule, we deduce

$$\sum_{h \in \mathcal{H}^m} \frac{1}{\xi}(\xi \circ h)|h'|^{\sigma/2} w_h \leq \sum_{h \in \mathcal{H}^m} \frac{1}{\xi}(\xi \circ h)|h'| |h'|^{(\sigma-2+\kappa)/2}.$$

Note that  $\sigma - 2 + \kappa \geq \kappa - \delta > 0$  by hypothesis, and we have  $|h'| \leq 1$  by the chain rule. Moreover, for any value of  $m$ , we may find at least one  $h \in \mathcal{H}^m$  with  $\|h'\|_\infty < 1$ , for example, by composing repeatedly  $t \mapsto \frac{1}{2+t}$ . Since  $\frac{1}{\xi}(\xi \circ h)|h'| > 0$ , we deduce

$$\rho_{\kappa-\delta} < \left\| \sum_{h \in \mathcal{H}^m} \frac{1}{\xi}(\xi \circ h)|h'| \right\|_\infty = 1.$$

Next, by using the rules (4.2), (4.3), (4.4), the bounded distortion property  $|h''| \ll |h'|$ , and simple computations, we obtain successively

$$\begin{aligned} \left\| \frac{1}{\xi} \right\|_{(\kappa)} &\ll 1, & \left\| |h'|^{s/2} \right\|_{(\kappa)} &\ll |s|^\kappa \|h'\|_\infty^{\sigma/2}, \\ \|\xi \circ h\|_{(\kappa)} &\leq \|h'\|_\infty^\kappa, & \left\| e^{i\langle \mathbf{t}, \phi \circ h \rangle} \right\|_{(\kappa)} &\ll \|h'\|_\infty^\kappa \|\mathbf{t}\|^{\kappa/\kappa_0} \|\phi\|_{(\kappa_0)}^{\kappa/\kappa_0}. \end{aligned}$$

In the last line, we used the definition (5.1). Grouping these bounds using (4.1), we deduce

$$\begin{aligned} \sum_{h \in \mathcal{H}^m} \left\| \frac{1}{\xi}(\xi \circ h)|h'|^{s/2} e^{i\langle \mathbf{t}, \phi \circ h \rangle} \right\|_{(\kappa)} &\ll 1 + |s|^\kappa + \sum_{h \in \mathcal{H}^m} \|h'\|_\infty^{\sigma/2+\kappa} \|\phi\|_{(\kappa_0)}^{\kappa/\kappa_0} \\ &\ll 1 + |s|^\kappa \end{aligned}$$

since  $\sigma/2 + \kappa \geq 1 + \kappa/2 \geq 1$ ,  $\|\phi\|_{(\kappa_0)}^{\kappa/\kappa_0} \leq 1 + \|\phi\|_{(\kappa_0)}^{\lambda_0}$  (by the definition (5.1)), and by our hypothesis (3.8). □

## 6 | MEROMORPHIC CONTINUATION

Following [112], define the generating series

$$\mathfrak{S}(\mathbf{t}, s) := \sum_{x \in \mathbb{Q} \cap (0,1]} q(x)^{-s} \exp(i\langle \mathbf{t}, S_\phi(x) \rangle),$$

where  $q(x)$  is the reduced denominator of  $x$ .

**Lemma 6.1.** *For  $\operatorname{Re}(s) > 2$  and  $\mathbf{t} \in \mathbb{R}^d$ , we have*

$$\mathfrak{S}(\mathbf{t}, s) = (\Pi_{s,\mathbf{t}}^{(0)} + \Pi_{s,\mathbf{t}}^{(1)} + \dots + \Pi_{s,\mathbf{t}}^{(m-1)})(\operatorname{Id} - \Pi_{s,\mathbf{t}})^{-1}[\mathbf{1}](1).$$

*Proof.* This is a straightforward extension of the computations in [4, equation (2.17)]. □

The aim of this section is to show the meromorphic continuation of  $\mathfrak{S}(s, \mathbf{t})$  to a half-plane  $\operatorname{Re}(s) \geq 2 - \delta$ . This will then be used in conjunction with Mellin transformation to provide an estimate with power-saving for  $\mathbb{E}_{\mathbb{Q}}(e^{i\langle \mathbf{t}, S(x) \rangle})$ . We use different arguments according to the size of  $\operatorname{Im}(s)$ , which is referred to as the height in what follows.

### 6.1 | Small height

The following lemma deals with  $\tau$  in a neighbourhood of 0. In this case, we use the spectral expansion of  $\Pi_{s,\mathbf{t}}$ .

**Lemma 6.2.** *There exists  $\delta, \tau_0, t_0 > 0$ , such that for all  $\sigma, \tau, t \in \mathbb{R}$  with  $|\sigma - 2| \leq \delta$ ,  $|\tau| \leq \tau_0$  and  $\|\mathbf{t}\| \leq t_0$ , the operator  $\Pi_{s,\mathbf{t}}$  acting on  $(H^\kappa, \|\cdot\|_{1,1})$  is quasi-compact, and for some  $\lambda(s, \mathbf{t}) \in \mathbb{C}$ , we have*

$$\Pi_{s,\mathbf{t}} = \lambda(s, \mathbf{t})\mathbb{P}_{s,\mathbf{t}} + \mathbb{N}_{s,\mathbf{t}}$$

where  $\mathbb{P}_{s,\mathbf{t}}$  is of rank 1,  $\mathbb{P}_{s,\mathbf{t}}\mathbb{N}_{s,\mathbf{t}} = \mathbb{N}_{s,\mathbf{t}}\mathbb{P}_{s,\mathbf{t}} = 0$ ,  $\mathbb{P}_{s,\mathbf{t}}^2 = \mathbb{P}_{s,\mathbf{t}}$ , and  $\operatorname{srd}(\mathbb{N}_{s,\mathbf{t}}) < 1 - \delta$ . Moreover, for each such fixed  $\mathbf{t}$ , the operators  $\Pi_{s,\mathbf{t}}$ ,  $\mathbb{P}_{s,\mathbf{t}}$ ,  $\mathbb{N}_{s,\mathbf{t}}$  and the eigenvalue  $\lambda(s, \mathbf{t})$  depend analytically on  $s$ .

*Proof.* This is a direct application of for example, [71, Theorem 2.3]; see also [69, chapter IV.3], and [4, p. 342]. □

### 6.2 | Moderate height

The following lemma is concerned with  $\tau$  of bounded size and away from 0. The main tool is again perturbation theory.

**Lemma 6.3.** *For all  $\tau_0, \tau_1 > 0$  with  $\tau_0 < \tau_1$ , there exists  $\delta, t_0 > 0$  such that for all  $t, \sigma, \tau \in \mathbb{R}$  with  $\|t\| \leq t_0$ ,  $\sigma \geq 2 - \delta$  and  $\tau_0 \leq |\tau| \leq \tau_1$ , we have*

$$\|\Pi_{s,t}\|_0 \leq 1 - \delta.$$

*Proof.* By (5.12) and the triangle inequality, the map  $\tau \mapsto \|\Pi_{2+i\tau,0}\|_0$  is continuous. By Lemma 5.2, we deduce that for some number  $\eta > 0$ , depending on  $\tau_0$  and  $\tau_1$ , we have the bound  $\|\Pi_{2+i\tau,0}\|_0 \leq 1 - \eta$ . By the perturbation bounds (5.10) and (5.11), we may pick  $\delta, t_0$  small enough so that  $\|\Pi_{s,t} - \Pi_{2+i\tau,0}\|_0 \leq \eta/2$ , and our claim follows.  $\square$

### 6.3 | Large height

The following lemma deals with the case of large enough  $|\tau|$ .

**Lemma 6.4.** *For some constants  $\delta, \tau_1, C > 0$ , whenever  $\sigma \geq 2 - \delta$ ,  $t \in \mathbb{R}^d$  with  $\|t\| \leq 1$ , and  $|\tau| \geq \tau_1$ , the operator  $\Pi_{s,t}$  acting on  $H^k$  has spectral radius  $\text{srd}(\Pi_{s,t}) < 1$ , and*

$$\sum_{j \geq 0} \|\Pi_{s,t}^j\|_{1,\tau} \ll |\tau|^{C|\sigma-2|} \log |\tau|.$$

For unbounded values of  $\tau$ , perturbation theory is not effective, instead we adapt the arguments of Dolgopyat [40] and Baladi–Vallée [4], which exploit the cancellation due to the varying argument of  $|h'|^{i\tau}$ . Compared with Baladi–Vallée’s setup, we make two modifications: we work with Hölder-continuous functions, rather than  $C^1$ , and the cost function is not assumed to be constant on each interval of the partition.

#### 6.3.1 | Sums over branches

We will require two estimates involving sums over inverse branches on  $T$ . Define, as in [4, equation (3.10)],

$$\Delta(h_1, h_2) := \inf_{x \in [0,1]} \left| \frac{h_1''(x)}{h_1'(x)} - \frac{h_2''(x)}{h_2'(x)} \right|,$$

Note that by the bounded distortion property, there exists  $\Delta_+ \geq 1$  such that

$$\Delta(h_1, h_2) \leq \Delta_+ \tag{6.1}$$

for all  $h_1, h_2 \in \mathcal{H}^*$ . The following property is the statement that condition UNI.(a) of Baladi–Vallée [4] holds for the Gauss map.

**Lemma 6.5.** *For some absolute constant  $\rho \in [0, 1)$ , we have uniformly in  $n \in \mathbb{N}$ ,  $h_1 \in \mathcal{H}^n$  and  $u \in [0, \Delta_+]$  that*

$$S(u) := \sum_{\substack{h_2 \in \mathcal{H}^n \\ \Delta(h_1, h_2) \leq u}} |h_2'(0)| \ll \rho^n + u.$$



*Proof.* See [4, Lemmas 1 and 6]; the main point is the construction of a dual dynamical system ([4, Section 3.4]) which encodes naturally the quantity  $\Delta(h_1, h_2)$ , and satisfies the dominant spectral bound (5.9). The dual map of the Gauss map is in fact the Gauss map.  $\square$

**Lemma 6.6.** *Under the assumption (3.8), uniformly for all  $n \in \mathbb{N}_{>0}, 0 \leq j \leq n - m$ , we have*

$$\sum_{h \in \mathcal{H}^n} |h'(0)| \left\| \phi|_{T^j \circ h(I)} \right\|_{(\kappa_0)}^{\lambda_0} \ll 1.$$

We recall that  $I = [0, 1]$ .

*Proof.* We decompose  $h = h_1 \circ h_2 \circ h_3$ , where  $h_1 \in \mathcal{H}^j, h_2 \in \mathcal{H}^m$  and  $h_3 \in \mathcal{H}^{n-j-m}$ . We have

$$|h'(0)| \ll |h'_1(0)h'_2(0)h'_3(0)|, \quad \left\| \phi|_{T^j \circ h(I)} \right\|_{(\kappa_0)} \leq \left\| \phi|_{h_2(I)} \right\|_{(\kappa_0)},$$

so that

$$\sum_{h \in \mathcal{H}^n} |h'(0)| \left\| \phi|_{T^j \circ h(I)} \right\|_{(\kappa_0)}^{\lambda_0} \ll \left( \sum_{h_1 \in \mathcal{H}^j} |h'_1(0)| \right) \left( \sum_{h_2 \in \mathcal{H}^m} |h'_2(0)| \left\| \phi|_{h_2(I)} \right\|_{(\kappa_0)}^{\lambda_0} \right) \left( \sum_{h_3 \in \mathcal{H}^{n-j-m}} |h'_3(0)| \right).$$

The sums over  $h_1$  and  $h_3$  are uniformly bounded by (5.8). The sum over  $h_2$  is finite by our hypothesis (3.8) and the triangle inequality (cf. 5.14).  $\square$

### 6.3.2 | Bound on the $L^2$ norm

We recall that  $A_\sigma$  is an upper-bound for the norm  $\|H_{\sigma,0}\|_0$  provided in (5.8).

**Lemma 6.7.** *For some  $\delta, t_0 > 0$  and  $\rho \in [0, 1)$ , whenever  $|\sigma - 2| \leq \delta, |\tau| \geq 1, \|t\| \leq t_0$  and  $\ell \in \mathbb{N}$ , we have*

$$\left( \int_{[0,1]} \left| \Pi_{s,t}^\ell[f] \right|^2 dv \right)^{1/2} \ll A_{2\sigma-2}^{m\ell/2} \left( \left( |\tau|^{-\kappa/2} + \rho^{m\ell/4} \right) \|f\|_0 + \rho^{\kappa m\ell/2} |\tau|^{-\kappa/2} \|f\|_1 \right).$$

*Proof.* Changing  $f$  to  $\bar{f}, t$  to  $-t$  and taking conjugates if necessary, we may assume that  $\tau \geq 0$ . Define, for all  $\ell \in \mathbb{N}_{>0}, \psi_\ell := \sum_{0 \leq j < \ell} \phi \circ T^{mj}$ , so that

$$\Pi_{s,t}^\ell[f] = \sum_{h \in \mathcal{H}^{m\ell}} e^{i\langle t, \psi_\ell \circ h \rangle} |h'|^{s/2} (f \circ h).$$

We note that for all  $h \in \mathcal{H}^{m\ell}$ , by (4.2) and (4.3), we have

$$\begin{aligned} \left\| e^{i\langle t, \psi_\ell \circ h \rangle} \right\|_{(\kappa)} &\leq \sum_{0 \leq j < \ell} \left\| T^{mj} \circ h \right\|_{(1)}^\kappa \left\| e^{i\langle t, \phi \rangle}|_{T^{mj} \circ h(I)} \right\|_{(\kappa)} \\ &\ll \sum_{0 \leq j < \ell} \rho^{m(\ell-j)\kappa} \left\| \phi|_{T^{mj} \circ h(I)} \right\|_{(\kappa_0)}^{\kappa/\kappa_0}. \end{aligned} \tag{6.2}$$

For  $h_1, h_2 \in \mathcal{H}^{m\ell}$ , let

$$g_{h_1, h_2} := e^{i\langle t, \psi_\ell \circ h_1 - \psi_\ell \circ h_2 \rangle} |h'_1 h'_2|^{\sigma/2} (f \circ h_1) \overline{(f \circ h_2)}.$$

This defines a function in  $H^\kappa$ . Expanding the square, we find

$$\int_{[0,1]} |\Pi_{s,t}^\ell[f]|^2 dv = \sum_{h_1, h_2 \in \mathcal{H}^{m\ell}} I(h_1, h_2), \quad I(h_1, h_2) := \int_0^1 g_{h_1, h_2}(x) \left| \frac{h'_1(x)}{h'_2(x)} \right|^{i\tau/2} dx.$$

We have, for all  $h_1, h_2 \in \mathcal{H}^{m\ell}$ , the trivial bound

$$|I(h_1, h_2)| \ll \|g_{h_1, h_2}\|_\infty. \quad (6.3)$$

On the other hand, for all  $h_1, h_2 \in \mathcal{H}^{m\ell}$  satisfying  $\Delta(h_1, h_2) > 0$ , we have from Lemma 4.1 the bound

$$|I(h_1, h_2)| \ll \frac{\|g_{h_1, h_2}\|_\infty}{|\tau|\Delta(h_1, h_2)} + \frac{\|g_{h_1, h_2}\|_{(\kappa)}}{(|\tau|\Delta(h_1, h_2))^\kappa}, \quad (6.4)$$

The norms are bounded, using (5.7), (4.1), (4.2) and (6.2), by

$$\|g_{h_1, h_2}\|_\infty \ll \|f\|_\infty^2 |h'_1(0)h'_2(0)|^{\sigma/2}, \quad (6.5)$$

$$\|g_{h_1, h_2}\|_{(\kappa)} \ll \|f\|_\infty |h'_1(0)h'_2(0)|^{\sigma/2} \left\{ \left( 1 + \sum_{h \in \{h_1, h_2\}} \sum_{0 \leq j < \ell} \rho^{m(\ell-j)\kappa} \|\phi|_{T^{mj} \circ h(I)}\|_{(\kappa_0)}^{\kappa/\kappa_0} \right) \|f\|_\infty + \rho^{\kappa m\ell} \|f\|_{(\kappa)} \right\}.$$

We write  $\|f\|_{(\kappa)} \|f\|_\infty \ll \|f\|_{(\kappa)}^2 + \|f\|_\infty^2 \ll \|f/\xi\|_{(\kappa)}^2 + \|f/\xi\|_\infty^2$ , which implies the variant

$$\|g_{h_1, h_2}\|_{(\kappa)} \ll |h'_1(0)h'_2(0)|^{\sigma/2} \left\{ \left( 1 + \sum_{h \in \{h_1, h_2\}} \sum_{0 \leq j < \ell} \rho^{m(\ell-j)\kappa} \|\phi|_{T^{mj} \circ h(I)}\|_{(\kappa_0)}^{\kappa/\kappa_0} \right) \|f\|_0^2 + \rho^{\kappa m\ell} \|f\|_1^2 \right\}. \quad (6.6)$$

Next, for all  $u \in [0, \Delta_+]$  (where we recall (6.1)), we have uniformly

$$K(u) := \max_{0 \leq j < \ell} \sum_{\substack{h_1, h_2 \in \mathcal{H}^{m\ell} \\ \Delta(h_1, h_2) \leq u}} |h'_1(0)h'_2(0)|^{\sigma/2} \left( 1 + \|\phi|_{T^{mj} \circ h_1(I)}\|_{(\kappa_0)}^{\lambda_0/2} \right)$$

$$\begin{aligned} &\ll \max_{0 \leq j < \ell} \left( \sum_{h_1, h_2 \in \mathcal{H}^{m\ell}} |h'_1(0)| (1 + \|\phi\|_{T^{mj} \circ h_1(I)} \|_{(\kappa_0)}^{\lambda_0}) |h'_2(0)|^{\sigma-1} \right)^{1/2} \\ &\quad \times \left( \sum_{\substack{h_1, h_2 \in \mathcal{H}^{m\ell} \\ \Delta(h_1, h_2) \leq u}} |h'_1(0)|^{\sigma-1} |h'_2(0)| \right)^{1/2} \\ &\ll A_{2\sigma-2}^{m\ell} (\rho^{m\ell/2} + u^{1/2}) \end{aligned}$$

by Lemmas 6.5 and 6.6. Let  $\eta \in (0, 1]$  be a parameter. We insert the bounds (6.5) and (6.6) in (6.3), (6.4), and we sum over  $(h_1, h_2)$ . When  $\Delta(h_1, h_2) \leq \eta$ , we use the trivial bound (6.3), otherwise we use (6.4). Using our bound on  $K(u)$  above, the symmetry  $h_1 \leftrightarrow h_2$ , the fact that  $\kappa/\kappa_0 \leq \lambda_0/2$ , and partial summation, we find

$$\begin{aligned} A_{2\sigma-2}^{-m\ell} \sum_{h_1, h_2 \in \mathcal{H}^{m\ell}} I(h_1, h_2) &\ll \|f\|_0^2 \left( K(\eta) + \frac{K(\Delta_+)}{\tau^\kappa} + \int_\eta^{\Delta_+} \left( \frac{1}{\tau u} + \frac{\kappa}{(\tau u)^\kappa} \right) \frac{K(u) du}{u} \right) \\ &\quad + \frac{\rho^{\kappa m\ell} \|f\|_1^2}{\tau^\kappa} \left( K(\Delta_+) + \kappa \int_\eta^{\Delta_+} \frac{K(u) du}{u^{\kappa+1}} \right) \\ &\ll \|f\|_0^2 \left( (\rho^{m\ell/2} + \eta^{1/2}) \left( 1 + \frac{1}{\tau\eta} + \frac{1}{(\tau\eta)^\kappa} \right) + \frac{1}{\tau^\kappa} \right) \\ &\quad + \|f\|_1^2 \rho^{\kappa m\ell} \left( \frac{1}{\tau^\kappa} + \frac{\rho^{m\ell/2} + \eta^{1/2}}{(\tau\eta)^\kappa} \right). \end{aligned}$$

Choosing  $\eta = 1/\tau$ , we obtain

$$\left( \int_{[0,1]} |\Pi_{s,t}^\ell[f]|^2 dv \right)^{1/2} \ll A_{2\sigma-2}^{m\ell} \left( (\tau^{-\kappa/2} + \rho^{m\ell/4}) \|f\|_0 + \tau^{-\kappa/2} \rho^{\kappa m\ell/2} \|f\|_1 \right)$$

as claimed. □

### 6.3.3 | Bound on the $L^\infty$ norm

Next, we transfer the  $L^2$  bound relative to the invariant measure into an  $L^\infty$  bound, following ideas of Dolgopyat [40] adapted to this context by Baladi and Vallée [4, section 3.3].

**Lemma 6.8.** *For some  $\rho \in [0, 1)$  and  $c_0, \delta, \tau_0 > 0$ , depending on  $\eta$  and  $\kappa$  at most, whenever*

$$\sigma \geq 2 - \delta, \quad \tau \geq \tau_0, \quad \|t\| \leq 1,$$

then letting  $n = \lfloor c_0 \log \tau \rfloor$ , we have

$$\|\Pi_{s,t}^n[f]\|_0 \leq \rho^n \|f\|_{1,\tau}.$$

*Proof.* By using the Cauchy–Schwarz inequality, as in [4, Lemma 1], for all  $x \in [0, 1]$  and  $f \in H^k$ , we have

$$\begin{aligned} \frac{|\Pi_{s,t}^k[f](x)|}{\xi(x)} &\leq \left( \sum_{h \in H^{mk}} |h'(x)|^{\sigma-1} \right)^{1/2} \left( \sum_{h \in H^{mk}} |h'(x)| (|f|^2 \circ h)(x) \right)^{1/2} \\ &\ll A_{2\sigma-2}^{mk/2} \left( \int_0^1 |f|^2 d\nu + \rho_1^{mk} \|f\|_0^2 \right)^{1/2} \end{aligned}$$

for some  $\rho_1 \in [0, 1]$  independent of  $k$ . We use this with  $f$  replaced by  $\Pi_{s,t}^\ell[f]$ , with  $k, \ell$  being any choice with  $k + \ell = n$  and  $\ell = k + O(1)$ . By Lemma 6.7 and the bound (5.8), we deduce that for all small enough  $\delta \geq 0$ , if  $\sigma \geq 2 - \delta$ , then

$$\|\Pi_{s,t}^n[f]\|_0 \ll e^{O(\delta n)} \left( (\tau^{-\kappa/2} + \rho_1^{mn/8}) \|f\|_0 + \rho_1^{mn\kappa/4} \tau^{-\kappa/2} \|f\|_1 \right).$$

By choosing  $n = c \log |\tau| + O(1)$  with  $c = 4(m|\log \rho_1|)^{-1}$ , and then  $\tau_0$  large enough and  $\delta$  small enough in terms of  $\kappa$ , we may ensure that

$$\|\Pi_{s,t}^n[f]\|_0 \leq \rho^n \|f\|_{1,\tau},$$

with  $\rho = \rho_1^{m\kappa/10} < 1$  and as claimed.  $\square$

### 6.3.4 | Proof of Lemma 6.4

Iterating the bound of Lemma 5.4, and using (5.8), we have for  $\delta$  small enough and all  $n \geq 0$ , the bound

$$\|\Pi_{s,t}^n[f]\|_1 \ll e^{O(\delta n)} |\tau|^\kappa \|f\|_0 + \rho^n \|f\|_1,$$

for some  $\rho \in [0, 1)$ . We replace  $f$  by  $\Pi_{s,t}^n[f]$  and use Lemmas 6.8 and 5.4. We find that for some constants  $\tau_0 \geq 0$ ,  $c_0 > 0$  and  $\rho \in [0, 1)$ , if  $\delta$  is small enough and  $n = \lfloor c_0 \log \tau \rfloor$ , then

$$\begin{aligned} \|\Pi_{s,t}^{2n} f\|_{1,\tau} &\ll (\rho^n + e^{O(\delta n)}) \|\Pi_{s,t}^n f\|_0 + |\tau|^{-\kappa} \rho^n \|\Pi_{s,t}^n f\|_1 \\ &\ll (\rho^{2n} + e^{O(\delta n)} \rho^n) \|f\|_0 + |\tau|^{-\kappa} (\rho^{2n} + e^{O(\delta n)} \rho^n) \|f\|_1. \end{aligned}$$

At the cost of choosing  $c_0$  large enough and  $\delta$  small enough in terms of the implied constants, we obtain

$$\|\Pi_{s,t}^{2n} f\|_{1,\tau} \leq \rho^{n/2} \|f\|_{1,\tau}.$$

By iterating, this bounds also holds for  $n = k \lfloor c_0 \log \tau \rfloor$ ,  $k \in \mathbb{N}$ , from which we deduce by Gelfand's inequality that  $\text{srd}(\Pi_{s,t}) \leq \rho^{1/4}$ , and  $\|(\text{Id} - \Pi_{s,t}^{2n})^{-1}\|_{1,\tau} \ll 1$ . Finally, from the bounds

$$\|(\text{Id} - \Pi_{s,t})^{-1}\|_{1,\tau} \leq \|(\text{Id} - \Pi_{s,t}^{2n})^{-1}\|_{1,\tau} \sum_{0 \leq j < 2n} \|\Pi_{s,t}^j\|_{1,\tau}$$

and  $\|\Pi_{s,t}^j\|_{1,\tau} \ll e^{O(|\sigma-2|j)}$ , we get the claimed result.

### 6.4 | Deduction of the meromorphic continuation

**Proposition 6.9.** *For some  $\tau_0, t_0, \delta > 0$ , and all  $\|\mathbf{t}\| \leq t_0$ , the function  $s \mapsto \mathfrak{S}(s, \mathbf{t})$ , initially only defined for  $\text{Re}(s) > 2$ , has a meromorphic continuation to the set*

$$H := \{s \in \mathbb{C}, s = \sigma + i\tau, \sigma \geq 2 - \delta\},$$

with possible poles occurring only for  $|\tau| < \tau_0$  and  $\lambda(s, \mathbf{t}) = 1$ . The meromorphic continuation of  $s \mapsto \mathfrak{S}(s, \mathbf{t})$  is bounded uniformly in  $\text{Re}(s) \geq 2 - \delta$  and  $|\tau| \geq \tau_0$  by

$$|\mathfrak{S}(s, \mathbf{t})| \ll |\tau|^{O(\max(0, 2-\sigma))} \log(|\tau| + 2). \tag{6.7}$$

More precisely, for  $|\tau| \leq \tau_0$ , the function

$$s \mapsto \mathfrak{S}(s, \mathbf{t}) - \frac{\lambda(s, \mathbf{t})}{1 - \lambda(s, \mathbf{t})} (\Pi_{s, \mathbf{t}}^{(0)} + \Pi_{s, \mathbf{t}}^{(1)} + \dots + \Pi_{s, \mathbf{t}}^{(m-1)}) \mathbb{P}_{s, \mathbf{t}}[\mathbf{1}](1) \tag{6.8}$$

has an analytic continuation to  $\sigma \geq 2 - \delta$  and  $|\tau| \leq \tau_0$ , and is uniformly bounded there.

*Proof.* By Lemma 6.1, we have for  $\text{Re}(s) > 1$

$$\mathfrak{S}(s, \mathbf{t}) = (\Pi_{s, \mathbf{t}}^{(0)} + \dots + \Pi_{s, \mathbf{t}}^{(m-1)}) \sum_{j \geq 0} \Pi_{s, \mathbf{t}}[\mathbf{1}](j). \tag{6.9}$$

Then Lemma 6.2 yields the analytic continuation of (6.8) for  $|\tau| \leq \tau_0$  for some  $\tau_0 > 0$ . Then Lemma 6.4 ensures the existence of  $\tau_1 > 0$  such that the sum over  $j$  in (6.8) converges uniformly over compacts in the region  $\sigma \geq 2 - \delta$  and  $|\tau| \geq \tau_1$ , and yields the bound (6.7). Finally, applying Lemma 6.3 with the values of  $\tau_0$  and  $\tau_1$  gives the same conclusion for  $\tau_0 \leq |\tau| \leq \tau_1$ . The conjunction of these three cases covers the whole half-plane  $H$ .  $\square$

## 7 | ASYMPTOTIC BEHAVIOUR OF THE LEADING EIGENVALUE

In this section, we study the behaviour, for small  $t$  and  $s$  close to 2, of the leading eigenvalue  $\lambda(s, \mathbf{t})$ . The estimates in this section will reduce the problem to the estimation as  $\mathbf{t} \rightarrow 0$  of the integral

$$\int_0^1 e^{i\langle \mathbf{t}, \phi(x) \rangle} \xi(x) dx,$$

where we recall the notation (3.6). We recall the hypotheses (3.7), (3.8).

### 7.1 | Perturbation theory and existence

Let

$$\mathfrak{d} := -m \int_0^1 \log(x) \xi(x) dx = \frac{m\pi^2}{12 \log 2}. \tag{7.1}$$

**Lemma 7.1.** *For all small enough  $\varepsilon > 0$ , there exists  $t_0 > 0$  such that whenever  $|s - 2| \leq \varepsilon$  and  $\|\mathbf{t}\| \leq t_0$ , we have*

$$\partial_{10}\lambda(s, \mathbf{t}) = -\mathfrak{d} + O(\varepsilon), \quad (7.2)$$

and

$$\lambda(s, \mathbf{t}) - 1 = (-\mathfrak{d} + O(\varepsilon))(s - 2) + O(\varepsilon). \quad (7.3)$$

*Proof.* Let  $f_{s,\mathbf{t}} = \mathbb{P}_{s,\mathbf{t}}[\xi]$  denote an eigenfunction of  $\Pi_{s,\mathbf{t}}$  associated with the eigenvalue  $\lambda(s, \mathbf{t})$ . By Lemma 5.3 and [71, Theorem 2.6, estimation of  $P_L$ ], we have

$$\|f_{s,\mathbf{t}} - \xi\|_\infty \ll_\varepsilon |s - 2| + \|\mathbf{t}\| + \|\mathbf{t}\|^{\alpha_0 - \varepsilon}. \quad (7.4)$$

On the other hand, differentiating the eigenvalue equation  $\mathbb{H}_{2,0}[g_{s,\mathbf{t}}f] = \lambda(s, \mathbf{t})f_{s,\mathbf{t}}$  and integrating with respect to the Lebesgue measure, we get

$$\partial_{10}\lambda(s, \mathbf{t}) \int_{[0,1]} f_{s,\mathbf{t}} \, d\nu = \int_{[0,1]} ((\log \mathfrak{Z})g_{s,\mathbf{t}}f_{s,\mathbf{t}} + (g_{s,\mathbf{t}} - \lambda(s, \mathbf{t}))\partial_{10}f_{s,\mathbf{t}}) \, d\nu.$$

Here we recall the notation (5.2). Setting  $(s, \mathbf{t}) = (2, \mathbf{0})$ , with  $f_{2,\mathbf{0}} = \xi$  and  $g_{2,\mathbf{0}} = 1$ , gives  $\partial_{10}\lambda(2, \mathbf{0}) = -\mathfrak{d}$ . Using the bound (7.4), we get the approximation (7.2).  $\square$

**Lemma 7.2.** *For all  $\eta > 0$ , there exists  $t_0 > 0$  and a unique function  $s_0 : [-t_0, t_0]^d \rightarrow \mathbb{C}$  such that  $s_0(\mathbf{0}) = 2$  and, for  $\|\mathbf{t}\| \leq t_0$ ,*

$$|s_0(\mathbf{t}) - 2| \leq \eta, \quad \lambda(s_0(\mathbf{t}), \mathbf{t}) = 1.$$

*Proof.* This follows from a general form of the implicit functions theorem, for example, [72, Theorem 1.1], whose hypotheses are satisfied by virtue of Lemma 7.1.  $\square$

In what follows we will not discuss the regularity of  $s_0(\mathbf{t})$  at each  $\mathbf{t}$ : we are only interested about its asymptotic behaviour around  $\mathbf{t} = \mathbf{0}$ . We will use results on effective perturbation theory of linear operators, which have been worked out recently in [71].

## 7.2 | The sub-CLT case

We first focus on the case where we do not aim at extracting a term of order 2 in the asymptotic expansion.

**Lemma 7.3.** *For  $\|\mathbf{t}\| \leq t_0$ , we have*

$$s_0(\mathbf{t}) - 2 = \frac{1}{\mathfrak{d}} \int_0^1 (e^{i\langle \mathbf{t}, \phi(x) \rangle} - 1)\xi(x) \, dx + O_\varepsilon(\|\mathbf{t}\|^2 + \|\mathbf{t}\|^{2\alpha_0 - \varepsilon}).$$

*Proof.* By [71, Theorem 2.6], we have

$$\begin{aligned} \lambda(s, \mathbf{t}) &= \int_0^1 \Pi_{s, \mathbf{t}} \xi(x) \, dx + O\left(\|\Pi_{s, \mathbf{t}} - \Pi_{2, 0}\|^2\right) \\ &= (s - 2) \int_0^1 \left[\frac{\partial}{\partial s} \Pi_{s, 0} \xi\right]_{s=2}(x) \, dx + \int_0^1 \Pi_{2, \mathbf{t}} \xi(x) \, dx + O_\varepsilon(|s - 2|^2 + \|\mathbf{t}\|^2 + \|\mathbf{t}\|^{2\alpha_0 - \varepsilon}) \\ &= (s - 2) \int_0^1 \left[\frac{\partial}{\partial s} \mathbb{H}^m[g_{s, 0} \xi]\right]_{s=2}(x) \, dx + \int_0^1 \mathbb{H}^m[e^{i\langle \mathbf{t}, \phi \rangle} \xi](x) \, dx \\ &\quad + O_\varepsilon(|s - 2|^2 + \|\mathbf{t}\|^2 + \|\mathbf{t}\|^{2\alpha_0 - \varepsilon}) \end{aligned} \tag{7.5}$$

by (5.2). Since  $\int \mathbb{H}[f] \, d\nu = \int f \, d\nu$  and  $\mathbb{H}[\xi] = \xi$ , the first integral is  $m \int_0^1 \log(x) \xi(x) \, dx = -\mathfrak{d}$ . The second equals  $\int_0^1 e^{i\langle \mathbf{t}, \phi(x) \rangle} \xi(x) \, dx$ . For  $\alpha = \min(\alpha_0, 1)$ , we have

$$\left| \int_0^1 (e^{i\langle \mathbf{t}, \phi \rangle} - 1) \xi \, d\nu \right| \leq \|\mathbf{t}\|^\alpha \int_0^1 \|\phi\|^\alpha \xi \, d\nu = \|\mathbf{t}\|^\alpha \int_0^1 \mathbb{H}[\|\phi\|^\alpha \xi] \, d\nu \ll \|\mathbf{t}\|^\alpha \tag{7.6}$$

by our hypothesis (3.7) and the triangle inequality (cf. (5.14)). Setting  $s = s_0(\mathbf{t})$ , we deduce  $s_0(\mathbf{t}) - 2 = O(\|\mathbf{t}\| + \|\mathbf{t}\|^{\alpha_0})$  by combining (7.5) and (7.6). Then another use of (7.5) yields our claimed estimate.  $\square$

### 7.3 | The CLT case

Our next goal is to extract the term of order  $\|\mathbf{t}\|^2$  in the analysis above. We assume throughout that  $\alpha_0 > 1$ . In order to describe the order 2 coefficients, we introduce the following notation. Recalling (5.2), let

$$\begin{aligned} \mu_\phi &:= \frac{1}{\mathfrak{d}} \int_0^1 \phi(x) \xi(x) \, dx, \\ \mathbb{K}[f] &:= \frac{1}{\xi} (\text{Id} - \mathbb{H}^m)^{-1} \mathbb{N}^m[f \xi], \end{aligned} \tag{7.7}$$

$$\begin{aligned} \psi &:= \phi + \mu_\phi \log \mathfrak{T}, \\ \chi &:= \mathbb{K}\psi. \end{aligned} \tag{7.8}$$

The well-definedness of  $\mu_\phi$  follows from our hypothesis  $\alpha_0 > 1$ . Note that  $\chi$  is bounded on  $[0, 1]$ , because

$$\begin{aligned} \|\mathbb{N}^m[\psi \xi]\|_\infty &\ll \int_{[0, 1]} \|\psi\| \, d\nu + \|\mathbb{H}^m[\psi \xi]\|_\infty \\ &\ll 1 + \int_{[0, 1]} \|\phi\| \, d\nu + \sum_{h \in \mathcal{H}^m} |h'(0)| \|\phi|_{h(\mathcal{I})}\|_\infty, \end{aligned}$$

which is finite by (3.7).

**Lemma 7.4.** *If  $\alpha_0 > 1$ , then*

$$s_0(\mathbf{t}) - 2 = \frac{1}{\mathfrak{d}} \int_0^1 (e^{i\langle \mathbf{t}, \phi(x) \rangle} - 1) \xi(x) dx - \mathbf{t}^T C_\phi \mathbf{t} + O_\varepsilon(\|\mathbf{t}\|^3 + \|\mathbf{t}\|^{\alpha_0+1-\varepsilon}), \quad (7.9)$$

with

$$C_\phi = \frac{1}{\mathfrak{d}} \int_0^1 \left( \frac{1}{2} (\psi - \phi) \cdot (\psi - \phi)^T + \phi \cdot (\psi - \phi)^T + \psi \cdot \chi^T \right) \xi d\nu. \quad (7.10)$$

Moreover, if  $\alpha_0 > 2$ , then

$$s_0(\mathbf{t}) - 2 = i\langle \mathbf{t}, \mu_\phi \rangle - \frac{1}{2} \mathbf{t}^T \Sigma_\phi \mathbf{t} + O_\varepsilon(\|\mathbf{t}\|^3 + \|\mathbf{t}\|^{\alpha_0-\varepsilon}), \quad (7.11)$$

with

$$\Sigma_\phi := \frac{1}{\mathfrak{d}} \int_0^1 (\psi + \chi - \chi \circ T^m) \cdot (\psi + \chi - \chi \circ T^m)^T \xi d\nu. \quad (7.12)$$

*Remark.* With the definition (7.12), it is clear that the matrix  $\Sigma_\phi$  is symmetric, positive semi-definite. It is definite if and only if the vectors  $\{(\psi + \chi - \chi \circ T^m)(x), x \in (0, 1)\}$  span the whole space  $\mathbb{R}^d$ . By our hypothesis (3.7), we have  $\int_0^1 \|\phi\| |\log \mathfrak{Z}| d\nu < \infty$  whenever  $\alpha_0 > 1$ , so that the matrix  $C_\phi$  is well defined in this case. Similarly, the matrix  $\Sigma_\phi$  is well defined whenever  $\alpha_0 > 2$ , since in that case  $\int_{[0,1]} \|\psi \cdot \psi^T\| d\nu \ll 1 + \int_{[0,1]} \|\phi\|^2 d\nu < \infty$ .

*Proof.* We extend the computations of Lemma 7.3, using our hypothesis on  $\alpha_0$  to expand the quantity  $g_{s,\mathbf{t}} = e^{i\langle \mathbf{t}, \phi \rangle} \mathfrak{Z}^{s-2}$  to order 2 at  $s = 2$  and order 1 at  $\mathbf{t} = 0$ . Let

$$\Delta_{s,\mathbf{t}} = \Pi_{s,\mathbf{t}} - \Pi_{2,0}.$$

We write  $e^{i\langle \mathbf{t}, \phi \rangle} = 1 + i\langle \mathbf{t}, \phi \rangle + O((\|\phi\| \|\mathbf{t}\|)^{\min(2, \alpha_0 - \varepsilon)})$ . Letting  $s = s_0(\mathbf{t}) = 2 + O(\|\mathbf{t}\|)$ , we obtain

$$\begin{aligned} \int_0^1 \Delta_{s,\mathbf{t}} \xi d\nu &= \int_0^1 (e^{i\langle \mathbf{t}, \phi \rangle} - 1) \xi d\nu - \mathfrak{d}(s-2) + i(s-2) \int_0^1 \langle \mathbf{t}, \phi \rangle (\log \mathfrak{Z}) \xi d\nu \\ &\quad + \frac{1}{2}(s-2)^2 \int_0^1 (\log \mathfrak{Z})^2 \xi d\nu + O(\|\mathbf{t}\|^3 + \|\mathbf{t}\|^{1+\alpha_0-\varepsilon}) \\ &= \int_0^1 (e^{i\langle \mathbf{t}, \phi \rangle} - 1) \xi d\nu - \mathfrak{d}(s-2) - \mathbf{t}^T C_{1,\phi} \mathbf{t} + O(\|\mathbf{t}\|^3 + \|\mathbf{t}\|^{1+\alpha_0-\varepsilon}), \end{aligned}$$

where  $C_{1,\phi} := \int_0^1 \phi \cdot (\psi - \phi)^T \xi d\nu + \frac{1}{2} \int_0^1 (\psi - \phi) \cdot (\psi - \phi)^T \xi d\nu$ . We use again [71, Theorem 2.6], getting

$$\lambda(s, \mathbf{t}) = 1 + \int_0^1 \Delta_{s,\mathbf{t}} \xi d\nu + \int_{[0,1]} \Delta_{s,\mathbf{t}} (\text{Id} - \mathbb{H}^m)^{-1} (\text{Id} - \mathbb{P}) \Delta_{s,\mathbf{t}} \xi d\nu + O\left(\|\Delta_{s,\mathbf{t}}\|^3\right).$$

By computations similar to (5.10), we have

$$\left\| \Delta_{s,\mathbf{t}}[f] - i\mathbb{H}^m[\langle \mathbf{t}, \phi \rangle f] - i\mathfrak{d}^{-1} \left( \int_0^1 \langle \mathbf{t}, \phi \rangle \xi d\nu \right) \mathbb{H}^m[\log \mathfrak{Z} f] \right\| \ll_\varepsilon (\|\mathbf{t}\|^{\alpha_0-\varepsilon} + \|\mathbf{t}\|^2) \|f\|_0.$$



Note that the left-hand side can be written  $\|\Delta_{s,t}[f] - i\langle \mathbf{t}, \mathbb{H}^m[\boldsymbol{\psi}f] \rangle\|_0$ . We deduce

$$\begin{aligned} & \int_{[0,1]} \Delta_{s,t}(\text{Id} - \mathbb{H}^m)^{-1}(\text{Id} - \mathbb{P})\Delta_{s,t}\xi \, d\nu \\ &= - \int_0^1 \langle \mathbf{t}, \boldsymbol{\psi} \rangle (\text{Id} - \mathbb{H}^m)^{-1}(\text{Id} - \mathbb{P})\mathbb{H}^m[\langle \mathbf{t}, \boldsymbol{\psi} \rangle \xi] \, d\nu + O_\varepsilon(\|\mathbf{t}\|^3 + \|\mathbf{t}\|^{1+\alpha_0-\varepsilon}). \end{aligned}$$

Since  $\mathbb{P}\mathbb{H}^m = \mathbb{P}$  and  $\mathbb{H}^m = \mathbb{P} + \mathbb{N}^m$ , we have  $(\text{Id} - \mathbb{H}^m)^{-1}(\text{Id} - \mathbb{P})\mathbb{H}^m[f\xi] = \xi\mathbb{K}[f]$  where we recall the definition (7.7). Therefore, we have

$$- \int_0^1 \langle \mathbf{t}, \boldsymbol{\psi} \rangle (\text{Id} - \mathbb{H}^m)^{-1}(\text{Id} - \mathbb{P})\mathbb{H}^m[\langle \mathbf{t}, \boldsymbol{\psi} \rangle \xi] \, d\nu = -\mathbf{t}^T C_{2,\boldsymbol{\phi}} \mathbf{t},$$

where  $\mathbf{u}^T$  denotes the transpose of the column vector  $\mathbf{u}$ , and  $C_{2,\boldsymbol{\phi}} = \int_{[0,1]} (\boldsymbol{\psi} \cdot \mathbb{K}[\boldsymbol{\psi}]^T)\xi \, d\nu$ . This proves (7.9) with  $C_\boldsymbol{\phi} = C_{1,\boldsymbol{\phi}} + C_{2,\boldsymbol{\phi}}$  as claimed.

To prove (7.11), we note that by the hypothesis  $\alpha_0 > 2$ , the quantity (7.12) is well defined. In order to expand it, we first note that, with the definition (7.8), that by construction

$$\mathbb{P}[\boldsymbol{\psi}\xi] = \mathbb{P}[\boldsymbol{\phi}\xi + \mu_\boldsymbol{\phi}\xi \log \mathfrak{Z}] = \left( \int \boldsymbol{\phi}\xi + \mathfrak{d}^{-1} \left( \int \boldsymbol{\phi}\xi \, d\nu \right) \left( \int (\log \mathfrak{Z})\xi \, d\nu \right) \right) \xi \, d\nu = 0$$

since  $\int (\log \mathfrak{Z})\xi = -\mathfrak{d}$ . Therefore

$$\boldsymbol{\chi}\xi = (\text{Id} - \mathbb{H}^m)^{-1}\mathbb{N}^m[\boldsymbol{\psi}\xi] = \sum_{j \geq 1} \mathbb{H}^{jm}[\boldsymbol{\psi}\xi] = \mathbb{H}^m[(\boldsymbol{\psi} + \boldsymbol{\chi})\xi].$$

By the property  $\int f(g \circ T^m) \, d\nu = \int \mathbb{H}^m[f]g \, d\nu$ , we have

$$\int (\boldsymbol{\psi} + \boldsymbol{\chi}) \cdot (\boldsymbol{\chi} \circ T^m)^T \xi \, d\nu = \int \mathbb{H}^m[(\boldsymbol{\psi} + \boldsymbol{\chi})\xi] \cdot \boldsymbol{\chi}^T \, d\nu = \int \boldsymbol{\chi} \cdot \boldsymbol{\chi}^T \xi \, d\nu.$$

Similarly, we have

$$\int (\boldsymbol{\chi} \circ T^m) \cdot (\boldsymbol{\chi} \circ T^m)^T \xi \, d\nu = \int \mathbb{H}^m[(\boldsymbol{\chi} \circ T^m)\xi] \cdot \boldsymbol{\chi}^T \, d\nu = \int \boldsymbol{\chi} \cdot \boldsymbol{\chi}^T \xi \, d\nu.$$

We deduce that, with the definition (7.12), we have

$$\begin{aligned} \Sigma_\boldsymbol{\phi} &= \int (\boldsymbol{\psi} + \boldsymbol{\chi}) \cdot (\boldsymbol{\psi} + \boldsymbol{\chi})^T \xi \, d\nu - 2 \int (\boldsymbol{\psi} + \boldsymbol{\chi})(\boldsymbol{\chi} \circ T^m) \, d\nu + \int (\boldsymbol{\chi} \circ T^m) \cdot (\boldsymbol{\chi} \circ T^m)^T \xi \, d\nu \\ &= \int \boldsymbol{\psi} \cdot \boldsymbol{\psi}^T \xi \, d\nu + 2 \int \boldsymbol{\psi} \cdot \boldsymbol{\chi}^T \xi \, d\nu. \end{aligned}$$

On the other hand, expanding the squares in (7.10), we find

$$2C_\boldsymbol{\phi} = \int \boldsymbol{\psi} \cdot \boldsymbol{\psi}^T \xi \, d\nu - \int \boldsymbol{\phi} \cdot \boldsymbol{\phi}^T \xi \, d\nu + 2 \int \boldsymbol{\psi} \cdot \boldsymbol{\chi}^T \xi \, d\nu.$$

The claimed formula (7.11) follows by the Taylor expansion  $e^{iu} = 1 + u + \frac{1}{2}u^2 + O(u^{\alpha_0-\varepsilon})$  with  $u = i\langle \mathbf{t}, \boldsymbol{\phi} \rangle$ . □

## 8 | PROOF OF THEOREM 3.1

Recall that  $\Omega_Q$  consists of the rationals in  $(0,1]$  of denominators at most  $Q$ , and let

$$\chi_Q(\mathbf{t}) := \sum_{x \in \Omega_Q} \exp(\langle i\mathbf{t}, S_\phi(x) \rangle). \quad (8.1)$$

**Proposition 8.1.** *For all  $\varepsilon > 0$ , there exists  $\delta, t_0 > 0$  such that for  $\|\mathbf{t}\| \leq t_0$ , we have*

$$\mathbb{E}_Q(e^{i\langle \mathbf{t}, S_\phi(x) \rangle}) = Q^{s_0(\mathbf{t})-2} \{1 + O_\varepsilon(Q^{-\delta} + \|\mathbf{t}\| + \|\mathbf{t}\|^{\alpha_0-\varepsilon})\}.$$

*Proof.* Recall that  $\chi_Q(\mathbf{t})$  was defined in (8.1), so that  $\mathbb{E}_Q(e^{i\langle \mathbf{t}, S_\phi(x) \rangle}) = \chi_Q(\mathbf{t})/\chi_Q(0)$ . Let  $\Omega \geq 1$  be a parameter, and  $w : \mathbb{R}_+ \rightarrow [0, 1]$  be a smooth function satisfying

$$\mathbf{1}_{[0,1]} \leq w \leq \mathbf{1}_{[0,1+\Omega^{-1}]}, \quad \|w^{(j)}\|_\infty \ll_j \Omega^j. \quad (8.2)$$

Then by a trivial bound on the contribution of  $q \in [Q, Q(1 + \Omega^{-1})]$ , we have

$$\chi_Q(\mathbf{t}) = O(\Omega^{-1}Q^2) + \tilde{\chi}_Q(\mathbf{t}), \quad \text{where } \tilde{\chi}_Q(\mathbf{t}) := \sum_{\substack{1 \leq a \leq q \\ (a,q)=1}} e^{i\langle \mathbf{t}, S_\phi(a/q) \rangle} w\left(\frac{q}{Q}\right). \quad (8.3)$$

By Perron's formula, we have

$$\tilde{\chi}_Q(\mathbf{t}) = \frac{1}{2\pi i} \int_{3-i\infty}^{3+i\infty} Q^s \mathfrak{G}(s, \mathbf{t}) \hat{w}(s) ds,$$

where the Mellin transform  $\hat{w}(s) = \int_0^\infty w(u)u^{s-1} du$  is defined for  $\text{Re}(s) > 0$ . We move the contour to the line  $\text{Re}(s) = 2 - \delta$ . If  $t_0, \delta > 0$  are small enough, then by Proposition 6.9, we encounter exactly one pole, at  $s = s_0(\mathbf{t})$ . By Cauchy's theorem, we deduce

$$\tilde{\chi}_Q(\mathbf{t}) = \text{Res}_{s=s_0(\mathbf{t})} (Q^s \mathfrak{G}(s, \mathbf{t}) \hat{w}(s)) + \frac{1}{2\pi i} \int_{2-\delta-i\infty}^{2-\delta+i\infty} Q^s \mathfrak{G}(s, \mathbf{t}) \hat{w}(s) ds. \quad (8.4)$$

For some absolute constant  $C \geq 1$ , Proposition 6.9 yields the bound

$$|\mathfrak{G}(s, \mathbf{t})| \ll (|\tau| + 1)^C.$$

On the other hand, we have  $|\hat{w}(s)| \ll_C \Omega^{C+2} |s|^{-C-2}$  for  $\text{Re}(s) \in [1/2, 3]$  by integration by parts and (8.2), so that

$$\left| \int_{2-\delta-i\infty}^{2-\delta+i\infty} Q^s \mathfrak{G}(s, \mathbf{t}) \hat{w}(s) ds \right| \ll \Omega^{C+2} Q^{2-\delta}. \quad (8.5)$$

Finally, we have by (6.8)

$$\text{Res}_{s=s_0(\mathbf{t})} (Q^s \mathfrak{G}(s, \mathbf{t}) \hat{w}(s)) = -\frac{Q^{s_0} \hat{w}(s_0)}{\partial_{10} \lambda(s_0, \mathbf{t})} \sum_{0 \leq j < m} \Pi_{s_0, \mathbf{t}}^{(j)} \mathbb{P}_{s_0, \mathbf{t}}[\mathbf{1}](1), \quad (8.6)$$

where we abbreviated  $s_0 = s_0(\mathbf{t})$  in the right-hand side. Again by [71, Theorem 1.6, estimation of  $P_L$ ] and Lemma 5.3, we have  $\|P_{s_0, \mathbf{t}} - \mathbb{P}_{2,0}\|_0 \ll \|\mathbf{t}\| + \|\mathbf{t}\|^{\alpha_0 - \varepsilon}$ , and therefore

$$\Pi_{s_0, \mathbf{t}}^{(j)} \mathbb{P}_{s_0, \mathbf{t}}[\mathbf{1}](1) = \frac{1}{2 \log 2} \{1 + O_\varepsilon(\|\mathbf{t}\| + \|\mathbf{t}\|^{\alpha_0 - \varepsilon})\}$$

for  $0 \leq j < m$ . The quantity  $\partial_{10} \lambda(s_0(\mathbf{t}), \mathbf{t})$  was estimated in (7.2). Finally, we have by (8.2)

$$\widehat{w}(s_0) = \frac{1}{2} \{1 + O_\varepsilon(\Omega^{-1} + \|\mathbf{t}\| + \|\mathbf{t}\|^{\alpha_0 - \varepsilon})\}.$$

Inserting these estimates in (8.6), we deduce

$$\operatorname{Res}_{s=s_0(\mathbf{t})} (Q^s \mathfrak{G}(s, \mathbf{t}) \widehat{w}(s)) = \frac{3}{\pi^2} Q^{s_0(\mathbf{t})} \{1 + O_\varepsilon(\Omega^{-1} + \|\mathbf{t}\| + \|\mathbf{t}\|^{\alpha_0 - \varepsilon})\}.$$

Grouping this (8.3)–(8.5), we conclude

$$\chi_Q(\mathbf{t}) = \frac{3}{\pi^2} Q^{s_0(\mathbf{t})} \{1 + O_\varepsilon(\Omega^{-1} + \Omega^{C+2} Q^{-\delta} + \|\mathbf{t}\| + \|\mathbf{t}\|^{\alpha_0 - \varepsilon})\}.$$

Our claim follows by optimizing  $\Omega = Q^{\delta/(C+3)}$  and dividing by  $\chi_Q(0)$ . □

*Proof of Theorem 3.1.* We use Proposition 8.1 along with Lemmas 7.3–7.4 and the value (7.1). □

## 9 | APPLICATIONS

When using the Berry–Esseen inequality, we will require a separate treatment of very small values of  $t$  in order to handle the error term  $O(Q^{-\delta})$  in Theorem 3.1 (the argument described in [52, Remark 3.8] is not readily adapted since part of this term originates from counting pairwise coprime numbers).

**Lemma 9.1.** *Suppose that the function  $\phi$  satisfies (3.7) and (3.8). Then we have*

$$\mathbb{E}_Q(e^{i\langle \mathbf{t}, S_\phi(x) \rangle}) = 1 + O(\|\mathbf{t}\|^{\alpha_0/3} \log Q). \tag{9.1}$$

*Proof.* For all  $n \geq 1$ , define  $c(n) := \sup_{x \in (\frac{1}{n+1}, \frac{1}{n})} \|\phi(x)\|^{\alpha_0/3}$ . Then with the terminology of [3, p. 750], the function  $c$  has strong moments to order 3. Setting  $a_j(x) = \lfloor 1/T^{j-1}(x) \rfloor$ , we deduce  $\mathbb{E}_Q(\sum_{j=1}^r c(a_j)) \ll \log Q$  by [3, Remark 1.2]. We conclude by taking expectations in the bound  $|e^{i\langle \mathbf{t}, S_\phi(x) \rangle} - 1| = O(\sum_{j=1}^r (\|\mathbf{t}\| \|\phi(T^{j-1}(x))\|)^{\alpha_0/3})$ . □

*Proof of Corollary 3.2.* In (3.11), we estimate the integral  $\mathfrak{F}_\phi(t)$  using [12, Proposition 2.1], which amounts here to integrating the Taylor expansion of  $e^{it\phi(x)}$  at  $t = 0$ . From (7.12), and with the notation (3.12), we obtain

$$U(t) = it\mu - \frac{t^2}{2} \sigma^2 + O_\varepsilon(|t|^{\min(3, \alpha_0 - \varepsilon)}), \tag{9.2}$$

where, with  $\psi(x) = \phi(x) + \mu \log x$ ,

$$\sigma = \frac{12 \log 2}{\pi^2} \int_0^1 (\psi(x) + \chi(x) - \chi(T(x)))^2 \frac{dx}{1+x}.$$

We recall that  $\chi$  is related to  $\psi$  by (7.8) (with  $m = 1$ ). It is clear that  $\sigma \geq 0$ . If  $\sigma = 0$ , then the integrand vanishes identically, and we would conclude that  $\phi = -\mu \log -\chi + \chi \circ T$ , contradicting our hypothesis. We use the Berry–Esseen theorem [110, theorem II.7.16], [42, equation XVI.(3.13)] with  $T = c(\log Q)^{1/2}$  for some parameter  $c > 0$  to be chosen. Let

$$f(\tau) := \mathbb{E}_Q \left( \exp \left( i\tau \frac{S_\phi(x) - \mu \log Q}{\sigma \sqrt{\log Q}} \right) \right), \quad g(\tau) := e^{-\tau^2/2}.$$

When  $|\tau| \leq Q^{-\delta}$ , we use Lemma 9.1 and a Taylor bound, getting

$$f(\tau) = 1 + O(|\tau|^{\alpha_0/3} (\log Q)^{1-\alpha_0/6} + |\tau| (\log Q)^{1/2}).$$

When  $Q^{-\delta} < |\tau| \leq (\log Q)^\varepsilon$ , we use Theorem 3.1, getting

$$\begin{aligned} f(\tau) &= \exp \left\{ -\frac{\tau^2}{2} + O \left( \frac{|\tau|}{(\log Q)^{1/2}} + \frac{|\tau|^{\alpha_0-\varepsilon}}{(\log Q)^{\alpha_0/2-1-\varepsilon/2}} + \frac{1}{Q^\delta} \right) \right\} \\ &= e^{-\tau^2/2} \left( 1 + O \left( \frac{|\tau|}{(\log Q)^{1/2}} + \frac{|\tau|^{\alpha_0-\varepsilon}}{(\log Q)^{\alpha_0/2-1-\varepsilon/2}} + \frac{1}{Q^\delta} \right) \right). \end{aligned}$$

Finally, when  $(\log Q)^\varepsilon < |\tau| \leq T$ , we get

$$\begin{aligned} f(\tau) &= \exp \left\{ -\frac{\tau^2}{2} \left( 1 + O \left( \frac{1}{|\tau| (\log Q)^{1/2}} + \frac{|\tau|^{\alpha_0-2-\varepsilon}}{(\log Q)^{\alpha_0/2-1-\varepsilon/2}} + \frac{1}{\tau^2 Q^\delta} \right) \right) \right\} \\ &= \exp \left\{ -\frac{\tau^2}{2} (1 + O(c^{\alpha_0-2-\varepsilon})) \right\} \end{aligned}$$

and so, for some small enough choice of  $c > 0$ , we have  $|f(\tau)| \leq e^{-\tau^2/3}$ .

We deduce

$$\begin{aligned} & \int_{-T}^T \left| \frac{f(\tau) - g(\tau)}{\tau} \right| d\tau \\ & \ll \int_0^{Q^{-\delta}} (\tau^{\alpha_0/3-1} (\log Q)^{1-\alpha_0/6} + (\log Q)^{1/2} + \tau) dt \\ & \quad + \int_{Q^{-\delta}}^{(\log Q)^\varepsilon} e^{-\tau^2/2} ((\log Q)^{-1/2} + \tau^{\alpha_0-1-\varepsilon} (\log Q)^{-\alpha/2+1+\varepsilon/2} + \tau^{-1} Q^{-\delta}) d\tau \\ & \quad + \int_{(\log Q)^\varepsilon}^T \tau^{-1} e^{-\tau^2/3} d\tau \\ & \ll (\log Q)^{-1/2} + (\log Q)^{-\alpha_0/2+1+\varepsilon}, \end{aligned}$$

and therefore by the Berry–Esseen inequality

$$\sup_{v \in \mathbb{R}} \left| \mathbb{P}_Q \left( \frac{S_\phi(x) - \mu \log Q}{\sigma \sqrt{\log Q}} \leq v \right) - \Phi(v) \right| \ll \frac{1}{T} + \int_{-T}^T \left| \frac{f(\tau) - g(\tau)}{\tau} \right| d\tau \ll \frac{1}{(\log Q)^{\min(1/2, \alpha_0/2 - 1 - \epsilon)}}$$

as claimed. □

### 9.1 | Central modular symbols

Let  $f(z) = \sum_{n \geq 1} a_n e(nz)$  be a non-zero primitive Hecke eigencuspform of weight  $k$  for  $SL(2, \mathbb{Z})$  with trivial multiplier. Note that  $k$  is necessarily even and  $k \geq 12$ .

Define, for all integer  $1 \leq m \leq k - 1$  and all  $x \in \mathbb{Q}$ , the modular symbol

$$\langle x \rangle_{f,m} := \frac{(2\pi i)^m}{(m-1)!} \int_x^{i\infty} f(z)(z-x)^{m-1} dz. \tag{9.3}$$

**Lemma 9.2.** *For  $m > k/2$ , the function  $x \mapsto \langle x \rangle_{f,m}$ , initially defined over  $\mathbb{Q}$ , can be extended to a bounded function in  $H^{1-\epsilon}(\mathbb{R})$  for any  $\epsilon \in (0, 1)$ .*

*Proof.* By Deligne’s bound [35], we have  $|a_n| \ll_{\epsilon,f} n^{(k-1)/2+\epsilon}$ . Therefore, the sum  $\sum_{n \geq 1} |a_n|/n^m$  is finite, and we deduce by Fubini’s theorem that for  $x \in \mathbb{Q}$ ,

$$\langle x \rangle_{f,m} = (-1)^m \sum_{n \geq 1} \frac{a_n e(nx)}{n^m},$$

and the left-hand side is now defined for  $x \in \mathbb{R}$ . By [63, Theorem 5.3], we have

$$\sum_{n \leq t} a_n e(nx) \ll_f t^{k/2} \log(2t) \quad (t \geq 1) \tag{9.4}$$

uniformly in  $x \in \mathbb{R}$ . Let  $x, x' \in \mathbb{R}$ ,  $\delta = |x - x'|$ , and for  $t \geq 1$ ,  $S(t) := \sum_{n \leq t} a_n (e(nx) - e(nx'))$ . Then using (9.4) and partial summation, we obtain  $|S(t)| \ll_{\epsilon,f} t^{k/2+\epsilon} \min(1, \delta t)$ , and so

$$\begin{aligned} \left| \langle x \rangle_{f,m} - \langle x' \rangle_{f,m} \right| &\ll_{\epsilon,f} \int_1^\infty t^{-m-1+k/2+\epsilon} \min(1, \delta t) dt \\ &\ll_{\epsilon,f} \delta + \delta^{m-k/2-\epsilon} \end{aligned}$$

as claimed, since  $m \geq k/2 + 1$ . □

**Lemma 9.3.** *For any  $\epsilon \in (0, 1)$ , some function  $\phi_f \in H^{1-\epsilon}([0, 1], \mathbb{C})$ , and all  $x \in \mathbb{Q} \setminus \{0\}$ , we have*

$$\langle x \rangle_{f,k/2} = \left\langle \frac{-1}{x} \right\rangle_{f,k/2} + \phi_f(x).$$

Moreover, we have  $\|\phi_f \circ h\|_{(1-\epsilon)} \ll 1$  uniformly for  $h \in \mathcal{H}^*$ .

We will actually only require the last bound for  $h \in \mathcal{H}$ .

*Proof.* For  $\text{Im}(z) > 0$ , define

$$\begin{aligned}\widehat{f}(z) &:= \sum_{n \geq 1} \frac{a_n}{n^{k-1}} e(nz) \\ &= \sum_{n \geq 1} a_n e(nz) \frac{(-2\pi i)^{k-1}}{(k-1)!} \int_0^{i\infty} \tau^{k-2} e^{2\pi i n \tau} d\tau \\ &= \frac{(-2\pi i)^{k-1}}{(k-1)!} \int_z^{i\infty} (\tau - z)^{k-2} f(\tau) d\tau.\end{aligned}$$

Using the modularity relation  $f(-1/z) = z^k f(z)$ , and changing variables  $\tau \rightarrow -1/\tau$ , we obtain

$$\begin{aligned}\frac{(k-1)!}{(-2\pi i)^{k-1}} z^{k-2} \widehat{f}(-1/z) &= z^{k-2} \int_{-1/z}^{i\infty} (\tau + 1/z)^{k-2} f(\tau) d\tau = \int_z^0 (-1/\tau + 1/z)^{k-2} (z\tau)^{k-2} f(\tau) d\tau \\ &= - \int_0^z (\tau - z)^{k-2} f(\tau) d\tau\end{aligned}$$

and so the period polynomial of  $f$

$$r_f(z) := \widehat{f}(z) - z^{k-2} \widehat{f}(-1/z) = \frac{(-2\pi i)^{k-1}}{(k-1)!} \int_0^{i\infty} (\tau - z)^{k-2} f(\tau) d\tau, \quad (\text{Im}(z) > 0),$$

is indeed a polynomial in  $z$  of degree at most  $k-2$ .

Let now  $x \in \mathbb{Q}_{>0}$ . As  $\delta \rightarrow 0$  with  $\delta > 0$ , we have

$$\begin{aligned}\widehat{f}(x(1+i\delta)) &= \frac{(-2\pi i)^{k-1}}{(k-1)!} \int_{x(1+i\delta)}^{i\infty} (\tau - x - ix\delta)^{k-2} f(\tau) d\tau \\ &= \frac{(-2\pi i)^{k-1}}{(k-1)!} \left( \int_x^{i\infty} - \int_x^{x(1+i\delta)} \right) (\tau - x - ix\delta)^{k-2} f(\tau) d\tau.\end{aligned}$$

The second integral is  $O_{M,x,f}(\delta^M)$  as  $\delta \rightarrow 0$  for any fixed  $M > 0$ , since  $f$  is a cusp form. Thus, by the binomial formula, we obtain

$$\widehat{f}(x(1+i\delta)) = \frac{(-1)^{k-1}}{(k-1)} \sum_{\ell=0}^{k-2} \frac{(2\pi x\delta)^\ell}{\ell!} \langle x \rangle_{f,k-1-\ell} + o_{x,f}(\delta^{k-2}).$$

In the same way, since  $-\frac{1}{x(1+i\delta)} = -\frac{1}{x}(1-i\delta')$  with  $\delta' = \delta/(1+i\delta)$ , so that  $\text{Re}(\delta) > 0$ , we have

$$\begin{aligned}(x(1+i\delta))^{k-2} \widehat{f}\left(-\frac{1}{x(1+i\delta)}\right) \\ = \frac{(-1)^{k-1}}{(k-1)} \sum_{\ell=0}^{k-2} (i\delta)^\ell \sum_{j=0}^{\ell} \frac{(-2\pi i)^j}{j!} \binom{k-2-j}{\ell-j} x^{k-2-j} \left\langle \frac{-1}{x} \right\rangle_{f,k-1-j} + o_{x,f}(\delta^{k-2}).\end{aligned}$$

With  $m := k/2 - 1$ , reading the coefficients of  $\delta^m$  on each side of the definition of  $r_f(x(1 + i\delta))$ , and since  $k$  is even, we deduce

$$\langle x \rangle_{f,1+m} - \sum_{j=0}^m c_{j,k} x^j \left\langle \frac{-1}{x} \right\rangle_{f,1+m+j} = -(k-1)(2\pi)^{m+1} i^m r_f^{(m)}(x),$$

$$c_{j,k} := j! \binom{m}{j} \binom{m+j}{j} (-2\pi i)^{-j}.$$

We single out the term  $j = 0$ . The function

$$\phi_f(x) := \sum_{j=1}^m c_{j,k} x^j \left\langle \frac{-1}{x} \right\rangle_{f,1+m+j} - (k-1)(2\pi)^{m+1} i^m r_f^{(m)}(x),$$

defines, by Lemma 9.2, a function in  $H^{1-\varepsilon}([\frac{1}{n+1}, \frac{1}{n}])$  for all  $\varepsilon \in (0, 1)$  and  $n \geq 1$ . The value  $c_{0,k} = 1$  proves our claimed formula. Finally, for  $h \in \mathcal{H}$ , by the rules (4.1), (4.2), (4.4) and 1-periodicity of  $x \mapsto \langle x \rangle_{f,1+m+j}$ , we have

$$\begin{aligned} \|\phi_f \circ h\|_{(1-\varepsilon)} &\ll_f \sum_{j=1}^m \|x \mapsto h(x)^j \langle x \rangle_{f,1+m+j}\|_{(1-\varepsilon)} + \|r_f^{(m)} \circ h\|_{(1-\varepsilon)} \\ &\ll_f \sum_{j=1}^m \left( \|(h^j)'\|_{\infty}^{\varepsilon} \|h^j\|_{\infty}^{1-\varepsilon} + \|h^j\|_{\infty} \right) + \|h'\|_{\infty}^{1-\varepsilon} \\ &\ll_f 1. \end{aligned}$$

By the rule (4.2) again, and since  $\|h\|_{(1)} \leq \|h'\|_{\infty} \ll 1$  for  $h \in \mathcal{H}^*$ , we deduce that the same bound  $\|\phi_f \circ h\|_{(1-\varepsilon)} \ll 1$  holds for  $h \in \mathcal{H}^*$ . □

*Proof of Theorems 2.3 and 2.4.* Iterating Lemma 9.3, we have for all  $x \in \mathbb{Q} \cap (0, 1]$ ,

$$\langle x \rangle_{f,k/2} = \sum_{j=1}^r \phi((-1)^{j-1} T^{j-1}(x)) + \langle 0 \rangle_{f,k/2}. \tag{9.5}$$

Note that changing the coordinates of the set  $\mathcal{R} \subset \mathbb{R}^2$  by an amount  $O(1/\sqrt{\log Q})$  in (2.2) does not alter the right-hand side, so that we may replace  $\langle x \rangle_{f,k/2}$  by  $\langle x \rangle_{f,k/2} - \langle 0 \rangle_{f,k/2}$ . For all  $t \in \mathbb{R}^2$ , identifying  $\mathbb{C} \simeq \mathbb{R}^2$  with basis  $(1, i)$ , we let

$$\chi(t) := \mathbb{E}_Q \left( \exp \left\{ \frac{i \langle t, \langle x \rangle_{f,k/2} - \langle 0 \rangle_{f,k/2} \rangle}{\sigma_f \sqrt{\log Q}} \right\} \right),$$

where we recall that  $\sigma_f$  was defined in Theorem 2.3. We apply Theorem 3.1 with  $m = 1$  and  $d = 2$ . The hypothesis (3.7) is satisfied with  $\alpha_0 = 4$ , since the function  $\phi_f$  is continuous on  $[0,1]$  by Lemma 9.3 and therefore bounded. The hypothesis (3.8) is satisfied for any  $\lambda_0 < \frac{1}{2-2\varepsilon}$ , by using Lemma 9.3 and noting that  $\|\phi_f|_{h(I)}\|_{(1-\varepsilon)} \ll |h'(0)|^{-1+\varepsilon} \|\phi_f \circ h\|_{(1-\varepsilon)}$ . Using (3.11) along with the

expressions (7.11) and (7.12), we obtain for some  $\mu_f \in \mathbb{R}^2$  and real  $2 \times 2$  matrix  $\Sigma_f$  the estimate

$$\chi(\mathbf{t}) = \exp \left\{ i \langle \mathbf{t}, \mu_f \rangle - \frac{1}{2} \mathbf{t}^T \Sigma_f \mathbf{t} + O \left( \frac{\|\mathbf{t}\| + \|\mathbf{t}\|^3}{\sqrt{\log Q}} + \frac{1}{Q^\delta} \right) \right\}.$$

To compute the variance, we appeal to the bound

$$\mathbb{E}_Q(|\langle x \rangle_{f,m} - \mu_f \log Q|^4) \ll (\log Q)^2. \quad (9.6)$$

This can be proved by shifting to the setting of [4], where the variable  $t$  is extended to a complex neighbourhood of the origin; the functions  $U, V$  defined in (3.10) are, in this case, analytic in  $t$  near the origin, by boundedness of  $\phi_j$ . Then by, for example, [15, theorem 25.12] and (9.6), we find

$$\mu_f = \lim_{Q \rightarrow \infty} \frac{\mathbb{E}_Q(\langle x \rangle_{f,k/2})}{\log Q}, \quad \Sigma_f = \lim_{Q \rightarrow \infty} \frac{\mathbb{E}_Q(\langle x \rangle_{f,k/2} \langle x \rangle_{f,k/2}^T)}{\sigma_f^2 \log Q}.$$

On the other hand, as  $Q \rightarrow \infty$ , we have the following asymptotic formulae, where now  $\langle x \rangle_{f,k/2}$  is interpreted as a complex number:

$$\mathbb{E}_Q(\langle x \rangle_{f,m}) = o(\log Q), \quad (9.7)$$

$$\mathbb{E}_Q(\langle x \rangle_{f,m}^2) = o(\log Q), \quad (9.8)$$

$$\mathbb{E}_Q(|\langle x \rangle_{f,m}|^2) \sim 2\sigma_f^2 \log Q. \quad (9.9)$$

These statements can be proven (in a stronger form) by standard methods, using orthogonality of additive characters, the approximate functional equation [65, Theorem 5.3] and Rankin–Selberg theory [63, Chapter 13.6]. The value  $\sigma_f$  appears as  $\sigma_f^2 = \text{Res}_{s=1} L(f \times \bar{f}, 1)$ , which is evaluated in [63, equation (13.52)]. Note that the analogues of (9.8) and (9.9) with a single average over numerator have recently been computed in [18]; in their result as stated, however, the denominator is assumed to be prime.

The equality (9.7) shows that  $\mu_f = 0$ . The equality (9.8) shows that the matrix  $\Sigma_f$  is a multiple of the identity, and the equality (9.9) then shows that  $\Sigma_f = \text{Id}$ . Using the Berry–Esseen inequality, along with Lemma 9.1, concludes the proof of Theorem 2.3.

To justify Theorem 2.4, we use again the analyticity of  $U, V$  defined in (3.10) in a neighbourhood of the origin. Hypothesis (4) of [61] is therefore satisfied in our case with  $\phi(n)$  replaced by  $\log Q$ , and (2.3) follows by [61, Theorem 1], taking  $t = \varepsilon \sqrt{\log Q}$ .  $\square$

## 9.2 | Central value of the Estermann function

*Proof of Theorem 2.1.* The Estermann function  $D$  corresponds to  $D_0$  in notation of [8]. For  $x \in \mathbb{Q}$ , it is the analytic continuation of

$$D(s, x) = \sum_{n \geq 1} \frac{e(nx)\tau(n)}{n^s},$$



initially defined for  $\text{Re}(s) > 1$ , evaluated at  $s = \frac{1}{2}$ . We recall that  $\tau(n)$  is the number of divisors of  $n$ . We use [8, Lemma 10], noting that the quantity  $v_{j-1}/v_j$  corresponds to  $T^{j-1}(x)$ . Therefore

$$D\left(\frac{1}{2}, x\right) = \zeta\left(\frac{1}{2}\right)^2 + \sum_{j=1}^r \phi_j(T^{j-1}(x)),$$

where

$$\begin{aligned} \phi_j(x) &= \frac{1}{2}x^{-1/2} \left( \log(1/x) + \gamma_0 - \log(8\pi) - \frac{\pi}{2} \right) \\ &\quad + (-1)^{j-1} i \frac{1}{2} x^{-1/2} \left( \log(1/x) + \gamma_0 - \log(8\pi) + \frac{\pi}{2} \right) + \zeta\left(\frac{1}{2}\right)^2 + \mathcal{E}((-1)^j x), \end{aligned}$$

and  $\mathcal{E}$ , which corresponds to  $\mathcal{E}(0, \cdot)$  in the notation of [8, p. 6900], is bounded and continuous. By comparing the cases  $N = 0$  and  $N = 1$  of [8, equation (3.17)], we have

$$\mathcal{E}(x) = \mathcal{E}_1(x) + \sum_{j \in \{1,2\}} \frac{(-1)^j}{j\pi} \Gamma\left(\frac{1}{2} + j\right)^2 \left(\frac{x}{2\pi i}\right)^j \left( D\left(\frac{1}{2} + j, -1/x\right) + \zeta\left(\frac{1}{2} + j\right)^2 / j \right), \tag{9.10}$$

where  $\mathcal{E}_1 \in C^1([0, 1])$ . For  $j \geq 1$ , the function  $D(\frac{1}{2} + j, \cdot)$  belongs to  $H^{1/2-\varepsilon}([0, 1], \mathbb{C})$  for all  $\varepsilon \in (0, 1/2)$ . We deduce that the right-hand side of (9.10) defines, for all  $n \geq 1$ , a function in  $H^{1/2-\varepsilon}([\frac{1}{n+1}, \frac{1}{n}], \mathbb{C})$ . By an argument identical to Lemma 9.3, we also have the bound  $\|\mathcal{E} \circ h\|_{(1/2-\varepsilon)} \ll 1$  for all  $h \in \mathcal{H}^2$ , and similarly for  $x \mapsto \mathcal{E}(-x)$ . This validates the hypothesis (3.8) with any  $\lambda_0 < 1$ .

On the other hand, setting  $\alpha_0 = 2 - \varepsilon$ , we have

$$\sum_{h \in \mathcal{H}} |h'(0)| \|\phi_j|_{h(I)}\|_\infty^{\alpha_0} \ll \sum_{n \geq 1} n^{-2} |n^{1/2} \log(2n)|^{\alpha_0} < \infty,$$

and so the hypothesis (3.7) holds. We apply Theorem 3.1 with  $m = 2, d = 2$ , and the above given value  $\alpha = 2 - \varepsilon$ . In the estimate (3.11), we evaluate the integrals by appealing to [12, Corollary 3.1]. We deduce that there is a constant  $\mu \in \mathbb{C}$  such that, letting  $\sigma = 1/\pi$  and

$$\chi(t) := \mathbb{E}_Q \left( \exp \left\{ i \operatorname{Re} \left( t \frac{D(\frac{1}{2}, x) - \mu \log Q}{\sigma (\log Q)^{1/2} (\log \log Q)^{3/2}} \right) \right\} \right), \quad (t \in \mathbb{C}),$$

we have

$$\chi(t) = \exp \left\{ -\frac{|t|^2}{2} + O\left(\frac{1}{Q^\delta} + \frac{|t| + |t|^2}{(\log \log Q)^{1-\varepsilon}}\right) \right\}. \tag{9.11}$$

We then obtain, by the Berry–Essen inequalities and the bound (9.1) for small frequencies, the statement of Theorem 2.1 up to the value of the expectation. We compute the expectation from the initial object, using the expression [65, equation (3.2)] for Ramanujan sums. For  $s \in \mathbb{C}$  with  $\text{Re}(s) > 1$ , we have

$$\sum_{\substack{1 \leq a \leq q \\ (a,q)=1}} D(s, a/q) = \zeta(s)^2 \sum_{\ell|q} \mu\left(\frac{q}{\ell}\right) \frac{\tau(\ell)}{\ell^{s-1}} \prod_{p^v|\ell} \left(1 - \frac{v}{v+1} p^{-s}\right),$$

and so by analytic continuation

$$\sum_{\substack{1 \leq a \leq q \\ (a,q)=1}} D\left(\frac{1}{2}, a/q\right) = \zeta\left(\frac{1}{2}\right)^2 \sum_{\ell|q} \mu\left(\frac{q}{\ell}\right) \tau(\ell) \ell^{1/2} \prod_{p^v \parallel \ell} \left(1 - \frac{v}{v+1} p^{-1/2}\right) = O_\varepsilon(q^{1/2+\varepsilon}).$$

We deduce that

$$\mathbb{E}_Q(D(\tfrac{1}{2}, x)) \ll_\varepsilon Q^{-1/2+\varepsilon}. \quad (9.12)$$

On the other hand, by (9.11), we have

$$\chi(t) = 1 + O_\varepsilon(Q^{-\delta} + (\log \log Q)^{-1/2+\varepsilon} t + t^2),$$

whereas expanding the exponential as  $e^{iu} = 1 + iu + O(u^{3/2})$  in the definition of  $\chi(t)$ , by (9.12), we get

$$\chi(t) = 1 + it\mu(\log Q)^{1/2}(\log \log Q)^{-3/2} + O(Q^{-1/3} + |t|^{3/2}((\log Q)^{1/2} + \mathbb{E}_Q(|D(\tfrac{1}{2}, x)|^{3/2}))).$$

Using the trivial bound  $|D(\tfrac{1}{2}, x)| \ll \sum_{j=1}^{r(x)} a_j(x)^{2/3}$  and Hölder's inequality, we find

$$\begin{aligned} \mathbb{E}_Q(|D(\tfrac{1}{2}, x)|^{3/2}) &\ll \mathbb{E}_Q\left(\left(\sum_{j=1}^{r(x)} 1\right)^{1/2} \left(\sum_{j=1}^{r(x)} a_j(x)\right)\right) \\ &\ll (\log Q)^{5/2} \end{aligned}$$

by the bound  $r(x) \ll \log(\text{denom}(x) + 1)$  and [117, Theorem]. Setting  $t = Q^{-\delta/2}$ , we obtain

$$\mu(\log Q)^{1/2}(\log \log Q)^{-3/2} = O_\varepsilon((\log \log Q)^{-1/2+\varepsilon})$$

and so  $\mu = 0$ , by letting  $Q \rightarrow \infty$ . □

### 9.3 | Large moments of continued fractions expansions

For all  $\lambda \geq 0$  and  $x \in \mathbb{Q} \cap (0, 1)$ , we recall that  $\Sigma_\lambda(x)$  was defined in (2.4). For  $0 < \alpha < 2$ , define

$$c_\alpha = \left(\frac{\Gamma(1-\alpha) \cos(\frac{\pi\alpha}{2})}{\pi^2/12}\right)^{1/\alpha}$$

and by continuity  $c_1 = \frac{6}{\pi}$ . Let

$$g_\alpha(x) := \begin{cases} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx - (c_\alpha |t|)^\alpha (1 - i \operatorname{sgn}(t) \tan(\frac{\pi\alpha}{2}))} dt, & (\alpha \neq 1) \\ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx - c_1 |t| (1 + i \frac{2}{\pi} \operatorname{sgn}(t) \log |t|)} dt, & (\alpha = 1) \end{cases}$$

be the probability distribution function of a stable law  $S_\alpha(c_\alpha, 1, 0)$  (see [102]), and

$$G_\alpha(v) := \int_{-\infty}^v g_\alpha(x) dx.$$

**Theorem 9.4.** *Let  $\lambda \geq 0$  and  $v \in \mathbb{R}$ , and for  $\lambda < 1$  define  $\mu_\lambda = \frac{12}{\pi^2} \sum_{n \geq 1} n^\lambda \log\left(\frac{(n+1)^2}{n(n+2)}\right)$ .*

(1) *If  $\lambda < 1/2$ , then with for some  $\sigma_\lambda > 0$ , we have*

$$\mathbb{P}_Q\left(\frac{\Sigma_{1/2}(x) - \mu_\lambda \log Q}{\sigma_\lambda \sqrt{\log Q}} \leq v\right) = \Phi(v) + O_\varepsilon\left(\frac{1}{(\log Q)^{\min(1/2, 1/(2\lambda)-1-\varepsilon)}}\right). \tag{9.13}$$

(2) *If  $\lambda = 1/2$ , then with  $\sigma = (\pi^2/6)^{-1/2}$ , we have*

$$\mathbb{P}_Q\left(\frac{\Sigma_{1/2}(x) - \mu_{1/2} \log Q}{\sigma \sqrt{\log Q \log \log Q}} \leq v\right) = \Phi(v) + O\left(\frac{1}{(\log \log Q)^{1-\varepsilon}}\right). \tag{9.14}$$

(3) *If  $1/2 < \lambda < 1$ , then*

$$\mathbb{P}_Q\left(\frac{\Sigma_\lambda(x) - \mu_\lambda \log Q}{(\log Q)^\lambda} \leq v\right) = G_{1/\lambda}(v) + O\left(\frac{1}{(\log \log Q)^{1-\varepsilon}}\right). \tag{9.15}$$

(4) *If  $\lambda = 1$ , then letting  $\gamma_0$  denote the Euler constant,*

$$\mathbb{P}_Q\left(\frac{\Sigma_1(x)}{\log Q} - \frac{\log \log Q - \gamma_0}{\pi^2/12} \leq v\right) = G_1(v) + O\left(\frac{1}{(\log Q)^{1-\varepsilon}}\right). \tag{9.16}$$

(5) *If  $\lambda > 1$ , then*

$$\mathbb{P}_Q\left(\frac{\Sigma_\lambda(x)}{(\log Q)^\lambda} \leq v\right) = G_{1/\lambda}(v) + O\left(\frac{1}{(\log \log Q)^{1-\varepsilon}}\right). \tag{9.17}$$

*In all four cases, the implied constant depends at most on  $\varepsilon$  and  $\lambda$ .*

Except for (9.15), we expect the error terms to be optimal up to an exponent  $\varepsilon$ . The estimate (9.17) is in accordance with results on the statistical distribution of  $\max_{1 \leq j \leq r} a_j$  [28, 59].

To prove Theorem 9.4, we note that  $\Sigma_\lambda(x) = S_{\phi_\lambda}(x)$  with  $\phi_\lambda(x) := \lfloor 1/x \rfloor^\lambda$ . The function  $\phi_\lambda$  satisfies the hypotheses of Theorem 3.1 with  $d = m = 1$ ,  $\alpha_0 = 1/\lambda - \varepsilon$ ,  $\kappa_0 = 1$ , and all small enough exponents  $\lambda_0 > 0$ .

### 9.3.1 | Case $\lambda < 1/2$

We wish to apply Corollary 3.2. To proceed, we need to show that  $\phi_\lambda$  is not of the shape  $c \log + f - f \circ T$ . Suppose that it were, let  $n \geq 1$  and  $x \in (0, 1)$  solve  $x = \frac{1}{n+x}$ . Evaluating at  $x$  yields  $n^\lambda \sim -c \log n$  as  $n \rightarrow \infty$ , a contradiction. Corollary 3.2 may be applied and yields (9.13).

### 9.3.2 | Case $\lambda = 1/2$

The estimates (3.9)–(3.11) hold, and the integral is evaluated in [12, Corollary 3.2]. With the notation  $\sigma = (\pi^2/6)^{-1/2}$  and

$$\chi_{1/2,Q}(t) := \mathbb{E}_Q \left( \exp \left\{ it \frac{\Sigma_{1/2}(x) - \mu_{1/2} \log Q}{\sigma \sqrt{\log Q \log \log Q}} \right\} \right),$$

we find that for  $|t| \leq \log \log Q$ ,

$$\chi_{1/2,Q}(t) = \exp \left\{ -\frac{3t^2}{\pi^2} + O \left( \frac{1}{Q^\delta} + \frac{|t|}{(\log Q)^{1/2-\varepsilon}} + \frac{t^2}{(\log \log Q)^{1-\varepsilon}} \right) \right\}.$$

On the other hand, by (9.1), we have  $\chi_{1/2,Q}(t) = 1 + O(|t|^{1/2} \log Q)$ . Inserting these two bounds in the Berry–Esseen theorem [110, theorem II.7.16] yields the claimed conclusion (9.14).

### 9.3.3 | Case $\lambda = 1$

We use the estimates (3.9)–(3.10). Define

$$\chi_{1,Q}(t) := \mathbb{E}_Q \left( \exp \left\{ it \left( \frac{\Sigma_1(x)}{\log Q} - \frac{\log \log Q - \gamma_0}{\pi^2/12} \right) \right\} \right).$$

Then for  $0 < t \leq (\log Q)^{1-\varepsilon}$ , we obtain by [12, Corollary 3.3]

$$\chi_{1,Q}(t) = \exp \left\{ -\frac{it}{\pi^2/12} (|\log t| - \pi i) + O \left( \frac{1}{Q^\delta} + \frac{|t|^{1-\varepsilon} + |t|^{2-\varepsilon}}{(\log Q)^{1-\varepsilon}} \right) \right\},$$

and we may again conclude by the Berry–Esseen inequality.

### 9.3.4 | Case $\lambda \notin \{1/2, 1\}$

Assume first  $\lambda > 1$ . Then we use the estimates (3.9)–(3.10). Define

$$\chi_{\lambda,Q}(t) := \mathbb{E}_Q \left( \exp \left\{ it \frac{\Sigma_\lambda(x)}{(\log Q)^\lambda} \right\} \right).$$

Then, by [12, Corollary 3.2], for  $0 \leq t \leq \log \log Q$ , we obtain

$$\chi_{\lambda,Q}(t) = \exp \left\{ -(c_{1/\lambda} t)^{1/\lambda} (1 - i \tan(\frac{\pi}{2\lambda})) + O \left( \frac{1}{Q^\delta} + \frac{t^{1/\lambda-\varepsilon}}{(\log Q)^{1-\varepsilon}} + \frac{t^{1/\lambda}}{(\log \log Q)^{1-\varepsilon}} \right) \right\}$$

and we may again conclude by the Berry–Esseen inequality and Lemma 9.1.

The case  $\lambda \in (1/2, 1)$  follows by identical computations, the shift by  $\mu_\lambda \log Q$  being accounted for by the linear term in the asymptotic evaluation of (3.11), as performed in [12, Corollary 3.2]

## 9.4 | Dedekind sums

*Proof of Theorem 2.7.* By [60, Theorem 1], we have for  $x \in \mathbb{Q} \cap (0, 1)$  the equality

$$s(x) = \delta_x + \frac{1}{12} \sum_{j=1}^r \phi_j(T^{j-1}(x)),$$

where  $|\delta_x| \leq \frac{5}{12}$  and  $\phi_j(x) := (-1)^{j-1} \lfloor 1/x \rfloor$ . Note that  $\phi_j$  depends only on the parity of  $j$ . Since changing  $v$  by an amount  $O(1/\log Q)$  does not affect the right-hand side of (2.7), we may replace  $s(x)$  by  $s(x) - \delta_x$ . Let

$$\chi(t) := \mathbb{E}_Q \left( \exp \left\{ it \frac{s(x) - \delta_x}{\log Q} \right\} \right).$$

Then Theorem 3.1 applies with  $d = 1$ ,  $m = 2$  and the functions  $\phi_j$  defined above, with  $\alpha_0 = 1 - \varepsilon$ . We use the expression (3.10) and refer to [12, Corollary 3.4] for the evaluation of the integral, obtaining

$$\chi(t) = \exp \left\{ -\frac{|t|}{2\pi} + O \left( \frac{1}{Q^\delta} + \frac{|t| + |t|^{1-\varepsilon}}{(\log Q)^{1-\varepsilon}} \right) \right\},$$

and we conclude again by the Berry–Esseen inequality.  $\square$

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# Modularity and value distribution of quantum invariants of hyperbolic knots

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## Abstract

We obtain an exact modularity relation for the  $q$ -Pochhammer symbol. Using this formula, we show that Zagier's modularity conjecture for a knot  $K$  essentially reduces to the arithmeticity conjecture for  $K$ . In particular, we show that Zagier's conjecture holds for hyperbolic knots  $K \neq 7_2$  with at most seven crossings. For  $K = 4_1$ , we also prove a complementary reciprocity formula which allows us to prove a law of large numbers for the values of the colored Jones polynomials at roots of unity. We conjecture a similar formula holds for all knots and we show that this is the case if one assumes a suitable version of Zagier's conjecture.

**Mathematics Subject Classification** 11B65 (primary) · 57M27 · 11F03 · 60F05 (secondary)

## 1 Introduction

Among knot invariants, the colored Jones polynomials  $\{J_{K,n}\}_{n \geq 2}$  and the Kashaev invariants  $\{\langle K \rangle_N\}_{N \geq 2}$  are of particular interest, by their relation to quantum field theory, and the geometry of hyperbolic manifolds [25,26,44]. We refer to e.g. [31,46] for their definitions; by [31], the two invariants are related by  $\langle K \rangle_N = J_{K,N}(e^{2\pi i/N})$ . We refer to [14,24,25] for more results and references on this topic.

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The Kashaev invariant is extended to a function on roots of unity by setting, for  $(h, k) = 1$ ,  $\mathcal{J}_{K,0}(e^{2\pi ih/k}) := J_{K,k}(e^{2\pi ih/k})$ . For fixed  $k$ , the values  $(J_{K,0}(e^{2\pi ih/k}))_{(h,k)=1}$  are simply the Galois conjugates of  $\langle K \rangle_N$  in  $\mathbb{Q}(e^{2\pi i/k})$ . In the case of  $K = 4_1$ , the simplest hyperbolic knot, we have explicitly

$$\mathcal{J}_{4_1,0}(q) = \sum_{r=0}^{\infty} |(1 - q)(1 - q^2) \cdots (1 - q^r)|^2 \tag{1.1}$$

for a root of unity  $q$ .<sup>1</sup> In general,  $\mathcal{J}_{K,0}(q)$  can be written as a multiple series of this kind, with each addend being a ratio of  $q$ -Pochhammer symbols of various indexes. See Sect. 2.4 for some more examples and the precise definition of  $\mathcal{J}_{K,0}(q)$  in the cases we will consider.

The *volume conjecture* [26] predicts that for any hyperbolic knot  $K$ ,

$$\lim_{N \rightarrow \infty} \frac{\log |\langle K \rangle_N|}{N} = \frac{\text{Vol}(K)}{2\pi}, \tag{1.2}$$

where  $\text{Vol}(K)$  is the hyperbolic volume of the complement of  $K$ . This is motivated by the analogy between the usual dilogarithm, which measures volumes of tetrahedra in the hyperbolic space, and the quantum dilogarithm, which are the building blocks of Kashaev’s invariant. A generalization for non-hyperbolic knots was formulated in [31, section 5], with the volume  $\text{Vol}(K)$  being replaced by a suitable multiple of the Gromov simplicial volumes of the complement of  $K$ .

This conjecture was extended in [22] and implicitly in [18] to a full asymptotic expansion, referred to as the *arithmeticity conjecture* in [12], whereas the corresponding question for the imaginary part of the logarithm is conjectured to involve the Chern-Simons invariant  $\text{cs}(K)$  of  $K$  [22,32]. The arithmeticity conjecture has been proved for all knots with up to seven crossings in [2,35–37]. We refer to [16,27,30,33,34] and the references therein for more results and information on the volume conjecture.

In [47], Zagier studies several examples of what is called “quantum modular forms”. Motivated by extensive numerical computations, he predicts that  $\mathcal{J}_{K,0}$  satisfies an approximate modularity property which relates, in the limit as  $x \rightarrow \infty$  among rationals of bounded denominator,  $\mathcal{J}_{K,0}(e^{2\pi i(ax+b)/(cx+d)})$  with  $\mathcal{J}_{K,0}(x)$  for any  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ . The constants involved are also considered from an algebraic point of view, and are expected to belong to simple algebraic extensions of the invariant trace field  $F_K$  of the knot (see e.g. [29, Chapter 3] for the definition). More specifically, given a hyperbolic knot  $K$ , the following conjecture is made (cf. also [19] and [15] where the coefficients in the series are analysed in detail).

**Conjecture 1** (Zagier’s modularity conjecture for  $K$ ) *For all  $\gamma \in \text{SL}_2(\mathbb{Z})$  such that<sup>2</sup>  $\alpha := \gamma(\infty) \in \mathbb{Q}$ , there exist  $C_K(\alpha) \in \mathbb{C}$  and a sequence  $(D_{K,n}(\alpha))_{n \geq 0}$  of*

<sup>1</sup> Notice that the series is in fact a finite sum at roots of unities.

<sup>2</sup> In what follows, a matrix in  $\text{SL}(2, \mathbb{R})$  acts on  $\mathbb{C}$  by homography.

complex numbers such that, for all  $M \in \mathbb{N}$  and  $x \in \mathbb{Q}$ , with  $x \rightarrow \infty$ , there holds

$$\frac{\mathcal{J}_{K,0}(e(\gamma(x)))}{\mathcal{J}_{K,0}(e(x))} = \left(\frac{2\pi}{\hbar}\right)^{3/2} e^{i\frac{\text{Vol}(K)-i\text{cs}(K)}{\hbar}} C_K(\alpha) \left( \sum_{0 \leq n < M} D_{K,n}(\alpha) \hbar^n + O(\hbar^M) \right),$$

$$\hbar := \frac{2\pi i}{x - \gamma^{-1}(\infty)}, \quad (1.3)$$

where  $e(x) := e^{2\pi i x}$  and the implied constant depends at most on  $\alpha$ , on the denominator of  $x$  and on  $M$ . Moreover, if  $F_{K,\alpha} := F_K(e(\alpha))$ , then:

- $C_K(\alpha)$  is a product of rational powers of elements of  $F_{K,\alpha}$ ;
- $D_{K,n}(\alpha) \in F_{K,\alpha}$  for  $n \geq 0$ .

In the case  $K = 4_1$ , Garoufalidis and Zagier [20] announced a proof of Theorem 1, and also numerically investigated the conjecture for other knots. The case of the  $4_1$  knot is special and rather simpler than that of other knots, due to the fact that in this case all the summands in the definition (1.1) of  $\mathcal{J}_{4_1,0}$  are positive. One can then use Laplace's method to extract the asymptotic expansion (1.3). In general, this positivity is not present and there is a remarkable amount of cancellation between the terms of the series. Indeed,  $\mathcal{J}_{K,0}(e(1/N))$  is typically exponentially smaller than the largest summands in its definition, and this prevents one from applying a direct estimation based on Laplace's method. We circumvent this serious obstacle by obtaining a new modularity relation, with a precise description of the holomorphic and periodic behaviour of the error terms, for the  $q$ -Pochhammer symbol. This symbol is of crucial importance in the theory of  $q$ -series and often appears in the theory of modular forms and combinatorics (see for example [7] and references therein). For  $r \in \mathbb{Z}_{\geq 0}$ , it is defined as

$$(q)_r := \prod_{j=1}^r (1 - q^j), \quad q \in \mathbb{C}. \quad (1.4)$$

When  $|q| < 1$ , one can also take  $r = +\infty$  and obtain the Dedekind  $\eta$ -function, an important example of a (half-integral weight) modular form defined as  $\eta(z) := e(z/24)(e(z))_\infty$ , with  $e(z) := e^{2\pi i z}$ . As such,  $\eta$  satisfies the relation

$$\eta(\gamma z) = \chi(\gamma)(cz + d)^{\frac{1}{2}} \eta(z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}),$$

for a certain "multiplier system"  $\chi$  (see [23, Section 2.8]). This modularity relation can be naturally extended to the partial product at root of unities. Indeed, denoting with  $\text{den}(\alpha)$  the reduced denominator of  $\alpha \in \mathbb{Q}$ , in Theorem 6 below we show that for  $\alpha \in \mathbb{Q}$ ,  $1 \leq r < \text{den}(\gamma\alpha)$  we have

$$e\left(\frac{\gamma\alpha}{24}\right) (e(\gamma\alpha))_r = \chi(\gamma) e\left(\frac{\alpha}{24}\right) (e(\alpha))_{r'} \psi_\gamma(\alpha, r) \quad (1.5)$$

for some  $1 \leq r' < \text{den}(\alpha)$  and where  $\psi_\gamma(\alpha, r)$  is an explicit function with suitable holomorphicity properties. We refer to Sect. 2 for the precise formulation of this reciprocity formula which we believe to be of independent interest. With this new tool,

$K$	$4_1$	$5_1$	$6_1$	$6_2$	$6_3$	$7_3$	$7_4$	$7_5$	$7_6$	$7_7$
$\nu_K$	0	1	2	-2	0	1	-3	-1	-1	-1

**Fig. 1** Values of  $\nu_K$

we can reduce Zagier’s modularity conjecture to a slightly modified form of the arithmeticity conjecture, thus showing that the two conjectures are “morally equivalent”. In particular, we are able to prove the conjecture for all hyperbolic knots  $K \neq 7_2$  with at most seven crossings, since for these the arithmeticity conjecture is known by works of Andersen and Hansen [2] (in the case  $K = 4_1$ ), Ohtsuki [35,36] (in the case  $K = 5_2$  and with 7 crossings) and Ohtsuki and Yokota [37] (in the case of 6 crossings). We recall that the fields  $F_K, F_{K,\alpha}$  were defined just before and in Conjecture 1.

**Theorem 1** *Let  $K \neq 7_2$  be a hyperbolic knot with at most 7 crossings. Then Conjecture 1 holds for  $K$ . The constant  $C_K(\alpha)$  has the shape*

$$C_K(\alpha) = e\left(\frac{\nu_K}{2} s(\alpha)\right) c^{\nu_K/2} \Lambda_{K,\alpha}^{1/c} \delta_K^{-1/2}, \tag{1.6}$$

where  $c$  is the denominator of  $\alpha$ ,  $\Lambda_{K,\alpha} \in F_{K,\alpha}$ ,  $\nu_K \in \mathbb{Z}$ , and  $\delta_K \in F_K$ ,  $\nu_K$  is given in Fig. 1 above and  $s(\alpha)$  is the Dedekind sum (see (2.4)).

**Remark 1** By the works [35–38], the number  $\pm 2i\delta_K^{-1}$  can be interpreted as the conjugate of a twisted Reidemeister torsion of  $K$ . Our method gives the constant term  $C_K \cdot D_{K,0}$  as an explicit product of algebraic numbers; in Remark 6 in Sect. 3 we give as examples its value in the cases of  $K = 4_1$  and  $K = 5_1$ .

**Remark 2** Recently, Calegari, Garoufalidis and Zagier [12] made a more precise conjecture on  $C_K(\alpha)$ , predicting it naturally factors as  $\mu_{K,8c} \cdot \varepsilon_K(\alpha)^{1/c} / \sqrt{\delta_K}$ , where  $c$  is the denominator of  $\alpha$ ,  $\mu_{K,8c}$  is a  $8c$  root of unity,  $\varepsilon_K(\alpha)$  is a unit of  $F_{K,\alpha}$  and  $\delta_K \in F_K$ . We do not at present have such a precise description of  $C_K(\alpha)$ . This would presumably require a fine understanding of the congruence sums (3.7).

By the work of Ohtsuki [36], the arithmeticity conjecture is known also for  $K = 7_2$  and we expect that our method would give Theorem 1 also in this case. However, the proof of this case is more involved, so we decided to exclude this case for simplicity. In any case, we want to stress again that the scope of our work is more general and suggests that any proof of the arithmeticity conjecture should be adaptable into a proof of the modularity conjecture via the use of the reciprocity relation (1.5).

The modularity and the volume conjectures likely don’t give the full picture on the symmetries of  $\mathcal{J}_{K,0}$  nor on its values at roots of unity. Indeed, in the case of  $K = 4_1$  we can show that Theorem 1 can be complemented by a second reciprocity formula relating  $X = \frac{h}{k}$  with  $X' = \frac{\bar{h}}{k}$ , where the overline indicates the multiplicative inverse modulo the denominator. This new reciprocity formula involves the “cotangent sum” (which appears also in the main term in the variation, effective for other ranges of the parameters, of the reciprocity formula given in Theorem 8 below)

$$c_0(h/k) := - \sum_{m=1}^{k-1} \frac{m}{k} \cot\left(\frac{\pi mh}{k}\right), \quad (h, k) = 1, \quad k \geq 1,$$

which is itself a quantum modular form [9] and has been widely studied due to its connection to the Báez–Duarte–Nyman–Beurling criterion for the Riemann hypothesis (see, for example, [3–5,43]).

**Theorem 2** *Let  $1 \leq h \leq k$  with  $(h, k) = 1$ . Then*

$$\frac{\mathcal{J}_{4_1,0}(e(\bar{h}/k))}{\mathcal{J}_{4_1,0}(e(\bar{k}/h))} = \exp\left(\frac{\text{Vol}(4_1)}{2\pi} \frac{k}{h} + \mathcal{E}(h, k)\right), \tag{1.7}$$

where

$$\mathcal{E}(h, k) = O\left(\frac{1}{k} \max_{0 \leq r' < h} \left| \sum_{1 \leq n \leq r'} \cot\left(\pi \frac{n\bar{k}}{h}\right) \frac{n}{h} \right| + \frac{1}{h} \left| c_0\left(\frac{\bar{k}}{h}\right) \right| + \log \frac{k}{h} + \frac{k}{h^2}\right). \tag{1.8}$$

This may be compared with (1.3). Note that  $\text{cs}(4_1) = 0$  (see e.g. [13]), so there is no corresponding contribution on the right-hand side of (1.7).

In this case as well, the reciprocity formula (1.7) stems from a corresponding relation for the  $q$ -Pochhammer symbol (see Theorem 7 below). Notice that, despite not giving a full asymptotic expansion, (1.7) is completely uniform. In particular, it permits to be successfully iterated for “typical” roots of unity, allowing us to deduce the following law of large numbers for  $\log \mathcal{J}_{4_1,0}$ .

**Theorem 3** *For  $\alpha \in \mathbb{Q} \cap (0, 1)$  with simple continued fraction expansion*

$$\alpha = [0; b_1, \dots, b_r] = \frac{1}{b_1 + \frac{1}{b_2 + \dots}},$$

where  $b_1, \dots, b_{r-1} \geq 1$  and  $b_r > 1$ , and let

$$r(\alpha) := r, \quad \Sigma(\alpha) := \sum_{\ell=1}^r b_\ell.$$

There is a function  $\varepsilon_1 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $\varepsilon_1(x) \rightarrow 0$  as  $x \rightarrow 0$ , such that for all  $\alpha \in \mathbb{Q} \cap (0, 1)$ ,

$$\log \mathcal{J}_{4_1,0}(e(\alpha)) = (1 + O(\varepsilon_1(r(\alpha)/\Sigma(\alpha)))) \frac{\text{Vol}(4_1)}{2\pi} \Sigma(\alpha). \tag{1.9}$$

In particular, there is a function  $\varepsilon_2 : \mathbb{N} \rightarrow \mathbb{R}_+$  with  $\varepsilon_2(N) \rightarrow 0$  as  $N \rightarrow \infty$ , such that for all roots of unity  $q$  of order  $n \leq N$ , with at most  $\varepsilon_2(N)N^2$  exceptions, one has

$$\log \mathcal{J}_{4_1,0}(q) = (1 + O(\varepsilon_2(N))) \frac{12}{\pi^2} \frac{\text{Vol}(4_1)}{2\pi} \log n \log \log n \tag{1.10}$$

as  $N \rightarrow \infty$ .



Equation (1.10) can be seen as a version of the volume conjecture (1.2) for typical roots of unity. Indeed, the volume conjecture provides the asymptotic behavior of  $\log \mathcal{J}_{4_1,0}$  at the root of unity  $e(1/N)$ , whereas our result gives the asymptotic for almost all roots of unity of denominator  $\leq N$ . Notice then in both cases the leading constant involves the hyperbolic volume  $\text{Vol}(4_1)$ , but the size of  $\mathcal{J}_{4_1,0}$  changes dramatically.

Equation (1.9) is stronger than (1.10), which will be readily deduced by [10], and gives an asymptotic formula in most cases, e.g. when  $\alpha$  is restricted to rational numbers with bounded  $r(\alpha)$  as the denominator of  $\alpha$  goes to infinity. In particular, it generalises the volume conjecture, which corresponds to the case  $\alpha = 1/N = [0; N]$ . It is very likely that the assumption  $\Sigma(\alpha)/r(\alpha) \rightarrow \infty$  cannot be removed in general. Indeed, if for example  $\alpha_n = F_{n-1}/F_n$  with  $F_n$  the  $n$ -th Fibonacci number so that  $\Sigma(\alpha_n) = r(\alpha_n) + 1 = n - 1$ , then Theorem 1.9 would give  $\log \mathcal{J}_{4_1,0}(e(\alpha_n)) \sim Cn$ , with  $C = \frac{\text{Vol}(4_1)}{2\pi} \approx 0.323 \dots$ , whereas numerically it appears that  $F(\alpha_n)$  grows like  $C'n$ , for  $C' \approx 1.1$  (cf. also [47, Figure 6]).

Our proof of Theorems 3 depends crucially on the positivity of the summands in (1.1) which is missing if  $K \neq 4_1$ . Nonetheless, we expect a similar result holds for all hyperbolic knots. Also, since  $\Sigma(\alpha)$  is distributed according to a stable law [10], we expect the same to hold for  $\log |\mathcal{J}_{K,0}(e(\alpha))|$  for any hyperbolic knot  $K$ . Stable distributions are attractors for independent and identically distributed random variables with variance non necessarily finite. In our case, the relevant stable law is  $S_1(\frac{6}{\pi}, 1, 0)$ , with density function  $c \mapsto f_1(c; \frac{6}{\pi}, 1, 0) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itc} e^{-\frac{6}{\pi}|t| + \frac{12}{\pi^2}t \log|t|} dt$  [41].

**Conjecture 2** *Let  $K$  be a hyperbolic knot. There exists a constant  $D_K \in \mathbb{R}$  such that for any interval  $[a, b] \subset \mathbb{R}$  there holds*

$$|Q_N|^{-1} \left| \left\{ q \in Q_N \left| \left( \frac{\log |\mathcal{J}_{K,0}(q)|}{\frac{\text{Vol}(K)}{2\pi} \log N} - \frac{12}{\pi^2} \log \log N - D_K \right) \in [a, b] \right. \right\} \right| = \int_a^b f_1(x; \frac{6}{\pi}, 1, 0) dx + o(1) \tag{1.11}$$

as  $N \rightarrow \infty$ , where  $Q_N$  is the set of roots of unity of order  $\leq N$ . In particular, there exists a function  $\varepsilon : \mathbb{N} \rightarrow \mathbb{R}_+$  with  $\varepsilon(N) \rightarrow 0$  as  $N \rightarrow \infty$ , such that

$$\log |\mathcal{J}_{K,0}(q)| = (1 + O(\varepsilon(N))) \frac{12}{\pi^2} \frac{\text{Vol}(K)}{2\pi} \log n \log \log n$$

for all roots of unity  $q$  of order  $n \leq N$ , with at most  $\varepsilon(N)N^2$  exceptions.

In [47] Zagier discusses the continuity with respect to the real topology of

$$H_{4_1}(h/k) := \log |\mathcal{J}_{4_1,0}(e(h/k))| - \log |\mathcal{J}_{4_1,0}(e(k/h))|$$

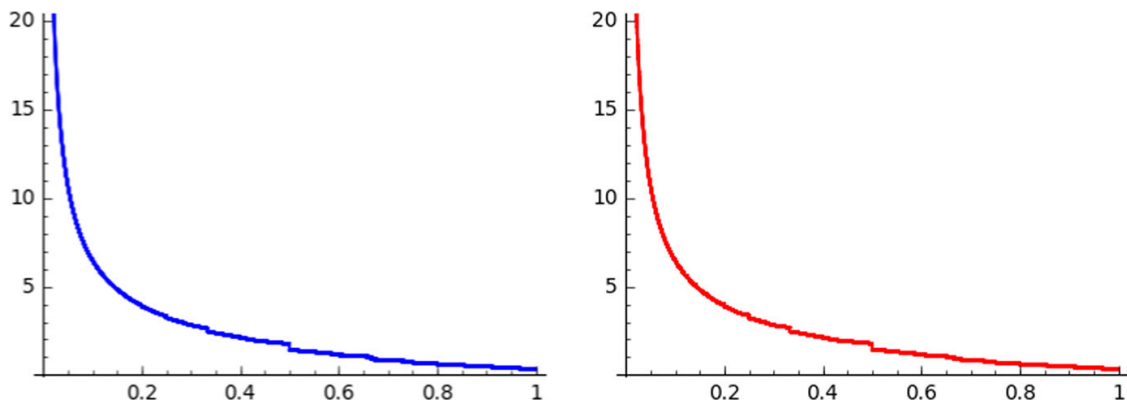


Fig. 2 Global graph of  $H_{4_1}$  and  $H_{4_1}^*$

and suggests that  $H_{4_1}$  is discontinuous but  $C^\infty$  from the right and the left at non-zero rationals<sup>3</sup> and continuous but not differentiable as one approaches irrational numbers. Using Lebesgue’s integrability condition and [10], one can easily show that this continuity condition together with a suitable continuity condition at zero implies Conjecture 2.

**Theorem 4** *Let  $K$  be a hyperbolic knot. Assume the following:*

- $H_K(h/k) := \log |\mathcal{J}_{K,0}(e(h/k))| - \log |\mathcal{J}_{K,0}(e(k/h))|$  has a limit as  $h/k$  tends to any positive irrational number;
- $H_K(h/k) - \frac{\text{Vol}(K)}{2\pi} \frac{k}{h} - \frac{3}{2} \log(k/h)$  is uniformly bounded.

Then Conjecture 2 holds with  $D_K = \frac{1-\gamma_0-\log 2}{\pi^2/12} + \frac{24}{\pi \text{Vol}(K)} \int_0^1 \frac{H_K(1/t) - \frac{\text{Vol}(K)}{2\pi t}}{1+t} dt$ , the function  $H_K$  being extended to  $\mathbb{R}_{>0}$  by taking limits over the rationals.

**Remark 3** One could replace the second assumption in Theorem 4 with the assumption that  $H_K(h/k) - \frac{\text{Vol}(K)}{2\pi} \frac{k}{h} - \frac{3}{2} \log(k/h)$  is left and right continuous as  $h/k$  approaches any rational number.

In the case of torus knots, the invariant  $\mathcal{J}_{K,0}$  can still be constructed and a formula of type (1.3) is expected to hold with  $\text{Vol}(K)$  replaced by 0. In this situation, the works [6,10] would suggest that  $\frac{\log |\mathcal{J}_{K,0}(q)|}{\sqrt{\log n}}$  becomes distributed according to a Gaussian law. In this case however, the conditions of Theorem 4 are not sufficient to conclude.

In view of Theorem 2, it is natural to wonder if also the function  $H_K^*(h/k) := \log |\mathcal{J}_{K,0}(e(\bar{h}/k))| - \log |\mathcal{J}_{K,0}(e(\bar{k}/h))|$  could be regular at irrational points (cf. Figs. 2, 3). We can answer this question in the negative in the case of  $K = 4_1$ .

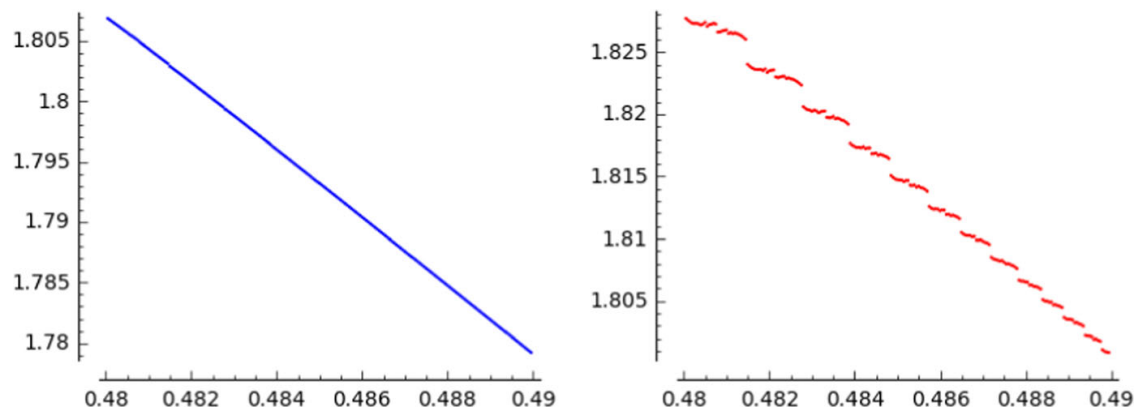
**Theorem 5** *For all  $x \in [0, 1]$  we have  $\limsup_{y \rightarrow x^\pm, y \in \mathbb{Q}} |H_{4_1}^*(y)| = +\infty$ .*

### 1.1 Outline of the paper and sketch of the proofs

Theorems 1 and 2 are both based on two new relations for the  $q$ -Pochhammer symbol, given in Theorem 6 (cf. (1.5)) and 7. These relations are proved in Sect. 2 and both

<sup>3</sup> This continuity property at rationals follows from the modularity conjecture but only when approaching a rational  $h/k = [0; b_1, \dots, b_r]$  with fractions essentially of the form  $[0; b_1, \dots, b_r, N]$  with  $N \rightarrow \infty$ ; and not, for example, with  $[0; b_1, \dots, b_r, N_1, N_2]$  with both  $N_1, N_2 \in \mathbb{N}$  going to infinity.





**Fig. 3** Graph of  $H_{4_1}$  and  $H_{4_1}^*$  around  $x = 0.485$

make use of the Abel–Plana summation formula [1, p.23], [40, p.408], [39, Chapter 8.3.1], which is a form of Euler–Maclaurin summation with an explicit form of the error term. In the case of Theorem 6, one starts by dividing the product in the definition (1.4) of  $(e(\gamma\alpha))_r$  into appropriate intervals and congruence classes. One then take the logarithm and apply the summation formula to the resulting sum of the function  $\log(1 - e(z))$ . As this function is close to a primitive of  $\pi \cot(\pi z)$ , which has poles at integers, then through a residue computation one eventually arrives to the dual object  $(e(\alpha))_{r'}$ . In the case of Theorem 7, the reciprocity relation for the  $q$ -Pochhammer symbol relates  $\frac{\bar{h}}{k}$  and  $\frac{\bar{k}}{h}$ . In this case one starts by applying the simple relation  $\frac{\bar{h}}{k} \equiv -\frac{\bar{k}}{h} + \frac{1}{hk} \pmod{1}$ . After some initial manipulations, one is lead to consider sums of the function  $\log(1 - \cot(\pi\xi) \tan(\pi\xi z)) + \cot(\pi\xi) \tan(\pi\xi z)$  for some choices of  $\xi \in (0, 1/2)$ . This is again performed using Abel–Plana summation formula followed by a careful analysis, with particular care needed in the reassembling of various main terms.

In both our reciprocity relations for the  $q$ -Pochhammer symbol, we show that the error terms extend to holomorphic functions of controlled growth. This is crucial for the applications to the modularity relations for the Kashaev invariant for knots other than  $4_1$ .

Once the reciprocity formulas for the  $q$ -Pochhammer symbol are established, the proofs of Theorems 1 and 2, given in Sects. 3 and 4 respectively, follow in similar ways. We first split the sums in the definition of the Kashaev invariant into congruence classes (and suitable intervals) and apply the reciprocity relations, reducing the problem to that of estimating certain sums of exponentials of linear combinations of dilogarithms. These sums are very similar to the ones one needs to consider for the the volume conjecture with only two relevant differences: the variables of summation range over some convex space rather than some larger cubic regions, and inside the exponential we have also some new error terms. For all the knots we consider the first difference is easily treated since, as shown in Lemma 10, the neglected terms are much smaller than the main terms (for other knots, such as  $7_2$ , this is no longer true and some of these neglected terms could be large, however we expect that a treatment as in [36, Section 8] could be used to show that the sum of these terms is in fact still small). In the case of Theorem 1, the second difference is also surpassed thanks the holomorphicity of the

error terms mentioned above, since the complex analytic methods of [35–37], using Poisson summation and the saddle-point method, go through essentially unchanged. In the case of Theorem 2, while we still have the holomorphicity of the error terms, the fact that the errors are not  $o(1)$  forces us to use positivity to avoid possible cancellations in the main terms, thus restricting the applicability to the  $4_1$  knot only.

Theorems 3, 4 and 5 are proved in Sect. 5 and all use the reciprocity formulas (1.3) and (1.7) (the latter being more crucial) in conjunction with the recent work [10] on the distribution of  $\Sigma(\alpha)$ . The difference between the reciprocity relations (1.3) and (1.7) can be better understood in terms of the continued fraction expansions  $[0; b_1, \dots, b_m]$  of  $h/k$  (for simplicity we assume  $m$  odd). Indeed, (1.3) relates the values of  $\mathcal{J}_{4_1,0}$  at  $e([0; b_1, \dots, b_m])$  and at  $e([0; b_\ell, \dots, b_m])$  provided that  $b_\ell \rightarrow \infty$  for some  $\ell \in \{1, \dots, m\}$  with all the other  $b_i$  bounded, whereas (1.7) relates the values of  $\mathcal{J}_{4_1,0}$  at  $e([0; b_1, \dots, b_m])$  and at  $e([0; b_1, \dots, b_{m-1}])$  provided that  $b_m \rightarrow \infty$  and that the other  $b_i$  are not too large (for example  $\log(b_i) = o(b_m)$  for all  $i < m$  would suffice). Because of its uniformity, (1.7) can be successfully iterated removing each time the last convergent  $b_m$  from  $\mathcal{J}_{4_1,0}(e([0; b_1, \dots, b_m]))$ . We keep doing so until we reach the last step for which we need to apply (1.3). In this process, we pick up a main term of  $\frac{\text{Vol}(4_1)}{2\pi} b_m$  at each step and thus arrive to (1.9). Equation (1.10) then follows by the law of large numbers for  $\Sigma(\alpha)$  established in [10].

Theorem 4 follows a similar line, with the difference that in this case the previous argument and Conjecture 2 give that  $\log \mathcal{J}_{4_1,0}(e(\alpha)) - \frac{\text{Vol}(4_1)}{2\pi} \Sigma(\alpha)$  can be well approximated by a differentiable function. The theorem then follows invoking again [10].

Finally, Theorem 5 follows via a simple argument from Theorem 8, a version of Theorem 2 which becomes useful when there is a middle partial quotient which is extremely large.

## Notation

Given  $D \subset \mathbb{R}$  and  $f : D \rightarrow \mathbb{R}$ , we write  $\|f\|_{\infty, D} := \sup_{t \in D} |f(t)|$ . Also, given  $t \in \mathbb{R}$ , we write  $\|t\| := \text{dist}(t, \mathbb{Z})$  and  $\{t\} := t - \lfloor t \rfloor$ , where  $\lfloor t \rfloor$  is the integer part of  $t$ . Given a property  $P$ , we define  $\mathbf{1}_P$  (or  $\mathbf{1}(P)$ ) to be 1 if the property  $P$  is satisfied and 0 otherwise.

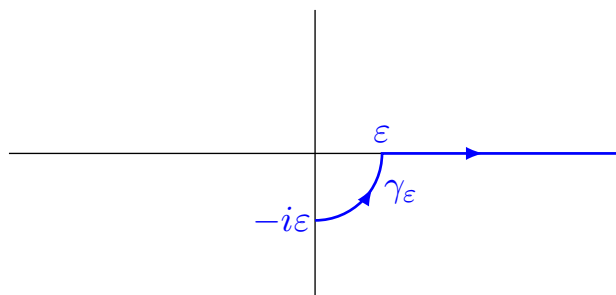
We use throughout the Landau notation  $f = O(g)$  to mean that there is an implicit constant  $C > 0$  such that  $|f| \leq Cg$  in the common domain of  $f$  and  $g$ . We also write  $f \ll g$  with the same meaning. All the implicit constants of the error terms are understood to be uniform in the various parameters unless otherwise indicated.

## 2 Two reciprocity formulae for the $q$ -Pochhammer symbol

### 2.1 Abel–Plana’s summation formula

Our argument is based on the Abel–Plana summation formula.

We denote by  $\gamma_\varepsilon$  the following integration contour.



**Lemma 1** Let  $\alpha, \beta, \beta' \in \mathbb{R}$  with  $\alpha \leq \beta < \beta'$ . Let  $f$  be an analytic function on a neighborhood of  $U := \{z \in \mathbb{C} \mid \alpha \leq \operatorname{Re}(z) \leq \beta'\} \setminus \{\alpha, \beta\}$ . Assume that the following holds:

- (1)  $f(z)$  is holomorphic at  $\beta$  if  $\beta$  is an integer, and otherwise  $f(z) = o(|z - \beta|^{-1})$  as  $z \rightarrow \beta$  with  $z \in U$ ,
- (2)  $f(x \pm iy) = o(e^{2\pi y}/y^2)$  as  $y \rightarrow +\infty$ , uniformly in  $x \in [\alpha, \beta]$ ,
- (3)  $f$  is integrable on  $(\alpha, \beta)$ .

Then we have

$$\sum_{\alpha < n \leq \beta} f(n) = \int_{\alpha}^{\beta} f(t)dt - C(f, \alpha) + C(f, \beta), \tag{2.1}$$

where

$$C(f, \alpha) := \lim_{\epsilon \rightarrow 0^+} \left( -i \int_{\gamma_{\epsilon}} \frac{f(\alpha + it)dt}{e(-\alpha)e^{2\pi t} - 1} + i \int_{\gamma_{\epsilon}} \frac{f(\alpha - it)dt}{e(\alpha)e^{2\pi t} - 1} \right).$$

**Proof** The arguments in [39, Chapter 8, eq. (3.01)] are readily adapted. □

For  $k \in \mathbb{Z}_{\geq 0}, k \neq 1$ , let  $\tilde{B}_k(t) = B_k(\{t\})$  where  $\{t\}$  is the fractional part of  $t$  and  $B_k$  is the  $k$ -th Bernoulli polynomial, and let  $\tilde{B}_1(t) = B_1(\{t\})$  for  $t \notin \mathbb{Z}$  and  $\tilde{B}_1(n) = 0$  for  $n \in \mathbb{Z}$ . We require the following computation.

**Lemma 2** For  $\ell \in \mathbb{Z}_{\geq 0}$  and  $v \in [0, 1)$ , we have

$$\int_0^{\infty} \operatorname{Im} \left( \frac{(-it)^{\ell}}{e(v)e^{2\pi t} - 1} \right) dt = \frac{(-1)^{\ell} \tilde{B}_{\ell+1}(v)}{2(\ell + 1)}.$$

Moreover, for all  $\omega \in \mathbb{C} \setminus (-\infty, 1]$ , we have

$$\int_0^{\infty} \frac{dt}{\omega e^{2\pi t} - 1} = \frac{-1}{2\pi} \log(1 - \omega^{-1})$$

with  $\log$  being the principal determination.

**Proof** The second claim is easy to prove by expanding the fraction as a power series in  $\omega$ , first for  $\omega \in (1, \infty]$ , and then by analytic continuation. To show the first claim,

first we note that for  $\ell = 0, v = 0$ , the fraction is a real number, and both sides evaluate to 0. We may therefore assume that  $\ell \geq 1$  or  $v \neq 0$ . Then

$$\begin{aligned} \int_0^\infty \operatorname{Im} \left( \frac{(-it)^\ell}{e(v)e^{2\pi t} - 1} \right) dt &= \frac{1}{(2\pi)^{\ell+1}} \operatorname{Im} \left( e(-v) (-i)^\ell \int_0^\infty \frac{t^\ell e^{-t} dt}{1 - e(-v)e^{-t}} \right) \\ &= \frac{\ell!}{(2\pi)^{\ell+1}} \operatorname{Im} \left( (-i)^\ell \sum_{n \geq 1} \frac{e(-nv)}{n^{\ell+1}} \right) \end{aligned}$$

by [21, eq. (3.411.6)]. We write  $\operatorname{Im}((-i)^\ell e(-nv)) = (-1)^{\ell+1} \cos(2\pi nv - \frac{\pi}{2}(\ell+1))$ , and conclude by the Fourier expansion of Bernoulli polynomials [21, eq. (9.622.1)]. □

### 2.2 First reciprocity formula for the $q$ -Pochhammer symbol

We fix the notations as follows. Let  $N, d \geq 1$  be coprime. Let  $\alpha = p/q$  in reduced form, and  $\gamma = \left(\frac{p \ \bar{q}}{q \ \bar{p}}\right)$  for some  $(\bar{p}, \bar{q})$  satisfying  $p\bar{p} + q\bar{q} = 1$ . Throughout the rest of Sect. 2.2, all error terms will be allowed to depend on  $d$  and  $\gamma$  (but not on  $N$ ). We write  $h = Np - d\bar{q}, k = Nq + d\bar{p}$ , so that

$$x = \frac{N}{d}, \quad \gamma(x) = \frac{Np - d\bar{q}}{Nq + d\bar{p}} = \frac{h}{k}$$

and notice that this implies that  $(h, k) = 1$  (since  $(h, k)$  divides  $pk - qh = d$  and  $(d, (h, k)) = (p, q) = 1$ ) and

$$\frac{h}{k} = \frac{p}{q} - \frac{d}{kq}, \tag{2.2}$$

$$\frac{d}{kq} + \frac{k}{dq} = \frac{N}{d} - \frac{h}{k} + \frac{p + \bar{p}}{q}. \tag{2.3}$$

We also recall from [23, p. 45] that the Dedekind sum  $s(\alpha)$  is defined by

$$s(\alpha) = s(p, q) := \sum_{n=1}^{q-1} \frac{n}{q} \left( \left( \frac{pn}{q} \right) \right), \quad \text{where } ((x)) = \{x\} - \frac{1}{2}, \tag{2.4}$$

For  $z \in (\mathbb{C} \setminus \mathbb{R}) \cup (0, 1)$ , we let

$$f(z) := \log(1 - e(z)) \tag{2.5}$$

taking the determination which is real on the positive imaginary axis. Notice that with this choice we have

$$f(z) = f(1 - z) + \pi i(2z - 1); \quad (2.6)$$

$$f(1 + z) = f(z), \text{ if } \operatorname{Im}(z) > 0; \quad f(1 + z) - f(z) = 2\pi i, \text{ if } \operatorname{Im}(z) < 0; \quad (2.7)$$

$$f(z) = \log(2 \sin(\pi z)) + i\pi(z - \frac{1}{2}) \text{ if } z \in (0, 1). \quad (2.8)$$

Moreover, if  $t \in (0, 1)$ , expanding the logarithm in its Taylor series

$$\sum_{g=1}^q f\left(\frac{g-t}{q}\right) = -\sum_{g=1}^q \sum_{m=1}^{\infty} \frac{e(-tm/q) e(mg/q)}{m} = \sum_{m=1}^{\infty} \frac{e(-mt)}{m} = f(1-t) \quad (2.9)$$

and the same formula holds for  $t \in (\mathbb{C} \setminus \mathbb{R}) \cup (0, 1)$  by analytic continuation. Finally, for  $\lambda \in (\mathbb{C} \setminus \mathbb{R}) \cup (0, 1)$ , we let

$$\mathfrak{L}(\lambda) = \int_0^\lambda f(1-t) dt - \frac{\pi i}{12}. \quad (2.10)$$

The function  $\mathfrak{L}$  is holomorphic in  $(\mathbb{C} \setminus \mathbb{R}) \cup (0, 1)$ . Note that the dilogarithm  $\operatorname{Li}_2(z) := -\int_0^z \frac{\log(1-u)}{u} du$  satisfies  $\operatorname{Li}_2(1) = \pi^2/6$ , so that whenever  $\operatorname{Im}(\lambda) < 0$ ,

$$\mathfrak{L}(\lambda) = \int_0^\lambda f(1-t) dt - \frac{\pi i}{12} = \frac{1}{2\pi i} \int_{e(-\lambda)}^1 \log(1-z) \frac{dz}{z} + \frac{\operatorname{Li}_2(1)}{2\pi i} = \frac{\operatorname{Li}_2(e(-\lambda))}{2\pi i}.$$

Before stating the main theorem of this section, we define

$$H_\kappa(u, v) := \lim_{\varepsilon \rightarrow 0^+} \left( i \int_{\gamma_\varepsilon} \frac{f(u - it\kappa) dt}{e(v)e^{2\pi t} - 1} - i \int_{\gamma_\varepsilon} \frac{f(u + it\kappa) dt}{e(-v)e^{2\pi t} - 1} \right) \quad (2.11)$$

for  $\kappa > 0$ ,  $u \in [0, 1]$ ,  $v \in \mathbb{R}/\mathbb{Z}$ . Notice that if  $v \notin \mathbb{Z}$  we have

$$H_\kappa(u, v) = i \int_0^\infty \left( \frac{f(u - it\kappa)}{e(v)e^{2\pi t} - 1} - \frac{f(u + it\kappa)}{e(-v)e^{2\pi t} - 1} \right) dt, \quad (2.12)$$

whereas if  $v \in \mathbb{Z}$  and  $0 < u < 1$  then

$$H_\kappa(u, 0) = -\frac{1}{2}f(u) + i \int_0^\infty \frac{f(u - it\kappa) - f(u + it\kappa)}{e^{2\pi t} - 1} dt, \quad (2.13)$$

as can be seen by isolating the contribution of the two circular paths, which are both  $\pi i/2$  times the residue at 0.

**Theorem 6** For  $1 \leq r < k$ , letting  $L := \lfloor rd/k \rfloor$ , and  $\lambda = \{rd/k\}$ , we have

$$\frac{(e(\gamma x))_r e(\frac{\gamma x}{24})}{(e(x))_L e(\frac{x}{24})} = \exp\left(\frac{\pi i(p + \bar{p})}{12q} - \pi i s(p, q) - \frac{\pi i}{4} + \frac{1}{2} \log \frac{k}{d} + \frac{k}{qd} \mathcal{E}(\lambda) + \mathcal{E}_r(\lambda, d/k)\right),$$

where  $s$  is the Dedekind sum [23, p. 45] and for  $\lambda \in [0, 1)$ ,  $\kappa > 0$  and, for  $s \in \mathbb{Z}/q\mathbb{Z}$ ,

$$\mathcal{E}_s(\lambda, \kappa) := -H_\kappa\left(\frac{\langle ps \rangle - \lambda}{q}, 0\right) - \sum_{\substack{g=1 \\ g \not\equiv ps \pmod{q}}}^q i \int_0^\infty \left( \frac{\mathfrak{f}(\frac{g-\lambda}{q} - it\kappa)}{e(\frac{g\bar{p}-s}{q})e^{2\pi t} - 1} - \frac{\mathfrak{f}(\frac{g-\lambda}{q} + it\kappa)}{e(-\frac{g\bar{p}-s}{q})e^{2\pi t} - 1} \right) dt, \tag{2.14}$$

with  $\langle n \rangle$  indicating the representative of  $n \pmod{q}$  in  $[1, q]$ . Moreover, for all  $s \pmod{q}$  and  $\kappa \in (0, 1]$ , the function  $\lambda \mapsto \mathcal{E}_s(\lambda, \kappa)$  is defined and holomorphic in the strip  $\{\text{Re}(\lambda) \in [0, 1)\}$ , and

$$\mathcal{E}_s(\lambda, \kappa) \ll |\log |1 - \lambda|| + \log(1/\kappa) \tag{2.15}$$

uniformly for  $\text{Re}(\lambda) \in [0, 1)$  and  $\text{Im}(\lambda) \ll 1$ .

**Remark 4** If  $\lambda \in (0, 1)$  or if  $s \not\equiv 0 \pmod{q}$  by (2.13) we can write  $\mathcal{E}_\lambda$  as

$$\mathcal{E}_s(\lambda, \kappa) := \frac{1}{2} f\left(\frac{\langle ps \rangle - \lambda}{q}\right) - \sum_{g=1}^q i \int_0^\infty \left( \frac{\mathfrak{f}(\frac{g-\lambda}{q} - it\kappa)}{e(\frac{g\bar{p}-s}{q})e^{2\pi t} - 1} - \frac{\mathfrak{f}(\frac{g-\lambda}{q} + it\kappa)}{e(-\frac{g\bar{p}-s}{q})e^{2\pi t} - 1} \right) dt. \tag{2.16}$$

We remark that if  $(q, d) = 1$  (i.e.  $(k, d) = 1$ ), one cannot have  $\{rd/k\} = 0$ .

**Remark 5** Notice that by (2.3) we have  $e(\frac{\gamma x}{24})/e(\frac{x}{24}) \cdot \exp(-\frac{\pi i(p+\bar{p})}{12q}) = \exp(-\frac{\pi i}{12}(\frac{d}{kq} + \frac{k}{dq}))$ .

In order to prove Theorem 6, we require some properties of the function  $H_\kappa(u, v)$ .

**Lemma 3** We have:

- (1)  $H_\kappa(1 - u, -v) + H_\kappa(u, v) = -2\pi i B_1(u) \tilde{B}_1(v) + \pi i \kappa \tilde{B}_2(v)$  for  $v \notin \mathbb{Z}$ ,
- (2)  $H_\kappa(1, v) - H_\kappa(0, v) = \mathfrak{f}(\{-v\})$  for  $v \notin \mathbb{Z}$ ,
- (3)  $H_\kappa(1, 0) = -\frac{\log(\kappa)}{2} - \frac{\pi i}{4} + \frac{\pi i \kappa}{12}$ .

**Proof** Let  $v \notin \mathbb{Z}$ . By (2.6), Lemma 2 and (2.8) we have

$$\begin{aligned} &H_\kappa(1 - u, -v) + H_\kappa(u, v) \\ &= i \int_0^\infty \left( \frac{f(1 - u - it\kappa)}{e(-v)e^{2\pi t} - 1} - \frac{f(1 - u + it\kappa)}{e(v)e^{2\pi t} - 1} \right) dt + \\ &\quad + i \int_0^\infty \left( \frac{f(u - it\kappa)}{e(v)e^{2\pi t} - 1} - \frac{f(u + it\kappa)}{e(-v)e^{2\pi t} - 1} \right) dt \\ &= - \int_0^\infty \left( \frac{(1 - 2u)\pi - 2\pi it\kappa}{e(-v)e^{2\pi t} - 1} - \frac{(1 - 2u)\pi + 2\pi it\kappa}{e(v)e^{2\pi t} - 1} \right) dt \\ &= 2\pi i(1 - 2u) \operatorname{Im} \int_0^\infty \frac{dt}{e(v)e^{2\pi t} - 1} - 4\pi i\kappa \operatorname{Im} \int_0^\infty \frac{-it dt}{e(v)e^{2\pi t} - 1} \\ &= -2\pi i B_1(u) \tilde{B}_1(v) + \pi i\kappa \tilde{B}_2(v). \end{aligned}$$

Also, by (2.7), we have

$$\begin{aligned} H_\kappa(1, v) - H_\kappa(0, v) &= i \int_0^\infty \left( \frac{f(1 - it\kappa)}{e(v)e^{2\pi t} - 1} - \frac{f(1 + it\kappa)}{e(-v)e^{2\pi t} - 1} \right) dt + \\ &\quad + i \int_0^\infty \left( \frac{f(-it\kappa)}{e(v)e^{2\pi t} - 1} - \frac{f(it\kappa)}{e(-v)e^{2\pi t} - 1} \right) dt \\ &= - \int_0^\infty \frac{2\pi}{e(v)e^{2\pi t} - 1} dt = f(\{-v\}). \end{aligned}$$

Finally, by (2.6) and (2.7) as  $\varepsilon \rightarrow 0^+$  we have

$$\begin{aligned} &i \int_{\gamma_\varepsilon} \frac{f(1 - it\kappa) dt}{e^{2\pi t} - 1} - i \int_{\bar{\gamma}_\varepsilon} \frac{f(1 + it\kappa) dt}{e^{2\pi t} - 1} \\ &= i \int_{\gamma_\varepsilon} \frac{f(it\kappa) dt}{e^{2\pi t} - 1} - i \int_{\bar{\gamma}_\varepsilon} \frac{f(it\kappa) dt}{e^{2\pi t} - 1} - \int_{\gamma_\varepsilon} \frac{\pi dt}{e^{2\pi t} - 1} + \int_0^\infty \frac{2\pi it\kappa dt}{e^{2\pi t} - 1} + o(1). \end{aligned}$$

The last two integrals can be easily computed and contribute  $-\frac{i\pi}{4} + \frac{\log(2\pi\varepsilon)}{2}$  and  $\frac{\pi i\kappa}{12}$  respectively. The contributions of the interval  $(\varepsilon, \infty)$  in the first two integral cancel out. Thus, since  $f(it\kappa) = \log(2\pi t\kappa) + o(1)$  as  $|t| \rightarrow 0$  with  $-\pi/2 < \arg t < \pi/2$ , we have that the first two integral contribute

$$i \int_{C_\varepsilon} \frac{\log(2\pi t\kappa) dt}{e^{2\pi t} - 1} = -\frac{\log(2\pi\varepsilon\kappa)}{2} + o(1)$$

where  $C_\varepsilon$  is the semicircle centered at the origin going from  $-\varepsilon$  to  $i\varepsilon$  counter-clockwise. We then have

$$H_\kappa(1, 0) = -\frac{\log \kappa}{2} - \frac{i\pi}{4} + \frac{\pi i\kappa}{12}.$$

□

**Lemma 4** Let  $0 < \kappa \leq 1$ ,  $\operatorname{Re}(u) \in [0, 1]$ ,  $v \in \mathbb{R}/\mathbb{Z}$  with  $u \neq 0, 1$ , and  $A \geq 1$ . Then

$$H_\kappa(u, v) = -\log |u| + O_{v,A}(\log(2/\kappa))$$

uniformly in  $u, \kappa$  with  $\operatorname{Im}(u) \leq A$ .

**Proof** The case  $v \notin \mathbb{Z}$ ,  $\operatorname{Re}(u) \in [0, 1]$  is an easy consequence of the bound

$$f(x + it) \ll |t| + |\log |t||, \quad (x \in [0, 1], t \in \mathbb{R}_{\neq 0}). \quad (2.17)$$

Now assume  $v \in \mathbb{Z}$  and  $\operatorname{Re}(u) \in (0, 1/2]$ . We recall (2.13). By (2.17) the contribution to the integral from the interval  $[2A, \infty)$  is  $O(\log(2/\kappa))$ . Next, we write

$$\begin{aligned} I(u, \kappa) &:= i \int_0^{2A} \frac{f(u - it\kappa) - f(u + it\kappa)}{e^{2\pi t} - 1} dt \\ &= -\pi i \int_0^{2A} \int_{u-it\kappa}^{u+it\kappa} (\cot(\pi z) + i) dz \frac{dt}{e^{2\pi t} - 1}. \end{aligned}$$

Note that  $\cot(\pi z) = 1/(\pi z) + O(1)$  uniformly for  $\operatorname{Re}(z) \in [0, 1/2]$ , and that  $\frac{1}{e^{2\pi t} - 1} = \frac{1}{2\pi t} + O(1)$  for  $t > 0$ . Therefore,

$$I(u, \kappa) = \frac{1}{2\pi i} \int_0^{2A} \int_{u-it\kappa}^{u+it\kappa} \frac{dz}{z} \frac{dt}{t} + O_A(\log(2/\kappa)).$$

Changing variables  $z \leftarrow |u|z$  and  $t \leftarrow t|u|/\kappa$ , we get

$$I(u, \kappa) = \frac{1}{2\pi i} \int_0^{2A\kappa/|u|} \int_{u'-it}^{u'+it} \frac{dz}{z} \frac{dt}{t} + O_A(\log(2/\kappa)),$$

where  $|u'| = 1$ . For  $t \leq 1/2$ , we may bound the  $z$ -integral by  $O(t)$ , while for  $t \geq 2$ , we have  $\int_{u'-it}^{u'+it} \frac{dz}{z} = \pi i + O(1/t)$ . We deduce

$$I(u, \kappa) = \frac{1}{2\pi i} \int_{1/2}^2 \int_{u'-it}^{u'+it} \frac{dz}{z} \frac{dt}{t} + O_A(\log(2/\kappa)) - \frac{1}{2} \log |u|.$$

The double integral here is bounded independently of  $u'$ , and so finally

$$I(u, \kappa) = O_A(\log(2/\kappa)) - \frac{1}{2} \log |u|.$$

We deduce  $H_\kappa(u, 0) = -\log |u| + O_A(\log(2/\kappa))$  for  $0 \leq \operatorname{Re}(u) \leq 1/2$ ,  $u \neq 0$ . On the other hand, by computations similar to Lemma 3, we get

$$H_\kappa(1 - u, 0) + H_\kappa(u, 0) = -f(u) - \frac{\pi i}{2}(2u - 1) - \frac{\pi i \kappa}{6},$$



from which we get the claimed behaviour for all  $u$ .  $\square$

**Proof of Theorem 6** For  $0 \leq \ell \leq L$ , let  $r_\ell = \ell k/d$ . We split  $(e(\gamma x))_r$  as  $(e(\gamma x))_r = \prod_{\ell=0}^L P_\ell^L$ , where for  $0 \leq \ell < L$ ,

$$P_\ell^L = \prod_{r_\ell < n \leq r_{\ell+1}} \left( 1 - e\left(\frac{nh}{k}\right) \right) \quad \text{and} \quad P_L^L = \prod_{r_\ell < n \leq r} \left( 1 - e\left(\frac{nh}{k}\right) \right).$$

First we focus on the case  $0 \leq \ell < L$ . By (2.2) and by periodicity we have

$$\begin{aligned} P_\ell^L &= \prod_{r_\ell < n \leq r_{\ell+1}} \left( 1 - e\left(\frac{np}{q} - \frac{nd}{kq}\right) \right) \\ &= \prod_{a=1}^q \prod_{\frac{r_\ell - a}{q} < m \leq \frac{r_{\ell+1} - a}{q}} \left( 1 - e\left(\frac{ap}{q} - \frac{d(a + mq)}{kq}\right) \right) \\ &= \prod_{a=1}^q \prod_{\frac{r_\ell - a}{q} < m \leq \frac{r_{\ell+1} - a}{q}} \left( 1 - e\left(\frac{ga}{q} - \frac{d(a + mq)}{kq}\right) \right), \end{aligned}$$

where  $g_a$  is the representative of the class  $pa \pmod{q}$  contained in  $[\ell + 1, \ell + q]$ , so that in the last line each  $e(\cdot)$  is computed at a number in  $(0, 1)$ . It follows that we can write

$$P_\ell^L = \exp \left( \sum_{a=1}^q \sum_{\frac{r_\ell - a}{q} < m \leq \frac{r_{\ell+1} - a}{q}} f\left(\frac{ga}{q} - \frac{d(a + mq)}{kq}\right) \right).$$

Also, we notice that  $0 < \frac{ga}{q} - \frac{d(r_{\ell+1} - a)}{kq} < 1$  whenever  $\frac{r_{\ell+1} - a}{q}$  is an integer. We can then apply Abel–Plana formula in the form of Lemma 1, whose conditions are easily verified. Note that

$$\begin{aligned} \sum_{a=1}^q \int_{\frac{r_\ell - a}{q}}^{\frac{r_{\ell+1} - a}{q}} f\left(\frac{ga}{q} - \frac{d(a + tq)}{kq}\right) dt &= \frac{k}{qd} \sum_{a=1}^q \int_{\ell}^{\ell+1} f\left(\frac{ga}{q} - \frac{t}{q}\right) dt \\ &= \frac{k}{qd} \sum_{g=1}^q \int_0^1 f\left(\frac{g-t}{q}\right) dt = \frac{k}{qd} \int_0^1 f(1-t) dt = 0 \end{aligned}$$

by (2.9). Therefore, by Lemma 1, Eq. (2.11) and the definition of  $r_\ell$ ,

$$P_\ell^L = \exp \left( \sum_{a=1}^q H_{d/k} \left( \frac{ga}{q} - \frac{\ell}{q}, \frac{a}{q} - \frac{\ell k}{qd} \right) - \sum_{a=1}^q H_{d/k} \left( \frac{ga}{q} - \frac{\ell+1}{q}, \frac{a}{q} - \frac{(\ell+1)k}{qd} \right) \right).$$

Now,  $\frac{k}{qd} = \frac{N}{d} + \frac{\bar{p}}{q}$  and  $1 \leq g_a - \ell \leq q$  with  $pa \equiv g_a \pmod{q}$ , thus by a change of variable and multiplying this equality over  $0 \leq \ell < L$ , we get

$$\begin{aligned} \prod_{\ell=0}^{L-1} P_\ell^L &= \exp \left( \sum_{\ell=0}^{L-1} \left( \sum_{g=1}^q H_{d/k} \left( \frac{g}{q}, \frac{g\bar{p}}{q} - \frac{\ell N}{d} \right) - \sum_{g=0}^{q-1} H_{d/k} \left( \frac{g}{q}, \frac{g\bar{p}}{q} - \frac{(\ell+1)N}{d} \right) \right) \right) \\ &= \exp \left( \sum_{\ell=0}^{L-1} \left( H_{d/k} \left( 1, -\frac{\ell N}{d} \right) - H_{d/k} \left( 0, -\frac{(\ell+1)N}{d} \right) \right) \right. \\ &\quad \left. + \sum_{g=1}^{q-1} \left( H_{d/k} \left( \frac{g}{q}, \frac{g\bar{p}}{q} \right) - H_{d/k} \left( \frac{g}{q}, \frac{g\bar{p}}{q} - \frac{LN}{d} \right) \right) \right). \end{aligned} \tag{2.18}$$

We treat  $P_L^L$  in the same way. First,

$$P_L^L = \exp \left( \sum_{a=1}^q \sum_{\substack{r_L - a \\ q} < m \leq \frac{r-a}{q}} f \left( \frac{g_a}{q} - \frac{d(a + mq)}{kq} \right) \right).$$

Note that by (2.9) and (2.10)

$$\begin{aligned} \sum_{a=1}^q \int_{(r_L - a)/q}^{(r-a)/q} f \left( \frac{g_a}{q} - \frac{d(a + qt)}{kq} \right) dt &= \frac{k}{qd} \sum_{a=1}^q \int_L^{L+\lambda} f \left( \frac{g_a}{q} - \frac{t}{q} \right) dt \\ &= \frac{k}{qd} \int_0^\lambda f(1 - t) dt = \frac{k}{qd} \mathfrak{L}(\lambda) + \frac{\pi ik}{12qd} \end{aligned}$$

where we recall that  $\lambda = \{rd/k\}$ . Applying Lemma 1, we therefore find

$$P_L^L = \exp \left( \frac{k}{qd} \mathfrak{L}(\lambda) + \frac{\pi ik}{12qd} + \sum_{g=1}^q \left( H_{d/k} \left( \frac{g}{q}, \frac{g\bar{p}}{q} - \frac{LN}{d} \right) - H_{d/k} \left( \frac{g-\lambda}{q}, \frac{g\bar{p}-r}{q} \right) \right) \right). \tag{2.19}$$

Multiplying the equalities (2.18) and (2.19) and recalling that  $(e(\gamma x))_r = \prod_{\ell=0}^L P_\ell^L$ , we obtain

$$\begin{aligned} (e(\gamma x))_r &= \exp \left( \frac{k}{qd} \mathfrak{L}(\lambda) + \frac{\pi ik}{12qd} + \sum_{g=1}^{q-1} H_{d/k} \left( \frac{g}{q}, \frac{g\bar{p}}{q} \right) - \sum_{g=1}^q H_{d/k} \left( \frac{g-\lambda}{q}, \frac{g\bar{p}-r}{q} \right) \right. \\ &\quad \left. + H_{d/k}(1, 0) + \sum_{1 \leq \ell \leq L} \left( H_{d/k} \left( 1, -\frac{\ell N}{d} \right) - H_{d/k} \left( 0, -\frac{\ell N}{d} \right) \right) \right). \end{aligned}$$

By Lemma 3 (1) we have

$$\begin{aligned} \sum_{g=1}^{q-1} H_{d/k}\left(\frac{g}{q}, \frac{g\bar{p}}{q}\right) &= -\pi i \sum_{g=1}^{q-1} \tilde{B}_1\left(\frac{g}{q}\right) \tilde{B}_1\left(\frac{g\bar{p}}{q}\right) + \frac{\pi id}{2k} \sum_{g=1}^{q-1} \tilde{B}_2\left(\frac{g}{q}\right) \\ &= -\pi i s(p, q) + \frac{\pi id}{12kq} - \frac{\pi id}{12k} \end{aligned}$$

since  $\sum_{g=0}^{q-1} B_2(g/q) = \frac{1}{6q}$  and  $B_2(0) = 1/6$ . Also, by Lemma 3 (3) and (2) we have  $H_{d/k}(1, 0) = \frac{\log(k/d)}{2} - \frac{\pi i}{4} + \frac{\pi id}{12k}$  and

$$\sum_{1 \leq \ell \leq L} \left( H_{d/k}\left(1, -\frac{\ell N}{d}\right) - H_{d/k}\left(0, -\frac{\ell N}{d}\right) \right) = \sum_{1 \leq \ell \leq L} f(\{\ell N/d\}).$$

Thus,

$$\begin{aligned} (e(\gamma x))_r &= (e(N/d))_L \exp\left(\frac{k}{qd} \mathfrak{L}(\lambda) + \frac{1}{2} \log \frac{k}{d} - \sum_{g=1}^q H_{d/k}\left(\frac{g-\lambda}{q}, \frac{g\bar{p}-r}{q}\right) + \right. \\ &\quad \left. - \pi i s(p, q) - \frac{\pi i}{4} + \frac{\pi id}{12kq} + \frac{\pi ik}{12qd}\right). \end{aligned}$$

By (2.3) and (2.12) we then obtain the claimed result. The bound (2.15) follows by Lemma 4. □

### 2.3 Sums of cotangents

In this section and the next, we introduce the following extension of the Landau  $O$ -symbol. Let  $D_1 \subset D_2 \subset \mathbb{C}$  be two sets given by the context, and  $g : D_2 \rightarrow \mathbb{R}_+$ . We will write

$$f(z) = \mathcal{O}_{(p_1, p_2, \dots)}^z(g(z)) \quad (z \in D_1) \tag{2.20}$$

whenever, for any fixed choice of the parameters  $p_1, p_2, \dots$ , there exists a function  $\varphi$  holomorphic on  $D_2$ , which satisfies  $\varphi(z) = f(z)$  for  $z \in D_1$ , and  $|\varphi(z)| \ll g(z)$  for  $z \in D_2$ , the implied constant being *uniform* in the parameters  $p_1, p_2, \dots$ . The additional information is the holomorphic behaviour in  $z$ , which may become useful when studying knots other than  $4_1$ . However we stress that, in the present work, a later obstacle (possible cancellation of main terms) imposes the restriction  $K = 4_1$ , and in this case, we do not require holomorphicity of error terms.

**Lemma 5** For  $0 < \alpha < \frac{1}{2}$ , we have

$$\cot(\pi\alpha) \tan(\pi\alpha x) \leq x \quad (0 < x \leq 1), \tag{2.21}$$

$$x(1 - (2\alpha)^2) < \cot(\pi\alpha) \tan(\pi\alpha x) \leq O_\varepsilon(x) \quad (0 < x \leq \frac{1}{\alpha}(\frac{1}{2} - \varepsilon)). \tag{2.22}$$

**Proof** Assume  $0 < x \leq \frac{1}{\alpha}(\frac{1}{2} - \varepsilon)$ . The function  $x \mapsto \frac{\tan(x\lambda)}{x}$  is increasing on  $(0, \frac{\pi}{2\lambda})$  for all  $\lambda > 0$ . It follows that  $\pi\alpha \leq \tan(\pi\alpha x)/x \leq O_\varepsilon(\alpha)$ , while  $\tan(\pi\alpha x)/x \leq \tan(\pi\alpha)$  if  $x \leq 1$ . Thus, it suffices to show the bounds

$$1 - (2\alpha)^2 < \pi\alpha \cot(\pi\alpha), \quad \tan(\pi\alpha) \cot(\pi\alpha) \leq 1, \quad \alpha \cot(\pi\alpha) \ll 1.$$

The second is trivial, while the third follows from elementary properties of cot. For the first, we note that by the Taylor expansion of cot [21, 1.411.7], we have  $\phi(\alpha) := \frac{1 - \pi\alpha \cot(\pi\alpha)}{\alpha^2} = \sum_{j \geq 0} c_j \alpha^{2j}$  for  $\alpha \in (0, 1)$ , where  $c_j > 0$ . In particular  $\phi$  is increasing, and we conclude by  $\phi(\frac{1}{2}) = 4$ . □

**Lemma 6** *Let  $\varepsilon > 0$ . For  $0 < \alpha \leq \frac{1}{2} - \varepsilon$  and  $|\operatorname{Re}(\alpha z)| \leq \frac{1}{2} - \varepsilon$  we have*

$$1 - \cot(\pi\alpha) \tan(\pi\alpha z) = (1 - z)(1 + O_\varepsilon(\alpha^2(|z|^2 + |z|))).$$

**Proof** If  $|z| \leq 1/2$ , expanding in Taylor series we obtain

$$\cot(\pi\alpha) \tan(\pi\alpha z) = \cot(\pi\alpha) \pi\alpha z (1 + O(\alpha^2|z|^2)) = z(1 + O(\alpha^2))(1 + O(\alpha^2|z|^2))$$

and the claimed result follows. If  $|z| \geq 1/2$ , we observe that the left hand side is equal to  $\cot(\pi\alpha)(\tan(\pi\alpha) - \tan(\pi\alpha z))$  and so, since  $\cot(\pi\alpha)\pi\alpha = 1 + O_\varepsilon(\alpha^2)$ , we are required to show that

$$\frac{\tan(\pi\alpha z) - \tan(\pi\alpha)}{\pi\alpha(z - 1)} = \frac{1}{\alpha z - \alpha} \int_\alpha^{\alpha z} \frac{dv}{\cos(\pi v)^2} = 1 + O_\varepsilon(|\alpha z|^2). \tag{2.23}$$

This follows for  $|\operatorname{Im}(\alpha z)| \leq 1$  by the estimate  $\cos(\pi v)^2 = \exp\{O_\varepsilon(|v|^2)\}$  for  $|\operatorname{Re}(v)| \leq \frac{1}{2} - \varepsilon$  and  $|\operatorname{Im}(v)| \leq 1$ . If  $|\operatorname{Im}(\alpha z)| > 1$ , then both the ratio  $\frac{1}{\alpha z - \alpha}$  and the integral are bounded, since  $|\cos(\pi v)| \gg e^{\pi|\operatorname{Im}(v)|}$ , so that (2.23) holds in this case as well. □

**Lemma 7** *Let  $\alpha \in \mathbb{R}$  with  $0 < |\alpha| < 1/2$ , and*

$$D_\alpha := \{z \in \mathbb{C} \mid \frac{-1}{3|\alpha|} < \operatorname{Re}(z) < 1\}.$$

*Then, taking the determination of the logarithm which is real on the real axis, the function given by*

$$\psi_\alpha(z) := \log(1 - \cot(\pi\alpha) \tan(\pi\alpha z)) + \cot(\pi\alpha) \tan(\pi\alpha z) \tag{2.24}$$

*is holomorphic on  $D_\alpha$ , where it satisfies*

$$\psi_\alpha(z) \ll |z|^2 + |z| |\log(1 - z)|, \quad \operatorname{Im}(\psi_\alpha(z)) \ll |z|^2, \tag{2.25}$$

$$\psi_\alpha(z) = \log(1 - z) + z + O_\varepsilon((|z|^2 + |z|^3)|\alpha|^2) \quad \text{if } |\alpha| \leq \frac{1}{2} - \varepsilon. \tag{2.26}$$

**Proof** We can assume  $\alpha \in (0, 1/2)$ , since  $\psi_\alpha$  is even in  $\alpha$ . For  $|\operatorname{Re}(z)| < \frac{1}{2\alpha}$ , let  $g_\alpha(z) := \cot(\pi\alpha) \tan(\pi\alpha z)$ . Since, for  $z = x + iy$ , we have

$$\tan(\pi\alpha z) = \frac{\tan(\pi\alpha x)(1 - \tanh^2(\pi\alpha y)) + i \tanh(\pi\alpha y)(1 + \tan^2(\pi\alpha x))}{1 + \tan^2(\pi\alpha x) \tanh^2(\pi\alpha y)}, \tag{2.27}$$

we deduce, for  $x, y$  real and  $|x| \leq 1$ ,

$$|\operatorname{Re}(g_\alpha(z))| \leq |\cot(\pi\alpha) \tan(\pi\alpha x)| \leq |x|. \tag{2.28}$$

by Lemma 5. Moreover, if  $x \leq 0$ , then  $\operatorname{Re}(g_\alpha(z)) \leq 0$ . Using (2.27) and the lower bound (2.22), we obtain  $|\operatorname{Re}(g_\alpha(z))| \gg \cot(\pi\alpha) \tan(\pi\alpha |x|) \gg_\varepsilon |x|$ . Thus, there exists  $c_\varepsilon > 0$  such that

$$\operatorname{Re}(1 - g_\alpha(z)) \geq \min(1 - x, 1 - c_\varepsilon x). \tag{2.29}$$

In particular, the function  $z \mapsto \psi_\alpha(z) = \log(1 - g_\alpha(z)) + g_\alpha(z)$  is well-defined and holomorphic in  $D_\alpha$ , and moreover  $\operatorname{Im}(\log(1 - g_\alpha(z))) < \pi/2$ .

We observe that if  $\alpha y \rightarrow \pm\infty$  (and so  $z \rightarrow \infty$ ), then  $g_\alpha(z) \sim \pm i \cot(\pi\alpha)$  uniformly in  $x$ , so that  $\psi_\alpha(z) \ll 1/\alpha \ll |z|$  and (2.25), (2.26) follow trivially. Thus, we can assume  $\alpha y \ll 1$ . Also, by (2.27) and Lemma 5 we obtain

$$|\operatorname{Im}(g_\alpha(z))| \ll |\cot(\pi\alpha)| (|\tan(\pi x\alpha)| + |\tanh(\pi\alpha y)|) \ll |x| + |y| \ll |z|. \tag{2.30}$$

In particular, recalling (2.28), we have  $g_\alpha(z) \ll |z|$ .

Now, assume  $\alpha \leq \frac{1}{2} - \varepsilon$ . By (2.27), and since  $\alpha y \ll 1$ , we have

$$|\operatorname{Im}(g_\alpha(z))| \geq |\cot(\pi\alpha) \tanh(\pi\alpha y)| \gg_\varepsilon |y|.$$

By (2.29), it then follows that  $|1 - g_\alpha(z)| \gg_\varepsilon \min(1 - x, 1 - c_\varepsilon x) + |y| \gg_\varepsilon |1 - z|$ , and so  $|u_\alpha(z)| \gg_\varepsilon 1$  for  $u_\alpha(z) := \frac{1 - g_\alpha(z)}{1 - z}$ . Thus,

$$\begin{aligned} \log(u_\alpha(z)) &= u_\alpha(z) - 1 + O_\varepsilon(|u_\alpha(z) - 1|^2) = z - g_\alpha(z) + z(u_\alpha(z) - 1) + O_\varepsilon(|u_\alpha(z) - 1|^2) \end{aligned}$$

and so (2.26) follows since  $u_\alpha(z) - 1 = O_\varepsilon(\alpha^2(|z|^2 + |z|))$  by Lemma 6 (and since  $\alpha z \ll 1$ ).

We now move to (2.25). Since  $g_\alpha(z) \ll |z|$ , then if  $|z| < \delta$  with  $\delta > 0$  sufficiently small, then by Taylor expansion one trivially has that (2.25) holds for  $|z| < \delta$ . By the assumption  $|\alpha y| \ll 1$  we have that (2.26) implies (2.25) for  $|z| \geq \delta$  and  $\alpha \leq \frac{1}{2} - \varepsilon$ . It follows that we can also assume  $\alpha \gg 1$  (and so also  $|z| \ll \alpha |z| \ll 1$ ).

Finally, we observe that under the above assumptions we have that (2.27) implies also

$$\operatorname{Re}(1 - g_\alpha(z)) \geq 1 - x(1 - \tanh^2(\pi\alpha y)) \gg 1 - x + y^2 \gg |1 - z|^2 \quad (x \geq 0),$$

which inequality  $\operatorname{Re}(1 - g_\alpha(z)) \gg |1 - z|^2$  is also trivially true if  $x < 0$ . The bound (2.26) follows since  $g_\alpha(z) \ll |z|$ ,  $\operatorname{Im}(\log(1 - g_\alpha(z))) < \pi/2$ , and  $\log(1 - w) + w \ll \min\{|w|^2, |w| + \log|1 - w|\}$  for  $w \in \mathbb{C} \setminus [1, \infty)$ .  $\square$

**Lemma 8** *Let  $\gamma \leq \delta$  and let  $a \in \mathbb{N}$  with  $\gamma \leq a - \frac{1}{2} \leq \delta$ . Let  $g(z)$  be holomorphic on a neighborhood of  $\gamma \leq \operatorname{Re}(z) \leq \delta$ , where it satisfies  $|g(z)| \leq C_1|z|^m + C_2$  for some  $m, C_1, C_2 \geq 0$ . Then, with  $z = r$ , we have*

$$\sum_{n=a}^r g(n) = \mathcal{O}_{(g,a)}^r(C_1((|a| + |z|)^{m+1} + 1) + C_2(|a| + |z| + 1)), \quad (r \in \mathbb{Z} \cap [a, \delta])$$

using the notation (2.20) with  $D_2 = \{z, \gamma + \frac{1}{2} \leq \operatorname{Re}(z) \leq \delta\}$ .

**Proof** We apply Lemma 1 with  $\alpha = a - \frac{1}{2}, \beta = r - \frac{1}{2}$ , we find

$$\begin{aligned} \sum_{n=a}^{r-1} g(n) &= \int_{a-\frac{1}{2}}^{r-\frac{1}{2}} g(z) dz \\ &\quad - i \int_0^\infty \frac{g(a - \frac{1}{2} + it) - g(a - \frac{1}{2} - it) - g(r - \frac{1}{2} + it) + g(r - \frac{1}{2} - it)}{e^{2\pi t} + 1} dt, \end{aligned}$$

Denoting by  $\varphi(r) - g(r)$  the right hand side, we immediately see that  $\varphi$  extend to a holomorphic function in  $\gamma + \frac{1}{2} \leq \operatorname{Re}(z) \leq \delta$  and that  $\varphi(z) \ll C_1((|a| + |z|)^{m+1} + 1) + C_2(|a| + |z| + 1)$  in this strip.  $\square$

**Lemma 9** *Let  $(h, k) = 1, 3 \leq h < k$  and  $r_0 \in \mathbb{Z}/h\mathbb{Z}$ . Then for all  $0 \leq r < k$  with  $r \equiv r_0 \pmod{h}$ , writing  $z = r/k$ , we have*

$$\begin{aligned} &\sum_{\substack{0 \leq n \leq r \\ h \nmid n}} \left( \log \left( 1 - \cot \left( \frac{\pi n \bar{k}}{h} \right) \tan \left( \frac{\pi n}{hk} \right) \right) + \cot \left( \frac{\pi n \bar{k}}{h} \right) \tan \left( \frac{\pi n}{hk} \right) \right) \\ &= \frac{k}{h} \left( \frac{\pi i (z^2 - z)}{2} + \mathfrak{L}(z) + \frac{\pi i}{12} + z \log(2\pi z/e) \right) \\ &\quad + \mathcal{O}_{(h,k,r_0)}^z((1 + |\log(1 - z)| + |z|^4)(1 + k/h^2 + \log(k/h))), \end{aligned}$$

using the notation (2.20) with  $D_2 = \{z \in \mathbb{C}, 0 \leq \operatorname{Re}(z) < 1\}$ . Moreover, the error term  $|\log(1 - z)|$  can be omitted if  $z \in [0, 1 - \frac{h}{k}(1 - \{\frac{r_0-k}{h}\})]$ .

**Proof** In this proof, the notation  $\mathcal{O}$  will stand for  $\mathcal{O}_{(k,h,r_0)}^z$  relative to the set  $D_2 = \{z, \operatorname{Re}(z) \in [0, 1)\}$ . We divide the sum into congruence classes  $n \equiv \ell k \pmod{h}, |\ell| < h/2$ , where the possible term  $\ell = h/2$  is excluded since the summand is zero in this case. With the notation (2.24), we write the sum to be computed as

$$\sum_{0 < |\ell| < h/2} S(\ell), \quad S(\ell) := \sum_{\substack{0 < n \leq r \\ n \equiv \ell k \pmod{h}}} \psi_{\ell/h} \left( \frac{n}{k\ell} \right).$$

We consider each  $\ell$  separately. We first consider  $\ell \neq 1$ . Let  $\delta_\ell := \{\frac{r_0 - k\ell}{h}\}$ , and

$$\alpha_\ell = \left\lceil \frac{-k\ell}{h} \right\rceil - \frac{1}{2}, \quad \beta_\ell(r) := \left\lfloor \frac{r - k\ell}{h} \right\rfloor + \frac{1}{2} = \frac{r - k\ell}{h} + \frac{1}{2} - \delta_\ell.$$

Then by Lemma 1, we have

$$S(\ell) = \sum_{\alpha_\ell < m \leq \beta_\ell(r)} \psi_{\ell/h}(1 + \frac{hm}{k\ell}) = I + C(\alpha_\ell) - C(\beta_\ell(r)),$$

where

$$I := \int_{\alpha_\ell}^{\beta_\ell(r)} \psi_{\ell/h}(1 + \frac{ht}{k\ell}) dt, \quad C(\gamma) := 2 \int_0^\infty \frac{\text{Im}(\psi_{\ell/h}(1 + \frac{h}{k\ell}(\gamma + it))) dt}{e^{2\pi t} + 1}.$$

Note that  $\beta_\ell(r) = \frac{r-r_0}{h} + \beta_\ell(r_0)$ , the right-hand side of which depends holomorphically on  $r$ . Splitting the integral as  $\int_{-k\ell/h}^{(z-\ell)k/h} + \int_{\alpha_\ell}^{-k\ell/h} + \int_{(z-\ell)k/h}^{\beta_\ell(r)}$  with  $z = r/k$ , we get

$$\begin{aligned} I &= \frac{k}{h} \int_0^z \psi_{\ell/h}(t/\ell) dt + \int_{\alpha_\ell + \frac{k\ell}{h}}^0 \psi_{\ell/h}(\frac{ht}{k\ell}) dt + \int_0^{\frac{1}{2} - \delta_\ell} \psi_{\ell/h}(\frac{z}{\ell} + \frac{ht}{k\ell}) dt \\ &= \frac{k}{h} \int_0^z \psi_{\ell/h}(t/\ell) dt + \mathcal{O}(\frac{1+|z|^2}{\ell^2}), \end{aligned}$$

by Lemma 7.

Next, we have

$$C(\beta_\ell(r)) = \frac{1}{2i} \sum_{\pm} \int_0^\infty \psi_{\ell/h} \left( \frac{z}{\ell} + \frac{h}{k\ell} \left( \frac{1}{2} - \delta_\ell \right) \pm i \frac{th}{k\ell} \right) \frac{dt}{e^{2\pi t} + 1}.$$

This also defines a holomorphic function of  $z$  for  $\text{Re}(z) \in [0, 1)$ , by Lemma 7. Since  $\ell \neq 1$ , it is bounded by  $\mathcal{O}(\frac{|z|^2+1}{\ell^2})$ .

Grouping the above discussion, we deduce for  $\ell \neq 1$  the estimate

$$S(\ell) = \frac{k}{h} \int_0^z \psi_{\ell/h}(t/\ell) dt + \mathcal{O}\left(\frac{1 + |z|^2}{\ell^2}\right). \tag{2.31}$$

Consider now the case  $\ell = 1$ . We recall the notation  $\delta_\ell$  from above. Since  $h \geq 3$ , by (2.26) and Lemma 8 we have

$$S(1) = \mathcal{O}\left(\frac{k}{h^3}(|z|^4 + 1)\right) + \sum_{\substack{0 < n \leq r \\ n \equiv k \pmod{h}}} \left(\frac{n}{k} + \log\left(1 - \frac{n}{k}\right)\right). \tag{2.32}$$

Let  $a \in \{1, \dots, h\}$  satisfy  $a \equiv k \pmod{h}$ ,  $q = \frac{k-a}{h}$  and  $g = \lfloor \frac{r-a}{h} \rfloor = \frac{r-a}{h} - \delta_1$ . In the sum, the integer  $m = \frac{k-n}{h}$  runs through  $\mathbb{Z} \cap [q - g, q]$ , so that

$$\begin{aligned} \sum_{\substack{0 < n \leq r \\ n \equiv k \pmod{h}}} \log\left(1 - \frac{n}{k}\right) &= (g + 1) \log\left(\frac{h}{k}\right) + \log \frac{\Gamma(q + 1)}{\Gamma(q - g)} \\ &= \int_0^{g+1} \left( \log\left(\frac{h}{k}\right) + \frac{\Gamma'}{\Gamma}(q + 1 - v) \right) dv \\ &= \int_0^{g+1} \left( \log\left(\frac{h}{k}(q + 1 - v)\right) - \frac{1}{q + 1 - v} \right. \\ &\quad \left. - \int_0^\infty \frac{\{s\} ds}{(s + q + 1 - v)^2} \right) dv \end{aligned}$$

by Stirling’s formula [42, Theorem II.0.12]. First, setting  $t = \frac{(v-1)h+a}{k}$ , we have

$$\begin{aligned} \int_0^{g+1} \log\left(\frac{h}{k}(q + 1 - v)\right) dv &= \frac{k}{h} \int_0^z \log(1 - t) dt + \frac{k}{h} \left[ \int_{\frac{a-h}{k}}^0 + \int_z^{z - \frac{h}{k} \delta_1} \right] \log(1 - t) dt \\ &= \frac{k}{h} \int_0^z \log(1 - t) dt + \mathcal{O}(|\log(1 - z)| + \log(1 + k/h)). \end{aligned}$$

Secondly, we have

$$\begin{aligned} \int_0^{g+1} \frac{dv}{q + 1 - v} &= \log\left(1 - z + \frac{h}{k} \delta_1\right) - \log\left(1 + \frac{h - a}{k}\right) = \mathcal{O}(|\log(1 - z)| + \log(1 + k/h)). \end{aligned}$$

Note that in both cases, as well as in the following computations, the error term  $|\log(1 - z)|$  can be omitted if  $z \in [0, 1 - \frac{h}{k}(1 - \delta_1)]$  (which is the case when  $z = r/k$ ). Finally,

$$\begin{aligned} \int_0^{g+1} \int_0^\infty \frac{\{s\} ds}{(s + q + 1 - v)^2} dv &= \int_0^\infty \frac{\{s\}(g + 1) ds}{(s + q - g)(s + q + 1)} \\ &= \int_0^\infty \{s\} \frac{1 + \frac{k}{h}z - \frac{a}{h} - \delta_1}{(s + \frac{k}{h}(1 - z) + \delta_1)(s + \frac{k-a}{h} + 1)} ds \\ &= \mathcal{O}(\log(1 + k/h) + \log(1 + |z|)). \end{aligned}$$

We turn to the contribution of the term  $n/k$  in (2.32). By a direct computation, we find

$$\sum_{\substack{0 < n \leq r \\ n \equiv k \pmod{h}}} \frac{n}{k} - \frac{k}{h} \int_0^z t dt = z\left(\frac{1}{2} - \delta_1\right) + 1 - \frac{h}{2k} \left(\frac{a}{h} + \delta_1\right) \left(\frac{a}{h} + 1 - \delta_1\right) = \mathcal{O}(|z| + 1).$$



On the other hand, we note that by (2.26),

$$\int_0^z (t + \log(1 - t)) dt = \int_0^z \psi_{1/h}(t) dt + \mathcal{O}\left(\frac{1 + |z|^4}{h^2}\right).$$

Grouping the above estimates, we deduce

$$S(1) = \frac{k}{h} \int_0^z \psi_{1/h}(t) dt + \mathcal{O}\left(\left(1 + \frac{k}{h^3} + \log(k/h)\right)(1 + |z|^4 + |\log(1 - z)|)\right). \quad (2.33)$$

We now sum the estimates (2.31), (2.33) over  $\ell$ , getting

$$\begin{aligned} \sum_{0 < |\ell| < h/2} S(\ell) &= \frac{k}{h} \int_0^z \sum_{0 < |\ell| < h^{2/3}} \psi_{\ell/h}(t/\ell) dt \\ &\quad + \mathcal{O}\left(\left(1 + \frac{k}{h^3} + \log(k/h)\right)(1 + |z|^4 + |\log(1 - z)|)\right). \end{aligned}$$

The main term is evaluated by (2.26) (for  $|\ell| \leq h/3$ ) and (2.25) (for  $h/3 < |\ell| < h/2$ ) as

$$\begin{aligned} &\sum_{0 < |\ell| \leq h/2} \int_0^z \psi_{\ell/h}(t/\ell) dt \\ &= \sum_{0 < |\ell| \leq h/3} \left( \int_0^z (\log(1 - t/\ell) + t/\ell) dt + \mathcal{O}\left(\frac{1 + |z|^4}{h^2}\right) \right) + \mathcal{O}\left(\frac{1 + |z|^3}{h}\right) \\ &= \sum_{0 < \ell < h/3} \int_0^z \log(1 - t^2/\ell^2) dt + \mathcal{O}\left(\frac{1 + |z|^4}{h}\right) \\ &= \sum_{\ell > 0} \int_0^z \log(1 - t^2/\ell^2) dt + \mathcal{O}\left(\frac{1 + |z|^4}{h}\right). \end{aligned}$$

For  $-1 \leq t \leq 1$  we have  $\prod_{\ell=1}^{\infty} (1 - t^2/\ell^2) = \frac{\sin(\pi t)}{\pi t}$ . Moreover, for  $0 \leq z < 1$  we have

$$\begin{aligned} \int_0^z \log(2 \sin(\pi t)) dt &= \int_0^z \log(-ie^{\pi i t} (1 - e^{-2\pi i t})) dt \\ &= \frac{\pi i (z^2 - z)}{2} + \int_0^z \log(1 - e^{-2\pi i t}) dt \\ &= \frac{\pi i (z^2 - z)}{2} + \mathfrak{L}(z) + \frac{\pi i}{12}, \end{aligned}$$

by (2.10). Collecting the previous estimates, we conclude that

$$\sum_{0 < |\ell| < h/2} S(\ell) = \frac{k}{h} \left( \frac{\pi i (z^2 - z)}{2} + \mathfrak{L}(z) + \frac{\pi i}{12} + z \log(2\pi z/e) \right) + \mathcal{O} \left( \left( 1 + \frac{k}{h^2} + \log(k/h) \right) (1 + |z|^4 + |\log(1 - z)|) \right),$$

as claimed. □

### 2.4 Second reciprocity formula for the $q$ -Pochhammer symbol

**Theorem 7** *Let  $4 \leq h < k$  with  $(h, k) = 1$  and  $r_0 \in \{0, \dots, h - 1\}$ . Let  $0 \leq r < k$  with  $r \equiv r_0 \pmod{h}$ . We have*

$$\frac{(e(-\bar{h}/k))_r e(-\frac{h}{24k})}{(e(\bar{k}/h))_{r_0} e(\frac{k}{24h})} = \exp \left( \frac{k}{h} \mathfrak{L}(r/k) - \frac{\pi}{k} \sum_{1 \leq n \leq r_0} \cot \left( \pi \frac{n\bar{k}}{h} \right) \frac{n}{h} + \frac{\pi}{k} \left\lfloor \frac{r}{h} \right\rfloor c_0 \left( \frac{\bar{k}}{h} \right) \right) + \mathcal{O}_{h,k,r_0}^z \left( (1 + |z|^4 + |\log(z(1 - z))|) \left( 1 + \log \left( \frac{k}{h} \right) + \frac{k}{h^2} \right) \right)$$

where  $z = r/k$ , using the notation (2.20) with domain  $D_2 = \{z \neq 0, \operatorname{Re}(z) \in [0, 1]\}$ . Moreover, the term  $|\log(z(1 - z))|$  can be omitted if  $z \in [\frac{r_0}{k}, 1 - \frac{h}{k}(1 - \{\frac{r_0 - k}{h}\})]$ .

**Proof** In this proof, the notation  $\mathcal{O}$  will stand for  $\mathcal{O}_{h,k,r_0}^z$  with respect to the domain  $D_2 = \{z \neq 0, \operatorname{Re}(z) \in [0, 1]\}$ . Applying the identity  $\frac{\bar{h}}{k} + \frac{\bar{k}}{h} \equiv \frac{1}{hk} \pmod{1}$  we have

$$(e(-\bar{h}/k))_r = \prod_{n=1}^r \left( 1 - e \left( \frac{n\bar{k}}{h} - \frac{n}{hk} \right) \right) = \mathcal{P} \cdot \prod_{\substack{n=1 \\ h \nmid n}}^r \left( 1 - e \left( \frac{n\bar{k}}{h} - \frac{n}{hk} \right) \right)$$

where  $\mathcal{P} := \prod_{n=1}^{\lfloor r/h \rfloor} (1 - e(-\frac{n}{k}))$ . Note that for  $x, y \in \mathbb{C}$ , we have

$$1 - e(y - x) = \frac{1}{2} (1 + e(-x))(1 - e(y))(1 - \tan(\pi x) \cot(\pi y)).$$

Thus, since  $\prod_{n=1}^{h-1} (1 - e(\frac{n\bar{k}}{h})) = h$ , we have

$$(e(-\bar{h}/k))_r = (e(\bar{k}/h))_{r_0} h^{\lfloor r/h \rfloor} \cdot \mathcal{P} \cdot \mathcal{M} \cdot \mathcal{L}, \tag{2.34}$$

where

$$\mathcal{M} = \prod_{\substack{n=1 \\ h \nmid n}}^r \frac{1 + e(-\frac{n}{hk})}{2}, \quad \mathcal{L} = \prod_{\substack{n=1 \\ h \nmid n}}^r \left( 1 - \cot \left( \frac{\pi n\bar{k}}{h} \right) \tan \left( \frac{\pi n}{hk} \right) \right).$$

First, we examine  $\mathcal{M}$ . We have

$$\mathcal{M} = \exp \left( - \sum_{\substack{n=1 \\ h \nmid n}}^r \frac{\pi i n}{hk} + \sum_{\substack{n=1 \\ h \nmid n}}^r \log \left( \cos \left( \frac{\pi n}{hk} \right) \right) \right).$$

We split the second sum as  $\sum_{1 \leq n \leq r} - \sum_{1 \leq n \leq r-r_0, h|n}$ . For any  $\varepsilon > 0$  we have  $\log(\cos(\frac{\pi w}{hk})) \ll_{\varepsilon} (\frac{|w|}{hk})^2$  for  $|\operatorname{Re}(w/(hk))| \leq (1 - \varepsilon)/2$  and so applying Lemma 8 we obtain, with  $z = r/k$ ,

$$\mathcal{M} = \exp \left( - \sum_{\substack{n=1 \\ h \nmid n}}^r \frac{\pi i n}{hk} + \frac{k}{h^2} \mathcal{O}(|z|^3 + 1) \right) = \exp \left( - \frac{\pi i (r^2 + r)}{2hk} + \frac{k}{h^2} \mathcal{O}(|z|^3 + 1) \right). \quad (2.35)$$

We then move to  $\mathcal{P} = \exp(\log \mathcal{P})$ . Taking the determination which is real on the negative imaginary axis, we have that  $\log(\frac{1-e(-w)}{2\pi i w})$  is holomorphic and  $\mathcal{O}_{\varepsilon}(|w|)$  for  $|\operatorname{Re}(w)| < 1 - \varepsilon$ . Thus, by Lemma 8,

$$\begin{aligned} \mathcal{P} &= \exp \left( \sum_{1 \leq n \leq (r-r_0)/h} \log \left( 2\pi i \frac{n}{k} \right) + \frac{k}{h^2} \mathcal{O}(|z|^2 + 1) \right) \\ &= \exp \left( \log \Gamma \left( 1 + \frac{r-r_0}{h} \right) + \frac{r-r_0}{h} \log \frac{2\pi i}{k} + \frac{k}{h^2} \mathcal{O}(|z|^2 + 1) \right). \end{aligned}$$

Write  $\log(\Gamma(1+w)) = (w+1/2)\log(w+1) - w + \mathcal{E}_1(w)$  with  $\mathcal{E}_1$  holomorphic and  $\mathcal{O}(|\log(w+1)|)$  on  $\operatorname{Re}(w) > -1$ . Abbreviating temporarily  $q = \frac{r-r_0}{h}$ , it follows that

$$\begin{aligned} \mathcal{P} &= \exp \left( \left( \frac{1}{2} + q \right) \log(1+q) + q \log \frac{2\pi i}{ke} + \mathcal{E}_1(1+q) + \frac{k}{h^2} \mathcal{O}(|z|^2 + 1) \right) \\ &= \exp \left( \frac{r}{h} \log \left( \frac{2\pi i r}{ke} \right) - \left\lfloor \frac{r}{h} \right\rfloor \log h + \mathcal{O} \left( (1 + \log(k/h) + \frac{k}{h^2})(1 + |z|^2 + |\log(z)|) \right) \right), \end{aligned} \quad (2.36)$$

since

$$\begin{aligned} &\left( \frac{1}{2} - \frac{r_0}{h} \right) \log \left( \frac{kz + h - r_0}{h} \right) + \frac{kz}{h} \log \left( 1 + \frac{h - r_0}{kz} \right) + \\ &- \frac{r_0}{h} \log \frac{2\pi i h}{ke} + \mathcal{E}_1 \left( \frac{kz + h - r_0}{h} \right) = \mathcal{O}(|z| + |\log(z)|). \end{aligned}$$

Note that the terms  $|\log(z)|$  can be omitted if  $z \in [\frac{r_0}{k}, 1)$ .

It remains to study  $\mathcal{L}$ . By Lemma 5 we have  $|\tan(\frac{\pi n}{hk}) \cot(\pi \frac{n\bar{k}}{h})| < 1$  and so we can write  $\mathcal{L} = \exp(\log \mathcal{L})$  with the principal determination. First, we consider

$$\sum_{\substack{1 \leq n \leq r \\ h \nmid n}} \tan\left(\pi \frac{n}{hk}\right) \cot\left(\pi \frac{n\bar{k}}{h}\right) = \sum_{\substack{1 \leq n \leq r \\ h \nmid n}} \frac{\pi n}{hk} \cot\left(\pi \frac{n\bar{k}}{h}\right) + \sum_{\substack{1 \leq n \leq r \\ h \nmid n}} \mathcal{E}_2\left(\frac{n}{hk}\right) \cot\left(\pi \frac{n\bar{k}}{h}\right),$$

where  $\mathcal{E}_2(z) := \tan(\pi z) - \pi z$ . Clearly,  $\mathcal{E}_2(z)$  is holomorphic and  $O_\varepsilon(|z|^2)$  in  $|\operatorname{Re}(z)| < (1 - \varepsilon)/2$ . Thus, dividing in congruence classes modulo  $h$ , the second summand above is

$$\mathcal{O}\left((1 + |z|^3) \sum_{\ell=1}^h \frac{1}{\ell} \frac{k}{h^2}\right) = \mathcal{O}\left((1 + |z|^3) \frac{k \log h}{h^2}\right).$$

Also,

$$\begin{aligned} \sum_{\substack{1 \leq n \leq r \\ h \nmid n}} \cot\left(\pi \frac{n\bar{k}}{h}\right) \frac{\pi n}{hk} &= \sum_{1 \leq n \leq r_0} \cot\left(\pi \frac{n\bar{k}}{h}\right) \frac{\pi n}{hk} + \sum_{\ell=1}^{\lfloor r/h \rfloor} \sum_{1 \leq n < h} \cot\left(\pi \frac{n\bar{k}}{h}\right) \frac{\pi(n + \ell h)}{hk} \\ &= \frac{\pi}{k} \sum_{1 \leq n \leq r_0} \cot\left(\pi \frac{n\bar{k}}{h}\right) \frac{n}{h} - \left\lfloor \frac{r}{h} \right\rfloor \frac{\pi}{k} c_0\left(\frac{\bar{k}}{h}\right) \end{aligned}$$

since  $\sum_{n=1}^{h-1} \cot(\pi \frac{n}{h}) = 0$ . Thus, by Lemma 9 we have

$$\begin{aligned} \mathcal{L} &= \exp\left(\frac{\pi i(r^2 - rk)}{2kh} + \frac{k}{h} \left(\mathfrak{L}(r/k) + \frac{\pi i}{12} + \frac{r}{k} \log\left(\frac{2\pi r}{ke}\right)\right)\right) \\ &\quad + \frac{\pi}{k} \sum_{1 \leq n \leq r_0} \cot\left(\pi \frac{n\bar{k}}{h}\right) \frac{n}{h} - \frac{\pi}{k} \left\lfloor \frac{r}{h} \right\rfloor c_0\left(\frac{\bar{k}}{h}\right) \\ &\quad + \mathcal{O}\left(\left(1 + \frac{k}{h^2} + \log(k/h)\right)(1 + |z|^4 + |\log(1 - z)|)\right), \end{aligned}$$

where the error term  $|\log(1 - z)|$  can be omitted if  $z \in [0, 1 - \frac{h}{k}(1 - \{\frac{r_0 - k}{h}\})]$ . The theorem then follows by (2.34)–(2.36), since  $\frac{h}{24k} = O(1)$ . □

### 3 Proof of Theorem 1

Throughout the rest of the section,  $K$  will denote any hyperbolic knot  $K \neq 7_2$  with at most 7 crossings.

We will use the same notation as in Sect. 2.2. In particular, all error terms will be allowed to depend on  $d$  and  $\gamma$ .

For  $n \in \mathbb{N}_{\geq 0}$  and  $\alpha = \frac{h}{k} \in \mathbb{Q}$ , with  $(h, k) = 1$ , we let

$$[\alpha]_n := k^{-1/2} (e(\alpha))_{n'}, \tag{3.1}$$

where  $n' \equiv n \pmod{k}$ ,  $0 \leq n' < k$ .

There exist  $m, m_1, \dots, m_4 \in \mathbb{N}$ ,  $\iota, \nu \in \mathbb{Z}$  and linear functions  $\ell_{i,j}(\mathbf{r}) = \sum_{u=1}^m \kappa_{i,j}(u)r_u$  with  $\kappa_{i,j} \in \{0, \pm 1\}$  such that

$$\begin{aligned} \mathcal{J}_K(x) &= \text{denom}(x)^\iota e(\nu x) \sum_{0 \leq r_1, \dots, r_m < k}^* \frac{\prod_{j=1}^{m_1} [x]_{\ell_{1,j}(\mathbf{r})} \prod_{j=1}^{m_2} \overline{[x]}_{\ell_{2,j}(\mathbf{r})}}{\prod_{j=1}^{m_3} [x]_{\ell_{3,j}(\mathbf{r})} \prod_{j=1}^{m_4} \overline{[x]}_{\ell_{4,j}(\mathbf{r})}} \\ &= \text{denom}(x)^\iota e(\nu x) \sum_{0 \leq r_1, \dots, r_m < k}^* \Pi_K([x]_{\ell_{i,j}(\mathbf{r})}), \end{aligned} \tag{3.2}$$

where  $\sum^*$  indicates that the sum is restricted to the terms with  $0 \leq \ell_{i,j}(\mathbf{r}) < k$  and, here and what follows, we put

$$\Pi_K(z_{i,j}) = \frac{\prod_{j=1}^{m_1} z_{1,j} \prod_{j=1}^{m_2} \overline{z_{2,j}}}{\prod_{j=1}^{m_3} z_{3,j} \prod_{j=1}^{m_4} \overline{z_{4,j}}}.$$

The Kashaev invariants for the knots under consideration has been given for example in [35–37]. In all these cases,  $m + 3$  coincides with the number of crossings of  $K$ , moreover

$$\iota = \frac{3 - m}{2}, \quad m_1 + m_2 + m_3 + m_4 = 3m - 1,$$

and the values of  $m_i, \ell_{i,j}$  are as in Fig. 4 below, where we used the formula  $[\alpha]_n \overline{[\alpha]}_{\text{denom}(\alpha) - n} = 1$  to write the Kashaev invariants given in [35–37] as in (3.2). Finally, different variants of the definition of the Kashaev invariant lead to slightly different values for  $\nu$  (cf. [35, p.677 footnote 4] and [45]). In the context of the modularity conjecture it is natural to always take  $\nu = 0$ , which we shall do in the following. This choice will lead to the expression (1.3) for the reciprocity formula, as conjectured by Zagier. Using (2.3) one can then easily deduce the suitable modified reciprocity formula corresponding to other choices of  $\nu$ .

We divide the sum over  $\mathbf{r}$  restricting the  $r_i$  into congruence classes modulo  $q$  and in intervals of length  $k/d$ :

$$\mathcal{J}_K(\gamma x) = \text{denom}(\gamma x)^{\frac{3-m}{2}} \sum_{L \in \{0, \dots, d-1\}^m} \sum_{s \pmod{q}} \mathcal{J}_K(\gamma, x; L, s) \tag{3.3}$$

$K$	$(m_j)_{1 \leq j \leq 4}$	$l_{i,j} = l_{i,j}(\mathbf{r})$
$4_1$	$(1, 1, 0, 0)$	$l_{1,1} = l_{2,1} = r$
$5_2$	$(0, 1, 2, 2)$	$l_{2,1} = r_1 + r_2, \quad l_{3,1} = r_1 + r_2$ $l_{3,2} = l_{4,1} = r_2, \quad l_{4,2} = r_1$
$6_1$	$(0, 2, 3, 3)$	$l_{2,1} = r_1 + r_2, \quad l_{2,2} = r_1 + r_2 + r_3$ $l_{3,1} = r_1, \quad l_{3,2} = r_1 + r_2, \quad l_{3,3} = r_1 + r_2 + r_3$ $l_{4,1} = r_1, \quad l_{4,2} = r_2, \quad l_{4,3} = r_3$
$6_2$	$(2, 1, 2, 3)$	$l_{1,1} = r_1, \quad l_{1,2} = r_2 + r_3, \quad l_{2,1} = r_1$ $l_{3,1} = l_{4,1} = r_2, \quad l_{3,2} = r_3, \quad l_{4,2} = r_1 - r_2, \quad l_{4,3} = r_2 + r_3$
$6_3$	$(1, 1, 3, 3)$	$l_{1,1} = l_{2,1} = r_2, \quad l_{3,1} = l_{4,1} = r_1$ $l_{3,2} = l_{4,2} = r_3, \quad l_{3,3} = r_2 - r_3, \quad l_{4,3} = r_2 - r_1$
$7_3$	$(2, 3, 3, 3)$	$l_{1,1} = l_{2,1} = r_2, \quad l_{1,2} = r_2 - r_1$ $l_{2,2} = l_{3,1} = r_2 - r_3, \quad l_{2,3} = l_{3,2} = r_2 - r_3 - r_4, \quad l_{3,3} = r_1$ $l_{4,1} = r_2 - r_1, \quad l_{4,2} = r_3, \quad l_{4,3} = r_4$
$7_4$	$(3, 0, 4, 4)$	$l_{1,1} = l_{4,1} = r_1 + r_2, \quad l_{1,2} = r_2 + r_3$ $l_{1,3} = l_{4,2} = r_3 + r_4$ $l_{3,1} = r_1, \quad l_{3,2} = l_{4,3} = r_2$ $l_{3,3} = l_{4,4} = r_3, \quad l_{3,4} = r_4$
$7_5$	$(2, 2, 3, 4)$	$l_{1,1} = l_{2,1} = r_3, \quad l_{1,2} = l_{4,1} = r_3 - r_4$ $l_{2,2} = l_{3,1} = r_2, \quad l_{3,2} = l_{4,2} = r_1, \quad l_{3,3} = r_4$ $l_{4,3} = r_2 - r_1$ $l_{4,4} = r_3 - r_2$
$7_6$	$(2, 1, 4, 4)$	$l_{1,1} = l_{4,1} = r_2, \quad l_{1,2} = r_3 + r_4$ $l_{2,1} = r_2 + r_3, \quad l_{3,1} = l_{4,2} = r_1, \quad l_{3,2} = r_2 - r_1$ $l_{3,3} = l_{4,3} = r_3, \quad l_{3,4} = l_{4,4} = r_4$
$7_7$	$(2, 1, 4, 4)$	$l_{1,1} = r_1 + r_2, \quad l_{1,2} = r_3 + r_4$ $l_{2,1} = r_2 + r_3, \quad \forall j \in \{1, 2, 3, 4\}, \quad l_{3,j} = l_{4,j} = r_j$

Fig. 4 Parameters of the Kashaev invariants

where, for  $\mathbf{L} = (L_1, \dots, L_m) \in \mathbb{Z}_{\geq 0}^m, \mathbf{s} = (s_1, \dots, s_m) \in (\mathbb{Z}/q\mathbb{Z})^m$  we write

$$\mathcal{J}_K(\gamma, x; \mathbf{L}, \mathbf{s}) = \sum_{\substack{0 \leq r_1, \dots, r_m < k, \\ [r_i d/k] = L_i, \forall i \\ r_i \equiv s_i \pmod{q}, \forall i}}^* \Pi_K([\gamma x]_{l_{i,j}(\mathbf{r})}).$$

By Theorem 6 and Remark 5, we have

$$[\gamma x]_r = [x]_{[rd/k]} A(p, q) \Phi_r(\{rd/k\}),$$

where for  $s \in \mathbb{Z}/q\mathbb{Z}, \lambda \in [0, 1)$ , we define

$$A(p, q) = e\left(-\frac{s(p, q)}{2} - \frac{1}{8}\right),$$

$$\Phi_s(\lambda) = \exp\left(\frac{k}{qd} \left(\mathfrak{L}(\lambda) + \frac{\pi i}{12}\right) + \frac{\pi i d}{12kq} + \mathcal{E}_s(\lambda, d/k)\right). \tag{3.4}$$

It follows that

$$\begin{aligned} \mathcal{J}_K(\gamma, x; \mathbf{L}, \mathbf{s}) &= A(p, q)^{m_1+m_4-m_2-m_3} \sum_{\substack{0 \leq r_1, \dots, r_m < k, \\ \lfloor r_i d/k \rfloor = L_i, \forall i \\ r_i \equiv s_i \pmod{q}, \forall i}}^* \Pi_K([x]_{\lfloor \ell_{i,j}(\mathbf{r})d/k \rfloor} \Phi_{\ell_{i,j}(\mathbf{s})}(\{\ell_{i,j}(\mathbf{r})d/k\})). \end{aligned}$$

Now, let  $\lambda_i = \{r_i d/k\}$ . The next lemma shows that the contribution of the terms for which  $\ell_{i,j}(\boldsymbol{\lambda}) \notin [0, 1)$  for some  $i, j$  is negligible.

**Lemma 10** *There exists  $\delta > 0$  such that*

$$\left| \Pi_K(\exp(\mathcal{L}(\ell_{i,j}(\boldsymbol{\lambda})))) \right| \leq \exp\left(\frac{\text{Vol}(K)}{2\pi} - \delta\right)$$

whenever  $\ell_{i,j}(\boldsymbol{\lambda}) \notin [0, 1)$  for some  $i, j$ .

We postpone the proof of Lemma 10 to Sect. 3.1. Since  $[x]_r \asymp 1$  for  $0 \leq r < d$ , applying (2.15) and the above lemma we obtain

$$\begin{aligned} \mathcal{J}_K(\gamma, x; \mathbf{L}, \mathbf{s}) &= A(p, q)^{m_1+m_4-m_2-m_3} \Pi_K([x]_{\ell_{i,j}(\mathbf{L})}) \mathcal{J}_K^*(\gamma, x; \mathbf{L}, \mathbf{s}) + \\ &+ O\left(k^{O(1)} \exp\left(\left(\frac{\text{Vol}(K)}{2\pi} - \delta\right) \frac{k}{qd}\right)\right), \end{aligned} \tag{3.5}$$

where

$$\mathcal{J}_K^*(\gamma, x; \mathbf{L}, \mathbf{s}) = \sum_{\substack{0 \leq r_1, \dots, r_m < k, \\ \lfloor r_i d/k \rfloor = L_i, \forall i \\ r_i \equiv s_i \pmod{q}, \forall i \\ \ell_{i,j}(\boldsymbol{\lambda}) \in [0, 1) \forall i, j}}^* \Pi_K(\Phi_{\ell_{i,j}(\mathbf{s})}(\{\ell_{i,j}(\mathbf{r})d/k\})).$$

We notice that the condition  $0 \leq \ell_{i,j}(\mathbf{r}) < k$ , which is implicit in the summation  $\sum^*$ , can be written as  $0 \leq \ell_{i,j}(\mathbf{L}) + \ell_{i,j}(\boldsymbol{\lambda}) < d$  and so, since  $\ell_{i,j}(\boldsymbol{\lambda}) \in [0, 1)$ , it is equivalent to  $0 \leq \ell_{i,j}(\mathbf{L}) < d$ . Assuming that  $\mathbf{L}$  satisfies this condition,  $\mathcal{J}_K^*(\gamma, x; \mathbf{L}, \mathbf{s})$  can be then rewritten as

$$\mathcal{J}_K^*(\gamma, x; \mathbf{L}, \mathbf{s}) = \sum_{\substack{0 \leq r_i - L_i k/d < k/d \forall i, \\ r_i \equiv s_i \pmod{q}, \forall i, \\ 0 \leq \ell_{i,j}(\mathbf{r} - k\mathbf{L}/d) < k/d \forall i, j}} \Pi_K(\Phi_{\ell_{i,j}(\mathbf{s})}(\{\ell_{i,j}(\mathbf{r})d/k\}))$$

where we used  $\{\ell_{i,j}(\mathbf{r})d/k\} = \ell_{i,j}(\mathbf{r}d/k - \mathbf{L})$  to rewrite the summation conditions. This sum is essentially the same sum which arises when taking  $d = 1, \gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ , i.e. the conjugate of the sum arising in the volume conjecture. The only difference in the general case is that  $\mathbf{r}$  is summed over a box with sides of length  $k/d$  and along arithmetic progressions modulo  $q$ . For fixed  $d, q$  these restrictions are negligible

from the analytical point of view, and the works [35–37] can be adapted. For  $z \in (\mathbb{C} \setminus \mathbb{R}) \cup (0, 1)$ , define

$$\psi_1(z) := f(z), \quad \psi_2(z) := f(1 - z), \quad \psi_3(z) := -f(z), \quad \psi_4(z) := -f(1 - z),$$

and let  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_m)$  be the solution described in [35, §5.1], [37, §3.3, §4.3, §5.3] [36, §3.3, §4.3, §5.3, §6.3, §7.3] (conjugated to agree with our definition of  $\mathcal{J}_K$ ) to the system of equations

$$\sum_{i=1}^4 \sum_{j=1}^{m_i} \kappa_{i,j}(u) \psi_i(1 - \ell_{i,j}(\boldsymbol{\mu})) = 0, \quad \forall u \in \{1, \dots, m\} \tag{3.6}$$

satisfying  $0 < \text{Re}(\ell_{i,j}(\boldsymbol{\mu})) < 1$  for all  $i, j$ . We write

$$v_i := \exp(\mu_i), \quad v_i^{1/q} := \exp(\mu_i/q).$$

It is known [35–37] that  $\mathbb{Q}(\mathbf{v}) = F_K$ , the trace field of  $K$ . It will be useful to denote

$$F_{K,q} := \mathbb{Q}(e(1/q), \mathbf{v}), \quad \tilde{F}_{K,q} := \mathbb{Q}(e(1/q), \mathbf{v}^{1/q}).$$

The following lemma will be proven in Sect. 3.2.

**Lemma 11** *Let  $\mathbf{L} \in \{0, \dots, d - 1\}^m$ ,  $\mathbf{s} \in (\mathbb{Z}/q\mathbb{Z})^m$  with  $0 \leq \ell_{i,j}(\mathbf{L}) < d$  for all  $i, j$ . Then for all  $N \geq 0$ , we have*

$$\begin{aligned} \mathcal{J}_K^*(x; \mathbf{L}, \mathbf{s}) &= \frac{1}{\mathfrak{D}^{1/2}} \left(\frac{2\pi i k}{qd}\right)^{m/2} \exp\left(\frac{\text{Vol}(K) - i \text{cs}(K)}{2\pi} \frac{k}{qd} + C(\mathbf{s})\right) \\ &\quad \left(\sum_{n=0}^N \omega_{\mathbf{s},n} \times \left(\frac{2\pi i qd}{k}\right)^n + O\left(\frac{qd}{k}\right)^{N+1}\right), \end{aligned}$$

where  $0 \neq \mathfrak{D} \in F_K$ ,  $\omega_{\mathbf{s},0} := 1$  and

$$C(\mathbf{s}) := \sum_{i=1}^4 \sum_{j=1}^{m_i} C_{i,j}(\mathbf{s}), \quad C_{i,j}(\mathbf{s}) := \sum_{g=1}^q B_1\left(\frac{\langle g\bar{p} - \ell_{i,j}(\mathbf{s}) \rangle}{q}\right) \psi_i\left(\frac{g - \ell_{i,j}(\boldsymbol{\mu})}{q}\right) \tag{3.7}$$

where  $B_1$  is the 1-st Bernoulli polynomial. Moreover, for all  $n \geq 1$ ,  $\omega_{\mathbf{s},n} \in \tilde{F}_{K,q}$ , and for all  $\sigma \in \text{Gal}(\tilde{F}_{K,q}/F_{K,q})$ , we have  $\sigma(\omega_{\mathbf{s},n}) = \omega_{\mathbf{s}-\bar{p}\mathbf{u},n}$  if  $\sigma$  is given by

$$\sigma(v_i^{1/q}) = v_i^{1/q} e(u_i/q) \quad (1 \leq i \leq m) \tag{3.8}$$

for some  $u_1, \dots, u_m \in \mathbb{Z}$ .



Applying Lemma 11, by (3.3) and (3.5), and recalling the condition  $0 \leq \ell_{i,j}(\mathbf{L}) < d$  for all  $i, j$ , we obtain

$$\frac{\mathcal{J}_K(\gamma x)}{\mathcal{J}_K(x)} = \frac{A(p, q)^{m_1+m_4-m_2-m_3}}{\mathfrak{D}^{1/2}} \left(\frac{k}{d}\right)^{\frac{3}{2}} e^{(-m/8)q^{-m/2}} \exp\left(\frac{\text{Vol}(K) - i \text{cs}(K)}{2\pi} \frac{k}{qd}\right) \\ \times \sum_{s \pmod{q}} \exp(C(s)) \left(\sum_{n=0}^N \omega_{s,n} \left(\frac{2\pi i q d}{k}\right)^n + O\left(\frac{1}{k}\right)^{N+1}\right),$$

for all  $N \geq 0$ .

Now,  $6qs(p, q)$  is an integer, so  $A(p, q)$  is a  $24q$  root of unity, and letting  $\nu_K = -m_1 - m_4 + m_2 + m_3$ , we have  $\nu_K \equiv m + 1 \pmod{2}$ . We deduce

$$e\left(\frac{\nu_K}{2} s(p, q) + \frac{\nu - m}{8}\right) = \pm e\left(\frac{\nu_K}{2} s(p, q) + \frac{1}{8}\right)\omega,$$

where  $\omega \in \{1, i\}$  is independent of  $\alpha$  and  $\nu_K$  is as in the following table.

Keeping track of the factor  $i$  implicit in  $\frac{2\pi}{h} = -i \frac{k}{dq}$ , the proof of Theorem 1 follows from the following lemma, upon possibly multiplying the value of  $\mathfrak{D}$  by  $-1$ .

**Lemma 12** *There exists  $0 \neq U \in F_K$  such that defining*

$$S_n := U^{1/2} \sum_{s \pmod{q}} \exp(C(s))\omega_{s,n}, \quad n \in \mathbb{N}_{\geq 0},$$

we have  $S_n^q \in F_{K,q}$  for all  $n \geq 0$ . If moreover  $S_n \neq 0$ , then for all  $n' \geq 0$  we have  $S_n^{-1}S_{n'} \in F_{K,q}$ .

Note that in Theorems 2.2 and 2.6 of [15], similar computations are carried out for coefficients of power series constructed by a different process, which are conjectured to match those in the modularity conjecture.

**Proof** For  $x \in (-1, 1)$  we have  $B_1(\langle x \rangle) = x - \frac{1}{2} + \mathbf{1}_{x \leq 0}$ . Thus,

$$C_{i,j}(s) = \sum_{g=1}^q B_1\left(\frac{\langle g - \ell_{i,j}(s) \rangle}{q}\right) \psi_i\left(\frac{\langle gp \rangle - \ell_{i,j}(\boldsymbol{\mu})}{q}\right) \\ = \sum_{g=1}^{\langle \ell_{i,j}(s) \rangle} \psi_i\left(\frac{\langle gp \rangle - \ell_{i,j}(\boldsymbol{\mu})}{q}\right) + \sum_{g=1}^q \psi_i\left(\frac{\langle gp \rangle - \ell_{i,j}(\boldsymbol{\mu})}{q}\right) \left(\frac{g}{q} - \frac{\langle \ell_{i,j}(s) \rangle}{q} - \frac{1}{2}\right) \\ = \sum_{g=1}^{\langle \ell_{i,j}(s) \rangle} \psi_i\left(\frac{\langle gp \rangle - \ell_{i,j}(\boldsymbol{\mu})}{q}\right) + \sum_{g=1}^q \psi_i\left(\frac{\langle gp \rangle - \ell_{i,j}(\boldsymbol{\mu})}{q}\right) \frac{g}{q} \\ - \psi_i(1 - \ell_{i,j}(\boldsymbol{\mu})) \left(\frac{\langle \ell_{i,j}(s) \rangle}{q} + \frac{1}{2}\right)$$

where in the last equality we used (2.9). Then,

$$C_{i,j}(s) = D_{i,j}(s) + \frac{1}{q}D'_{i,j} - \psi_i(1 - \ell_{i,j}(\boldsymbol{\mu}))\left(\frac{\ell_{i,j}(s)}{q} + \frac{1}{2}\right), \tag{3.9}$$

with

$$D_{i,j}(s) := \sum_{g=1}^{\langle \ell_{i,j}(s) \rangle} \psi_i\left(\frac{\langle gp \rangle - \ell_{i,j}(\boldsymbol{\mu})}{q}\right) + \psi_i(1 - \ell_{i,j}(\boldsymbol{\mu}))\frac{\ell_{i,j}(s) - \langle \ell_{i,j}(s) \rangle}{q}, \tag{3.10}$$

$$D'_{i,j} := \sum_{g=1}^q \psi_i\left(\frac{\langle gp \rangle - \ell_{i,j}(\boldsymbol{\mu})}{q}\right)g. \tag{3.11}$$

It follows that

$$\begin{aligned} & \sum_{s \pmod q} \exp(C(s))\omega_{s,n} \\ &= \exp\left(\frac{1}{q} \sum_{i=1}^4 \sum_{j=1}^{m_i} D'_{i,j} - \frac{1}{2}E\right) \sum_{s \pmod q} \exp\left(\sum_{i=1}^4 \sum_{j=1}^{m_i} D_{i,j}(s)\right)\omega_{s,n} \end{aligned} \tag{3.12}$$

where we used that, by (3.6),

$$E = 2 \sum_{i=1}^4 \sum_{j=1}^{m_i} \psi_i(1 - \ell_{i,j}(\boldsymbol{\mu}))\left(\frac{\ell_{i,j}(s)}{q} + \frac{1}{2}\right) = \sum_{i=1}^4 \sum_{j=1}^{m_i} \psi_i(1 - \ell_{i,j}(\boldsymbol{\mu}))$$

is independent of  $s$ . In particular, writing  $U = \exp(E)$ , by the definition of  $\psi_i$ , (3.10), (3.11) and Lemma 11, we have  $S_n^q \in \tilde{F}_{K,q}$ . The extension  $\tilde{F}_{K,q}/F_{K,q}$  is Galois and  $\text{Gal}(\tilde{F}_{K,q}/F_{K,q})$  consists of automorphisms of the form (3.8). Thus, it suffices to show that  $S_n^q$  is invariant under any such automorphism  $\sigma$ . Now, by Lemma 11,

$$\sigma(S_n^q) = \exp\left(\frac{1}{q} \sum_{i=1}^4 \sum_{j=1}^{m_i} D'^{\sigma}_{i,j}\right) \sum_{s \pmod q} \exp\left(\sum_{i=1}^4 \sum_{j=1}^{m_i} D^{\sigma}_{i,j}(s)\right)\omega_{s-\bar{p}\mathbf{u},n}$$

where

$$\begin{aligned} D^{\sigma}_{i,j}(s) &:= \sum_{g=1}^{\langle \ell_{i,j}(s) \rangle} \psi_i\left(\frac{\langle gp - \ell_{i,j}(\mathbf{u}) \rangle - \ell_{i,j}(\boldsymbol{\mu})}{q}\right) - \psi_i(1 - \ell_{i,j}(\boldsymbol{\mu}))\frac{\ell_{i,j}(s) - \langle \ell_{i,j}(s) \rangle}{q}, \\ (D'_{i,j})^{\sigma} &:= \sum_{g=1}^q \psi_i\left(\frac{\langle gp - \ell_{i,j}(\mathbf{u}) \rangle - \ell_{i,j}(\boldsymbol{\mu})}{q}\right)g. \end{aligned} \tag{3.13}$$

The same computation as above gives

$$D_{i,j}^\sigma(s) + \frac{1}{q} D'_{i,j}{}^\sigma - \psi_i(1 - \ell_{i,j}(\boldsymbol{\mu})) \left( \frac{\ell_{i,j}(s)}{q} + \frac{1}{2} \right) = C_{i,j}^\sigma(s) \quad (3.14)$$

where

$$C_{i,j}^\sigma(s) := \sum_{g=1}^q B_1 \left( \frac{\langle g - \ell_{i,j}(s) \rangle}{q} \right) \psi_i \left( \frac{\langle gp - \ell_{i,j}(\mathbf{u}) \rangle - \ell_{i,j}(\boldsymbol{\mu})}{q} \right) \quad (3.15)$$

so that one finds

$$\sigma(S_n^q) = \exp(-qE/2) \left( \sum_{s \pmod{q}} \exp \left( \sum_{i=1}^4 \sum_{j=1}^{m_i} C_{i,j}^\sigma(s) \right) \omega_{s - \bar{p}\mathbf{u},n} \right)^q.$$

By the change of variables  $g \rightarrow g + \bar{p}\ell_{i,j}(\mathbf{u})$  one obtains  $C_{i,j}^\sigma(s) = C(s - \bar{p}\mathbf{u})$ , so that, after the change of variables  $s \rightarrow s + \bar{p}\mathbf{u}$ , one obtains  $\sigma(S_n^q) = S_n^q$ , and so  $S_n^q \in F_{K,q}$ , as desired.

Now, assume  $S_{n'} \neq 0$ . By (3.12) we have

$$S_n^{-1} S_{n'} = \left( \sum_{s \pmod{q}} \exp \left( \sum_{i=1}^4 \sum_{j=1}^{m_i} D_{i,j}(s) \right) \omega_{s,n} \right)^{-1} \sum_{s \pmod{q}} \exp \left( \sum_{i=1}^4 \sum_{j=1}^{m_i} D_{i,j}(s) \right) \omega_{s,n'}.$$

and so  $S_n^{-1} S_{n'} \in \tilde{F}_{K,q}$ . Moreover, given an automorphism  $\sigma$  as in (3.8), one shows as above that  $\sigma(S_n^{-1} S_{n'}) = S_n^{-1} S_{n'}$  and so  $S_n^{-1} S_{n'} \in F_{K,q}$ .  $\square$

**Remark 6** The constant term in Theorem 1 can be worked out from the arguments above as an explicit product of algebraic numbers. In the case  $K = 4_1$ , we obtain

$$\begin{aligned} C_{4_1}(\alpha) D_{4_1,0}(\alpha) &= c \delta_{4_1}^{-1/2} \Lambda_{4_1,\alpha}^{1/c}, \\ \delta_{4_1} &= i\sqrt{3}, \\ \Lambda_{4_1,\alpha}^{1/c} &= \left( \prod_{g=1}^c |\omega_g|^{2g/c} \right) \sum_{r=1}^c \prod_{g=1}^r |\omega_g|^2, \end{aligned}$$

where  $\omega_g = 1 - e(g\alpha - \frac{5}{6c})$ . In the case of  $K = 5_2$ , let  $\tau \approx 0.665 + 0.562i$  solve  $\tau^3 - \tau + 1 = 0$ , and let  $\mu_1 \approx 0.224 + 0.045i$  and  $\mu_2 \approx 0.164 - 0.067i$  be such

that  $e(\mu_1) = \tau^2$  and  $e(\mu_2) = \tau^2 + \tau$ . Then

$$\begin{aligned}
 C_{5_2}(\alpha)D_{5_2,0}(\alpha) &= e\left(\frac{s(\alpha)}{2}\right)c^{1/2}\delta_{5_2}^{-1/2}\Lambda_{5_2,\alpha}^{1/c}, \\
 \delta_{5_2} &= 3\tau - 2\tau^2, \\
 \Lambda_{5_2,\alpha}^{1/c} &= e\left(\mu_1\frac{c+1}{2c}\right)\left(\prod_{g=1}^c\omega_g^{-g/c}\vartheta_g^{-2g/c}\right)\times \\
 &\quad \times \sum_{r_1,r_2=1}^c e\left(\frac{\mu_1(r_1+r_2)+\mu_2r_1}{c} + \frac{r_1}{2} - \frac{\alpha}{2}r_1(1+r_1+2r_2)\right) \\
 &\quad \times \left(\prod_{g=1}^{r_1}\omega_g^{-1}\right)\left(\prod_{g=1}^{r_2}\vartheta_g^{-2}\right),
 \end{aligned}$$

where  $\omega_g = 1 - e(-g\alpha + \frac{\mu_1}{c})$  and  $\vartheta_g = 1 - e(g\alpha - \frac{\mu_2}{c})$ , and the logarithms are taken with principal determination. Note that  $F_{5_2} = \mathbb{Q}(\tau)$  (see [35]).

### 3.1 Proof of Lemma 10

For  $\lambda \in \mathbb{R}$ , let  $\Lambda$  denote the Lobachevsky function

$$\Lambda(\lambda) := -\operatorname{Re}\left(\mathcal{L}(\lambda)\right) = -\int_0^{\{\lambda\}} \log(2 \sin(\pi t)) dt.$$

where the last equality follows by (2.8) and (2.10). Note that  $\Lambda$  is 1-periodic and odd. We need to bound

$$W_K(\lambda) := -\sum_{i=1}^2 \sum_{j=1}^{m_i} \Lambda(\ell_{i,j}(\lambda)) + \sum_{i=3}^4 \sum_{j=1}^{m_i} \Lambda(\ell_{i,j}(\lambda))$$

for all  $\lambda \in [0, 1)^m$  such that  $\ell_{i,j}(\lambda) \notin [0, 1)$  for some  $(i, j)$ . Define

$$M = \Lambda(1/6) \in [0.16, 0.162].$$

We will require the following simple inequalities: for  $\alpha, \beta \in [0, 1]$ ,

$$|\Lambda(\alpha)| \leq M, \tag{3.16}$$

$$\Lambda(\alpha) \leq 0, \quad (\alpha \geq \frac{1}{2}), \tag{3.17}$$

$$2(\Lambda(\alpha) + \Lambda(\beta)) - \Lambda(\alpha + \beta) \leq 4\Lambda(\frac{1}{4}) < 0.59, \quad (\alpha + \beta \leq 1), \tag{3.18}$$

$$2(\Lambda(\alpha) + \Lambda(\beta)) - \Lambda(\alpha + \beta) \leq M, \quad (\alpha + \beta \geq 1). \tag{3.19}$$

$$2(\Lambda(\alpha) + \Lambda(\beta)) - \Lambda(\alpha + \beta) \leq 0.45, \quad (\alpha + \beta \leq 1, \text{ and } \alpha \geq \frac{1}{2}). \tag{3.20}$$

$$\Lambda(\alpha) - \Lambda(\beta) \leq 0.23, \quad (\alpha \leq \frac{1}{2} \leq \beta, \text{ and } \beta \leq 2\alpha), \tag{3.21}$$

The bound (3.21) is proved by optimizing at  $\beta = \min(\frac{5}{6}, 2\alpha)$ . The maximum is achieved at a point  $\alpha$  where  $\rho = \sin(\pi\alpha)$  solves  $\rho(1 - \rho^2) = \frac{1}{8}$ . Similarly, the bound (3.20) is proved by maximizing at  $\alpha$  for which  $\rho = \cos(\pi\alpha)$  solves  $\rho = 2(1 - \rho^2)$ .

We consider the situation case by case; in each case, we work under the extra assumption that for some  $(i, j)$ , we have  $\ell_{i,j}(\lambda) \notin [0, 1)$ .

- Case  $K = 5_2$ . We have

$$W_K(\lambda) = \Lambda(\lambda_1) + 2\Lambda(\lambda_2).$$

A bound of  $2M \leq \frac{\text{Vol}(K)}{2\pi} - 0.12$  is enough. Assume  $\lambda_1 + \lambda_2 \geq 1$ . Then  $\lambda_i \geq \frac{1}{2}$  for some  $i \in \{1, 2\}$ , so that using (3.16) and (3.17), we get

$$W_K(\lambda) \leq 2M$$

- Case  $K = 6_1$ . We have

$$W_K(\lambda) = 2\Lambda(\lambda_1) + \Lambda(\lambda_2) + \Lambda(\lambda_3).$$

A bound of  $3M \leq \frac{\text{Vol}(K)}{2\pi} - 0.01$  is enough. Thus, by (3.17), we may assume  $\lambda_i \leq \frac{1}{2}$  for all  $i \in \{1, 2, 3\}$ . Assume  $\lambda_1 + \lambda_2 + \lambda_3 \geq 1$ . Then by concavity of  $\Lambda$  on  $[0, \frac{1}{2}]$ , we have  $W_K(\lambda) \leq 2\Lambda(\lambda_1) + 2\Lambda(\frac{1}{2}(\lambda_2 + \lambda_3))$ . The bound (3.21) can then be applied with  $(\alpha, \beta) = (\frac{1}{2}(\lambda_2 + \lambda_3), 1 - \lambda_1)$ , and yields  $W_K(\lambda) \leq 0.46 \leq 3M$ . We find in all cases

$$W_K(\lambda) \leq 3M.$$

- Case  $K = 6_2$ . We have

$$W_K(\lambda) = -2\Lambda(\lambda_1) + 2\Lambda(\lambda_2) + \Lambda(\lambda_3) + \Lambda(\lambda_1 - \lambda_2).$$

A bound of  $4M \leq \frac{\text{Vol}(K)}{2\pi} - 0.05$  is enough. Because of (3.17), we may assume  $\lambda_1 > \frac{1}{2}$  and  $\lambda_2 \leq \frac{1}{2}$ . Then the case  $\lambda_1 \leq \lambda_2$  is excluded, and we may assume  $\lambda_2 + \lambda_3 \geq 1$ . Then  $\lambda_3 \geq \frac{1}{2}$ , and so by (3.17) and (3.18) (with  $(\alpha, \beta) = (1 - \lambda_1, \lambda_2)$ ), we obtain  $W_K(\lambda) \leq 4\Lambda(\frac{1}{4})$ . We obtain in all cases

$$W_K(\lambda) \leq 4M.$$

- Case  $K = 6_3$ . We have

$$W_K(\lambda) = -2\Lambda(\lambda_2) + 2\Lambda(\lambda_1) + 2\Lambda(\lambda_3) + \Lambda(\lambda_2 - \lambda_3) + \Lambda(\lambda_2 - \lambda_1).$$

A bound of  $5M \leq \frac{\text{Vol}(K)}{2\pi} - 0.09$  is enough. By symmetry, we may assume  $\lambda_2 \leq \lambda_3$ . Suppose first  $\lambda_2 \geq \frac{1}{2}$ . Then  $\lambda_2 - \lambda_3 \in [-\frac{1}{2}, 0]$ , so by (3.17),  $\Lambda(\lambda_2 - \lambda_3) \leq 0$

and  $\Lambda(\lambda_3) \leq 0$ . We deduce  $W_K(\boldsymbol{\lambda}) \leq 5M$  by (3.16). Suppose on the other hand that  $\lambda_2 < \frac{1}{2}$ . If  $\lambda_3 \geq \frac{1}{2}$ , then by (3.18), (3.19) with  $(\alpha, \beta) = (\lambda_1, 1 - \lambda_2)$ , we find  $W_K(\boldsymbol{\lambda}) \leq 4\Lambda(\frac{1}{4}) + M \leq 5M$ . If, finally,  $\lambda_3 < \frac{1}{2}$ , then by (3.17) we have  $\Lambda(\lambda_2 - \lambda_3) \leq 0$ , so that  $W_K(\boldsymbol{\lambda}) \leq 5M$ . In all cases, we find

$$W_K(\boldsymbol{\lambda}) \leq 5M.$$

- Case  $K = 7_3$ . We have

$$W_K(\boldsymbol{\lambda}) = -2\Lambda(\lambda_2) + \Lambda(\lambda_1) + \Lambda(\lambda_3) + \Lambda(\lambda_4).$$

A bound of  $4M \leq \frac{\text{Vol}(K)}{2\pi} - 0.08$  is enough. Thus, by (3.17), we may assume  $\lambda_j < \frac{1}{2}$  for all  $j \in \{1, 3, 4\}$ , and  $\lambda_2 > \frac{1}{2}$ . Assume that  $\lambda_2 \leq \lambda_3 + \lambda_4$ . By (3.17) and the concavity of  $\Lambda$  on  $[0, \frac{1}{2}]$ , we have

$$W_K(\boldsymbol{\lambda}) \leq -2\Lambda(\lambda_2) + \Lambda(\lambda_3) + \Lambda(\lambda_4) + M \leq 2\left(\Lambda\left(\frac{\lambda_3 + \lambda_4}{2}\right) - \Lambda(\lambda_2)\right) + M.$$

By (3.21) with  $\beta = \frac{1}{2}(\lambda_3 + \lambda_4)$ , we obtain  $W_K(\boldsymbol{\lambda}) \leq 0.46 + M \leq 4M$ . We deduce that in all cases,

$$W_K(\boldsymbol{\lambda}) \leq 4M.$$

- Case  $K = 7_4$ . We have

$$W_K(\boldsymbol{\lambda}) = -\Lambda(\lambda_2 + \lambda_3) + \Lambda(\lambda_1) + 2\Lambda(\lambda_2) + 2\Lambda(\lambda_3) + \Lambda(\lambda_4).$$

A bound of  $5M \leq \frac{\text{Vol}(K)}{2\pi} - 0.01$  is enough. By (3.17) and (3.16) we may assume without loss of generality  $\lambda_2, \lambda_3 < \frac{1}{2}$ . Moreover, by (3.18) (note that  $0.59 \leq 4M$ ), we always have  $W_K(\boldsymbol{\lambda}) \leq 4M + \Lambda(\lambda_1) + \Lambda(\lambda_4)$ , and so we may assume  $\lambda_1, \lambda_4 < \frac{1}{2}$  as well. But then, neither of the cases  $\lambda_i + \lambda_j \geq 1$  can occur for  $(i, j) \in \{(1, 2), (2, 3), (3, 4)\}$ . We find in all cases that

$$W_K(\boldsymbol{\lambda}) \leq 5M.$$

- Case  $K = 7_5$ . We have

$$W_K(\boldsymbol{\lambda}) = -2\Lambda(\lambda_3) + 2\Lambda(\lambda_1) + \Lambda(\lambda_4) + \Lambda(\lambda_2 - \lambda_1) + \Lambda(\lambda_3 - \lambda_2).$$

A bound of  $6M \leq \frac{\text{Vol}(K)}{2\pi} - 0.05$  is enough. We may assume  $\lambda_3 > \frac{1}{2}$  and  $\lambda_1, \lambda_4 < \frac{1}{2}$ . Then the case  $\lambda_3 \leq \lambda_4$  is excluded. Assume next  $\lambda_2 \leq \lambda_1$ . Then  $\lambda_2 - \lambda_1 \in (-\frac{1}{2}, 0]$ , so that  $\Lambda(\lambda_2 - \lambda_1) \leq 0$ . Similarly, if  $\lambda_3 \leq \lambda_2$ , then  $\lambda_2 > \frac{1}{2}$ , and  $\Lambda(\lambda_3 - \lambda_2) \leq 0$ . In all cases, we find by (3.16) and (3.17) that

$$W_K(\boldsymbol{\lambda}) \leq 6M.$$

- Case  $K = 7_6$ . We have

$$W_K(\boldsymbol{\lambda}) = -\Lambda(\lambda_3 + \lambda_4) - \Lambda(\lambda_2 + \lambda_3) + 2\Lambda(\lambda_1) + \Lambda(\lambda_2 - \lambda_1) \\ + 2\Lambda(\lambda_3) + 2\Lambda(\lambda_4).$$

A bound of  $0.45 + 4M \leq \frac{\text{Vol}(K)}{2\pi} - 0.03$  is enough; note that  $0.59 + 3M < 0.45 + 4M$ . In particular, we may assume that  $\lambda_1 < \frac{1}{2}$ , since otherwise, by (3.17) and (3.18), (3.19), we get  $W_K(\boldsymbol{\lambda}) \leq 0.59 + 2M$ . Assume first  $\lambda_3 + \lambda_4 \geq 1$ . Then by (3.19), we obtain  $W_K(\boldsymbol{\lambda}) \leq 5M$ , which is acceptable. Next, assume that  $\lambda_2 + \lambda_3 \geq 1$ . If  $\lambda_3 \leq \frac{1}{2}$ , then  $\lambda_2 + \lambda_3 \in [1, \frac{3}{2}]$  and by (3.19) we have  $W_K(\boldsymbol{\lambda}) \leq 0.59 + 3M$ . If on the other hand  $\lambda_3 > \frac{1}{2}$ , then by (3.20), we get  $W_K(\boldsymbol{\lambda}) \leq 0.45 + 4M$ . Both bounds are acceptable. Finally, assume that  $\lambda_2 \leq \lambda_1$ . Then  $\lambda_2 - \lambda_1 \in (-\frac{1}{2}, 0]$ , and so  $\Lambda(\lambda_2 - \lambda_1) \leq 0$ , and we again obtain  $W_K(\boldsymbol{\lambda}) \leq 0.59 + 3M$ . We find in all cases that

$$W_K(\boldsymbol{\lambda}) \leq 0.45 + 4M.$$

- Case  $K = 7_7$ . We have

$$W_K(\boldsymbol{\lambda}) = 2(\Lambda(\lambda_1) + \Lambda(\lambda_2) + \Lambda(\lambda_3) + \Lambda(\lambda_4)) - \Lambda(\lambda_1 + \lambda_2) \\ - \Lambda(\lambda_2 + \lambda_3) - \Lambda(\lambda_3 + \lambda_4).$$

Assume  $\lambda_i + \lambda_j \geq 1$  for some  $(i, j) \in \{(1, 2), (2, 3), (3, 4)\}$ . Then by (3.19) and (3.16), we have

$$W_K(\boldsymbol{\lambda}) \leq 7M \leq \frac{\text{Vol}(K)}{2\pi} - 0.08.$$

Summarizing the above, we find that in all cases considered for  $K$ , there holds

$$E_K := \sup\{W_K(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in [0, 1]^n, \exists i, j, \ell_{i,j}(\boldsymbol{\lambda}) \notin [0, 1]\} \leq \frac{\text{Vol } K}{2\pi} - 0.01. \quad (3.22)$$

This proves Lemma 10.

**Remark 7** The analogue of Lemma 10 for the knot  $7_2$  is false as stated. It is likely that this obstacle can be lifted by processing the contour integral arguments underlying Lemma 11 more carefully (see [36, Remark 8.1]). For sake of clarity, and since our main point is rather to stress how the modularity conjecture can be reduced to the arithmeticity conjecture, we chose to omit the case  $K = 7_2$ .

### 3.2 Proof of Lemma 11

**Remark 8** We have  $\overline{\mathfrak{L}(\lambda)} = -\mathfrak{L}(1 - \lambda)$  for  $\lambda \in [0, 1)$  and  $\overline{f(z)} = f(1 - \bar{z})$  for  $z \in (\mathbb{C} \setminus \mathbb{R}) \cup (0, 1)$ . In particular we can write the conjugates of  $\Phi_r(\lambda)$  and  $g_d(\lambda)$

given in (3.4) and (2.16) ((2.14) if  $\lambda = 0$ ) as

$$\begin{aligned} \overline{\Phi_r(\lambda)} &= \exp\left(-\frac{k}{qd}\left(\mathfrak{L}(1-\lambda) + \frac{\pi i}{12}\right) - \frac{\pi i d}{12kq} + \mathcal{E}_r^*(\lambda, d/k)\right) \\ \mathcal{E}_r^*(\lambda, d/k) &:= \frac{1}{2}f\left(\frac{q - \langle pr \rangle + \lambda}{q}\right) + \sum_{g=1}^q i \int_0^\infty \left(\frac{f\left(\frac{q-g+\lambda}{q} - i\frac{dt}{k}\right)}{e\left(-\frac{g\bar{p}-r}{q}\right)e^{2\pi t} - 1} \right. \\ &\quad \left. - \frac{f\left(\frac{q-g+\lambda}{q} + i\frac{dt}{k}\right)}{e\left(\frac{g\bar{p}-r}{q}\right)e^{2\pi t} - 1}\right) dt, \quad (\lambda \neq 0) \end{aligned}$$

and  $\mathcal{E}_r^*(0, d/k) = \overline{\mathcal{E}_r(0, d/k)}$ . In particular, we can extend  $\overline{\Phi_r(\lambda)}$  to a holomorphic function of  $\lambda$  in the strip  $0 < \text{Re}(\lambda) < 1$ .

In the following lemmas we give some properties of the expansion of  $\mathcal{E}_r$  and  $\mathcal{E}_r^*$ .

**Lemma 13** Assume  $\text{Re}(\lambda) \in (\varepsilon, 1 - \varepsilon)$  with  $\varepsilon \in (0, 1/2)$  and let  $s \in \mathbb{Z}/q\mathbb{Z}$ . Then, for all  $M \geq 0$  uniformly in  $\lambda$  we have

$$\begin{aligned} \mathcal{E}_s(\lambda, d/k) &= \sum_{\ell=0}^M \left(\frac{qd}{k}\right)^\ell \mathcal{E}_{s,\ell}(\lambda) + O_\varepsilon\left(\frac{qd}{k}\right)^{M+1}, \\ \mathcal{E}_s^*(\lambda, d/k) &= \sum_{\ell=0}^M \left(\frac{qd}{k}\right)^\ell \mathcal{E}_{s,\ell}^*(\lambda) + O_\varepsilon\left(\frac{qd}{k}\right)^{M+1}, \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} \mathcal{E}_{0,\ell}(\lambda) &:= \sum_{g=1}^q f\left(\frac{g-\lambda}{q}\right) B_1\left(\frac{\langle g\bar{p}-s \rangle}{q}\right), \quad \mathcal{E}_{0,\ell}^*(\lambda) := \sum_{g=1}^q f\left(\frac{g-\lambda}{q}\right) B_1\left(\frac{\langle g\bar{p}-s \rangle}{q}\right), \\ \mathcal{E}_{s,\ell}(\lambda) &:= \frac{(-1)^\ell}{q^\ell(\ell+1)!} \sum_{g=1}^q f^{(\ell)}\left(\frac{g-\lambda}{q}\right) \tilde{B}_{\ell+1}\left(\frac{g\bar{p}-s}{q}\right), \quad \ell \geq 1, \\ \mathcal{E}_{s,\ell}^*(\lambda) &:= \frac{(-1)^\ell}{q^\ell(\ell+1)!} \sum_{g=1}^q f^{(\ell)}\left(\frac{1-g+\lambda}{q}\right) \tilde{B}_{\ell+1}\left(\frac{g\bar{p}-s}{q}\right), \quad \ell \geq 1. \end{aligned}$$

**Proof** For  $\text{Re}(\lambda) \in (\varepsilon, 1 - \varepsilon)$  we have  $\inf_{g \in \mathbb{Z}} \left\| \frac{g-\lambda}{q} \right\| \gg_\varepsilon 1$ . Thus, since

$$f'(z) = \pi(\cot(\pi z) + i), \quad f^{(v)}(z) = (-1)^{v-1}(v-1)! \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^v}, \quad (z \notin \mathbb{Z}, v \geq 2),$$

we have that  $f^{(v)}\left(\frac{g-\lambda+it}{q}\right) \ll_{q,v,\varepsilon} (1 + |t|^\varepsilon)$  for all  $v \geq 0, t \in \mathbb{R}$ . The Lemma then follows immediately by expanding in Taylor series and applying Lemma 2.  $\square$



**Lemma 14** Assume  $\operatorname{Re}(\tilde{\lambda}) \in (0, 1)$  and let  $s \in \mathbb{Z}/q\mathbb{Z}$ . Let  $h_{s,\ell}$  be either  $\mathcal{E}_{s,\ell}$  or  $\mathcal{E}_{s,\ell}^*$ . Then, for all  $\ell, M \geq 0$ , and all  $\lambda$  in a neighborhood of  $\tilde{\lambda}$ , we have

$$h_{s,\ell}(\lambda) = \sum_{v=0}^M (2\pi i)^{\ell+v} C_{s,\ell,v}(\tilde{\lambda})(\lambda - \tilde{\lambda})^v + O(|\lambda - \tilde{\lambda}|^{M+1}), \quad (3.24)$$

where, for  $(v, \ell) = (0, 0)$ ,

$$C_{0,0}(\tilde{\lambda}) = \sum_{g=1}^q f\left(\frac{g - \tilde{\lambda}}{q}\right) B_1\left(\frac{\langle g\bar{p} - s \rangle}{q}\right), \quad \text{if } h_{s,\ell} = \mathcal{E}_{s,\ell} \quad (3.25)$$

$$C_{0,0}(\tilde{\lambda}) = \sum_{g=1}^q f\left(\frac{q - g + \tilde{\lambda}}{q}\right) B_1\left(\frac{\langle g\bar{p} - s \rangle}{q}\right), \quad \text{if } h_{s,\ell} = \mathcal{E}_{s,\ell}^*. \quad (3.26)$$

Moreover, for  $(v, \ell) \neq (0, 0)$  we have  $C_{s,\ell,v}(\tilde{\lambda}) \in \mathbb{Q}(e(\frac{1}{q}), e(\frac{\tilde{\lambda}}{q}))$ ; also if

$$\sigma \in \operatorname{Gal}(\mathbb{Q}(e(\frac{1}{q}), e(\frac{\tilde{\lambda}}{q}))/\mathbb{Q}(e(\frac{1}{q}), e(\tilde{\lambda})))$$

is such that  $\sigma(e(\tilde{\lambda}/q)) = e(\tilde{\lambda} + u)/q$  for some  $u \in \{0, \dots, q-1\}$ , then

$$\sigma(C_{s,\ell,v}(\tilde{\lambda})) = C_{s-u\bar{p},\ell,v}(\tilde{\lambda}). \quad (3.27)$$

**Proof** The Eqs. (3.24)–(3.26) follow immediately by Taylor expansion. Moreover, if  $\ell \geq 1, v \geq 0$  and  $h_{r,\ell} = \mathcal{E}_{r,\ell}$  then

$$C_{r,\ell,v}(\tilde{\lambda}) = \frac{(-1)^\ell}{q^\ell(\ell+1)!} \frac{1}{q^v v!} \sum_{g=1}^q (2\pi i)^{\ell+v} \mathfrak{f}^{(\ell+v)}\left(\frac{g - \tilde{\lambda}}{q}\right) \tilde{B}_{\ell+1}\left(\frac{g\bar{p} - r}{q}\right),$$

so that  $C_{r,\ell,v}(\tilde{\lambda}) \in \mathbb{Q}(e(\frac{1}{q}), e(\frac{\tilde{\lambda}}{q}))$  since  $f'(z) = \pi(\cot(\pi z) + i) = \frac{2i}{1-e(-z)}$  and (3.27) follows by the change of variables  $g \rightarrow g + j$ . The case  $\ell = 0, v \geq 1$  and the analogous property for  $\tilde{\mathcal{E}}_{r,\ell}$  can be proven in the same way.  $\square$

**Proof of Lemma 11** As mentioned in the introduction, for the knots under consideration, the asymptotic expansion stated in Lemma 11 will be essentially reduced to a proof of the asymptotic expansion in the volume conjecture for those knots. Thus, we shall frequently refer to [35–37] where this asymptotic expansion was proven for hyperbolic knots with 5, 6 and 7 crossings. The case of the knot  $4_1$  is easier, since there is a dominant critical point on  $(0, 1)$ , and the method of stationary phase can be applied, similarly as in the proof of Theorem 2 below. Thus, we will focus here on the case  $K \neq 4_1$ . Recall also that we assume  $K \neq 7_2$ .

By Remark 8, for  $0 \leq \ell_{i,j}(\mathbf{L}) < d$  we can write  $J_K^*$  as

$$\mathcal{J}_K^*(x; \mathbf{L}, s) = \sum_{\substack{rd/k - \mathbf{L} \in \mathcal{D}, \\ r_i \equiv s_i \pmod{q}, \forall i}} \exp\left(\frac{k}{qd} V_{s,d/k}(d\mathbf{r}/k - \mathbf{L})\right),$$

where  $\mathcal{D} = \{\mathbf{n} \in [0, 1)^m \mid \ell_{i,j}(\mathbf{n}) \in [0, 1) \forall i, j\}$ ,

$$V_{s,\kappa}(\mathbf{n}) := \hat{V}(\mathbf{n}) + q\kappa \left( U_{s,\kappa}(\mathbf{n}) + \frac{\pi i(m_1 + m_4 - m_2 - m_3)}{12q} \kappa \right)$$

and

$$\begin{aligned} \hat{V}(\mathbf{n}) &= \sum_{j=1}^{m_1} \left( \mathcal{L}(\ell_{1,j}(\mathbf{n})) + \frac{\pi i}{12} \right) - \sum_{j=1}^{m_2} \left( \mathcal{L}(1 - \ell_{2,j}(\mathbf{n})) + \frac{\pi i}{12} \right) \\ &\quad - \sum_{j=1}^{m_3} \left( \mathcal{L}(\ell_{3,j}(\mathbf{n})) + \frac{\pi i}{12} \right) + \sum_{j=1}^{m_4} \left( \mathcal{L}(1 - \ell_{4,j}(\mathbf{n})) + \frac{\pi i}{12} \right), \\ U_{s,\kappa}(\mathbf{n}) &= \sum_{j=1}^{m_1} \mathcal{E}_s(\ell_{1,j}(\mathbf{n}), \kappa) + \sum_{j=1}^{m_2} \mathcal{E}_s^*(\ell_{2,j}(\mathbf{n}), \kappa) - \sum_{j=1}^{m_3} \mathcal{E}_s(\ell_{3,j}(\mathbf{n}), \kappa) \\ &\quad - \sum_{j=1}^{m_4} \mathcal{E}_s^*(\ell_{4,j}(\mathbf{n}), \kappa). \end{aligned}$$

The function  $\hat{V}(\mathbf{n})$  coincides, up to conjugation, with the limiting value of the potential function of the hyperbolic structure of the knot complement given in [35, (10)], [37, p. 297, p. 308, p. 322] and [36, p. 12, p. 26, p. 38, p. 50, p. 63]. We remark that the expressions for  $\hat{V}$  given there differ from the one we have here, however it is easy to see that the two expressions actually coincide upon using the dilogarithm identity (or the formula (2.6))

$$\mathcal{L}(1 - \lambda) + \mathcal{L}(\lambda) = -\pi i B_2(\lambda).$$

In [35, Lemma 2.1], [37, §3.2, §4.2, §5.2] and [36, §3.2, §4.2, §5.2, §6.2, §7.2] it was shown that for all knots under consideration  $\text{Re}(\hat{V})$  is smaller than  $\frac{\text{Vol}(K)}{2\pi}$  on the boundary of  $\mathcal{D}$ . More precisely, there exists a domain  $\mathcal{D}' \subset \mathcal{D}$  with  $\text{dist}(\mathcal{D}', \partial\mathcal{D}) > 0$  such that  $\text{Re}(\hat{V}(\mathbf{n})) < \frac{\text{Vol}(K)}{2\pi} - \delta'$  for all  $\mathbf{n} \in \mathcal{D} \setminus \mathcal{D}'$  and some  $\delta' > 0$ . Thus, by (2.15), we have

$$\begin{aligned} \mathcal{J}_K^*(\gamma, x; \mathbf{L}, s) &= \sum_{\substack{rd/k - \mathbf{L} \in \mathcal{D}', \\ r_i \equiv s_i \pmod{q}, \forall i}} \exp\left(\frac{k}{qd} V_{s,d/k}(d\mathbf{r}/k - \mathbf{L})\right) \\ &\quad + O\left(k^m \exp\left(\left(\frac{\text{Vol}(K)}{2\pi} - \delta'\right) \frac{k}{qd}\right)\right). \end{aligned}$$

We now apply Poisson summation formula in the form of [35, Proposition 4.6] (with  $k/d$  playing the role of  $N$  of [35]). Note that our sum are restricted to arithmetic progressions modulo  $q$ ; since  $q$  is fixed, this does not affect the argument. By [35, Lemma 5.1], [37, Lemma 3.4, 4.3 and 5.2] and [36, Lemma 3.2, 4.2, 5.2, 6.2 and 7.2] we have that  $\hat{V}(\mathbf{n}) - \frac{\text{Vol}(K)}{2\pi}$  satisfies the conditions (41)–(42) of [35, Proposition 4.6] and by [35, Remark 4.8] and Lemma 13 we can apply Proposition 4.6 of [35] to  $V_{s,L,d/k}(\mathbf{n})$  rather than  $\hat{V}(\mathbf{n})$ . We find

$$\begin{aligned} \mathcal{J}_K^*(\gamma, x; \mathbf{L}, s) &= \left(\frac{k}{qd}\right)^m \iint_{\mathcal{D}'} \exp\left(\frac{k}{qd} V_{s,d/k}(z)\right) dz \\ &\quad + O\left(\exp\left(\left(\frac{\text{Vol}(K)}{2\pi} - \delta''\right) \frac{k}{qd}\right)\right), \end{aligned} \quad (3.28)$$

for some  $\delta'' > 0$ , where the extra factor  $q^{-m}$  comes from the restriction to the congruence classes. One can then apply the saddle point method in the form of [35, Proposition 3.5 and Remark 3.6] as done in [35, p. 705–706 and §5.2] (cf. also [46]), [37, p. 297 and §3.5; p. 309 and §4.5; p. 47 and §5.5] and [36, p. 13 and §3.5; p. 26 and §4.5; p. 39 and §5.5; p. 50 and §6.5; p. 64 and §7.5]. Notice that both  $V_{s,d/k}(z)$  and the corresponding functions studied in these papers converge uniformly to  $\hat{V}$ , so the same computations apply. We then find that for all  $N \geq 1$  the first summand on the right of (3.28) is equal to

$$\begin{aligned} &\left(\frac{2\pi k}{qd}\right)^{m/2} \frac{1}{\det(-\text{Hess})^{1/2}} \exp\left(\frac{k}{qd} \hat{V}(\boldsymbol{\mu}) + C(s)\right) \\ &\quad \times \left(1 + \sum_{n=1}^N \omega_{s,n} \left(\frac{2\pi i qd}{k}\right)^n + O\left(\frac{1}{k^{N+1}}\right)\right) \end{aligned}$$

where  $\boldsymbol{\mu}$  is a critical point of  $\hat{V}(z)$  [and thus satisfies (3.6)] with  $0 < \text{Re}(\ell_{i,j}(\boldsymbol{\mu})) < 1$  for all  $i, j$ , such that  $\hat{V}(\boldsymbol{\mu}) = \frac{\text{Vol}(K) - i \text{cs}(K)}{2\pi}$  and Hess is the Hesse matrix of  $\hat{V}$  at  $\boldsymbol{\mu}$ . In particular,  $\det(-\text{Hess}) \in (\pi i)^m F_K$ . By Lemma 14 the coefficients  $\omega_{s,n}$  are in  $\tilde{F}_{K,q}$  for all  $n \geq 1$  and if  $\sigma$  is as in (3.8), then  $\sigma(\omega_{s,n}) = \omega_{s-\bar{p}\mathbf{u},n}$ . Finally, by (3.25), (3.26) we have that  $C(s)$  is as in (3.7).  $\square$

## 4 Proof of Theorems 2

We proceed in a similar way as in the proof of Theorem 1. For  $x = \bar{h}/k$ , with  $(h, k) = 1$  and  $1 \leq h < k$ , we write  $\mathcal{J}_{4_1}(x)$  as in (3.2) and we divide the sum into congruence classes modulo  $h$ , that is we write

$$\mathcal{J}_{4_1}(\bar{h}/k) = k^{\frac{3-m}{2}} \sum_{0 \leq s \leq h-1} \mathcal{J}_{4_1}(\bar{h}/k; s) \quad (4.1)$$

where, for  $0 \leq s \leq h - 1$ , we write

$$\mathcal{J}_{4_1}(\bar{h}/k; s) = \sum_{\substack{0 \leq r < k, \\ r \equiv s \pmod{h}}} |[\bar{h}/k]_r|^2. \tag{4.2}$$

with  $[\cdot]_r$  as in (3.1). We apply Theorem 7, which for  $0 \leq s < h, s \equiv r \pmod{h}$ , gives

$$|[\bar{h}/k]_r|^2 = |[\bar{k}/h]_s|^2 \Phi_s^\dagger(r/k) \exp\left(-\frac{2\pi r}{hk} c_0(\bar{k}/h) + \mathcal{E}_s^\dagger(r/k)\right) \tag{4.3}$$

for some  $\mathcal{E}_s^\dagger(r/k)$  satisfying  $|\mathcal{E}_s^\dagger(r/k)| \ll E(h, k)$  for all  $0 \leq r < k$ , where

$$E(h, k) := 1 + \log \frac{k}{h} + \frac{k}{h^2} + \max_{r'=0, \dots, h-1} \left| \sum_{1 \leq n \leq r'} \cot\left(\pi \frac{n\bar{k}}{h}\right) \frac{n}{hk} \right|,$$

(note that if  $0 \leq r_0 < h$  with  $r_0 \equiv r \pmod{q}$ , then  $\frac{r}{k} \in [\frac{r_0}{k}, 1 - \frac{h}{k}(1 - \{\frac{r_0-k}{h}\})$ ), and where for  $\lambda \in [0, 1]$ , we define

$$\Phi_s^\dagger(\lambda) = \exp\left(2\frac{k}{h} \operatorname{Re}\left(\mathcal{L}(\lambda)\right)\right) = \exp\left(\frac{k}{h} \int_0^\lambda \log(4 \sin(\pi t)^2) dt\right),$$

by (2.8) and (2.10). By positivity, it follows that

$$\mathcal{J}_{4_1}(\bar{h}/k; s) = \exp\left(O(E(h, k) + |c_0(\bar{k}/h)|/h)\right) |[\bar{k}/h]_s|^2 \sum_{\substack{0 \leq r < k \\ r \equiv s \pmod{h}}} \Phi_s^\dagger(r/k). \tag{4.4}$$

Now, the function  $\lambda \rightarrow \int_0^\lambda \log(4 \sin(\pi t)^2) dt$  is continuous on  $[0, 1]$  and it has a unique maximum in this interval, located at  $\lambda = 5/6$ . Moreover, it can be expanded in a neighborhood of this point as

$$\int_0^\lambda \log(4 \sin(\pi t)^2) dt = \frac{\operatorname{Vol}(4_1)}{2\pi} - \pi\sqrt{3}\left(\lambda - \frac{5}{6}\right)^2 + O(|\lambda - \frac{5}{6}|^3). \tag{4.5}$$

since  $\int_0^{5/6} \log(4 \sin(\pi t)^2) dt = \frac{\operatorname{Vol}(4_1)}{2\pi}$ . It follows that

$$\begin{aligned} \sum_{\substack{0 \leq r < k \\ r \equiv s \pmod{h}}} \Phi_s^\dagger(r/k) &= \sqrt{\frac{k}{h\sqrt{3}}} \exp\left(\frac{\operatorname{Vol}(4_1)}{2\pi} \frac{k}{h}\right) (1 + O(h/k)) \\ &= \exp\left(\frac{\operatorname{Vol}(4_1)}{2\pi} \frac{k}{h} + O(\log(1 + k/h))\right), \end{aligned} \tag{4.6}$$

where the first step is justified in a standard way [17, p. 517], e.g. by smoothly restricting the sum to the terms where  $r/k$  is in a neighborhood of  $5/6$ , using (4.5) and

applying the Poisson summation formula. The second step instead follows immediately by positivity.

By (4.1), (4.4) and (4.6) we then find

$$\mathcal{J}_{4_1}(\bar{h}/k) = \mathcal{J}_{4_1}(\bar{k}/h) \exp(O(E(h, k) + |c_0(\bar{k}/h)|/h)), \quad (4.7)$$

as desired.

With a similar argument we can prove the following theorem, which deals with the case where the dominating term on the right hand side of (4.3) is  $\exp(-\frac{2\pi r}{hk}c_0(\bar{k}/h))$ .

**Theorem 8** *Let  $1 \leq h \leq k$  with  $(h, k) = 1$  and assume  $c_0(\bar{k}/h) < 0$ . Then,*

$$\begin{aligned} \log |\mathcal{J}_{4_1,0}(e(\bar{h}/k))| &= \log |\mathcal{J}_{4_1,0}(e(\bar{k}/h))| - \frac{2\pi}{h} c_0(\bar{k}/h) \\ &+ O\left(\frac{k}{h} + \max_{0 \leq r' < h} \left| \sum_{1 \leq n \leq r'} \cot\left(\pi \frac{n\bar{k}}{h}\right) \frac{n}{hk} \right|\right). \end{aligned} \quad (4.8)$$

To prove (4.8) first we observe we can assume  $k > 2h$  since otherwise the result is trivial. Also, we observe that, bounding trivially  $\Phi_s^\dagger$ , one can write (4.3) as

$$|[\bar{h}/k]_r|^2 = |[\bar{k}/h]_s|^2 \exp\left(-\frac{2\pi r}{hk}c_0(\bar{k}/h) + O(E(h, k) + k/h)\right)$$

and so

$$\mathcal{J}_{4_1}(\bar{h}/k; s) = \exp(O(E(h, k) + k/h)) |[\bar{k}/h]_s|^2 \sum_{\substack{0 \leq r < k \\ r \equiv s \pmod{h}}} \exp\left(-\frac{2\pi r}{hk}c_0(\bar{k}/h)\right).$$

Now, if  $x \geq 0$ , we have  $\sum_{0 \leq n \leq m} e^{nx} = e^{mx+O(m)}$ , and thus if  $x$  is large the sum is roughly dominated by the last term. Then for  $c_0(\bar{k}/h) < 0$ , we have

$$\sum_{\substack{0 \leq r < k \\ r \equiv s \pmod{h}}} \exp\left(-\frac{2\pi r}{hk}c_0(\bar{k}/h)\right) = \exp\left(-\frac{2\pi r'}{hk}c_0(\bar{k}/h) + O(k/h)\right)$$

where  $r'$  is the maximum integer satisfying  $0 \leq r' < k$  with  $r' \equiv s \pmod{h}$ . Then,  $k-h \leq r' < k$  and so in particular  $\frac{r'}{hk}c_0(\bar{k}/h) = \frac{1}{h}c_0(\bar{k}/h) + O(\frac{1}{k}|c_0(\bar{k}/h)|)$  and the result follows.

## 5 Proof of Theorem 3

Before starting, we state some basic properties of continued fractions (see [28] for a reference). Given  $h/k \in \mathbb{Q} \cap (0, 1)$  with  $h, k \in \mathbb{N}$ ,  $(h, k) = 1$ , we denote by  $[0; b_1, \dots, b_r]$  the continued fraction expansion of  $h/k$ . Then for  $0 \leq s \leq r$  the

convergents of  $h/k$  are the fractions  $[0; b_1, \dots, b_s] = \frac{u_s}{v_s}$  with  $(u_s, v_s) = 1$  (as usual  $u_0/v_0 = 0/1, v_{-1} := 0, u_{-1} := 1$ ); the  $v_s$  are called the partial quotients. The partial quotients satisfy the bounds  $v_s \ll 2^{-s/2}, v_{r-s} \ll k2^{-s/2}$  for  $0 \leq s \leq r$ , and  $v_s/v_{s-1} \leq b_s + 1$  for  $1 \leq s \leq r$ . Also,  $r \ll \log k$ . For all  $1 \leq s \leq r$  we have  $v_s u_{s-1} - v_{s-1} u_s = (-1)^s$  and so  $\frac{\overline{v_{s-1}}}{v_s} \equiv (-1)^{s+1} \frac{u_s}{v_s} \pmod{1}$ . Moreover, if  $1 \leq h' \leq k$  is such that  $h' \equiv (-1)^{r+1} \overline{h} \pmod{k}$ , then the Euclid algorithm on  $h'$  and  $k$  can be written as

$$\begin{aligned} v_r &= k, & v_{r-1} &= h', \\ v_{\ell+1} &= b_{\ell+1} v_\ell + v_{\ell-1}, & \ell &= 0, \dots, r-1. \end{aligned} \tag{5.1}$$

The following technical result, proved in [11], will be needed in the proof of Theorem 3.

**Lemma 15** [11, Theorem 1] *Let  $1 \leq h < k$  with  $(h, k) = 1$ . Let  $v_0, \dots, v_r$  be the partial quotients of  $h/k$ . Then uniformly for  $1 \leq c \leq k$ , we have*

$$\sum_{1 \leq n \leq c} \cot\left(\pi \frac{nh}{k}\right) \frac{n}{k} \ll \sum_{m=0}^{r-1} v_{m+1} \log(v_{m+1}/v_m) + k.$$

**Proof of Theorem 3** Since  $J$  is even, applying repeatedly the reciprocity formula (1.7) and the recurrence relation (5.1), we see that for all  $0 \leq s \leq r$

$$\begin{aligned} \log \mathcal{J}_{4_1,0}(e(h/k)) &= \log \mathcal{J}_{4_1,0}(e(\overline{v_{r-1}}/v_r)) \\ &= C \sum_{\ell=s+1}^r \left( \frac{v_\ell}{v_{\ell-1}} + E(v_{\ell-1}/v_\ell) \right) + \log \mathcal{J}_{4_1,0}(e(\overline{v_{s-1}}/v_s)), \end{aligned}$$

with  $C = \frac{\text{Vol}(4_1)}{2\pi}$  and  $E$  satisfying (1.8). Now, by Lemma 15

$$\sum_{\ell=1}^r \max_{r'=0, \dots, v_{\ell-1}-1} \left| \sum_{1 \leq n \leq r'} \cot\left(\pi \frac{n\overline{v_{\ell-2}}}{v_{\ell-1}}\right) \frac{n}{v_{\ell-1}^2} \right| \ll \sum_{\ell=1}^r \sum_{m=0}^{\ell-2} \frac{v_{m+1}}{v_{\ell-1}} \log(v_{m+1}/v_m) + r,$$

since  $\frac{\overline{v_{\ell-2}}}{v_{\ell-1}} = (-1)^\ell \frac{u_{\ell-1}}{v_{\ell-1}} \pmod{1}$  and so its partial quotients are  $v_0, \dots, v_{\ell-1}$ . The second sum is  $O(1)$ , whereas changing the order of summation and using  $v_{\ell-1-n} \ll v_{\ell-1} 2^{-n/2}$ , for  $n \geq 0$ , we obtain

$$\begin{aligned} \sum_{\ell=1}^r \sum_{m=0}^{\ell-2} \frac{v_{m+1}}{v_{\ell-1}} \log(v_{m+1}/v_m) &\ll \sum_{m=0}^{r-2} \sum_{\ell=m+2}^r 2^{(m-\ell)/2} \log(v_{m+1}/v_m) \\ &\ll \sum_{m=0}^{r-2} \log(v_{m+1}/v_m). \end{aligned}$$

Now, we fix an  $\varepsilon > 1/k$  and we take  $s$  to be the least integer in  $\{1, \dots, r\}$  such that  $v_s \geq 1/\varepsilon$  and notice that, since  $v_m \gg 2^{m/2}$ , we have  $s = O_\varepsilon(1)$ . Then, the above

computations and (1.8) give

$$\begin{aligned} \log \mathcal{J}_{4_1,0}(e(h/k)) &= \sum_{\ell=s+1}^r \frac{v_\ell}{v_{\ell-1}} (C + O(\sqrt{\varepsilon})) + \sum_{\ell=1}^r O(\log(v_\ell/v_{\ell-1})) \\ &\quad + \log \mathcal{J}_{4_1,0}(e(\pm u_s/v_s)). \end{aligned}$$

since  $\frac{\bar{v}_{s-1}}{v_s} = \pm \frac{u_s}{v_s} \pmod{1}$  for  $\pm 1 := (-1)^{s-1}$ . Now, if  $s \geq 1$ , we have  $\pm u_s/v_s = \gamma(b_s)$ , where  $\gamma := \begin{pmatrix} \pm u_{s-1} & u_{s-2} \\ \pm v_{s-1} & v_{s-2} \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ . Notice that by definition of  $s$  all entries of  $\gamma$  are bounded by  $1/\varepsilon$ . Thus, by Theorem 1, we have

$$\log \mathcal{J}_{4_1,0}(e(\pm u_s/v_s)) = Cb_s + O(\log b_s) + O_\varepsilon(1) = C \sum_{\ell=1}^s b_\ell + O(\log b_s) + O_\varepsilon(1),$$

since  $s, v_{\ell-1} = O_\varepsilon(1)$  and so also  $b_\ell = O_\varepsilon(1)$  for all  $\ell \leq s-1$ . Then, since  $v_\ell/v_{\ell-1} = b_\ell + O(1)$  we find

$$\begin{aligned} \log \mathcal{J}_{4_1,0}(e(h/k)) &= (C + O(\sqrt{\varepsilon})) \sum_{\ell=1}^r (b_\ell + O(\log b_\ell)) + O(r) + O_\varepsilon(1). \end{aligned}$$

Finally, we observe that  $b_\ell + O(\log b_\ell) = b_\ell(1 + O(\sqrt{\varepsilon})) + O(1/\varepsilon)$ . Thus,

$$\begin{aligned} \log \mathcal{J}_{4_1,0}(e(h/k)) &= (C + O(\sqrt{\varepsilon})) \sum_{\ell=1}^r b_\ell + O_\varepsilon(r) \\ &= \Sigma(h/k) \left( C + O(\sqrt{\varepsilon}) + O_\varepsilon \left( \frac{r(h/k)}{\Sigma(h/k)} \right) \right). \end{aligned}$$

By hypothesis  $r(h/k)/\Sigma(h/k) \rightarrow 0$  and so (1.9) follows by letting  $\varepsilon \rightarrow 0^+$  sufficiently slowly. Equation (1.10) then follows immediately from [10, Corollary 1.4].  $\square$

**Proof of Theorem 4** We extend  $H_K$  to a function on  $\mathbb{R}_{>0}$  by setting  $H_K(x) := \lim_{y \rightarrow x, y \in \mathbb{Q}} H_K(y)$  for all  $x \notin \mathbb{R}_{>0} \setminus \mathbb{Q}$ . By hypothesis  $H_K(x)$  is well defined and it is easy to prove that  $H_K$  is continuous on  $\mathbb{R} \setminus \mathbb{Q}$ . Then, setting  $\psi(x) := H_K(x) - \frac{\text{Vol}(K)}{2\pi} \frac{1}{x} - \frac{3}{2} \log(1/x)$ , for  $x > 0$ , and  $\psi(0) = 0$ , we have that  $\psi(x)$  is bounded and continuous almost everywhere on  $[0, 1]$ . Thus, by Lebesgue's integrability condition,  $\psi$  is Riemann-integrable on  $[0, 1]$ . In particular, for all  $\varepsilon > 0$  there exist a differentiable function  $\psi_\varepsilon : [0, 1] \rightarrow \mathbb{R}$  such that  $\|\psi - \psi_\varepsilon\|_{\infty, [0,1]} \leq \varepsilon$ .

By definition  $\log \mathcal{J}_{K,0}(e(h/k)) - \log \mathcal{J}_{4_1,0}(e(k/h)) = \psi(h/k) + \frac{\text{Vol}(K)}{2\pi} \frac{k}{h} + \frac{3}{2} \log(k/h)$ . Thus, proceeding as in the proof of Theorem 3 using this formula instead

of Theorem 2, we obtain

$$\begin{aligned} \log \mathcal{J}_{4_1,0}(e(\bar{h}/k)) &= \sum_{\ell=1}^r \left( \frac{\text{Vol}(K)}{2\pi} \frac{v_\ell}{v_{\ell-1}} + \frac{3}{2} \log(v_\ell/v_{\ell-1}) + \psi(v_{\ell-1}/v_\ell) \right) \\ &= \phi_\varepsilon(h/k) + O(\varepsilon \log k), \end{aligned}$$

where

$$\phi_\varepsilon(h/k) := \sum_{\ell=1}^r \left( \frac{\text{Vol}(K)}{2\pi} \frac{v_\ell}{v_{\ell-1}} + \frac{3}{2} \log(v_\ell/v_{\ell-1}) + \psi_\varepsilon(v_{\ell-1}/v_\ell) \right).$$

Letting  $T(x) = \{1/x\}$  for  $x \in (0, 1]$ , we note that for  $2 \leq s \leq r$ , we have  $\frac{v_{s-1}}{v_s} = T^{r-s}(h'/k)$ , whereas  $\frac{v_0}{v_1} = T^{r-1}(h'/k) + \mathbf{1}_{b_1=1}$  (the contribution of  $\mathbf{1}_{b_1=1}$  being negligible). We apply [10] (Theorem 1.3 with  $\lambda = 1$ , complemented by Theorem 1.2 with  $\gamma(x) = \{1/x\}$ ), and obtain that the estimate (1.11) holds with  $\log \mathcal{J}_{4_1,0}(e(h/k))$  replaced by  $\phi_\varepsilon(h/k)$  and

$$D_K = \frac{1 - \gamma_0 - \log 2}{12/\pi^2} + \frac{3\pi}{\text{Vol}(K)} + \frac{24}{\pi \text{Vol}(K)} \int_0^1 \frac{\psi_\varepsilon(1/x) dx}{1+x},$$

and with an error term  $o_\varepsilon(N^2)$ . The result then follows by letting  $\varepsilon \rightarrow 0^+$  sufficiently slowly with respect to  $N$ , and making the change of variables  $h/k \rightarrow h'/k$  on the left hand side of (1.11). □

**Proof of Corollary 5** We prove the result in the case where  $x \in [0, 1] \setminus \mathbb{Q}$  and for  $y \rightarrow x^-$ , the other cases being analogous. Let  $x = [0; b_1, b_2, \dots]$  and let  $h/k = [0; b_1, b_2, \dots, b_{2n}, X, Y]$  for some  $X, Y \in \mathbb{N}_{>0}$ . Then,  $h/k \rightarrow x^-$  as  $n \rightarrow \infty$ , uniformly in  $X > b_{2n+1}$  and  $Y$ . We have  $k/h \equiv [0; b_2, \dots, b_{2n}, X, Y] \pmod{1}$  and so by [8, (1.2)-(1.3) and Lemma 4]

$$\frac{1}{h} c_0(\bar{k}/h) = \frac{1}{\pi} \sum_{\ell=1}^{2n+1} \frac{(-1)^\ell \log(v_{\ell-1}/v_\ell)}{v_{\ell-1}} + O(n)$$

uniformly in  $X, Y$ , where  $v_i$  denotes the partial quotient of  $[0; b_2, \dots, b_{2n}, X, Y]$ . Now, let  $B = 2 + \max_{1 \leq i \leq 2n} b_i$ . Then, for  $\ell < 2n$ , we have  $v_\ell/v_{\ell-1} < B$ , whereas  $v_{2n}/v_{2n-1} = X + O(1)$  and  $v_{2n+1}/v_{2n} = Y + O(1)$ . Also,  $v_{2n} = Xv_{2n-1} + v_{2n-2}$  and  $v_{2n-2} \leq v_{2n-1} \leq B^{2n-1}$ . It follows, that

$$\frac{\pi}{h} c_0(\bar{k}/h) = \frac{\log Y}{Xv_{2n-1} + v_{2n-2}} - \frac{\log X}{v_{2n-1}} + O(n \log B) = -\frac{\log X}{v_{2n-1}}(1 + o(1))$$



as  $n, X, Y \rightarrow \infty$  under the constraint  $YnB^{2n} = o(\log X)$ . By (4.8) and [11, Theorem 1] we then have

$$H_{4_1}^*(h/k) = -\frac{\log X}{v_{2n-1}}(1 + o(1))(1 + O(1/Y)) + O\left(Y + \frac{\log X}{Yv_{2n-1}}\right)$$

and this goes to  $-\infty$  as  $n, X, Y \rightarrow \infty$  with  $YnB^{2n} = o(\log X)$ .  $\square$

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